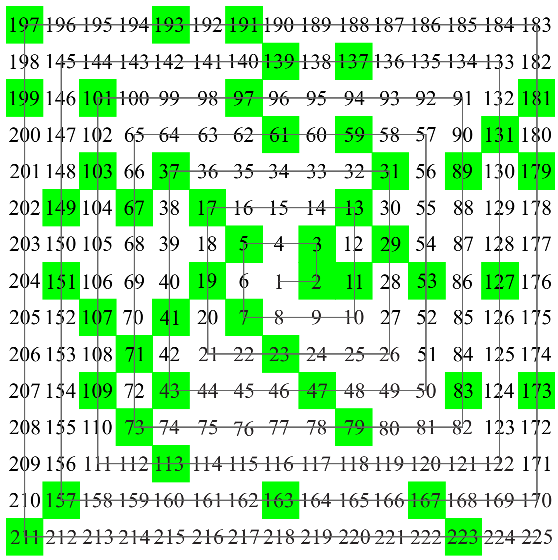
Composites in the Ulam Spiral

In 1963, Stanisław Ulam plotted the natural numbers in a rectilinear spiral array and observed that prime values in the resulting *Ulam spiral* tend to cluster on certain diagonal lines [1]. The Ulam spiral also contains arbitrarily large square patches consisting entirely of composite numbers.



The nth element on the diagonal ray $1, 9, 25, 49, \ldots$ in the Ulam spiral is $(2n-1)^2$. The $d \times d$ block with $(2n+3)^2$ as its upper-left corner is a $d \times d$ matrix whose entries are quadratic polynomials in n with positive integer coefficients and constant terms greater than 1. For example, $d = 3$ yields

$$A(n) = \begin{bmatrix} (2n+3)^2 & (2n+3)^2+1 & (2n+5)^2+2 \\ (2n+5)^2-1 & (2n+5)^2 & (2n+5)^2+1 \\ (2n+7)^2-2 & (2n+7)^2-1 & (2n+7)^2 \end{bmatrix} \equiv \begin{bmatrix} 9 & 10 & 27 \\ 24 & 25 & 26 \\ 47 & 48 & 49 \end{bmatrix} \pmod{n}.$$

Let ℓ denote the least common multiple of the constant terms of these d^2 polynomials. For example, $\ell = \operatorname{lcm}\{9, 10, 24, 25, 26, 27, 47, 48, 49\} = 323{,}341{,}200$ when $d = 3$. Since $(2\ell+3)^2 > \ell$, each entry of $A(\ell)$ is larger than ℓ and divisible by a prime factor of ℓ. Thus, $A(\ell)$ contains only composite numbers.

REFERENCES

[1] Gardner, M. (1964). Mathematical games: The remarkable lore of the prime numbers. *Sci. Amer.* 210(3): 120–128.

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