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Factorization length distribution for affine semigroups II: Asymptotic behavior for numerical semigroups with arbitrarily many generators ☆

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A B S T R A C T

For numerical semigroups with a specified list of (not necessarily minimal) generators, we obtain explicit asymptotic expressions, and in some cases quasipolynomial/quasirational representations, for all major factorization length statistics. This involves a variety of tools that are not standard in the subject, such as algebraic combinatorics (Schur polynomials), probability theory (weak convergence of measures, characteristic functions), and harmonic analysis (Fourier transforms of distributions). We provide instructive examples which demonstrate the power and generality of our techniques. We also

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Homogeneous symmetric function

highlight unexpected consequences in the theory of homogeneous symmetric functions.

1. Introduction

In what follows, $\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the set of nonnegative integers. A *numerical semigroup* $S \subset \mathbb{N}$ is an additive subsemigroup containing 0. We write

$$S = \langle n_1, n_2, \ldots, n_k \rangle = \{a_1 n_1 + a_2 n_2 + \cdots + a_k n_k \, : \, a_1, a_2, \ldots, a_k \in \mathbb{N}\}$$

for the numerical semigroup generated by distinct positive $n_1 < \cdots < n_k$ in $\mathbb{N}$. Each numerical semigroup S admits a finite generating set. Moreover, there is a unique generating set that is minimal with respect to containment [58]. We always assume S has finite complement in $\mathbb{N}$ or, equivalently, $\gcd(n_1, n_2, \ldots, n_k) = 1$, and that the generators $n_1, n_2, \ldots, n_k$ are listed in increasing order. We do not assume that $n_1, n_2, \ldots, n_k$ minimally generate S.

A *factorization* of $n \in S$ is an expression

$$n = a_1 n_1 + a_2 n_2 + \cdots + a_k n_k$$

of n as a sum of generators of S, which we represent here using the k-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$. The *length* of the factorization $\mathbf{a}$ is

$$\|\mathbf{a}\| = a_1 + a_2 + \cdots + a_k.$$

The *length multiset of* n, denoted $\mathsf{L}[\![n]\!]$, is the multiset with a copy of $\|\mathbf{a}\|$ for each factorization $\mathbf{a}$ of n. Recall that a *multiset* is a set in which repetition is taken into account; that is, its elements can occur multiple times. In particular, the cardinality $|\mathsf{L}[\![n]\!]|$ of $\mathsf{L}[\![n]\!]$ equals the number of factorizations of n.

It is well known that all sufficiently large $n \in \mathbb{N}$ belong to S when the generators are relatively prime. The largest integer that does not belong to S, called its *Frobenius number*, has been studied extensively in the literature [57]. As an extension of this, the s-Frobenius numbers (i.e., the largest integer with at most s factorizations) have also been studied [30], as has an analogous question for rings of integers [32]. More recently, J. Bourgain and Ya. G. Sinai [16], among others [2,25], investigated the asymptotic behavior of the Frobenius number, as did V.I. Arnold [4] in the context of estimating the number of factorizations of elements of S.

Factorizations and their lengths have been studied extensively under the broad umbrella of factorization theory [22,42,60] (see [41] for a thorough introduction). Investigations usually concern sets of lengths (i.e., without repetition), including asymptotic structure theorems [31,37,40,52] as well as specialized results spanning numerous families of rings and semigroups from number theory [7,8,17], algebra [5,6] and elsewhere (see the survey [38] and the references therein). Several combinatorially-flavored invariants have also been studied (e.g., elasticity [3,45], the delta set [23,44], and the catenary degree [36,39]) to obtain more refined comparisons of length sets across different settings [21]. Numerical semigroups have received particular attention [14,43,54], in part due to their suitability for computation [27,35] and the availability of machinery from combinatorial commutative algebra [49,53] (see [51] for background on the latter). Additionally, factorizations of numerical semigroup elements arise naturally in discrete optimization as solutions to knapsack problems [26,56] as well as in algebraic geometry and commutative algebra [1,10].

One of the crowning achievements in factorization theory is the *structure theorem for sets of length*, which in this setting states that for any numerical semigroup S, there exist constants $d, M > 0$ such that for all sufficiently large elements $n \in S$, the length set $L(n)$ is an arithmetic sequence from which some subset of the first and last M elements are removed [41]. As a consequence, most invariants derived from factorization length focus on extremal lengths.

We consider here asymptotic questions surrounding length multisets of numerical semigroups. This question was initially studied in [34] for three-generated numerical semigroups, where a closed form for the limiting distribution was obtained via careful combinatorial arguments for bounding factorization-length multiplicities. This approach proved difficult, if not impossible, when four or more generators are allowed and [34] ended with many questions unanswered.

Theorem 1 below, our main result, answers almost all questions about the asymptotic properties of important statistical quantities associated to factorization lengths in numerical semigroups. It relates asymptotic questions about factorization lengths to properties of an explicit probability distribution, which permits us to obtain numerous asymptotic predictions in closed form. Our theorem recovers the key results from [34] on three-generated semigroups, and generalizes them to semigroups with an arbitrary number of generators.

For what follows, we require some algebraic terminology. The *complete homogeneous symmetric polynomial* of degree p in the k variables $x_1, x_2, \ldots, x_k$ is

$$h_p(x_1, x_2, \ldots, x_k) \quad = \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_p \leq k} x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_p},$$

the sum of all degree p monomials in $x_1, x_2, \ldots, x_k$. A *quasipolynomial* of degree d is a function $f : \mathbb{Z} \to \mathbb{C}$ of the form

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_1(n)n + c_0(n),$$

in which the coefficients $c_1(n), c_2(n), \ldots, c_d(n)$ are periodic functions of n [11]. A *quasirational function* is a quotient of two quasipolynomials. The cardinality of a set X is denoted $|X|$.

Theorem 1. *Let $S = \langle n_1, n_2, \ldots, n_k \rangle$, in which $k \geq 3$, $\gcd(n_1, n_2, \ldots, n_k) = 1$, and $n_1 < n_2 < \cdots < n_k$.*

(a) *For real $\alpha < \beta$,*

$$\lim_{n\to\infty} \frac{|\{\ell \in \mathsf{L}[\![n]\!] : \ell \in [\alpha n, \beta n]\}|}{|\mathsf{L}[\![n]\!]|} = \int_{\alpha}^{\beta} F(t)\, dt,$$

where $F : \mathbb{R} \to \mathbb{R}$ is the probability density function

$$F(x) := \frac{(k-1)n_1 n_2 \cdots n_k}{2} \sum_{r=1}^{k} \frac{|1 - n_r x|(1 - n_r x)^{k-3}}{\prod_{j\neq r}(n_j - n_r)}.$$

The support of F is $\left[\frac{1}{n_k}, \frac{1}{n_1}\right]$.

(b) *For $p \in \mathbb{N}$, the pth moment of F is*

$$\int_0^1 t^p F(t)\, dt = \binom{p+k-1}{p}^{-1} h_p\left(\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}\right).$$

(c) *For any continuous function $g : (0,1) \to \mathbb{C}$,*

$$\lim_{n\to\infty} \frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\ell \in \mathsf{L}[\![n]\!]} g\left(\frac{\ell}{n}\right) = \int_0^1 g(t)F(t)\, dt.$$

In Theorem 1a, observe that F is a piecewise-polynomial function of degree $k-2$ that is $(k-3)$-times continuously differentiable, but not everywhere differentiable $k-2$ times. In particular, its smoothness increases as the number of generators increases. This is characteristic of the Curry–Schoenberg B-spline from computer-aided design [24], of which the function F is a special case; this connection is discussed in much greater detail in [15].

The explicit nature and broad generality of Theorem 1 permit strikingly accurate asymptotic predictions, often in closed form, of virtually every statistical quantity related to factorization lengths when considered with multiplicity. For example, Theorem 1 immediately predicts the number of factorizations of n, the moments of the factorization-length multiset $\mathsf{L}[\![n]\!]$, its mean, standard deviation, median, mode, skewness, and so forth (see Section 2). The flexibility afforded by Theorem 1c permits us to address quantities such as the harmonic and geometric mean factorization length, which would previously have been beyond the scope of standard semigroup-theoretic techniques.

The proof of Theorem 1 is contained in Section 6. It involves a variety of tools that are not standard fare in the numerical semigroup literature. For example, weak convergence of probability measures, Fourier transforms of distributions, and the theory of characteristic functions come into play. In addition, Theorem 1 builds upon two other results, described below, whose origins are in complex variables (Theorem 2) and algebraic combinatorics (Theorem 3).

Theorem 2, whose proof is deferred until Section 4, concerns a quasipolynomial representation for the pth power sum of the factorization lengths of n (the main ingredient for the pth moment of $F(x)$). Although this result is of independent interest to the numerical semigroup community, its true power emerges when combined with Theorems 1 and 3.

Theorem 2. *Let $S = \langle n_1, n_2, \ldots, n_k \rangle$, in which $k \geq 3$, $\gcd(n_1, n_2, \ldots, n_k) = 1$, and $n_1 < n_2 < \cdots < n_k$. For $p \in \mathbb{N}$,*

$$\sum_{\ell \in \mathsf{L}[\![n]\!]} \ell^p = \frac{p!}{(k+p-1)!(n_1 n_2 \cdots n_k)} h_p\left(\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}\right) n^{k+p-1} + w_p(n),$$

in which $w_p(n)$ is a quasipolynomial of degree at most $k+p-2$ whose coefficients have period dividing $\operatorname{lcm}(n_1, n_2, \ldots, n_k)$.

Our next result, whose proof is in Section 5, is an exponential generating function identity. Although its derivation involves a bit of algebraic combinatorics and the result itself might seem a bit of a digression, this identity is a crucial ingredient to the proof of Theorem 1.

Theorem 3. *Let $x_1, x_2, \ldots, x_k \in \mathbb{C} \backslash \{0\}$ be distinct. For $z \in \mathbb{C}$,*

$$\sum_{p=0}^{\infty} \frac{h_p(x_1, x_2, \ldots, x_k)}{(p+k-1)!} z^{p+k-1} = \sum_{r=1}^{k} \frac{e^{x_r z}}{\prod_{j \neq r}(x_r - x_j)}.$$

There are several unexpected consequences of our work to the realm of symmetric functions. For example, Theorem 12 in Section 3 provides a novel probabilistic interpretation of the complete homogeneous symmetric polynomials. This not only recovers a well-known positivity result (Corollary 17), it also provides a natural method to extend the definition of $h_p(x_1, x_2, \ldots, x_k)$ to nonintegral p.

We are optimistic that Theorem 1 will prove to be a standard tool in the study of numerical semigroups; statistical results about factorization lengths that before appeared intractable are now straightforward consequences of Theorem 1. We devote all of Section 2 to applications and examples of our results. Sections 4, 5, and 6 contain the proofs of Theorems 2, 3, and 1 respectively. We wrap up in Section 7 with some closing remarks.

2. Applications and examples

This section consists of a host of examples and applications of Theorems 1 and 2. We avoid the traditional corollary-proof format, which would soon become overbearing, in favor of a more leisurely and less staccato pace. In particular, we demonstrate how a wide variety of factorization-length statistics, some frequently considered and others more exotic, can be examined using our methods. The following examples and commentary illustrate the effectiveness of our techniques as well as their implementation.

We begin in Subsection 2.1 with a brief rundown of fundamental factorization-length statistics, giving closed-form formulas for the asymptotic behavior when convenient. In Subsection 2.2, we recover all of the key results of [34] on three-generator numerical semigroups. Subsection 2.3 contains explicit formulas, all of them novel, for asymptotic statistics in four-generated semigroups. Numerical semigroups with more generators and related phenomena are discussed in Subsection 2.4.

2.1. Factorization-length statistics

Fix $S = \langle n_1, n_2, \ldots, n_k \rangle$, where as always we assume that $\gcd(n_1, n_2, \ldots, n_k) = 1$. The quasipolynomial or quasirational functions mentioned below all have $\mathbb{Q}$-valued coefficients with periods dividing $\operatorname{lcm}(n_1, n_2, \ldots, n_k)$. For each key factorization-length statistic we provide an explicit, asymptotically equivalent expression when available. We say that $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$ and $f(n) = O(g(n))$ if there is a constant C such that $|f(n)| \leq C|g(n)|$ for sufficiently large $n \in \mathbb{N}$.

(a) **Number of Factorizations.** Theorem 2 with $p = 0$ implies that the cardinality $|\mathsf{L}[\![n]\!]|$ of the factorization length multiset $\mathsf{L}[\![n]\!]$ is a quasipolynomial and

$$|\mathsf{L}[\![n]\!]| \;=\; \frac{n^{k-1}}{(k-1)!(n_1 n_2 \cdots n_k)} + O(n^{k-2}). \tag{4}$$

(b) **Moments.** Theorem 2 and (4) imply that the *pth factorization length moment*

$$m_p(n) \;:=\; \frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\ell \in \mathsf{L}[\![n]\!]} \ell^p \;\sim\; \binom{p+k-1}{p}^{-1} h_p\left(\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}\right) n^p \tag{5}$$

is quasirational.

(c) **Mean.** The preceding implies that the *mean factorization length*

$$m_1(n) \;:=\; \frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\ell \in \mathsf{L}[\![n]\!]} \ell \;\sim\; \frac{n}{k}\left(\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}\right)$$

is quasirational. It is asymptotically linear as $n \to \infty$ and its slope is the reciprocal of the harmonic mean of the generators of S.

(d) **Variance and standard deviation.** The factorization length *variance*, given by $\sigma^2(n) := m_2(n) - (m_1(n))^2$, is quasirational by (b). From (5), we have

$$\sigma^2(n) \ \sim \ \frac{n^2}{k^2(k+1)}\left((k-1)\sum_{i=1}^{k}\frac{1}{n_i^2} - 2\sum_{i<j}\frac{1}{n_i n_j}\right).$$

The *standard deviation* is then $\sigma(n)$, the square root of the variance.

(e) **Median.** Theorem 1a ensures that the *median* factorization length satisfies

$$\mathsf{Median}\,\mathsf{L}[\![n]\!] \ \sim \ \beta n,$$

in which $\beta \in [0,1]$ is the unique positive real number so that $\int_0^\beta F(t)\,dt = \frac{1}{2}$. Note that it was already demonstrated in [34] that β can be irrational, even when $k = 3$, in which case the median cannot be quasirational in n.

(f) **Mode.** Since the function F is known to be unimodal (see [24, Thm. 1] and [15]), the *mode* factorization length satisfies

$$\mathsf{Mode}\,\mathsf{L}[\![n]\!] \ \sim \ n \operatorname{argmax} F(x),$$

in which $\operatorname{argmax} F(x)$ is the unique value in $[0,1]$ at which F assumes its absolute maximum.

(g) **Skewness.** The factorization length *skewness* is

$$\mathsf{Skew}\,\mathsf{L}[\![n]\!] := \frac{1}{|\mathsf{L}[\![n]\!]|}\sum_{\ell\in\mathsf{L}[\![n]\!]}\left(\frac{\ell - m_1(n)}{\sigma(n)}\right)^3 = \frac{m_3(n) - 3m_1(n)\sigma^2(n) - m_1(n)^3}{\sigma^3(n)},$$

the third centered moment. In light of (b), (c), and (d), an explicit asymptotic formula for $\mathsf{Skew}\,\mathsf{L}[\![n]\!]$ can be given, although we refrain from doing so.

(h) **Min / Max.** The minimum and maximum factorization lengths satisfy

$$\mathsf{Max}\,\mathsf{L}[\![n]\!] \ \sim \ \frac{n}{n_1} \qquad \text{and} \qquad \mathsf{Min}\,\mathsf{L}[\![n]\!] \ \sim \ \frac{n}{n_k}.$$

This follows from Theorem 1a since the distribution $F(t)$ is supported on $[1/n_k, 1/n_1]$ and places mass on any open neighborhood of its endpoints (it is known that $\mathsf{Max}\,\mathsf{L}[\![n]\!]$ and $\mathsf{Min}\,\mathsf{L}[\![n]\!]$ are linear quasipolynomials with leading coefficients $1/n_1$ and $1/n_k$, respectively [9, Theorems 4.2 and 4.3]).

(i) **Harmonic mean.** The *harmonic mean* factorization length satisfies

$$H(n) \ := \ \frac{|\mathsf{L}[\![n]\!]|}{\sum_{\ell\in\mathsf{L}[\![n]\!]}\ell^{-1}} \ \sim \ \frac{n}{\int_0^1 t^{-1}F(t)\,dt}.$$

The integral is taken over $[0,1]$ for convenience; since F is supported on $[1/n_k, 1/n_1]$, the integrand vanishes at $t = 0$.

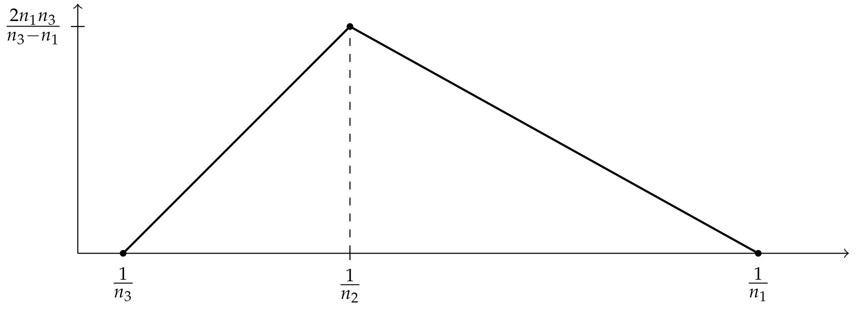


Fig. 1. The asymptotic length distribution function $F(x)$ for a three-generated semigroup $S = \langle n_1, n_2, n_3\rangle$ is a triangular distribution on $[1/n_3, 1/n_1]$ with peak of height $2n_1n_3/(n_3 - n_1)$ at $1/n_2$.

(j) **Geometric mean.** The *geometric mean* factorization length satisfies

$$G(n) \;:=\; \Big(\prod_{\ell \in \mathsf{L}[\![n]\!]} \ell\Big)^{\frac{1}{|\mathsf{L}[\![n]\!]|}} \;\sim\; n e^{\int_0^1 (\log t)F(t)\,dt}$$

since

$$\begin{aligned}\log G(n) &= \frac{1}{|\mathsf{L}[\![n]\!]|}\sum_{\ell\in\mathsf{L}[\![n]\!]} \log \ell = \frac{1}{|\mathsf{L}[\![n]\!]|}\sum_{\ell\in\mathsf{L}[\![n]\!]} \Big(\log\frac{\ell}{n} + \log n\Big)\\ &= \log n + \frac{1}{|\mathsf{L}[\![n]\!]|}\sum_{\ell\in\mathsf{L}[\![n]\!]} \log\frac{\ell}{n} \;\sim\; \log n + \int_0^1 (\log t)F(t)\,dt.\end{aligned}$$

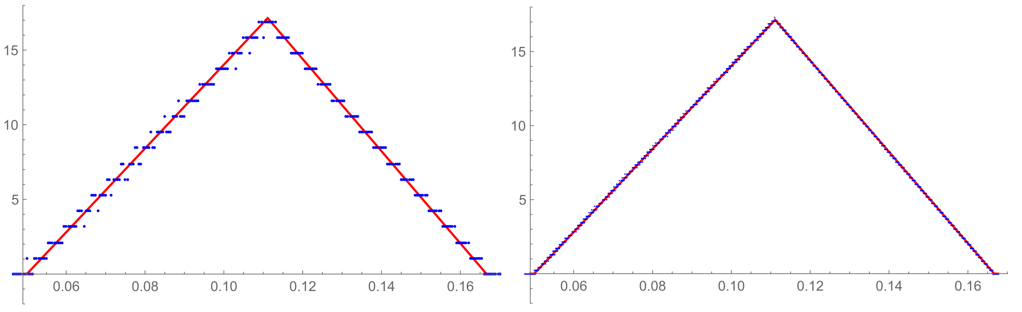
For the sake of uniformity, we often prefer to use the more explicit notation $\mathsf{Mean}\,\mathsf{L}[\![n]\!]$, $\mathsf{Median}\,\mathsf{L}[\![n]\!]$, $\mathsf{Mode}\,\mathsf{L}[\![n]\!]$, $\mathsf{Var}\,\mathsf{L}[\![n]\!]$, $\mathsf{StDev}\,\mathsf{L}[\![n]\!]$, $\mathsf{HarMean}\,\mathsf{L}[\![n]\!]$, $\mathsf{Skew}\,\mathsf{L}[\![n]\!]$, and $\mathsf{GeoMean}\,\mathsf{L}[\![n]\!]$, instead of distinctive symbols, such as $\mu(n)$ or $\sigma(n)$.

2.2. Three generators: triangular distribution

The asymptotic behavior of factorization lengths in three-generator semigroups was studied in [34] with other methods. Theorem 1 recovers all of the main results from that paper.

For $S = \langle n_1, n_2, n_3\rangle$, the function $F(x)$ of Theorem 1 is a triangular distribution; see Fig. 1. Indeed, letting $k = 3$ and $(a, b, c) = (\frac{1}{n_3}, \frac{1}{n_1}, \frac{1}{n_2})$ in Theorem 1 we obtain

$$F(x) = \begin{cases} 0 & \text{if } x \le a,\\ \dfrac{2(x-a)}{(b-a)(c-a)} & \text{for } a \le x \le c,\\ \dfrac{2(b-x)}{(b-a)(b-c)} & \text{for } c < x \le b,\\ 0 & \text{if } x \ge b.\end{cases} \tag{6}$$



(a) $n = 2{,}000$ **(b)** $n = 10{,}000$

Fig. 2. Normalized histogram of the length multiset $\mathsf{L}[\![n]\!]$ (blue) and graph of the length distribution function $F(x)$ (red) for $S = \langle 6, 9, 20\rangle$. For $i \in \mathbb{N}$, a blue dot occurs above i/n at height equal to the multiplicity of i in $\mathsf{L}[\![n]\!]$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

This is the familiar triangular distribution on $[a, b]$ with peak of height $2/(b-a)$ at $c \in (a, b)$ [28, Ch. 40], [50, Ch. 1]. As predicted in the comments after Theorem 1, the distribution function is continuous but not everywhere differentiable. The standard properties of the triangular distribution provide us with the asymptotic behavior of lengths in three-generated semigroups:

$$\operatorname{Mean} \mathsf{L}[\![n]\!] \sim \frac{n}{3}\left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\right),$$

$$\operatorname{Median} \mathsf{L}[\![n]\!] \sim n \cdot \begin{cases} \dfrac{1}{n_3} + \sqrt{\dfrac{1}{2}\left(\dfrac{1}{n_1} - \dfrac{1}{n_3}\right)\left(\dfrac{1}{n_2} - \dfrac{1}{n_3}\right)} & \text{if } \dfrac{1}{n_2} \geq \dfrac{1}{2}\left(\dfrac{1}{n_1} + \dfrac{1}{n_3}\right), \\ \dfrac{1}{n_1} - \sqrt{\dfrac{1}{2}\left(\dfrac{1}{n_1} - \dfrac{1}{n_3}\right)\left(\dfrac{1}{n_1} - \dfrac{1}{n_2}\right)} & \text{if } \dfrac{1}{n_2} < \dfrac{1}{2}\left(\dfrac{1}{n_1} + \dfrac{1}{n_3}\right), \end{cases}$$

$$\operatorname{Mode} \mathsf{L}[\![n]\!] \sim \frac{n}{n_2},$$

$$\operatorname{Var} \mathsf{L}[\![n]\!] \sim \frac{n^2}{18}\left(\frac{1}{n_1^2} + \frac{1}{n_2^2} + \frac{1}{n_3^2} - \frac{1}{n_1 n_2} - \frac{1}{n_2 n_3} - \frac{1}{n_3 n_1}\right), \quad \text{and}$$

$$\operatorname{Skew} \mathsf{L}[\![n]\!] \sim \frac{\sqrt{2}\left(\frac{1}{n_1} + \frac{1}{n_3} - \frac{2}{n_2}\right)\left(\frac{2}{n_1} - \frac{1}{n_3} - \frac{1}{n_2}\right)\left(\frac{1}{n_1} - \frac{2}{n_3} + \frac{1}{n_2}\right)}{5\left(\frac{1}{n_1^2} + \frac{1}{n_2^2} + \frac{1}{n_3^2} - \frac{1}{n_1 n_2} - \frac{1}{n_1 n_3} - \frac{1}{n_2 n_3}\right)^{3/2}}.$$

The harmonic and geometric means can also be worked out in closed form; the interested reader may wish to pursue the matter further.

Example 7. Consider the McNugget semigroup $S = \langle 6, 9, 20\rangle$. The normalized histogram of the length multiset $\mathsf{L}[\![n]\!]$ rapidly approaches the corresponding triangular distribution with parameters $(a, b, c) = (\frac{1}{20}, \frac{1}{9}, \frac{1}{6})$; see Fig. 2. The asymptotic formulae furnished

Table 1
Actual versus predicted statistics (rounded to two decimal places) for $\mathsf{L}[\![10^5]\!]$, the multiset of factorization lengths of 100,000, in $S = \langle 6, 9, 20\rangle$.

Statistic	Actual	Predicted	Statistic	Actual	Predicted
Mean $\mathsf{L}[\![10^5]\!]$	10925.14	10925.93	HarMean $\mathsf{L}[\![10^5]\!]$	10359.00	10359.86
Median $\mathsf{L}[\![10^5]\!]$	10970	10970.61	GeoMean $\mathsf{L}[\![10^5]\!]$	10650.22	10651.03
Mode $\mathsf{L}[\![10^5]\!]$	{11109,11110,11111}	11111.11	Skew $\mathsf{L}[\![10^5]\!]$	−0.046593	−0.046592
StDev $\mathsf{L}[\![10^5]\!]$	2382.40	2382.35	Min /Max $\mathsf{L}[\![10^5]\!]$	5000/16662	5000.00/16666.67

by our results perform admirably in estimating key factorization-length statistics; see Table 1.

2.3. Four generators: piecewise quadratic

For $S = \langle n_1, n_2, n_3, n_4\rangle$, the function $F(x)$ of Theorem 1 is piecewise quadratic and can be worked out in closed form:

$$F(x) = 3n_1n_2n_3n_4 \times \begin{cases} 0 & \text{if } x < \frac{1}{n_4}, \\ \dfrac{(1-n_4x)^2}{(n_4-n_1)(n_4-n_2)(n_4-n_3)} & \text{if } \frac{1}{n_4} \le x \le \frac{1}{n_3}, \\ f(x) & \text{if } \frac{1}{n_3} \le x \le \frac{1}{n_2}, \\ \dfrac{(1-n_1x)^2}{(n_2-n_1)(n_3-n_1)(n_4-n_1)} & \text{if } \frac{1}{n_2} \le x \le \frac{1}{n_1}, \\ 0 & \text{if } x > \frac{1}{n_1}, \end{cases}$$

in which

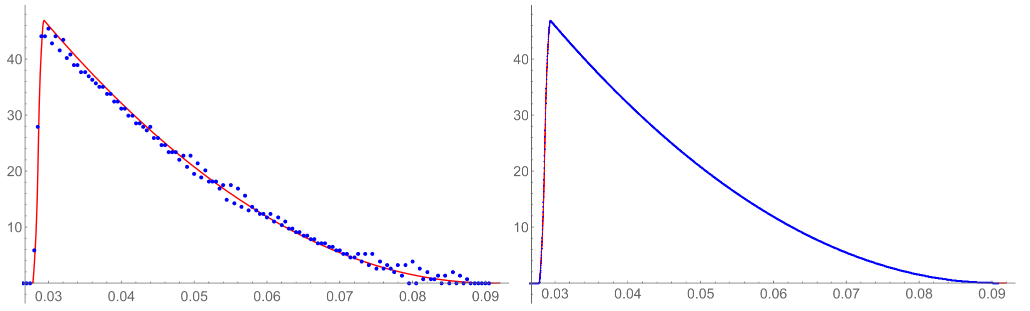
$$f(x) = \frac{(n_1n_2n_3 + n_1n_2n_4 - n_1n_3n_4 - n_2n_3n_4)x^2 - 2(n_1n_2 - n_3n_4)x + (n_1 + n_2 - n_3 - n_4)}{(n_3-n_1)(n_3-n_2)(n_4-n_1)(n_4-n_2)}.$$

We remark that this explicit formula for the length distribution function completely answers the open problem suggested at the end of [34]. As predicted by the comments after Theorem 1, F is continuously differentiable but not twice differentiable. Moreover, one can see that F is unimodal and that its absolute maximum is attained in $(\frac{1}{n_3}, \frac{1}{n_2})$. A few computations reveal that

$$\mathsf{Mean}\,\mathsf{L}[\![n]\!] \sim \frac{n}{4}\left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4}\right),$$

$$\mathsf{Mode}\,\mathsf{L}[\![n]\!] \sim \left(\frac{n_1n_2 - n_3n_4}{n_1n_2n_3 + n_1n_2n_4 - n_1n_3n_4 - n_2n_3n_4}\right)n, \quad \text{and}$$

$$\mathsf{Var}\,\mathsf{L}[\![n]\!] \sim \frac{n^2}{80}\left(4\sum_{i-1}^{4}\frac{1}{n_i^2} - 2\sum_{i<j}\frac{1}{n_in_j}\right).$$



(a) $n = 2{,}000$ **(b)** $n = 50{,}000$

Fig. 3. Normalized histogram of the length multiset $\mathsf{L}[\![n]\!]$ (blue) and graph of the length distribution function $F(x)$ (red) for $S = \langle 11, 34, 35, 36\rangle$. For each $i \in \mathbb{N}$ a blue dot occurs above i/n at height equal to the multiplicity of i in $\mathsf{L}[\![n]\!]$.

The asymptotic median factorization length is not so amenable to closed-form expression, although it is easily computed for specific semigroups as we see below.

Example 8. For $S = \langle 11, 34, 35, 36\rangle$, we have

$$F(x) = 1413720 \begin{cases} 0 & \text{if } x \le \frac{1}{36}, \\ \frac{1}{50}(36x-1)^2 & \text{if } \frac{1}{36} \le x \le \frac{1}{35}, \\ \frac{1}{600}\left(-15073x^2 + 886x - 13\right) & \text{if } \frac{1}{35} \le x \le \frac{1}{34}, \\ \frac{1}{13800}(11x-1)^2 & \text{if } \frac{1}{34} \le x \le \frac{1}{11}, \\ 0 & \text{if } x > \frac{1}{11}; \end{cases}$$

see Fig. 3. Elementary computation confirms that the median of the distribution function $F(x)$ occurs in $[\frac{1}{34}, \frac{1}{11}]$. For $x \in [\frac{1}{34}, \frac{1}{11}]$, we find that

$$\int_0^x F(t)\,dt = \frac{11}{115}(43197x^3 - 11781x^2 + 1071x - 22)$$

attains the value $\frac{1}{2}$ at precisely one point, namely $\frac{1}{11}(1 - \sqrt[3]{\frac{115}{714}}) \approx 0.041$. Thus,

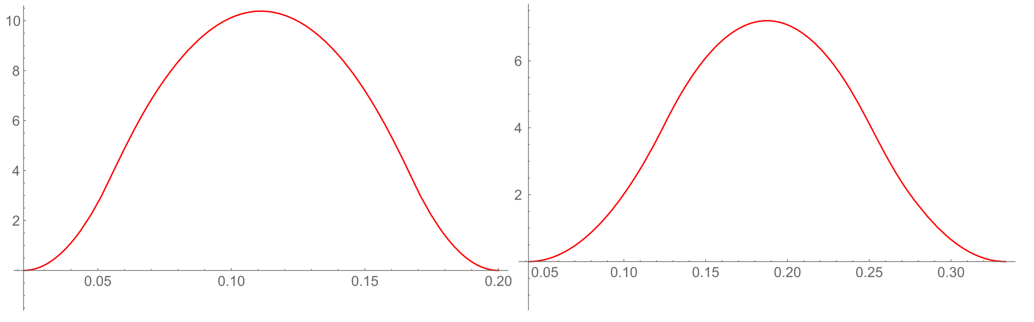
$$\operatorname{Median} \mathsf{L}[\![n]\!] \sim \frac{1}{11}\left(1 - \sqrt[3]{\frac{115}{714}}\right) n.$$

Table 2 provides factorization-length statistics for $\mathsf{L}[\![10^5]\!]$ and the strikingly accurate approximations furnished by our results.

Example 9. The factorization-length skewness, being expressible in terms of the first and third moments, and the variance, can be given in closed form:

Table 2
Actual versus predicted statistics (rounded to two decimal places) for $\mathsf{L}[\![10^5]\!]$, the multiset of factorization lengths of 100,000, in $S = \langle 11, 34, 35, 36 \rangle$.

Statistic	Actual	Predicted	Statistic	Actual	Predicted
$\mathsf{Mean}\,\mathsf{L}[\![10^5]\!]$	4417.31	4416.76	$\mathsf{HarMean}\,\mathsf{L}[\![10^5]\!]$	4130.30	4130.03
$\mathsf{Median}\,\mathsf{L}[\![10^5]\!]$	4145	4144.69	$\mathsf{GeoMean}\,\mathsf{L}[\![10^5]\!]$	4266.46	4266.06
$\mathsf{Mode}\,\mathsf{L}[\![10^5]\!]$	2939	2939.03	$\mathsf{Skew}\,\mathsf{L}[\![10^5]\!]$	0.8594802	0.8594804
$\mathsf{StDev}\,\mathsf{L}[\![10^5]\!]$	1207.84	1207.14	$\mathsf{Min}\,/\mathsf{Max}\,\mathsf{L}[\![10^5]\!]$	2778/9082	2777.78/9090.91



(a) $S = \langle 5, 6, 18, 45 \rangle$ **(b)** $S = \langle 3, 4, 8, 24 \rangle$

Fig. 4. Highly symmetric distribution functions $F(x)$ for two numerical semigroups S chosen by virtue of an Egyptian-fraction identity; see Example 9.

$$\mathsf{Skew}\,\mathsf{L}[\![n]\!] \sim \frac{2\sqrt{5}(a+b-c-d)(a-b+c-d)(a-b-c+d)}{\left(3(a^2+b^2+c^2+d^2)-2(ab+ac+bc+ad+bd+cd)\right)^{3/2}},$$

in which $(a, b, c, d) = (\frac{1}{n_1}, \frac{1}{n_2}, \frac{1}{n_3}, \frac{1}{n_4})$. In particular, $\mathsf{Skew}\,\mathsf{L}[\![n]\!]$ tends to zero (that is, F tends to be highly symmetric) if and only if one of the following occurs:

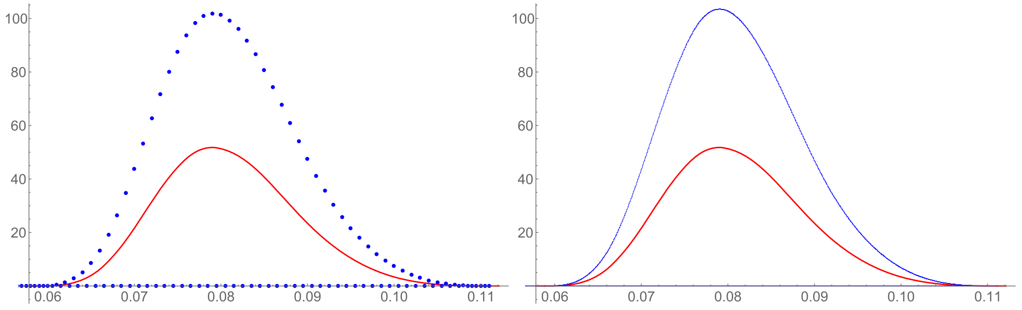
$$\frac{1}{n_1} + \frac{1}{n_3} = \frac{1}{n_2} + \frac{1}{n_4} \qquad \text{or} \qquad \frac{1}{n_1} + \frac{1}{n_4} = \frac{1}{n_2} + \frac{1}{n_3}.$$

For example, the equalities

$$\frac{1}{5} + \frac{1}{45} = \frac{1}{6} + \frac{1}{18} \qquad \text{and} \qquad \frac{1}{3} + \frac{1}{24} = \frac{1}{4} + \frac{1}{8}$$

yield two numerical semigroups with highly symmetric length distribution functions; see Fig. 4. This highlights another connection between Egyptian fractions and the statistical properties of length distributions in numerical semigroups [34].

Similar computations can be carried out for semigroups with more generators, although it becomes rapidly less rewarding to search for answers in closed form as the number of generators increases. We leave the details and particulars of such computations to the reader.



(a) $n = 2{,}000$ **(b)** $n = 50{,}000$

Fig. 5. Normalized histogram of the length multiset $\mathsf{L}[\![n]\!]$ (blue) and graph of the length distribution function $F(x)$ (red) for $S = \langle 9, 11, 13, 15, 17\rangle$. For each $i \in \mathbb{N}$ a blue dot occurs above i/n at height equal to the multiplicity of i in $\mathsf{L}[\![n]\!]$. Since 2,000 and 50,000 are even, there are no factorizations of odd length. Thus, the red curve is half the height of the upper curve suggested by the blue points.

2.4. Additional examples

In this section, we give two final examples. The first points out a curious, but easily explained, phenomenon related to the constant

$$\delta = \gcd(n_k - n_{k-1}, n_{k-1} - n_{k-2}, \ldots, n_2 - n_1),$$

which arises in the semigroup literature as the minimum element of the *delta set* (see [20] for more on this invariant). Since $n_i \equiv n_j \pmod{\delta}$ for every i, j, it follows that δ is the smallest distance that can occur between distinct factorization lengths of n, meaning all factorization lengths of a given n are equivalent modulo δ. If $\delta > 1$, then this causes “gaps” between positive values in the length multiset. The question of decomposing $\mathsf{L}[\![n]\!]$ along arithmetic sequences is treated in [33].

Example 10. Let $S = \langle 9, 11, 13, 15, 17\rangle$, for which $\delta = 2$. If n is even, then every element of $\mathsf{L}[\![n]\!]$ is even, and if n is odd, then every element of $\mathsf{L}[\![n]\!]$ is odd. The corresponding length distribution function is

$$F(x) = \frac{109395}{32}\begin{cases} 0 & \text{if } x < \frac{1}{17}, \\ (17x-1)^3 & \text{if } \frac{1}{17} \le x < \frac{1}{15}, \\ 3 - 129x + 1833x^2 - 8587x^3 & \text{if } \frac{1}{15} \le x < \frac{1}{13}, \\ -3 + 105x - 1209x^2 + 4595x^3 & \text{if } \frac{1}{13} \le x < \frac{1}{11}, \\ (1-9x)^3 & \text{if } \frac{1}{11} \le x < \frac{1}{9}, \\ 0 & \text{if } \frac{1}{9} \le x, \end{cases}$$

which appears to be half the height of the upper curve suggested by the blue dots in Fig. 5 since the factorization lengths of n all have identical parity. In other words, the

Table 3
Actual versus predicted statistics (rounded to two decimal places) for $\mathsf{L}[\![10^5]\!]$, the multiset of factorization lengths of 100,000, in $S = \langle 9, 11, 13, 15, 17 \rangle$.

Statistic	Actual	Predicted	Statistic	Actual	Predicted
Mean $\mathsf{L}[\![10^5]\!]$	8088.80	8088.67	HarMean $\mathsf{L}[\![10^5]\!]$	8019.043	8018.96
Median $\mathsf{L}[\![10^5]\!]$	8038	8037.53	GeoMean $\mathsf{L}[\![10^5]\!]$	8053.75	8053.64
Mode $\mathsf{L}[\![10^5]\!]$	7904	7904.25	Skew $\mathsf{L}[\![10^5]\!]$	0.32812710	0.32812712
StDev $\mathsf{L}[\![10^5]\!]$	757.14	756.89	Min /Max $\mathsf{L}[\![10^5]\!]$	11110/5884	11111.11/5882.35

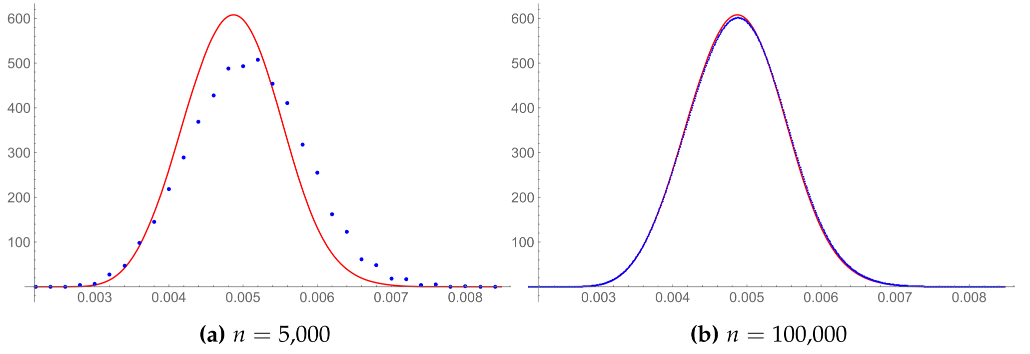


Fig. 6. Normalized histogram of the length multiset $\mathsf{L}[\![n]\!]$ (blue) and graph of the length distribution function $F(x)$ (red) for the numerical semigroup $\langle 118, 150, 162, 175, 182, 258, 373, 387, 456 \rangle$. For each $i \in \mathbb{N}$ a blue dot occurs above i/n at height equal to the multiplicity of i in $\mathsf{L}[\![n]\!]$.

Table 4
Actual versus predicted statistics (rounded to two decimal places) for $\mathsf{L}[\![10^5]\!]$ in $S = \langle 118, 150, 162, 175, 182, 258, 373, 387, 456 \rangle$.

Statistic	Actual	Predicted	Statistic	Actual	Predicted
Mean $\mathsf{L}[\![10^5]\!]$	488.30	487.30	HarMean $\mathsf{L}[\![10^5]\!]$	479.46	478.64
Median $\mathsf{L}[\![10^5]\!]$	488	486.74	GeoMean $\mathsf{L}[\![10^5]\!]$	483.92	483.00
Mode $\mathsf{L}[\![10^5]\!]$	488	487.08	Skew $\mathsf{L}[\![10^5]\!]$	0.09692	0.09699
StDev $\mathsf{L}[\![10^5]\!]$	65.01	64.29	Min /Max $\mathsf{L}[\![10^5]\!]$	221/846	219.30/847.46

upper curve suggested by the blue dots in Fig. 5 must be "averaged out" by δ to produce the red curve, which depicts $F(x)$. The predictions afforded by our methods in this case, as outlined in Table 3, are still surprisingly accurate.

We conclude with one final example that demonstrates the impressive estimates our techniques afford for a numerical semigroup with nine generators.

Example 11. Consider $S = \langle 118, 150, 162, 175, 182, 258, 373, 387, 456 \rangle$, which has nine generators. Describing the length-distribution statistics of such a semigroup is far beyond the realm of previously-established techniques. We spare the reader the display of the explicit length-generating function; suffice it to say, F is a piecewise polynomial function of degree 7 (see Fig. 6). A few computations with a computer algebra system provide accurate approximations to the relevant statistics; see Table 4.

3. Complete homogeneous symmetric polynomials

There are several unexpected consequences of our work to the realm of symmetric functions. The next theorem provides a probabilistic interpretation of the complete homogeneous symmetric polynomials and a means to extend their definition to nonintegral degrees. From this result we recover a well-known positivity result (Corollary 17).

The proof of Theorem 12 was recently refined [15] using the theory of splines.

Theorem 12. *Let $x_1 < x_2 < \ldots < x_k$ be real numbers. Then*

$$H(x; x_1, x_2, \ldots, x_k) = \frac{k-1}{2} \sum_{r=1}^{k} \frac{|x_r - x|(x_r - x)^{k-3}}{\prod_{j \neq r}(x_r - x_j)} \tag{13}$$

is a probability distribution on $\mathbb{R}$ with support $[x_1, x_k]$. Moreover, for $p \in \mathbb{N}$,

$$h_p(x_1, x_2, \ldots, x_k) = \binom{p+k-1}{p} \int_{x_1}^{x_k} t^p H(t; x_1, x_2, \ldots, x_k)\, dt. \tag{14}$$

Proof. By continuity, we may assume $x_1, x_2, \ldots, x_k \in \mathbb{Q}$. In fact, it is not hard to see that we can further assume

$$x_1 = \tau + \frac{m}{n_k}, \qquad x_2 = \tau + \frac{m}{n_{k-1}}, \ldots, \qquad x_k = \tau + \frac{m}{n_1},$$

in which $\tau \in \mathbb{Z}$, $m \in \mathbb{N}$, and $n_1, n_2, \ldots, n_k \in \mathbb{N}$ satisfy $\gcd(n_1, n_2, \ldots, n_k) = 1$ and $n_1 < n_2 < \ldots < n_k$. Observe that

$$\begin{aligned} H(x; x_1, x_2, \ldots, x_k) &= H(x-\tau; x_1-\tau, x_2-\tau, \ldots, x_k-\tau) \\ &= \frac{1}{m} H\left(\frac{x-\tau}{m}; \frac{x_1-\tau}{m}, \frac{x_2-\tau}{m}, \ldots, \frac{x_k-\tau}{m}\right) \\ &= \frac{1}{m} H\left(\frac{x-\tau}{m}; \frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}\right) \\ &= \frac{1}{m} F\left(\frac{x-\tau}{m}\right), \end{aligned}$$

in which F denotes the function from Theorem 1a; to see this compare (45) with (13). Since F is a probability distribution on $\mathbb{R}$ supported on $[\frac{1}{n_k}, \frac{1}{n_1}]$, we conclude that $H(x; x_1, x_2, \ldots, x_k)$ is a probability distribution on $\mathbb{R}$ supported on $[x_1, x_k]$. Let ν denote the corresponding probability measure, which satisfies $\nu(A) = \int_A H(t; x_1, x_2, \ldots, x_k)\, dt$ for all Borel sets $A \subseteq \mathbb{R}$, and let

$$\varphi_\nu(z) = \int_{x_1}^{x_k} e^{itz}\, d\nu(t) = \sum_{p=0}^{\infty} m_p(\nu) \frac{(iz)^p}{p!} \tag{15}$$

denote the corresponding characteristic function. Then,

$$\begin{aligned}\varphi_\nu(z) &= \big(H(x;x_1,x_2,\ldots,x_k)\big)\widehat{\ }(z)\\ &= (k-1)!\sum_{r=1}^{k}\frac{e^{ix_r z}}{(iz)^{k-1}\prod_{j\neq r}(x_r-x_j)} && \text{(by (44))}\\ &= (k-1)!\sum_{p=0}^{\infty}\frac{h_p(x_1,x_2,\ldots,x_k)}{(p+k-1)!}(iz)^p && \text{(by Theorem 3)}\\ &= \sum_{p=0}^{\infty}\binom{p+k-1}{p}^{-1}h_p(x_1,x_2,\ldots,x_k)\frac{(iz)^p}{p!}. && (16)\end{aligned}$$

For $p\in\mathbb{N}$, compare (15) and (16) and obtain (14). □

As an immediate corollary, we obtain a short proof of the positive-definiteness of complete homogeneous symmetric functions of even degree. This dates back to D.B. Hunter [47] (a somewhat stronger version was recently obtained by T. Tao [63]).

Corollary 17. *If $x_1,x_2,\ldots,x_k\in\mathbb{R}\backslash\{0\}$, then $h_{2d}(x_1,x_2,\ldots,x_k)\geq 0$.*

Proof. By symmetry and continuity, we may assume $x_1<x_2<\cdots<x_k$. Then

$$h_{2d}(x_1,x_2,\ldots,x_k)=\binom{2d+k-1}{2d}\int_{x_1}^{x_k}t^{2d}H(t;x_1,x_2,\ldots,x_k)\,dt>0. \quad \square$$

In light of Theorem 12, we can define

$$h_z(x_1,x_2,\ldots,x_k):=\frac{(z+k-1)\cdots(z+1)}{(k-1)!}\int_{\mathbb{R}}t^zH(t;x_1,x_2,\ldots,x_k)\,dt \tag{18}$$

for nonintegral z. Since $H(t;x_1,x_2,\ldots,x_k)$ is a symmetric function of the variables $x_1,x_2,\ldots,x_k$, it follows that $h_z(x_1,x_2,\ldots,x_k)$ is a symmetric function. Moreover, $H(t;x_1,x_2,\ldots,x_k)$ is piecewise polynomial and hence the right-hand side of (18) is explicitly computable. Thus, (18) provides a natural notion of complete homogeneous symmetric polynomials of arbitrary degree. This complements recent work of T. Tao, who developed a notion of symmetric functions in a fractional number of variables [64]. Tao's approach, inspired by work of Bennett–Carbery–Tao on the multilinear restriction and Kakeya conjectures from harmonic analysis [12], is also based upon a probabilistic framework.

Example 19. For $k=3$ and distinct $a,b,c\in\mathbb{R}$, we have

$$H(x;a,b,c)=\frac{|a-x|}{(b-a)(c-a)}+\frac{|b-x|}{(a-b)(c-b)}+\frac{|c-x|}{(a-c)(b-c)}$$

and

$$h_z(a,b,c) = \frac{a^{z+2}(b-c) + b^{z+2}(c-a) + c^{z+2}(a-b)}{(a-b)(a-c)(b-c)}.$$

As expected, if we apply (18) for $z \in \mathbb{N}$ we obtain the complete homogeneous symmetric polynomials

$$h_0(a,b,c) = 1, \quad h_1(a,b,c) = a+b+c, \quad h_2(a,b,c) = a^2+b^2+c^2+ab+bc+ca,$$

and so forth. For $a,b,c > 0$, we obtain curious symmetric functions such as

$$\begin{aligned}
h_{\frac{1}{2}}(a,b,c) &= \frac{a^{\frac{5}{2}}(b-c) + b^{\frac{5}{2}}(c-a) + c^{\frac{5}{2}}(a-b)}{(a-b)(a-c)(b-c)}, \\
h_{-\frac{1}{2}}(a,b,c) &= \frac{\sqrt{a}\sqrt{b} + \sqrt{a}\sqrt{c} + \sqrt{b}\sqrt{c}}{(\sqrt{a}+\sqrt{b})(\sqrt{a}+\sqrt{c})(\sqrt{b}+\sqrt{c})}, \\
h_{-\frac{3}{2}}(a,b,c) &= -\frac{1}{(\sqrt{a}+\sqrt{b})(\sqrt{a}+\sqrt{c})(\sqrt{b}+\sqrt{c})}, \\
h_{-\frac{5}{2}}(a,b,c) &= \frac{\frac{a-b}{\sqrt{c}} + \frac{b-c}{\sqrt{a}} + \frac{c-a}{\sqrt{b}}}{(a-b)(a-c)(b-c)}, \\
h_{-3}(a,b,c) &= \frac{1}{abc}, \\
h_{-4}(a,b,c) &= \frac{ab+ac+bc}{a^2b^2c^2}.
\end{aligned}$$

We also note that $h_{-1}(a,b,c) = h_{-2}(a,b,c) = 0$.

For negative and positive rational values of z, one can explicitly describe $h_z(x_1, x_2, \ldots, x_k)$ in terms of Schur functions, though we do not wish to be drawn too far afield here. We intend to take this subject up in a subsequent publication.

4. Proof of Theorem 2

The first part of the proof concerns a certain two-variable generating function (Subsection 4.1). Next comes a lengthy residue computation (Subsection 4.2). A few power series computations complete the proof (Subsection 4.3).

4.1. Generating function

Fix $S = \langle n_1, n_2, \ldots, n_k \rangle$ with $\gcd(n_1, n_2, \ldots, n_k) = 1$ and consider the generating function

$$\begin{aligned} g(z,w) &:= \prod_{i=1}^{k} \frac{1}{1-wz^{n_i}} \\ &= \prod_{i=1}^{k}(1+wz^{n_i}+w^2z^{2n_i}+\cdots) \\ &= \sum_{a_1,a_2,\ldots,a_k\geq 0} w^{a_1+a_2+\cdots+a_k}\, z^{a_1n_1+a_2n_2+\cdots+a_kn_k} \\ &= \sum_{n=0}^{\infty} z^n \sum_{\ell=0}^{\infty}(\#\text{ of factorizations of } n \text{ of length } \ell)w^\ell \\ &= \sum_{n=0}^{\infty} z^n \sum_{\ell\in\mathsf{L}[\![n]\!]} w^\ell. \end{aligned} \tag{20}$$

Then

$$\left(w\frac{\partial}{\partial w}\right)^p g(z,w) = \sum_{n=0}^{\infty} z^n \sum_{\ell\in\mathsf{L}[n]} w^\ell \ell^p$$

and hence

$$\Lambda_p(n) := \sum_{\ell\in\mathsf{L}[\![n]\!]} \ell^p$$

is the coefficient of z^n in the series expansion of

$$G(z) := \left(w\frac{\partial}{\partial w}\right)^p g(z,w)\bigg|_{w=1}. \tag{21}$$

To make use of this we require the following lemma.

Lemma 22. *For $p\in\mathbb{N}$,*

$$\frac{\partial^p}{\partial w^p} g(z,w) = p!\left(\prod_{b=1}^{k}\frac{1}{1-wz^{n_b}}\right) h_p\left(\frac{z^{n_1}}{1-wz^{n_1}},\cdots,\frac{z^{n_k}}{1-wz^{n_k}}\right). \tag{23}$$

Proof. We proceed by induction. The base case $p=0$ is (20). For the inductive step, suppose that (23) holds for some $p\in\mathbb{N}$. Then

$$\begin{aligned} \frac{1}{p!}\frac{\partial^{p+1}}{\partial w^{p+1}} g(z,w) &= \frac{\partial}{\partial w}\left(\frac{1}{p!}\frac{\partial^p}{\partial w^p}g(z,w)\right) \\ &= \frac{\partial}{\partial w}\left(\prod_{b=1}^{k}\frac{1}{1-wz^{n_b}}\right) h_p\left(\frac{z^{n_1}}{1-wz^{n_1}},\cdots,\frac{z^{n_k}}{1-wz^{n_k}}\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\prod_{b=1}^{k} \frac{1}{1-wz^{n_b}}\right)\left(\sum_{i=1}^{k} \frac{z^{n_i}}{1-wz^{n_i}}\right) h_p\left(\frac{z^{n_1}}{1-wz^{n_1}}, \cdots, \frac{z^{n_k}}{1-wz^{n_k}}\right) \\
&\qquad + \left(\prod_{b=1}^{k} \frac{1}{1-wz^{n_b}}\right) \frac{\partial}{\partial w} h_p\left(\frac{z^{n_1}}{1-wz^{n_1}}, \cdots, \frac{z^{n_k}}{1-wz^{n_k}}\right) \\
&= (p+1)\left(\prod_{b=1}^{k} \frac{1}{1-wz^{n_b}}\right) h_{p+1}\left(\frac{z^{n_1}}{1-wz^{n_1}}, \cdots, \frac{z^{n_k}}{1-wz^{n_k}}\right).
\end{aligned}
$$

The final equality follows from counting how many times each term appears when the line before it is expanded. □

The *Stirling number of the second kind*, denoted $\left\{ n \atop i \right\}$, counts the number of partitions of $[n] := \{1, 2, \ldots, n\}$ into i nonempty subsets. These numbers satisfy

$$
\left\{ n+1 \atop i \right\} = i \left\{ n \atop i \right\} + \left\{ n \atop i-1 \right\}
$$

and

$$
\left(x \frac{d}{dx}\right)^p = \sum_{i=0}^{p} \left\{ p \atop i \right\} x^i \frac{d^i}{dx^i}, \tag{24}
$$

which holds for $p \in \mathbb{N}$ [18,19,65].

From (21), and then (24) and (23), we obtain

$$
\begin{aligned}
G(z) &= \left(w \frac{\partial}{\partial w}\right)^p g(z,w)\Big|_{w=1} \\
&= \sum_{i=0}^{p} \left\{ p \atop i \right\} w^i \frac{\partial^i g}{\partial w^i}\Big|_{w=1} \\
&= \sum_{i=0}^{p} \left\{ p \atop i \right\} i!\, w^i \left(\prod_{j=1}^{k} \frac{1}{1-wz^{n_j}}\right) h_i\left(\frac{z^{n_1}}{1-wz^{n_1}}, \cdots, \frac{z^{n_k}}{1-wz^{n_k}}\right)\Big|_{w=1} \\
&= \sum_{i=0}^{p} \left\{ p \atop i \right\} i! \left(\prod_{j=1}^{k} \frac{1}{1-z^{n_j}}\right) h_i\left(\frac{z^{n_1}}{1-z^{n_1}}, \cdots, \frac{z^{n_k}}{1-z^{n_k}}\right).
\end{aligned} \tag{25}
$$

Thus, $G(z)$ is a rational function in z, all of whose poles are certain Lth roots of unity, in which

$$
L := \operatorname{lcm}(n_1, n_2, \ldots, n_k).
$$

Each $1 - z^{n_i}$ factors as a product of n_i distinct linear factors, one of which is $1 - z$. Consequently, 1 is a pole of $G(z)$ of order $k + p$; this arises from the summand corresponding to $i = p$. Moreover, $k + p$ is the maximum possible order for a pole of $G(z)$, and 1 is the unique pole of this order. Indeed, $\gcd(n_1, n_2, \ldots, n_k) = 1$ ensures that the only common root of $1 - z^{n_1}, 1 - z^{n_2}, \ldots, 1 - z^{n_k}$ is 1.

Thus, $\Lambda_p(n)$ is a complex linear combination of terms of the form

$$n^{r-1}\omega^n,\ n^{r-2}\omega^n, \ldots,\ \omega^n,$$

in which ω is a pole of $G(z)$ of order at most r (see [11, Ch. 1] for an overview of this method). The unique pole of $G(z)$ of highest order is 1, which has order $k + p$. Thus, there exist periodic functions $a_0, a_1, \ldots, a_{k+p-1} : \mathbb{N} \to \mathbb{C}$ with periods dividing L such that

$$\Lambda_p(n) = a_{k+p-1}(n)n^{k+p-1} + a_{k+p-2}(n)n^{k+p-2} + \cdots + a_1(n)n + a_0(n).$$

This establishes the desired quasipolynomial representation. It remains to show

$$a_{k+p-1}(n) = \frac{p!}{(k+p-1)!(n_1 n_2 \cdots n_k)} h_p\left(\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}\right)$$

and that the periodic functions $a_i(n)$ assume only rational values.

4.2. A residue computation

Since $G(z)$ has a pole of order $k + p$ at 1, we have

$$G(z) = \frac{C}{(1-z)^{k+p}} + u(z) \tag{26}$$

for some constant C and some rational function u, all of whose poles are Lth roots of unity with order at most $k + p - 1$. In particular,

$$u(z) = \sum_{n=0}^{\infty} w(n) z^n$$

for some quasipolynomial $w(n)$ of degree at most $k + p - 2$ and period dividing L.

The only summand in (25) that has a pole at 1 of order $k + p$ is the term that corresponds to $i = p$. The summands in (25) with $0 \le i \le p - 1$ satisfy

$$\lim_{z \to 1} (1-z)^{k+p} \underbrace{\left(\prod_{j=1}^{k} \frac{1}{1 - z^{n_j}}\right) h_i\left(\frac{z^{n_1}}{1 - z^{n_1}}, \cdots, \frac{z^{n_k}}{1 - z^{n_k}}\right)}_{\text{has a pole at } z=1 \text{ of order } k+i \le k+p-1} = 0.$$

Consequently,

$$
\begin{aligned}
C &= \lim_{z\to 1}(1-z)^{k+p}G(z)\\
&= \lim_{z\to 1}\sum_{i=0}^{p}\left\{ {p \atop i} \right\} i!\,(1-z)^{k+p}\left(\prod_{j=1}^{k}\frac{1}{1-z^{n_j}}\right) h_i\left(\frac{z^{n_1}}{1-z^{n_1}},\dots,\frac{z^{n_k}}{1-z^{n_k}}\right)\\
&= p!\lim_{z\to 1}(1-z)^{k+p}\left(\prod_{j=1}^{k}\frac{1}{1-z^{n_j}}\right) h_p\left(\frac{z^{n_1}}{1-z^{n_1}},\dots,\frac{z^{n_k}}{1-z^{n_k}}\right)\\
&= p!\lim_{z\to 1}\left(\prod_{j=1}^{k}\frac{1}{1+\cdots+z^{n_j-1}}\right) h_p\left(\frac{z^{n_1}}{1+\cdots+z^{n_1-1}},\dots,\frac{z^{n_k}}{1+\cdots+z^{n_k-1}}\right)\\
&= \frac{p!}{n_1n_2\cdots n_k}h_p\left(\frac{1}{n_1},\frac{1}{n_2},\dots,\frac{1}{n_k}\right). \qquad (27)
\end{aligned}
$$

4.3. Completing the proof

Observe that

$$
\begin{aligned}
\frac{1}{(1-z)^{k+p}} &= \sum_{n=0}^{\infty}\binom{n+k+p-1}{k+p-1}z^n\\
&= \sum_{n=0}^{\infty}\frac{(n+k+p-1)\cdots(n+1)}{(k+p-1)!}z^n\\
&= \frac{1}{(k+p-1)!}\sum_{n=0}^{\infty}\big(n^{k+p-1}+v(n)\big)z^n,
\end{aligned}
$$

in which $v(n)$ is a quasipolynomial of degree $k+p-2$ with integer coefficients. Together with (26) and (27), we obtain

$$
\begin{aligned}
G(z) &= \frac{p!}{n_1\cdots n_k}h_p\left(\frac{1}{n_1},\frac{1}{n_2},\dots,\frac{1}{n_k}\right)\cdot\frac{1}{(1-z)^{k+p}}+u(z)\\
&= \frac{p!}{(k+p-1)!n_1\cdots n_k}h_p\left(\frac{1}{n_1},\dots,\frac{1}{n_k}\right)\sum_{n=0}^{\infty}(n^{k+p-1}+v(n))z^n+\sum_{n=0}^{\infty}w(n)z^n.
\end{aligned}
$$

Thus,

$$
\Lambda_p(n) = \frac{p!}{(k+p-1)!n_1n_2\cdots n_k}h_p\left(\frac{1}{n_1},\frac{1}{n_2},\dots,\frac{1}{n_k}\right)n^{k+p-1}+q(n),
$$

in which $q(n)$ is a quasipolynomial of degree at most $k+p-2$ whose coefficients have periods dividing L. Additionally, since $v(n)$ and $w(n)$ both have rational coefficients, so must $q(n)$. This completes the proof. □

5. Proof of Theorem 3

We wish to prove the exponential generating function identity

$$\sum_{p=0}^{\infty} \frac{h_p(x_1, x_2, \ldots, x_k)}{(p+k-1)!} z^{p+k-1} = \sum_{r=1}^{k} \frac{e^{x_r z}}{\prod_{j \neq r}(x_r - x_j)}, \tag{28}$$

valid for $z \in \mathbb{C}$. We first show that the power series on the left-hand side of (28) has an infinite radius of convergence (Subsection 5.1). Then we reduce (28) to an identity that links complete homogeneous symmetric polynomials to the determinants of certain Vandermonde-like matrices (Subsection 5.2). A brief excursion into algebraic combinatorics (Subsection 5.3) finishes off the proof.

5.1. Radius of convergence

Fix distinct $x_1, x_2, \ldots, x_k \in \mathbb{C}\backslash\{0\}$. We claim that the radius of convergence of the power series

$$\sum_{p=0}^{\infty} \frac{h_p(x_1, x_2, \ldots, x_k)}{(p+k-1)!} z^{p+k-1} \tag{29}$$

is infinite. This ensures that (28) is an equality of entire functions. The ordinary generating function for the complete homogeneous symmetric polynomials is

$$\sum_{p=0}^{\infty} h_p(x_1, x_2, \ldots, x_k) z^p = \prod_{i=1}^{k} \frac{1}{1 - x_i z}; \tag{30}$$

see [62]. The radius of convergence of the preceding power series is the distance from 0 to the closest pole $1/x_1, 1/x_2, \ldots, 1/x_k$. Consequently, the Cauchy–Hadamard formula [59, p. 55] yields

$$\limsup_{p\to\infty} |h_p(x_1, x_2, \ldots, x_k)|^{\frac{1}{p}} = \max\{|x_1|, |x_2|, \ldots, |x_k|\}.$$

Since

$$\lim_{p\to\infty} \big((p+k-1)!\big)^{\frac{1}{p}} \geq \lim_{p\to\infty} (p!)^{\frac{1}{p}} \geq \lim_{p\to\infty} \left(\left(\tfrac{p}{3}\right)^{\frac{p}{3}}\right)^{\frac{1}{p}} = \lim_{p\to\infty} \left(\frac{p}{3}\right)^{\frac{1}{3}} = \infty,$$

a second appeal to the Cauchy–Hadamard formula tells us that the radius of convergence R of (29) satisfies

$$\frac{1}{R} = \limsup_{p\to\infty} \left(\frac{h_p(x_1, x_2, \ldots, x_k)}{(p+k-1)!}\right)^{\frac{1}{p}} = \frac{\max\{|x_1|, |x_2|, \ldots, |x_k|\}}{\lim_{p\to\infty} ((p+k-1)!)^{\frac{1}{p}}} = 0.$$

Thus, the radius of convergence of (29) is infinite.

5.2. A Vandermonde-like determinant

The determinant of the $k \times k$ *Vandermonde matrix*

$$V(x_1, x_2, \ldots, x_k) := \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-1} \end{bmatrix}$$

is

$$\det V(x_1, x_2, \ldots, x_k) = \prod_{1 \leq i < j \leq n} (x_j - x_i);$$

see [46, p. 37]. In what follows, $V(x_1, \ldots, \widehat{x_r}, \ldots, x_k)$ denotes the $(k-1) \times (k-1)$ Vandermonde matrix obtained from $V(x_1, x_2, \ldots, x_k)$ by removing x_r (do not confuse the carat with the Fourier transform). Cofactor expansion and the linearity of the determinant in the final column of a matrix reveals that

$$\begin{aligned}
&\det V(x_1, x_2, \ldots, x_k) \sum_{r=1}^{k} \frac{e^{x_r z}}{\prod_{j \neq r}(x_r - x_j)} \\
&= \sum_{r=1}^{k} (-1)^{k-r} \frac{\det V(x_1, x_2, \ldots, x_k)}{\left(\prod_{j<r}(x_r - x_j)\right)\left(\prod_{j>r}(x_j - x_r)\right)} e^{x_r z} \\
&= \sum_{r=1}^{k} (-1)^{k-r} \det V(x_1, \ldots, \widehat{x_r}, \ldots, x_k) e^{x_r z} \\
&= \det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-2} & e^{x_1 z} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-2} & e^{x_2 z} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-2} & e^{x_k z} \end{bmatrix} \\
&= \sum_{i=0}^{\infty} \frac{z^i}{i!} \det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-2} & x_1^i \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-2} & x_2^i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-2} & x_k^i \end{bmatrix} \\
&= \sum_{p=0}^{\infty} \frac{z^{p+k-1}}{(p+k-1)!} \det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-2} & x_1^{p+k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-2} & x_2^{p+k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-2} & x_k^{p+k-1} \end{bmatrix}.
\end{aligned}$$

We reindexed the final sum to reflect the fact that the matrices in the second-to-last line have repeated columns for $i = 0, 1, \ldots, k-2$ and hence have vanishing determinant. To establish (28), and hence Theorem 3, it suffices to show that

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-2} & x_1^{p+k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-2} & x_2^{p+k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-2} & x_k^{p+k-1} \end{bmatrix} = h_p(x_1, \ldots, x_k) \det V(x_1, \ldots, x_k). \tag{31}$$

5.3. Some algebraic combinatorics

To establish (31) requires a small amount of algebraic combinatorics. We briefly review the notation and results necessary for this purpose; the interested reader may consult [61] for complete details.

Let $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_k)$ denote the integer partition

$$p = \lambda_1 + \lambda_2 + \cdots + \lambda_k,$$

in which $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$. To such a partition we associate the polynomial

$$a_{(\lambda_1+k-1, \lambda_2+k-2, \ldots, \lambda_k)}(x_1, x_2, \ldots, x_k) := \det \begin{bmatrix} x_1^{\lambda_1+k-1} & x_2^{\lambda_1+k-1} & \ldots & x_k^{\lambda_1+k-1} \\ x_1^{\lambda_2+k-2} & x_2^{\lambda_2+k-2} & \ldots & x_k^{\lambda_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_k} & x_2^{\lambda_k} & \ldots & x_k^{\lambda_k} \end{bmatrix},$$

which is an alternating function of the variables $x_1, x_2, \ldots, x_k$ (interchanging any two of the variables changes the sign of the determinant). As an alternating polynomial, the preceding is divisible by

$$\begin{aligned} a_{(k-1,k-2,\ldots,0)}(x_1, x_2, \ldots, x_k) &= \det \begin{bmatrix} x_1^{k-1} & x_2^{k-1} & \ldots & x_n^{k-1} \\ x_1^{k-2} & x_2^{k-2} & \ldots & x_n^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix} \\ &= (-1)^{\binom{n}{2}} \det V(x_1, x_2, \ldots, x_k). \end{aligned}$$

The *Schur polynomial* in the variables $x_1, x_2, \ldots, x_k$ corresponding to the partition λ is

$$s_\lambda(x_1, x_2, \ldots, x_k) := \frac{a_{(\lambda_1+k-1, \lambda_2+k-2, \ldots, \lambda_k+0)}(x_1, x_2, \ldots, x_k)}{a_{(k-1,k-2,\ldots,0)}(x_1, x_2, \ldots, x_k)};$$

this is Jacobi's bialternant identity (which is itself a special case of the famed Weyl character formula). We now prove (31). If we consider the partition

$$\lambda = (p, \underbrace{0, 0, \ldots, 0}_{k-1 \text{ zeros}}),$$

then it is well known that $s_\lambda(x_1, x_2, \ldots, x_k) = h_p(x_1, x_2, \ldots, x_k)$, and hence

$$\begin{aligned}
&\det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-2} & x_1^{p+k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-2} & x_2^{p+k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-2} & x_k^{p+k-1} \end{bmatrix} \\
&\qquad = (-1)^{\binom{n}{2}} \det \begin{bmatrix} x_1^{p+k-1} & x_2^{p+k-1} & \ldots & x_k^{p+k-1} \\ x_1^{k-2} & x_2^{k-2} & \ldots & x_k^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix} \\
&\qquad = (-1)^{\binom{n}{2}} a_{(p+k-1,k-2,\ldots,0)}(x_1, x_2, \ldots, x_k) \\
&\qquad = (-1)^{\binom{n}{2}} s_\lambda(x_1, x_2, \ldots, x_k) a_{(k-1,k-2,\ldots,0)}(x_1, x_2, \ldots, x_k) \\
&\qquad = h_p(x_1, x_2, \ldots, x_k) \det V(x_1, x_2, \ldots, x_k).
\end{aligned}$$

This establishes (31) and concludes the proof of Theorem 3. □

6. Proof of Theorem 1

The proof of Theorem 1 uses a few tools, such as weak convergence and Fourier transforms of measures, that are not standard in the study of numerical semigroups. Since these ideas are not required to apply Theorem 1 and are not used elsewhere in the paper, we introduce the required concepts as needed and make no attempt to state definitions and lemmas in the greatest possible generality.

We begin with the necessary background on moments of probability measures (Subsection 6.1), Fourier transforms of measures (Subsection 6.2), and characteristic functions (Subsection 6.3). We then prove a power series convergence lemma (Subsection 6.4) to set up our use of characteristic functions. We introduce a family of singular measures (Subsection 6.5) that converge weakly to the desired probability measure. This is established using the method of characteristic functions and Lévy's continuity theorem (Subsection 6.6). We wrap things up with a dose of Fourier inversion and some detailed computations (Subsection 6.7).

6.1. Measures and moments

A *Borel measure* is a measure defined on the Borel σ-algebra, the σ-algebra of subsets of $\mathbb{R}$ generated by the open sets. Every subset of $\mathbb{R}$ we consider in this paper is a Borel set. Let ν be a *probability measure* on $[0, 1]$; that is, ν is a Borel measure on $[0, 1]$ such that $\nu([0,1]) = 1$ and $\nu(A) \geq 0$ for every Borel set $A \subseteq [0,1]$. For $p \in \mathbb{N}$, the pth *moment* of ν is

$$m_p(\nu) = \int_0^1 t^p \, d\nu(t).$$

The moments of ν are uniformly bounded since

$$0 \leq m_p(\nu) \leq \int_0^1 d\nu(t) = \nu\big([0,1]\big) = 1. \tag{32}$$

A probability measure on $[0,1]$ is completely determined by its moments [13, Thm. 30.1].

Let ν_n be a sequence of probability measures on $[0,1]$. Then ν_n *converges weakly* to a measure ν on $[0,1]$, denoted by $\nu_n \to \nu$, if any of the following equivalent conditions hold [13, Thm. 25.8, Thm. 30.2]:

(a) $\lim\limits_{n\to\infty} \int_0^1 f(t)\, d\nu_n(t) = \int_0^1 f(t)\, d\nu(t)$ for every continuous function on $[0,1]$.
(b) $m_p(\nu_n) \to m_p(\nu)$ for all $p \in \mathbb{N}$.
(c) $\nu_n(A) \to \nu(A)$ for every Borel set $A \subseteq [0,1]$ for which $\nu(\partial A) = 0$; that is, ν places no mass on the boundary of A.

The equivalence of (a) and (b) follows from the Weierstrass approximation theorem: every continuous function on $[0,1]$ is uniformly approximable by polynomials. The weak limit of a sequence of probability measures is a probability measure and the limit measure is unique [13, pp. 336-7].

6.2. The Fourier transform

The *Fourier transform* of a probability measure ν on $[0,1]$ is

$$\widehat{\nu}(z) := \int_0^1 e^{itz}\, d\nu(t). \tag{33}$$

This may differ in appearance from what the reader is accustomed to. Normally one integrates over $\mathbb{R}$ in (33), but that is unnecessary here because ν is supported on $[0,1]$. We adhere to the positive sign in the exponent of the integrand in (33), which is standard in probability theory [13, Sect. 26]. Consequently, the reader should be aware of potential sign discrepancies between what follows and formulas from their favored sources.

The *inverse Fourier transform* of a suitable $f : \mathbb{R} \to \mathbb{C}$ is

$$\check{f}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} f(t) e^{-ixt}\, dt.$$

With much additional work, the inverse Fourier transform can be defined on distributions ("generalized functions"). A friendly introduction to Fourier transforms of distributions is [55, Ch. 4]. Using the method of finite parts, one can show

$$\left(\widecheck{\frac{e^{iat}}{t^n}}\right)(x) = \frac{|x-a|(x-a)^{n-2}}{2i^n (n-1)!}, \qquad \text{for } n \geq 2. \tag{34}$$

This follows from [29, Ex. 9, p. 340] or [48, Table A-6] and the standard translation identity [29, eq. (9.29)]. The method of finite parts for treating highly singular functions as distributions is discussed in [29, pp. 324-5].

6.3. Characteristic functions

The *characteristic function* φ_ν of a probability measure ν on $[0, 1]$ is the Fourier transform of ν:

$$\begin{aligned} \varphi_\nu(z) := \widehat{\nu}(z) &= \int_0^1 e^{itz}\, d\nu(t) = \int_0^1 \sum_{p=0}^{\infty} \frac{(itz)^p}{p!}\, d\nu(t) \\ &= \sum_{p=0}^{\infty} \frac{(iz)^p}{p!} \int_0^1 t^p\, d\nu(t) = \sum_{p=0}^{\infty} m_p(\nu) \frac{(iz)^p}{p!}. \end{aligned}$$

The interchange of sum and integral is permissible because for each fixed $z \in \mathbb{C}$ the series involved converges uniformly for $t \in [0, 1]$. Since $|m_p(\nu)| \leq 1$ for all $p \in \mathbb{N}$, comparison with the exponential series ensures that the series above has an infinite radius of convergence and hence φ_ν is an entire function. Moreover,

$$|\varphi_\nu(x)| = \left| \int_0^1 e^{itx}\, d\nu(t) \right| \leq \int_0^1 d\nu(t) = 1 \tag{35}$$

for all $x \in \mathbb{R}$ since ν is a probability measure on $[0, 1]$. If ν_1 and ν_2 are probability measures and $\varphi_{\nu_1} = \varphi_{\nu_2}$, then $\nu_1 = \nu_2$ [13, Thm. 26.2].

Under certain circumstances, we can recover a probability measure from its characteristic function [13, pp. 347-8].

Lemma 36 *(Inversion theorem). If $\int_{\mathbb{R}} |\varphi_\nu(t)|\, dt$ is finite, then $F_\nu := \widecheck{\varphi_\nu}$ is a bounded continuous function and $\nu(A) = \int_A F_\nu(t)\, dt$ for every Borel set A.*

The following theorem of Lévy relates weak limits of probability measures to pointwise convergence of the corresponding characteristic functions [13, Thm. 26.3].

Lemma 37 *(Lévy's continuity theorem). Let ν_n be probability measures on $[0,1]$ such that φ_{ν_n} converges pointwise on $\mathbb{R}$ to a continuous function φ. Then*

(a) *$\varphi = \varphi_\nu$ for some probability measure ν on $[0,1]$;*
(b) *$\nu_n \to \nu$ (weak convergence of measures);*
(c) *$m_p(\nu) = \lim_{p\to\infty} m_p(\nu_n)$ for all $p \in \mathbb{N}$.*

6.4. A power series lemma

We ultimately plan to consider a sequence of probability measures ν_n to which we will apply Lemma 37. To show that the associated sequence of characteristic functions converges we need the following lemma.

Lemma 38. *Suppose that $|a_p(n)| \leq 1$ for all $n, p \in \mathbb{N}$. If $\lim_{n\to\infty} a_p(n) = a_p$ for each p, then $\sum_{p=0}^{\infty} a_p(n)\frac{z^p}{p!}$ converges locally uniformly on $\mathbb{C}$ to $\sum_{p=0}^{\infty} a_p \frac{z^p}{p!}$*

Proof. Fix $R > 0$. Let $N \in \mathbb{N}$ be so large that

$$\sum_{p=N}^{\infty} \frac{R^p}{p!} < \frac{\epsilon}{4}.$$

Let $M \in \mathbb{N}$ be such that

$$n \geq M \qquad \Longrightarrow \qquad |a_p(n) - a_p|\frac{R^p}{p!} < \frac{\epsilon}{2N} \quad \text{for } 0 \leq p \leq N-1.$$

If $|z| \leq R$, then

$$\begin{aligned} \left|\sum_{p=0}^{\infty} a_p(n)\frac{z^p}{p!} - \sum_{p=0}^{\infty} a_p\frac{z^p}{p!}\right| &\leq \sum_{p=0}^{N-1} |a_p(n) - a_p|\frac{|z|^p}{p!} + \sum_{p=N}^{\infty} |a_p(n) - a_p|\frac{|z|^p}{p!} \\ &\leq \sum_{p=0}^{N-1} |a_p(n) - a_p|\frac{R^p}{p!} + 2\sum_{p=N}^{\infty} \frac{R^p}{p!} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, the convergence is uniform on $|z| \leq R$. Since $R > 0$ was arbitrary, the convergence is locally uniform on $\mathbb{C}$. □

6.5. The measures ν_n

Fix $S = \langle n_1, n_2, \ldots, n_k \rangle$. Consider the probability measures

$$\nu_n = \frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\ell \in \mathsf{L}[\![n]\!]} \delta_{\ell/n} \tag{39}$$

for $n \in \mathbb{N}$, wherein δ_x denotes the point mass at x. Since

$$\frac{1}{n_k} \leq \min_{\ell \in \mathsf{L}[\![n]\!]} \frac{\ell}{n} \quad \text{and} \quad \max_{\ell \in \mathsf{L}[\![n]\!]} \frac{\ell}{n} \leq \frac{1}{n_1}, \tag{40}$$

the support of each ν_n is contained in $[\frac{1}{n_k}, \frac{1}{n_1}] \subset [0,1]$. The pth moment of ν_n is

$$m_p(\nu_n) = \int_0^1 t^p \, d\nu_n(t) = \frac{1}{\mathsf{L}[\![n]\!]} \sum_{\ell \in \mathsf{L}[\![n]\!]} \left(\frac{\ell}{n}\right)^p .$$

Theorem 2, (4), and the preceding imply that

$$\lim_{n\to\infty} m_p(\nu_n) = \binom{p+k-1}{p}^{-1} h_p\left(\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}\right).$$

6.6. The function φ

For notational simplicity we let $x_i = 1/n_i$ for $i \in \{1, \ldots, k\}$. The bound (32) and Lemma 38 ensure that the characteristic functions φ_{ν_n} converge locally uniformly (and hence pointwise) on $\mathbb{C}$ to

$$\varphi(z) = \sum_{p=0}^{\infty} \binom{p+k-1}{p}^{-1} h_p(x_1, x_2, \ldots, x_k) \frac{(iz)^p}{p!} \tag{41}$$

$$= (k-1)! \sum_{p=0}^{\infty} \frac{h_p(x_1, x_2, \ldots, x_k)}{(p+k-1)!} (iz)^p$$

$$= (k-1)! \sum_{r=1}^{k} \frac{e^{ix_r z}}{(iz)^{k-1} \prod_{j\neq i}(x_r - x_j)}, \tag{42}$$

in which the final equality is Theorem 3. A glance at (41), or Theorem 3 itself, tells us that the apparent singularity in (42) at $z = 0$ is removable. In particular, φ is an entire function and $\varphi(0) = 1$.

Lemma 37 (Lévy's Continuity theorem) provides a probability measure ν such that $\nu_n \to \nu$ and $\varphi = \varphi_\nu$. From (35) we see that φ is bounded on $\mathbb{R}$. Moreover,

$$\int_1^{\infty}\left|\frac{e^{ix_r t}}{t^{k-1}}\right| dt + \int_{-\infty}^{-1}\left|\frac{e^{ix_r t}}{t^{k-1}}\right| dt \leq 2\int_1^{\infty}\frac{dt}{t^2} = 2$$

and hence (42) implies that $\int_{\mathbb{R}} |\varphi(t)|\, dt$ is finite. Lemma 36 implies that $F_\nu = \widecheck{\varphi_\nu}$ is a bounded continuous function such that

$$\nu(A) = \int_A F_\nu(t)\, dt \tag{43}$$

for all Borel sets $A \subseteq [0,1]$. In what follows, we let $F := F_\nu$.

6.7. Completion of the proof

We maintain the convention that $x_i = 1/n_i$ for $i \in \{1,\ldots,k\}$. From the preceding discussion we have

$$\begin{aligned}
F(x) &= \widecheck{\varphi_\nu}(x) \\
&= \left((k-1)!\sum_{r=1}^{k}\frac{e^{ix_r z}}{(iz)^{k-1}\prod_{j\neq r}(x_r - x_j)}\right)^{\vee}(x) && \text{(by (42))} \quad (44)\\
&= \frac{(k-1)!}{i^{k-1}}\sum_{r=1}^{k}\frac{1}{\prod_{j\neq r}(x_r - x_j)}\widecheck{\left(\frac{e^{ix_r z}}{z^{k-1}}\right)}(x) \\
&= \frac{(k-1)!}{2(i^{k-1})^2(k-2)!}\sum_{r=1}^{k}\frac{|x-x_r|(x-x_r)^{k-3}}{\prod_{j\neq r}(x_r - x_j)} && \text{(by (34))}\\
&= \frac{k-1}{2}\sum_{r=1}^{k}\frac{|x_r - x|(x_r - x)^{k-3}}{\prod_{j\neq r}(x_r - x_j)} && (45)\\
&= \frac{k-1}{2}\sum_{r=1}^{k}\frac{|\frac{1}{n_r} - x|(\frac{1}{n_r} - x)^{k-3}}{\prod_{j\neq r}(\frac{1}{n_r} - \frac{1}{n_j})} \\
&= \frac{(k-1)n_1 n_2\cdots n_k}{2}\sum_{r=1}^{k}\frac{|1-n_r x|(1-n_r x)^{k-3}}{\prod_{j\neq r}(n_j - n_r)}. && (46)
\end{aligned}$$

For $k \geq 3$, induction and the definition of the derivative confirm that $|x|x^{k-3}$ is $k-3$ times continuously differentiable on $\mathbb{R}$, but is not differentiable $k-2$ times at $x = 0$. Since the $n_1, n_2, \ldots, n_k$ are distinct and because the zeros of $1 - n_r x$ belong to $[0,1]$, we conclude that F is $k-3$ times continuously differentiable on $[0,1]$, but not differentiable $k-2$ times there.

Let $[\alpha,\beta] \subseteq [0,1]$. Observe that $\partial[\alpha,\beta] = \{\alpha,\beta\}$, so (43) implies

$$\nu(\{\alpha,\beta\}) = \nu(\{\alpha\}) + \nu(\{\beta\}) = \int_{\alpha}^{\alpha} F(t)\,dt + \int_{\beta}^{\beta} F(t)\,dt = 0.$$

Characterization (c) of the weak convergence $\nu_n \to \nu$ (Subsection 6.1) implies

$$\begin{aligned}
\lim_{n\to\infty} \frac{|\{\ell : \mathsf{L}[\![n]\!] : \ell \in [\alpha n, \beta n]\}|}{|\mathsf{L}[\![n]\!]|} &= \lim_{n\to\infty} \frac{|\{\ell : \mathsf{L}[\![n]\!] : \frac{\ell}{n} \in [\alpha,\beta]\}|}{|\mathsf{L}[\![n]\!]|} \\
&= \lim_{n\to\infty} \nu_n\big([\alpha,\beta]\big) && \text{by (39)} \\
&= \nu\big([\alpha,\beta]\big) \\
&= \int_{\alpha}^{\beta} F(t)\,dt.
\end{aligned}$$

Now observe that F is supported on $[1/n_k, 1/n_1]$ since (40) ensures that

$$\nu([a,b]) = \lim_{n\to\infty} \nu_n([a,b]) = 0$$

for any interval $[a,b]$ that does not intersect $[1/n_k, 1/n_1]$. Consequently, characterization (a) of the weak convergence $\nu_n \to \nu$ and Lemma 36 yield

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\ell \in \mathsf{L}[\![n]\!]} g(\ell/n) &= \lim_{n\to\infty} \int_0^1 g(t)\,d\nu_n(t) \\
&= \int_0^1 g(t)\,d\nu(t) \\
&= \int_0^1 g(t)F(t)\,dt
\end{aligned}$$

for any continuous function $g : (0,1) \to \mathbb{C}$ (since F is supported on $[1/n_k, 1/n_1] \subset (0,1)$, the values of g outside of this interval are irrelevant). This concludes the proof of Theorem 1. $\square$

7. Concluding remarks

Theorem 1 (which depends upon Theorems 2 and 3) appears to answer all questions about the asymptotic behavior of factorization length multisets in numerical semigroups.

However, there are a few issues it does not immediately address which suggest several avenues for further exploration.

Although the length distribution function F provided by Theorem 1 is explicit, it is no longer amenable to symbolic computation when a semigroup has a large number of generators. That is, one typically does not expect closed-form answers in terms of $n_1, n_2, \ldots, n_k$.

We have shown analytically that F is unimodal for $k = 3, 4$, and as previously mentioned, a general proof was subsequently obtained in [15]. Finally, the numerical examples in Section 2 suggest relatively rapid convergence in parts (a) and (c) of Theorem 1. It would be of some interest to prove this analytically. However, the techniques involved in the proof of these theorems do not appear to readily admit quantitative estimates.

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