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General Section

Multidimensional Schinzel-type theorems for multiplicative functions near prime tuples ☆

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Assuming Dickson's conjecture, we obtain multidimensional analogues of recent results on the behavior of certain multiplicative arithmetic functions near twin-prime arguments. This is inspired by analogous unconditional theorems of Schinzel undertaken without primality assumptions.

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1. Introduction

Our aim is to generalize recent results on the behavior of certain multiplicative functions near twin-prime arguments and also several related theorems of Schinzel undertaken without primality assumptions. In particular, we obtain multidimensional Schinzel-type results for more general multiplicative functions, in which prime pairs are replaced with prime tuples and the additive offsets from the prime arguments are essentially arbitrary. Consequently, the present work subsumes and generalizes many results from [8,10,11].

Despite a flurry of recent activity [12–14,22], the existence of infinitely many twin primes is still conjectural. Consequently, results involving twin primes and, more generally, prime tuples, must rely on unproven conjectures. Dickson's conjecture is one of the weakest widely-believed conjectures that implies the twin prime conjecture [1,5,15]. It is far weaker than the celebrated Bateman–Horn conjecture, which concerns polynomials of arbitrary degree and makes asymptotic predictions [1–3].

Dickson's Conjecture. *If $f_1, f_2, \ldots, f_k \in \mathbb{Z}[t]$ are linear polynomials with positive leading coefficients and $f = f_1 f_2 \cdots f_k$ does not vanish identically modulo any prime, then $f_1(t), f_2(t), \ldots, f_k(t)$ are simultaneously prime infinitely often.*

Before stating our main results, we briefly survey some of the relevant literature. In what follows, φ denotes the Euler totient function. In 2017, Garcia, Kahoro, and Luca showed that the Bateman–Horn conjecture implies $\varphi(p-1) \geqslant \varphi(p+1)$ for a majority of twin-primes pairs $p, p+2$ and that the reverse inequality holds for a small positive proportion of the twin primes [8]. This bias disappears if only p is assumed to be prime [9]. Analogues for prime pairs were obtained in 2018 by Garcia, Luca, and Schaaff [10]. Although preliminary numerical evidence suggested that $\varphi(p+1)/\varphi(p-1)$ might remain bounded as $p, p+2$ runs over the twin primes, Garcia, Luca, Shi, and Udell proved that Dickson's conjecture implies that these quotients are dense in $[0, \infty)$ [11].

The motivation for multidimensional generalizations of these results goes back to Schinzel, who obtained many similar results without primality restrictions. For example, [18, Thm. 1] ensures that

$$\left\{ \left(\frac{\varphi(n+1)}{\varphi(n)}, \frac{\varphi(n+2)}{\varphi(n+1)}, \ldots, \frac{\varphi(n+d)}{\varphi(n+d-1)} \right) : n \in \mathbb{N} \right\} \text{ is dense in } [0,\infty)^d. \tag{1.1}$$

The same result holds with the sum-of-divisors function σ in place of φ (Schinzel quips "Theorem 2 is obtained from Theorem 1 by replacing the letter φ with σ"). The seminal result in this direction is Schinzel's 1954 observation that

$$\left\{ \frac{\varphi(n+1)}{\varphi(n)} : n = 1, 2, \ldots \right\} \text{ is dense in } [0, \infty), \tag{1.2}$$

a variant of an obscure result of Somayajulu [21]. This density result inspired later work of Schinzel, Sierpiński, Erdős, and others [6,7,17–20] (see also [16, Ch. 1]).

Before stating our main result, we require a few words about notation. In what follows, $\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of natural numbers, $\mathbb{Z}$ the set of integers, and $\mathbb{R}$ the set of real numbers. We let $\mathbb{P} = \{2, 3, 5, 7, 11, \ldots\}$ denote the set of prime numbers; the symbol p always refers to a prime number. An m-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{Z}$ is *admissible* if there does not exist a $p \in \mathbb{P}$ such that $\alpha_1, \alpha_2, \ldots, \alpha_m$ form a complete residue system modulo p. This ensures that no congruence obstruction prevents the linear polynomials $x - \alpha_1, x - \alpha_2, \ldots, x - \alpha_m$ from being simultaneously prime infinitely often.

Our main theorem is both a broad multidimensional generalization of the results of [11] and a version of Schinzel's theorem (1.1) with primality restrictions.

Theorem 1. *Let f be a positive multiplicative function such that*

(a) $\lim_{p\to\infty} f(p) = 1$, *and*
(b) $\prod_{p\in\mathbb{P}} f(p)$ *is not absolutely convergent,*

let

$$\lim_{n\to\infty} \frac{h(n+1)}{h(n)} = \kappa \in (0, \infty),$$

and let $g = fh$. For any distinct $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{Z}$ and admissible $(\beta_1, \beta_2, \ldots, \beta_m)$ with $\alpha_i \neq \beta_j$ for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m$, Dickson's conjecture implies both

$$\left\{ \left(\frac{g(n+\alpha_2)}{g(n+\alpha_1)}, \frac{g(n+\alpha_3)}{g(n+\alpha_1)}, \ldots, \frac{g(n+\alpha_d)}{g(n+\alpha_1)} \right) : n+\beta_1, n+\beta_2, \ldots, n+\beta_m \in \mathbb{P} \right\} \tag{1.3}$$

and

$$\left\{ \left(\frac{g(n+\alpha_1)}{g(n+\alpha_2)}, \frac{g(n+\alpha_2)}{g(n+\alpha_3)}, \ldots, \frac{g(n+\alpha_{d-1})}{g(n+\alpha_d)} \right) : n+\beta_1, n+\beta_2, \ldots, n+\beta_m \in \mathbb{P} \right\} \tag{1.4}$$

are dense in $[0, \infty)^{d-1}$.

Theorem 1 also holds for $m = 0$; that is, without primality restrictions. To see this, choose β such that $\alpha_i \neq \beta$ for each $1 \leqslant i \leqslant d$ and observe that if tuples of ratios of functions of n are dense in $[0, \infty)^{d-1}$ when $n + \beta$ is prime, these ratios will still be dense in $[0, \infty)^{d-1}$ without primality assumptions. However, our proof relies on Dickson's conjecture and does not adapt to the $m = 0$ case unconditionally. It is a nontrivial task to remove this assumption, and we do not attempt to do so here.

The main ingredient in the proof of Theorem 1 is the following result, which is of independent interest (despite its more technical statement) since it generalizes several results from [11] that do not fall under the umbrella of Theorem 1.

Theorem 2. *Let f be a positive multiplicative function such that*

(a) $\lim_{p\to\infty} f(p) = 1$, *and*
(b) for some infinite subset $\mathbb{S} \subseteq \mathbb{P}$, $\prod_{p\in\mathbb{S}} f(p)$ *diverges to* 0.

For distinct $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{Z}$ *and an admissible* $(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{Z}^m$ *with* $\alpha_i \neq \beta_j$ *for all* i, j, *Dickson's conjecture implies that*

$$\big\{\big(f(n+\alpha_1), f(n+\alpha_2), \ldots, f(n+\alpha_d)\big) : n+\beta_1, n+\beta_2, \ldots, n+\beta_m \in \mathbb{P}\big\}$$

is dense in $[0, r]^d$, *in which*

$$r = \min\left\{ f\left(\prod_{j=1}^{\pi(m+d)} p_j^{x_j}\right) : 0 \leqslant x_j \leqslant \lfloor \log_2 d \rfloor + 1 \right\}. \tag{1.5}$$

Here $\pi(x) = \sum_{p\leqslant x} 1$ *denotes the prime-counting function.*

This paper is organized as follows. In Section 2 we present a wide array of examples and applications of Theorems 1 and 2. We present several necessary lemmas in Section 3 before proceeding to Section 4, which concerns the proof of Theorem 2. The proof of Theorem 1 is contained in Section 5. We conclude with several remarks and open problems in Section 6.

2. Examples and applications

We demonstrate that a wide variety of known and novel results follow from Theorems 1 and 2. Since there are so many consequences of these theorems, we split the following list of examples into one-dimensional and multidimensional categories. In particular, we highlight some striking numerical examples which illustrate that our method of proof narrows down the search for suitable prime tuples to the extent that the relevant computations are feasible on a standard laptop computer.

2.1. One-dimensional results

Before we recover all of the main one-dimensional results from [11], we first direct the reader to Table 1, which contains a few curious examples. As usual, we assume the truth of Dickson's conjecture.

Example 3. Theorem 1 with $f(n) = \varphi(n)/n$, $h(n) = n$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = 0$, and $\beta_2 = 2$, implies [11, Thm. 1]:

$$\left\{\frac{\varphi(p+1)}{\varphi(p-1)} : p, p+2 \in \mathbb{P}\right\} \quad \text{is dense in } [0, \infty).$$

Table 1
Here $p, p+6, p+12, p+18$ are prime and $\varphi(p+a_2)/\varphi(p+a_1)$ closely approximates a fundamental mathematical constant. Underlined digits agree with those of the constant in question.

ξ	a_1, a_2	p	$\frac{\varphi(p+a_2)}{\varphi(p+a_1)}$
γ	$-1, 1$	956741578168649510380109489907527800001	$\underline{0.57721566490153}0\ldots$
π	5, 16	1202984018066602651149425007990 1	$\underline{3.14159265535}5768\ldots$
e	11, 16	1067848087143349818099951 91	$\underline{2.71828182}788915\ldots$

Example 4. Apply Theorem 1 to $f(n) = \sigma(n)/n$, in which $\sigma(n) = \sum_{d|n} d$, with the same h, α_i, and β_i, as in the previous example, and obtain [11, Thm. 4a]:

$$\left\{ \frac{\sigma(p+1)}{\sigma(p-1)} : p, p+2 \in \mathbb{P} \right\} \quad \text{is dense in } [0, \infty).$$

Example 5. Since $\limsup_{p\to\infty} \varphi(p+1)/\varphi(p) = \frac{1}{2}$, we do not expect a straightforward prime version of Schinzel's result (1.2) (it is known unconditionally that $\{\varphi(p+1)/\varphi(p) : p \in \mathbb{P}\}$ is dense in $[0, \frac{1}{2}]$ [11, Thm. 2]). However, we can obtain shifted Schinzel-type results with primality restrictions, such as

$$\left\{ \frac{\varphi(n+1)}{\varphi(n)} : n+7, n+9 \in \mathbb{P} \right\} \quad \text{is dense in } [0, \infty).$$

Example 6. Let $f(n) = \varphi(n)/n$ in Theorem 2 with $\alpha_1 = 1$, $\beta_1 = 0$, and $\beta_2 = 2$. Then $m + d = 3$, $\pi(m+d) = 2$, and $\lfloor \log_2 d \rfloor + 1 = 1$. Since

$$r = \min \left\{ \frac{\varphi(2 \cdot 3)}{2 \cdot 3}, \frac{\varphi(2)}{2}, \frac{\varphi(3)}{3}, \frac{1}{1} \right\} = \frac{1}{3},$$

Theorem 2 implies $\{\varphi(p+1)/(p+1) : p, p+2 \in \mathbb{P}\}$ is dense in $[0, \frac{1}{3}]$ and hence we recover [11, Thm. 3]:

$$\left\{ \frac{\varphi(p+1)}{\varphi(p)} : p, p+2 \in \mathbb{P} \right\} \quad \text{is dense in } [0, \tfrac{1}{3}].$$

Example 7. Let $f(n) = n/\sigma(n)$ and use the same parameters as in the previous example. Since $r = \min\{\frac{6}{\sigma(6)}, \frac{2}{\sigma(2)}, \frac{3}{\sigma(3)}, \frac{1}{1}\} = \frac{1}{2}$, Theorem 2 implies $\{\frac{p+1}{\sigma(p+1)} : p, p+2 \in \mathbb{P}\}$ is dense in $[0, \frac{1}{2}]$ and we recover [11, Thm. 4c]:

$$\left\{ \frac{\sigma(p+1)}{\sigma(p)} : p, p+2 \in \mathbb{P} \right\} \quad \text{is dense in } [2, \infty).$$

Table 2
Here $p+\beta_1, p+\beta_2, \ldots$ are prime and $(\frac{\varphi(p+a_2)}{\varphi(p+a_1)}, \frac{\varphi(p+a_3)}{\varphi(p+a_1)})$ closely approximates (ξ_1, ξ_2). Underlined digits agree with those of the constants in question. Note that 43, 67, and 163 are the largest Heegner numbers; 5, 8, and 13 are Fibonacci numbers; 561, 1105, and 1729 are the first three Carmichael numbers; and 196884 is the coefficient of q in the Fourier expansion of the j-invariant (as in Monstrous Moonshine).

ξ_1	ξ_2	$\beta_1, \beta_2, \ldots$	$\alpha_1, \alpha_2, \alpha_3$	p	$\frac{\varphi(p+a_2)}{\varphi(p+a_1)}$	$\frac{\varphi(p+a_3)}{\varphi(p+a_1)}$
$\sqrt{2}$	$\sqrt{3}$	0,2	43, 67, 163	751184478449 416099048649 570893527818 494096598933 189697746350 453399017762 020065304443 6211	1.4142135 623730950 488016888 50439...	1.7320508 075688772 935283112 31013...
$\frac{\sqrt{5}+1}{2}$	$\frac{\sqrt{5}-1}{2}$	0, 2	5, 8, 13	130084391444 506326722340 792832995109 955572053763 06391	1.6180339 88749894 870...	0.6180339 88749894 8557...
$\int_0^1 x^x \, dx$	$\int_0^1 \frac{1}{x^x} \, dx$	0, 10, 12, 64, 88	561, 1105, 1729	918845569650 372195012106 105368325979 588394815789 872234894985 9809	0.7834305 107121345 08863...	1.291285 99706266 35404396 9...
$e/10$	$\pi/10$	0, 2, 56, 80, 196884	314, 159, 265	961359758712 644806513809 803026043276 8517	0.2718281 831735333 2418...	0.3141592 658358261 2337...

2.2. Higher-dimensional generalizations

Theorem 1 permits higher-dimensional generalizations of the key results of [11]. These are prime analogues of Schinzel's seminal result (1.1). Table 2 displays a variety of appealing examples. In what follows, we assume the truth of Dickson's conjecture.

Example 8. Apply Theorem 1 to $f(n) = \varphi(n)/n$ and deduce that both

$$\left\{ \left(\frac{\varphi(n+\alpha_1)}{\varphi(n+\alpha_2)}, \frac{\varphi(n+\alpha_2)}{\varphi(n+\alpha_3)}, \ldots, \frac{\varphi(n+\alpha_{d-1})}{\varphi(n+\alpha_d)} \right) : n+\beta_1, n+\beta_2, \ldots, n+\beta_m \in \mathbb{P} \right\}$$

and

$$\left\{ \left(\frac{\varphi(n+\alpha_2)}{\varphi(n+\alpha_1)}, \frac{\varphi(n+\alpha_3)}{\varphi(n+\alpha_1)}, \ldots, \frac{\varphi(n+\alpha_d)}{\varphi(n+\alpha_1)} \right) : n+\beta_1, n+\beta_2, \ldots, n+\beta_m \in \mathbb{P} \right\}$$

are dense in $[0,\infty)^{d-1}$ for any distinct $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{Z}$ and admissible $(\beta_1, \beta_2, \ldots, \beta_m)$ with $\alpha_i \neq \beta_j$ for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m$. These are prime analogues of Schinzel's theorem (1.1). In a similar manner, these results hold for σ as well.

Example 9. Let

$$f(n) = \exp\left(\sum_{p\in\mathbb{P}} \frac{\nu_p(n)}{p}\right)$$

and $h(n) = 1$, in which ν_p is the p-adic valuation. Then $f(p) = e^{1/p} \to 1$ and

$$\prod_{p\in\mathbb{P}} f(p) = \prod_{p\in\mathbb{P}} \exp\left(\frac{1}{p}\right) = \exp\left(\sum_{p\in\mathbb{P}} \frac{1}{p}\right)$$

diverges. Then Theorem 1 implies

$$\left\{\left(\frac{f(n+\alpha_1)}{f(n+\alpha_2)}, \frac{f(n+\alpha_2)}{f(n+\alpha_3)}, \ldots, \frac{f(n+\alpha_{d-1})}{f(n+\alpha_d)}\right) : n+\beta_1, n+\beta_2, \ldots, n+\beta_m \in \mathbb{P}\right\}$$

and

$$\left\{\left(\frac{f(n+\alpha_2)}{f(n+\alpha_1)}, \frac{f(n+\alpha_3)}{f(n+\alpha_1)}, \ldots, \frac{f(n+\alpha_d)}{f(n+\alpha_1)}\right) : n+\beta_1, n+\beta_2, \ldots, n+\beta_m \in \mathbb{P}\right\}$$

are dense in $[0,\infty)^{d-1}$ for any distinct $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{Z}$ and admissible $(\beta_1, \beta_2, \ldots, \beta_m)$ with $\alpha_i \neq \beta_j$ for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m$.

3. Preliminaries

The following lemma is essentially due to Schinzel [18, Lem. 1], except that here we insist upon the extra condition $\ell_k > n_k$ and we consider $0 < C < 1$ instead of $C > 1$. We provide the proof here because of these modifications.

Lemma 10. *Let u_n denote an infinite sequence of real numbers such that*

$$\lim_{n\to\infty} u_n = 0 \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1.$$

For each $0 < C < 1$ and strictly increasing sequence n_k in $\mathbb{N}$, there exists $\ell_k \in \mathbb{N}$ such that

$$\ell_k > n_k \quad \textit{for } k = 1, 2, 3\ldots \qquad \textit{and} \qquad \lim_{k\to\infty} \frac{u_{\ell_k}}{u_{n_k}} = C.$$

Proof. For $k \in \mathbb{N}$, let $\ell_k \geqslant n_k$ be the least natural number such that

$$\frac{u_{l_k}}{u_{n_k}} \leqslant C.$$

Such a number exists because $\lim_{n\to\infty} u_n = 0$. Furthermore, $\ell_k > n_k$ because $C < u_{n_k}/u_{n_k} = 1$ by assumption. The minimality of ℓ_k ensures that

$$C < \frac{u_{\ell_k - 1}}{u_{n_k}} \tag{3.1}$$

and hence

$$C\frac{u_{\ell_k}}{u_{\ell_k-1}} = \left(C\frac{u_{n_k}}{u_{\ell_k-1}}\right)\frac{u_{\ell_k}}{u_{n_k}} < \frac{u_{\ell_k}}{u_{n_k}} \leqslant C.$$

Thus,

$$\lim_{k\to\infty}\frac{u_{\ell_k}}{u_{n_k}} = C. \quad \square$$

The next lemma is a generalization of [11, Lem. 5] (see also [4, Prop. 8.8]).

Lemma 11. *Let f be a positive multiplicative function such that $\lim_{p\to\infty} f(p) = 1$ and $\prod_{p\in\mathbb{S}} f(p)$ diverges to zero for some $\mathbb{S} \subset \mathbb{P}$. For any finite subset $\mathbb{P}' \subset \mathbb{P}$,*

$$\{f(n) : n \text{ squarefree, } p \nmid n \text{ for all } p \in \mathbb{P}'\} \quad \text{is dense in } [0, 1].$$

Proof. Let q_i denote the ith smallest prime in the infinite set $\mathbb{S}\backslash\mathbb{P}'$. Define

$$u_n = \prod_{i=1}^{n} f(q_i),$$

which tends to zero as $n \to \infty$ and satisfies

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty} f(q_{n+1}) = 1.$$

Let n_k be an increasing sequence in $\mathbb{N}$ and $0 < C < 1$. Lemma 10 provides a sequence ℓ_n in $\mathbb{N}$ such that

$$\ell_k > n_k \quad \text{for } k = 1, 2, \ldots \qquad \text{and} \qquad \lim_{k\to\infty}\frac{u_{\ell_k}}{u_{n_k}} = C.$$

Then $w_k = \prod_{i=n_k+1}^{\ell_k} q_i$ is squarefree, not divisible by any element of $\mathbb{P}'$, and satisfies

$$f(w_k) = f\left(\prod_{i=n_k+1}^{\ell_k} q_i\right) = \prod_{i=n_k+1}^{\ell_k} f(q_i) = \frac{\prod_{i=1}^{\ell_k} f(q_i)}{\prod_{i=1}^{n_k} f(q_i)} = \frac{u_{\ell_k}}{u_{n_k}} \to C. \quad \square$$

Lemma 12. *Let f be a positive multiplicative function such that $\lim_{p\to\infty} f(p) = 1$ and $\prod_{p\in\mathbb{S}} f(p)$ diverges to zero for some $\mathbb{S} \subset \mathbb{P}$. For any finite $\mathbb{P}' \subset \mathbb{P}$, the set of d-tuples $(f(w_1), f(w_2), \ldots, f(w_d))$ such that*

(a) $w_1, w_2, \ldots, w_d \in \mathbb{N}$ *are squarefree and pairwise relatively prime, and*
(b) $p \nmid w_i$ *for* $p \in \mathbb{P}'$ *and* $1 \leqslant i \leqslant d$,

is dense in $[0,1]^d$

Proof. We proceed by induction on d. If $I_1 \subset [0,1]$ is an open interval, Lemma 11 provides a squarefree w_1 such that $f(w_1) \in I_1$ and $p \nmid w_1$ for all $p \in \mathbb{P}'$. Let $I_1, I_2, \ldots, I_d \subset [0,1]$ be open intervals and suppose that there are squarefree, pairwise relatively prime $w_1, w_2, \ldots, w_{d-1}$ such that $p \nmid w_i$ for all $p \in \mathbb{P}'$ and $f(w_i) \in I_i$ for $1 \leqslant i \leqslant d-1$. Let $\mathbb{P}''$ be the union of $\mathbb{P}'$ with the set of divisors of $w_1, w_2, \ldots, w_{d-1}$. Lemma 11 provides a squarefree w_d, coprime to $w_1, w_1, \ldots, w_{d-1}$, such that $(f(w_1), f(w_2), \ldots, f(w_d)) \in I_1 \times I_2 \times \cdots \times I_d$ and $p \nmid w_d$ for each $p \in \mathbb{P}'' \supset \mathbb{P}'$. This concludes the induction. □

The next lemma provides a simple method to pass between results about sets of the form (1.4) and (1.3).

Lemma 13. *The function* $\Phi : (0,\infty)^{d-1} \to (0,\infty)^{d-1}$ *defined by*

$$\Phi(y_1, y_2, \ldots, y_{d-1}) = \left(\frac{1}{y_1}, \frac{y_1}{y_2}, \frac{y_2}{y_3}, \ldots, \frac{y_{d-2}}{y_{d-1}}\right)$$

is a homeomorphism.

Proof. Since Φ is continuous, it suffices to observe that the continuous function $\Psi : (0,\infty)^{d-1} \to (0,\infty)^{d-1}$

$$\Psi(x_1, x_2, \ldots, x_{d-1}) = \left(\frac{1}{x_1}, \frac{1}{x_1 x_2}, \frac{1}{x_1 x_2 x_3}, \ldots, \frac{1}{\prod_{i=1}^{d-1} x_i}\right)$$

inverts Φ. □

4. Proof of Theorem 2

We break the proof of Theorem 2 into a number of subsections for clarity. This organization highlights the particular parameters involved at each stage.

4.1. Initial setup and outline

Suppose f is a positive multiplicative function satisfying hypotheses (a) and (b) of Theorem 2. Let $\alpha_1, \alpha_2, \ldots \alpha_d \in \mathbb{Z}$ be distinct, let $(\beta_1, \beta_2, \ldots, \beta_m)$ be an admissible m-tuple with $\alpha_i \neq \beta_j$ for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m$. Define $L = \pi(m+d)$ and let r be given by (1.5).

It suffices to show that for each $\varepsilon > 0$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_d) \in [0, r]^d$, there is an $n \in \mathbb{N}$ such that $n + \beta_j$ is prime for $1 \leqslant j \leqslant m$ and $f(n + \alpha_i) \in (\xi_i(1-\varepsilon), \xi_i(1+\varepsilon))$ for each $1 \leqslant i \leqslant d$.

4.2. The integers $b_1, b_2, \ldots, b_L$

Since $(\beta_1, \beta_2, \ldots, \beta_m)$ is an admissible m-tuple,

$$P(t) = (t+\beta_1)(t+\beta_2)\cdots(t+\beta_m)$$

does not vanish identically modulo any prime. Consequently, for each p_j with $j = 1, 2, \ldots, L$, there is some $b_j \in \mathbb{Z}$ such that $P(b_j) \not\equiv 0 \pmod{p_j}$ and hence

$$p_j \nmid b_j + \beta_i \quad \text{for } 1 \leqslant i \leqslant m. \tag{4.1}$$

4.3. The exponents $x_{i,j}$

Let

$$s = \lfloor \log_2 d \rfloor + 1$$

and observe that $p_j^s \geqslant 2^s > d$ for each $j \in \mathbb{N}$. Since there are precisely p_j^s multiples of p_j modulo p_j^{s+1}, there is an $e_j \in \mathbb{Z}$ such that

$$\alpha_i + e_j p_j + b_j \not\equiv 0 \pmod{p_j^{s+1}} \quad \text{for } 1 \leqslant i \leqslant d.$$

Define

$$x_{i,j} = \max\left\{ y \,:\, p_j^y \,|\, \alpha_i + e_j p_j + b_j \right\} \tag{4.2}$$

and observe that $x_{i,j} \leqslant s$ for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant L$.

4.4. The intervals $I_1, I_2, \ldots, I_d$

For $i = 1, 2, \ldots, d$, define

$$I_i = \left(\xi_i \frac{1-\varepsilon}{f(\prod_{j=1}^L p_j^{x_{i,j}})}, \xi_i \frac{1+\varepsilon}{f(\prod_{j=1}^L p_j^{x_{i,j}})} \right) \cap (0,1). \tag{4.3}$$

Since $\xi_i \in [0, r]$ and $0 < r \leqslant f(\prod_{j=1}^L p_j^{x_{i,j}})$, it follows that each I_i is nonempty.

4.5. The natural numbers $w_1, w_2, \ldots, w_d$

Define

$$\mathbb{P}' = \{p_1, p_2, \ldots, p_L\} \cup \left\{ p \,:\, p \,\Big|\, \left(\prod_{i=1}^{d} \alpha_i \right) \left(\prod_{\substack{1 \leqslant i \leqslant d \\ 1 \leqslant j \leqslant m}} (\alpha_i - \beta_j) \right) \left(\prod_{1 \leqslant i < j \leqslant d} (\alpha_i - \alpha_j) \right) \right\}. \tag{4.4}$$

Lemma 12 provides pairwise relatively prime $w_1, w_2, \ldots, w_d \in \mathbb{N}$ such that $f(w_i) \in I_i$ and $p \nmid w_i$ for all $p \in \mathbb{P}'$ and $1 \leqslant i \leqslant d$.

4.6. The natural number c

Since $p_1, p_2, \ldots, p_L, w_1, \ldots, w_d$ are pairwise relatively prime, the Chinese remainder theorem yields $c \in \mathbb{N}$ such that

$$c \equiv e_j p_j + b_j \pmod{p_j^{s+1}} \qquad \text{for } 1 \leqslant j \leqslant L, \tag{4.5}$$

$$c \equiv w_i - \alpha_i \pmod{w_i^2} \qquad \text{for } 1 \leqslant i \leqslant d. \tag{4.6}$$

4.7. The polynomials

Define

$$h_0(t) = \left(\prod_{j=1}^{d} w_j^2 \right) \left(\prod_{j=1}^{L} p_j^{s+1} \right) t + c \tag{4.7}$$

and

$$h_i(t) = h_0(t) + \beta_i \qquad \text{for } 1 \leqslant i \leqslant m, \tag{4.8}$$

$$g_i(t) = \frac{h_0(t) + \alpha_i}{w_i \prod_{j=1}^{L} p_j^{x_{i,j}}} \qquad \text{for } 1 \leqslant i \leqslant d. \tag{4.9}$$

4.8. Integer coefficients

By construction, $h_1, h_2, \ldots, h_m \in \mathbb{Z}[t]$. Let us verify that $g_1, g_2, \ldots, g_d \in \mathbb{Z}[t]$. From (4.7) and (4.9), we have

$$g_i(t) = \frac{(\prod_{j=1}^{d} w_j^2)(\prod_{j=1}^{L} p_j^{s+1})}{w_i \prod_{j=1}^{L} p_j^{x_{i,j}}} t + \frac{c + \alpha_i}{w_i \prod_{j=1}^{L} p_j^{x_{i,j}}}. \tag{4.10}$$

The coefficient of t is an integer since $x_{i,j} \leqslant s$ for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant L$. For the constant term, first observe that each $w_i \mid c + \alpha_i$ by (4.6). The definition (4.2) of $x_{i,j}$ ensures that $p_j^{x_{i,j}} \mid \alpha_i + e_j p_j + b_j$ and (4.5) implies

$$\alpha_i + e_j p_j + b_j \equiv \alpha_i + c \pmod{p_j^{s+1}}. \tag{4.11}$$

Consequently, $\prod_{j=1}^{L} p_j^{x_{i,j}} \mid c + \alpha_i$. Since $p_1, p_2, \ldots, p_L, w_1, \ldots, w_d$ are pairwise relatively prime, the constant term in (4.10) is an integer. Thus, $g_1, g_2, \ldots, g_d \in \mathbb{Z}[t]$.

4.9. Nonvanishing modulo small primes

Consider

$$F(t) = \left(\prod_{i=1}^{m} h_i(t)\right)\left(\prod_{i=1}^{d} g_i(t)\right) \in \mathbb{Z}[t] \tag{4.12}$$

(the first product excludes h_0) and observe that $\deg F = m + d$. We claim that F does not vanish modulo any of $p_1, p_2, \ldots, p_L$. Since $x_{i,\ell} \leqslant s$,

$$p_\ell \;\middle|\; \frac{\left(\prod_{j=1}^{d} w_j^2\right)\left(\prod_{j=1}^{L} p_j^{s+1}\right)}{w_i \prod_{j=1}^{L} p_j^{x_{i,j}}}$$

and hence $h_0(t) \equiv c \pmod{p_\ell}$ by (4.7). The definition (4.2) of $x_{i,\ell}$ and (4.11) imply

$$x_{i,\ell} = \max\{y : p_\ell^y \,|\, (c + \alpha_i)\}$$

and therefore the constant term in (4.10) is not divisible by p_ℓ. Thus,

$$g_i(t) = \frac{h_0(t) + \alpha_i}{w_i \prod_{j=1}^{L} p_j^{x_{i,j}}} \equiv \frac{c + \alpha_i}{w_i \prod_{j=1}^{L} p_j^{x_{i,j}}} \not\equiv 0 \pmod{p_\ell}.$$

Since (4.5) implies that $c \equiv b_\ell \pmod{p_\ell}$, it follows from (4.1) that

$$h_i(t) = h_0(t) + \beta_i \equiv c + \beta_i \equiv b_\ell + \beta_i \not\equiv 0 \pmod{p_\ell}.$$

Consequently, F does not vanish modulo any of $p_1, p_2, \ldots, p_L$.

4.10. Nonvanishing modulo large primes

Suppose toward a contradiction that F vanishes identically modulo some prime $p \notin \{p_1, p_2, \ldots, p_L\}$. Observe that $p > m + d = \deg F$ since $L = \pi(m + d)$. The fully-factored presentation (4.12) ensures that some linear factor of F vanishes identically modulo p.

The definitions (4.7), (4.8), and (4.9) ensure that the leading coefficient of each linear factor of F divides $(\prod_{j=1}^{d} w_j^2)(\prod_{j=1}^{L} p_j^{s+1})$. Thus, $p \mid w_k$ for some $1 \leqslant k \leqslant d$. Our construction (4.6) of c ensures that $c \equiv w_k - \alpha_k \equiv -\alpha_k \pmod{w_k}$ and hence

$$h_0(t) \equiv c \equiv -\alpha_k \pmod{p}. \tag{4.13}$$

The construction of w_k implies $\gcd(w_k, \beta_i - \alpha_k) = 1$ since no prime in the set $\mathbb{P}'$ defined by (4.4) divides w_k. Thus, for $i = 1, 2, \ldots, m$ and all $t \in \mathbb{Z}$,

$$h_i(t) \equiv \beta_i + c \equiv \beta_i - \alpha_k \not\equiv 0 \pmod{p}.$$

Since $p \nmid w_i$ for $i \neq k$, (4.13) implies that for all $t \in \mathbb{Z}$,

$$g_i(t) = \frac{h_0(t) + \alpha_i}{w_i \prod_{j=1}^{L} p_j^{x_{i,j}}} \equiv \frac{-\alpha_k + \alpha_i}{w_i \prod_{j=1}^{L} p_j^{x_{i,j}}} \not\equiv 0 \pmod{p}$$

because $\gcd(w_i, \alpha_i - \alpha_k) = 1$ since no prime in $\mathbb{P}'$ divides w_i. Now consider the case $i = k$, for which $p \mid w_k$. Then (4.6) ensures that

$$\frac{c + \alpha_k}{w_k} \equiv 1 \pmod{w_k} \qquad \text{and hence} \qquad \frac{c + \alpha_k}{w_k} \equiv 1 \pmod{p}.$$

For all $t \in \mathbb{Z}$, (4.7) and (4.9) imply

$$g_k(t) \equiv \prod_{j=1}^{L} p_j^{-x_{i,j}} \not\equiv 0 \pmod{p}.$$

Since no linear factor of F vanishes identically, we have reached a contradiction. Consequently, F does not vanish identically modulo any $p \notin \{p_1, p_2, \ldots, p_L\}$.

4.11. Conclusion

Dickson's conjecture provides infinitely many t such that

$$\begin{aligned} &h_i(t) \text{ is prime for } 1 \leqslant i \leqslant m, \\ &g_j(t) \text{ is prime for } 1 \leqslant j \leqslant d, \\ &g_j(t) > \max\{w_j, p_L\} \text{ for } 1 \leqslant j \leqslant d. \end{aligned}$$

Let $n = h_0(t)$ for any such t. Then (4.8) and (4.9) imply

$$\begin{aligned} n + \beta_i &= h_i(t) && \text{for } 1 \leqslant i \leqslant m, \\ n + \alpha_i &= g_i(t) w_i \prod_{j=1}^{L} p_j^{x_{i,j}} && \text{for } 1 \leqslant j \leqslant d. \end{aligned}$$

Since $g_i(t)$, w_i, and $\prod_{j=1}^{L} p_j^{x_{i,j}}$ are pairwise relatively prime for each $1 \leqslant i \leqslant d$,

$$f(n + \alpha_i) = f\left(g_i(t) w_i \prod_{j=1}^{L} p_j^{x_{i,j}}\right) = f\big(g_i(t)\big) f(w_i) f\left(\prod_{j=1}^{L} p_j^{x_{i,j}}\right)$$

because f is multiplicative. Condition (a) asserts that $\lim_{p\to\infty} f(p) = 1$, so $f(g_i(t)) = 1 + o(1)$ as t increases. By definition, each $f(w_i) \in I_i$, the open interval defined by (4.3) Consequently, if t is sufficiently large

$$f(n+\alpha_i) \in f\bigg(\prod_{j=1}^{L} p_j^{x_{i,j}}\bigg) I_i = \big(\xi_i(1-\varepsilon), \xi_i(1+\varepsilon)\big)$$

for $1 \leqslant i \leqslant d$. Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\Big\{\big(f(n+\alpha_1), f(n+\alpha_2), \ldots, f(n+\alpha_d)\big) : n+\beta_1, n+\beta_2, \ldots, n+\beta_m \in \mathbb{P}\Big\}$$

is dense in $[0,r]^d$. □

5. Proof of Theorem 1

Suppose that f is a positive multiplicative function such that

(a) $\lim_{p\to\infty} f(p) = 1$, and
(b) $\prod_p f(p)$ is not absolutely convergent,

$\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{Z}$ are distinct, and $(\beta_1, \beta_2, \ldots, \beta_m)$ is an admissible m-tuple with $\alpha_i \neq \beta_j$ for all i, j. Suppose that

$$\lim_{n\to\infty} \frac{h(n+1)}{h(n)} = \kappa \in (0,\infty)$$

and define $g = fh$. Since

$$\frac{g(n+\alpha_i)}{g(n+\alpha_j)} = \frac{f(n+\alpha_i)}{f(n+\alpha_j)} \cdot \frac{h(n+\alpha_i)}{h(n+\alpha_j)},$$

and

$$\lim_{n\to\infty} \frac{h(n+\alpha_i)}{h(n+\alpha_j)} = \kappa^{\alpha_i-\alpha_j},$$

to prove the density of either (1.4) or (1.3) in $[0,\infty)^{d-1}$, it suffices to consider the case in which h is identically 1.

Condition (b) ensures that there is an $\mathbb{S} \subseteq \mathbb{P}$ such that $\prod_{p\in\mathbb{S}} f(p)$ diverges to 0 or ∞. Assume Dickson's conjecture and apply Theorem 2 to f or $1/f$, respectively, and conclude that

$$S = \big\{\big(f(n+\alpha_1), f(n+\alpha_2), \ldots, f(n+\alpha_d)\big) : n+\beta_1, n+\beta_2, \ldots, n+\beta_m \in \mathbb{P}\big\} \quad (5.1)$$

is dense in $[0,r]^d$ or $[r,\infty)^d$ for some $r>0$. By possibly considering $1/f$ in place of f, we may assume that S is dense in $[0,r]^d$. Let $(\xi_1,\xi_2,\ldots,\xi_d)\in[0,r]^d$ and set

$$\rho=\max\{\xi_1,\xi_2,\ldots,\xi_d\}.$$

Given $\varepsilon>0$, let $\delta>0$ be such that

$$1-\varepsilon<\frac{1-\delta}{1+\delta}<\frac{1+\delta}{1-\delta}<1+\varepsilon.$$

Select $x_1\in(0,r/\rho)\cap(0,r)$ and define $x_i=x_1\xi_i$ for $2\leqslant i\leqslant d$. Thus,

$$0<x_i=x_1\xi_i<\left(\frac{r}{\rho}\right)\rho=r\quad\text{for } 1\leqslant i\leqslant d$$

and hence $(x_1,x_2,\ldots,x_d)\in(0,r)^d$. Since S is dense in $[0,r]^d$, there is an $n\in\mathbb{N}$ such that $n+\beta_1,n+\beta_2,\ldots,n+\beta_m$ are prime and

$$|f(n+\alpha_i)-x_i|<\delta x_i\quad\text{for } 1\leqslant i\leqslant d.$$

Consequently,

$$\frac{f(n+\alpha_{i-1})}{f(n+\alpha_1)}<\frac{x_{i-1}(1+\delta)}{x_1(1-\delta)}=\xi_{i-1}\frac{1+\delta}{1-\delta}<\xi_{i-1}(1+\varepsilon),$$

and

$$\frac{f(n+\alpha_{i-1})}{f(n+\alpha_1)}>\frac{x_{i-1}(1-\delta)}{x_1(1+\delta)}=\xi_{i-1}\frac{1-\delta}{1+\delta}>\xi_{i-1}(1-\varepsilon)$$

for $2\leqslant i\leqslant d$. In particular,

$$\frac{f(n+\alpha_{i-1})}{f(n+\alpha_1)}\in\big((1-\varepsilon)\xi_{i-1},(1+\varepsilon)\xi_{i-1}\big)\quad\text{for } i=2,3,\ldots,d-1,$$

and hence the set (1.3) is dense in $[0,\infty)^{d-1}$. Lemma 13 provides the corresponding result for the set (1.4). This completes the proof Theorem 1. □

6. Further research

We have focused on primality constraints of the form $n+\beta_1,n+\beta_2,\ldots,n+\beta_m\in\mathbb{P}$ and simple shifts $n+\alpha_1,n+\alpha_2,\ldots,n+\alpha_d$ in the arguments of the multiplicative function. One can consider more general conditions. For this, Dickson's conjecture (which concerns only linear polynomials) no longer suffices. However, Schinzel's Hypothesis H should permit a generalization [19]. Going further, the Bateman–Horn conjecture might even provide asymptotic estimates [1–3].

Problem 14. Generalize Theorems 1 and 2 to include polynomial primality constraints $P_1(n), P_2(n), \ldots, P_m(n) \in \mathbb{P}$.

Problem 15. Generalize Theorems 1 and 2 so that the arguments $n+\alpha_1, n+\alpha_2, \ldots, n+\alpha_d$ are replaced by polynomial functions of n.

Obviously, it would be of interest to generalize in both directions simultaneously. The interplay between the two conditions is likely to be nontrivial since already Theorem 1 requires that $\alpha_i \neq \beta_j$ for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m$. Example 5 shows that this restriction is, at least in some cases, necessary.

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