Existence and weak-strong uniqueness of solutions to the Cahn-Hilliard-Navier-Stokes-Darcy system in superposed free flow and porous media

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Abstract

We study a diffuse interface model for two-phase flows of similar densities in superposed free flow and porous media. The model consists of the Navier-Stokes-Cahn-Hilliard system in free flow and the Darcy-Cahn-Hilliard system in porous media coupled through a set of domain interface boundary conditions. These domain interface boundary conditions include the nonlinear Lions interface condition and the linear Beavers-Joseph-Saffman-Jones interface condition. We establish global existence of weak solutions in three dimension. We also show that the strong solution if exists agrees with the weak solutions.

Keywords: Navier-Stokes, Cahn-Hilliard, Darcy, diffuse interface model, well-posedness, superposed free flow and porous media
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1. Introduction

Multiphase flows are important to various engineering processes. In many applications such as contaminant transport in karst aquifers, oil recovery, the development of sinkholes, the biogeochemical processes in hyporheic zone of river

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beds, the proton exchange membrane fuel cell technology and cardiovascular modeling, multiphase flows in conduits/channels and in porous media interact with each other, and therefore need to be considered together. See Fig. 2.1 for an illustration of the coupled domain. In this article, we aim to study the well-posedeness of a diffuse interface model for multiphase flows in conduits and porous media where the Navier-Stokes-Cahn-Hilliard equations (NSCH) are coupled with the Darcy-Cahn-Hilliard equations (DCH) through a set of domain interface boundary conditions.

The well-posedness of either the NSCH system or the DCH system in single domains has been intensively investigated in recent years. Boyer in [1] studies existence and uniqueness as well as asymptotic stability of solutions of the NSCH system with both regular and degenerate mobility. Global (weak solutions, strong solutions in 2D) and local well-posedness (strong solutions in 3D), and regularity of solutions are further examined by Abels [2] and more recently by Giorgini et al in [3] for the NSCH system of singular free energy densities and matched densities, see [4, 5, 6, 7] for results regarding the NSCH type equations with general densities. Long time behavior of solutions to the NSCH system can be found in [8, 9, 10]. As for the DCH system (also referred to as Cahn-Hilliard-Hele-Shaw), the global existence of weak solutions is first established by Feng and Wise in [11]. Wang and Zhang [12] establish the existence and uniqueness of regular solutions (global in 2D and local in 3D) for the DCH system of variable viscosities, cf. [13] for the study on long-time behavior. Global well-posedness (resp., local) is also established by Zhao et. al. [14] in 2D (resp., 3D) for the DCH system modeling tumour growth, see also [15, 16, 17, 18, 19, 20]. The CHD system with the singular potential has been extensively analysed by Giorgini et al in [16, 21].

The diffuse interface model for two-phase flows in the coupled conduit and porous media setting is first derived by Han et al. in [22] via Onsager's extremum principle. The derivation only takes into account the irreversible part of the dynamics resulting in the coupling of the Stokes-Cahn-Hiliard equations in conduit and the DCH system in porous media. The existence and uniqueness of global weak finite energy solutions is shown in [23], see [24] for numerical methods solving the coupled system. A numerical model consisting of the NSCH system and Richards equation in a coupled free flow and porous media system is proposed by Chen et al. [25] in which the well-posedness is not analysed.

In this article, we propose a diffuse interface model for two-phase flows in the superposed free flow and porous media where the free flow is necessarily governed by the Navier-Stokes equations. The model comprises the NSCH system in free flow (hence incorporating the reversible dynamics) and the DCH system in porous media coupled via a set of domain interface boundary conditions. We establish the global existence of weak solutions in three dimension. Moreover, provided that there exists a strong solution (not established in this article), we show that the strong solution agrees with weak solutions (weak-strong uniqueness). These results are in parallel to those in [23] for the Cahn-Hilliard-Stokes-Darcy model. Central to our analysis is the utilization of the Lions interface boundary condition, cf. (2.10), which states that the stress in the normal direction to the domain interface including the dynamic pressure in free flow is balanced by the flow pressure in porous media. As a consequence one can show that the model obeys an energy law which implies the necessary a priori estimates for compactness argument. Compared to the work [23] for the Cahn-Hilliard-Stokes-Darcy system, the adoption of Navier-Stokes equations and the nonlinear Lions interface boundary conditions introduces extra nonlinearity and strong coupling among the equations. For establishing the existence of weak solutions we develop a divide-and-conquer strategy by taking advantage of the coupling of the equations via the chemical potential (an idea from [26]), and by application of the Leray-Schauder principle. We should also emphasize that the coupling between Cahn-Hilliard-Navier-Stokes system and Cahn-Hilliard-Darcy system poses new challenge for analysis. For instance, the uniqueness of weak solution in two dimensions and (local) existence of strong solutions remain open, even for the case of Cahn-Hilliard-Stokes-Darcy system [23].

There is a vast literature on single phase flows in the context of coupled free flow and porous media. Interested readers can refer to [27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. The rest of the article is organized as follows. In Section 2, we present the Cahn-Hilliard-Navier-Stokes-Darcy model, introduce the weak formulation and state the main theorem of the article. We prove existence of weak solutions in Section 3 based on solutions to a time-discrete elliptic system and compactness arguments. In Section 4 we establish the weak-strong uniqueness result. We give a brief derivation of the model in the Appendix.

2. The Cahn-Hilliard-Navier-Stokes-Darcy system and main result

In this section, we present the Cahn-Hilliard-Navier-Stokes-Darcy model (CH-NSD) for two phase flows of matched densities in superposed free flow and porous media; then we introduce the weak formulation of the model; finally we state the main results of this article. We will focus on the three dimensional case with the understanding that similar result holds for the two dimensional domain.

2.1. The CH-NSD system

The physical setting of the problem is that there is a mixture of two fluids (say oil and water) occupying the free flow region and porous media region. Through the domain interface of the two regions fluid in the two systems can exchange. Detailed discussion of the physical background and the derivation of the CH-NSD model are given in the Appendix. We also refer to [22] for a similar model (Cahn-Hilliard-Stokes-Darcy) where the Navier-Stokes equations are replaced by the Stokes equation equipped with some linear interface boundary conditions.

We consider a bounded domain $\Omega = \Omega_c \bigcup \Omega_m \subset \mathbb{R}^3$ of $C^{2,1}$ boundary $\partial\Omega$, where Ω_c is the free-flow region and Ω_m is the porous media region. Let $\partial\Omega_c$ and $\partial\Omega_m$, which are assumed to be Lipschitz continuous, denote the boundaries of Ω_c and Ω_m, respectively. Let $\Gamma = \partial\Omega_m \cap \partial\Omega_c$, $\Gamma_m = \partial\Omega_m \backslash \Gamma$, and $\Gamma_c = \partial\Omega_c \backslash \Gamma$. A two-dimensional geometry is shown in Figure 2.1 for illustration.

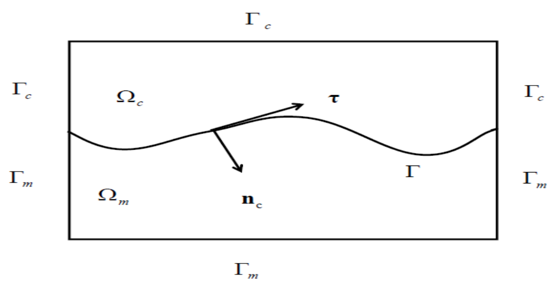

Figure 2.1: A sketch of the porous median domain Ω_m, fluid domain Ω_c, and the interface Γ.

For analysis purpose, we take the background density ρ_0 and the gravitational constant g to be unity throughout the rest of the article. Define $f(\phi) = F'(\phi)$ where $F(\phi)$ is a double-well polynomial: $F(\phi) = \frac{1}{4\epsilon}(\phi^2 - 1)^2$ with ϵ a measure of the capillary width of the thin interfacial region between two fluids. Throughout this article, $\phi|_{\Omega_i} = \phi_i$ $(i = c, m)$, and ϕ_i represents the phase function (order parameter) in Ω_i $(i = c, m)$, which attains distinct values (approximately -1 and 1) in the pure phases away from thin interfacial region and varies smoothly over this interfacial region, for distinguishing the fluid phases.

The Cahn-Hilliard-Navier-Stokes-Darcy model (cf. the Appendix for a derivation of the model) for two-phase superposed free flow and porous media comprises:
the Darcy-Cahn-Hilliard (DCH) equations in porous media Ω_m

$$\boldsymbol{u}_m = -\frac{\Pi}{\nu(\phi_m)}(\nabla p_m - w_m \nabla \phi_m), \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u}_m = 0, \tag{2.2}$$

$$\frac{\partial \phi_m}{\partial t} + \boldsymbol{u}_m \cdot \nabla \phi_m - \nabla \cdot (M(\phi_m) \nabla w_m) = 0, \tag{2.3}$$

$$w_m = -\gamma \epsilon \triangle \phi_m + \gamma f(\phi_m), \tag{2.4}$$

the Navier-Stokes-Cahn-Hilliard (NSCH) equations in free flow Ω_c

$$\frac{\partial \boldsymbol{u}_c}{\partial t} + (\boldsymbol{u}_c \cdot \nabla)\boldsymbol{u}_c - \nabla \cdot \mathbb{T}(\boldsymbol{u}_c, p_c) - w_c \nabla \phi_c = 0, \tag{2.5}$$

$$\nabla \cdot \boldsymbol{u}_c = 0, \tag{2.6}$$

$$\frac{\partial \phi_c}{\partial t} + \boldsymbol{u}_c \cdot \nabla \phi_c - \nabla \cdot (M(\phi_c) \nabla w_c) = 0, \tag{2.7}$$

$$w_c = -\gamma \epsilon \triangle \phi_c + \gamma f(\phi_c), \tag{2.8}$$

subject to the following domain interface boundary conditions on Γ

$$\boldsymbol{u}_c \cdot \mathbf{n}_c = -\boldsymbol{u}_m \cdot \mathbf{n}_m, \tag{2.9}$$

$$-\mathbf{n}_c \cdot (\mathbb{T}(\boldsymbol{u}_c, p_c) \cdot \mathbf{n}_c) + \frac{1}{2}(\boldsymbol{u}_c \cdot \boldsymbol{u}_c) = p_m, \tag{2.10}$$

$$-\boldsymbol{\tau}_j \cdot (\mathbb{T}(\boldsymbol{u}_c, p_c) \cdot \mathbf{n}_c) = \frac{\alpha_B \nu(\phi_c)}{\sqrt{\mathrm{tr}(\Pi)}} \boldsymbol{\tau}_j \cdot \boldsymbol{u}_c, \quad j = 1, 2, \tag{2.11}$$

$$\phi_m = \phi_c, \quad w_c = w_m, \tag{2.12}$$

$$\nabla \phi_c \cdot \mathbf{n}_c = -\nabla \phi_m \cdot \mathbf{n}_m, \quad \nabla w_c \cdot \mathbf{n}_c = -\nabla w_m \cdot \mathbf{n}_m, \tag{2.13}$$

and the following initial and boundary conditions

$$\boldsymbol{u}_m \cdot \mathbf{n}_m|_{\Gamma_m} = 0, \quad \nabla \phi_m \cdot \mathbf{n}_m|_{\Gamma_m} = 0, \quad \nabla w_m \cdot \mathbf{n}_m|_{\Gamma_m} = 0, \tag{2.14}$$

$$\boldsymbol{u}_c|_{\Gamma_c} = 0, \quad \nabla \phi_c \cdot \mathbf{n}_c|_{\Gamma_c} = 0, \quad \nabla w_c \cdot \mathbf{n}_c|_{\Gamma_c} = 0, \tag{2.15}$$

$$\phi_i(0, \boldsymbol{x}) = \phi_i^0(\boldsymbol{x}), \ i = c, m, \quad \boldsymbol{u}_c(0, \boldsymbol{x}) = \boldsymbol{u}_c^0(x), \tag{2.16}$$

where $\mathbf{n}_c = -\mathbf{n}_m$ is the unit outer normal vector relative to Ω_c, cf. the illustration in Fig. 2.1.

In the model for $i = c, m$, $\mathbf{u}_i$ are the fluid velocity; p_i are the pressure; ϕ_i are the order parameters; w_i are the chemical potentials. In addition, we denote by Π the permeability matrix of the porous media, ν the viscosity, M the scalar mobility function, γ the mixing energy density coefficient proportional to surface tension, $\mathbb{T}(\boldsymbol{u}_c, p_c) = 2\nu(\phi_c)\mathbb{D}(\boldsymbol{u}_c) - p_c\mathbb{I}$ the Cauchy stress tensor with $\mathbb{D}(\boldsymbol{u}_c) = \frac{1}{2}(\nabla \boldsymbol{u}_c + \nabla^T \boldsymbol{u}_c)$ the rate of deformation tensor and $\mathbb{I}$ the 3×3 identity matrix. In the domain interface boundary conditions (2.9)–(2.13), α_B is an empirical friction coefficient, $tr(\Pi)$ is the trace of Π, $\boldsymbol{\tau}_j$ $(j = 1, 2)$ denote mutually orthogonal unit tangential vectors to the interface Γ. We may also use P_τ to denote the orthogonal projection onto Γ. The domain interface boundary condition (2.10) expresses the balance of force (including the dynamic pressure) in the normal direction of the interface, also known as the Lions interface boundary condition. The Navier slip condition (2.11) is the celebrated Beavers-Joseph-Saffman-Jones (BJS) interface condition[40].

One can verify that the CH-NSD system satisfies an energy law.

Proposition 2.1. *Let* $(\boldsymbol{u}_m, \boldsymbol{u}_c, \phi, w)$ *be a smooth solution to the initial boundary value problem* (2.1)-(2.16) *with*

$$\phi = \begin{cases} \phi_c & in \quad \Omega_c \\ \phi_m & in \quad \Omega_m \end{cases}, \quad w = \begin{cases} w_c & in \quad \Omega_c \\ w_m & in \quad \Omega_m \end{cases}.$$

Then $(\boldsymbol{u}_m, \boldsymbol{u}_c, \phi, w)$ *satisfies the following energy law:*

$$\frac{d}{dt}E(\boldsymbol{u}_c, \phi) = -\mathcal{D}(t) \leq 0, \tag{2.17}$$

where the total energy E *and the dissipation function* $\mathcal{D}$ *are defined as*

$$\begin{aligned} E(\boldsymbol{u}_c, \phi) :=& \frac{1}{2}\int_{\Omega_c} |\boldsymbol{u}_c|^2 dx + \frac{\gamma\epsilon}{2}\int_{\Omega_c} |\nabla\phi_c|^2 + \frac{\gamma\epsilon}{2}\int_{\Omega_m} |\nabla\phi_c|^2 \\ &+ \gamma\int_{\Omega_c} F(\phi_c)dx, + \gamma\int_{\Omega_m} F(\phi_m)dx, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \mathcal{D}(t) :=& \int_{\Omega_c} M|\nabla w_c|^2 dx + \int_{\Omega_m} M|\nabla w_m|^2 dx + \int_{\Omega_c} 2\nu|\mathbb{D}(\boldsymbol{u}_c)|^2 dx \\ &+ \int_{\Omega_m} \nu|\Pi^{-\frac{1}{2}}\boldsymbol{u}_m|^2 dx + \int_{\Gamma} \frac{\alpha_B \nu}{\sqrt{tr(\Pi)}}|P_\tau \boldsymbol{u}_c|^2 ds. \end{aligned} \tag{2.19}$$

2.2. The weak formulation

We now provide the weak formulation of the Cahn-Hilliard-Navier-Stokes-Darcy model (2.1)-(2.13). We use the standard notation for the Sobolev space $W^{m,k}(\Omega)$, where m is a nonnegative integer and $1 \leq k \leq \infty$. Let $H^m(\Omega) = W^{m,2}(\Omega)$ with the norm $\|\cdot\|_{H^m}$ and the semi norm $|\cdot|_{H^m}$, and $L^k(\Omega) = W^{0,k}(\Omega)$ with the norm $\|\cdot\|_{L^k}$. The norm $\|\cdot\|_{L^\infty}$ denotes the essential supremum. Set $\boldsymbol{V} = [H_0^1(\Omega)]^3 = \{\boldsymbol{v} \in [H^1(\Omega)]^3 : \boldsymbol{v}|_{\partial\Omega} = 0\}$. Define the space

$$\dot{L}^2(\Omega_i) := \left\{ v \in L^2(\Omega_i) \middle| \int_{\Omega_i} v dx = 0 \right\}.$$

Furthermore, we denote $\dot{H}^1(\Omega_i) = H^1(\Omega_i) \cap \dot{L}^2(\Omega_i)$, which is a Hilbert space with inner product $(u, v)_{H^1} = \int_{\Omega_i} \nabla u \cdot \nabla v \, d\boldsymbol{x}$ due to the classical Poincaré inequality for functions with zero mean. Its dual space is simply denoted by $(\dot{H}^1(\Omega_i))'$. For our coupled system, the spaces that we utilize are

$$\begin{aligned}
&\boldsymbol{X}_c = \{\boldsymbol{v} \in [H^1(\Omega_c)]^3, \boldsymbol{v} = 0 \text{ on } \Gamma_c\},\\
&\boldsymbol{X}_{c,div} = \left\{\boldsymbol{v} \in \boldsymbol{X}_c \middle| \nabla\cdot\boldsymbol{v} = 0 \text{ in } \Omega_c\right\},\\
&\boldsymbol{X}_{m,div} = \left\{\boldsymbol{v} \in [L^2(\Omega_m)]^3 \middle| \nabla\cdot\boldsymbol{v} = 0, \boldsymbol{v}\cdot\mathbf{n}_m = 0 \text{ on } \Gamma_m\right\},\\
&Y = \left\{\varphi \in H^1(\Omega)\right\}, \quad Q_i = \dot{H}^1(\Omega_i), \quad Y_i = H^1(\Omega_i), \quad i = c, m.
\end{aligned}$$

P_τ denotes the projection onto the tangent space on Γ, i.e.

$$P_\tau \boldsymbol{u} = \sum_{j=1}^{2} (\boldsymbol{u}\cdot\boldsymbol{\tau}_j)\boldsymbol{\tau}_j.$$

For the domain Ω_i $(i = c, m)$, $(\cdot,\cdot)_i$ denotes the L^2 inner product on the domain Ω_j indicated by the subscript of integrated functions, and $\langle\cdot,\cdot\rangle_\Gamma$ denotes the L^2 inner product on the interface Γ. For convenience, we define the inner product on $L^2(\Omega)$: for $\forall u \in L^2(\Omega)$, $v \in L^2(\Omega)$

$$(u, v) = (u_c, v_c)_c + (u_m, v_m)_m, \tag{2.20}$$

where $u_c = u|_{\Omega_c}$ and $u_m = u|_{\Omega_m}$, and denote

$$\mathbf{L}^2(\Omega_i) = [L^2(\Omega_i)]^3, \quad \mathbf{H}^k(\Omega_i) = [H^k(\Omega_i)]^3, \quad i = c, m.$$

We now introduce the weak formulation for the Cahn-Hilliard-Navier-Stokes-Darcy model, similar to the weak form defined in [23] for the Cahn-Hilliard-Stokes-Darcy system.

Definition 2.1. $(\boldsymbol{u}_c, \boldsymbol{u}_m, p_m, \phi, w)$ *is called a weak solution to the Cahn-Hilliard-Navier-Stokes-Darcy system* (2.1)-(2.16) *if*

$$\boldsymbol{u}_c \in L^\infty\left(0, T; \mathbf{L}^2(\Omega_c)\right) \cap L^2\left(0, T; \boldsymbol{X}_{c,div}\right), \tag{2.21}$$

$$\boldsymbol{u}_m \in L^2\left(0, T; \boldsymbol{X}_{m,div}\right), \tag{2.22}$$

$$p_m \in L^{8/5}\left(0, T; Q_m\right), \tag{2.23}$$

$$\phi \in L^\infty\left(0, T; H^1(\Omega)\right) \cap L^2\left(0, T; H^3(\Omega)\right), \tag{2.24}$$

$$w \in L^2\left(0,T;H^1(\Omega)\right), \tag{2.25}$$

$$\boldsymbol{u}_c \in W^{1,4/3}\big(0,T;(\mathbf{H}^1(\Omega_c))'\big), \quad \phi \in W^{1,8/5}\big(0,T;(H^1(\Omega))'\big), \tag{2.26}$$

and for almost all $t \in (0,T)$ there hold

$$\left\langle \frac{\partial \phi}{\partial t}, \psi \right\rangle + (\boldsymbol{u}\cdot\nabla\phi, \psi) + (M(\phi)\nabla w, \nabla\psi) = 0, \quad \forall\ \psi \in Y, \tag{2.27}$$

$$(w,\omega) - \gamma\epsilon(\nabla\phi,\nabla\omega) - \gamma(f(\phi),\omega) = 0, \quad \forall\ \omega \in Y, \tag{2.28}$$

$$\begin{aligned}\left\langle \frac{\partial \boldsymbol{u}_c}{\partial t}, \boldsymbol{v}_c \right\rangle_c = &\left(\frac{\Pi}{\nu(\phi_m)} w_m \nabla\phi_m, \nabla q_m\right)_m + (w_c\nabla\phi_c, \boldsymbol{v}_c)_c \\ &- \left((\boldsymbol{u}_c\cdot\nabla)\,\boldsymbol{u}_c, \boldsymbol{v}_c\right)_c - a((\boldsymbol{u}_c,p_m),(\boldsymbol{v}_c,q_m)), \\ &\forall\ \boldsymbol{v}_c \in \boldsymbol{X}_{c,div},\ q_m \in Q_m,\end{aligned} \tag{2.29}$$

$$(\boldsymbol{u}_m, \boldsymbol{v}_m)_m = \left(\frac{\Pi}{\nu(\phi_m)}(-\nabla p_m + w_m\nabla\phi_m), \boldsymbol{v}_m\right)_m, \ \forall\ \boldsymbol{v}_m \in \mathbf{L}^2(\Omega_m), \tag{2.30}$$

where

$$\begin{aligned} a((\boldsymbol{u}_c,p_m),(\boldsymbol{v}_c,q_m)) = &(2\nu(\phi_c)\mathbb{D}(\boldsymbol{u}_c), \mathbb{D}(\boldsymbol{v}_c))_c + \left(\frac{\Pi}{\nu(\phi_m)}\nabla p_m, \nabla q_m\right)_m \\ &+ \left\langle \frac{\alpha_B \nu(\phi_c)}{\sqrt{tr(\Pi)}} P_\tau \boldsymbol{u}_c, P_\tau \boldsymbol{v}_c \right\rangle_\Gamma \\ &- \langle \boldsymbol{u}_c\cdot\mathbf{n}_c, q_m\rangle_\Gamma + \left\langle p_m - \frac{1}{2}(\boldsymbol{u}_c\cdot\boldsymbol{u}_c), \boldsymbol{v}_c\cdot\mathbf{n}_c \right\rangle_\Gamma, \end{aligned} \tag{2.31}$$

and $\boldsymbol{u}_c|_{t=0} = \boldsymbol{u}_c^0$, $\phi|_{t=0} = \phi^0$.

Remark 2.1. *Through interpolation, one has $\phi \in C(0,T;L^2(\Omega))$ and $\boldsymbol{u}_c \in C_w\big(0,T;\boldsymbol{L}^2(\Omega_c)\big)$ Hence the initial conditions in Definition 2.1 make sense.*

2.3. The main result

The following conditions on the problem parameters will be assumed throughout the article, cf. [23]:

(i) $M(\phi) \in C^1(\mathbb{R})$, $m_1 \le M(s) \le m_2$ and $|M'(s)| \le \tilde{m}$ for $s \in \mathbb{R}$, where m_1, m_2 and $\tilde{m}$ are positive constants.

(ii) $\nu \in C^1(\mathbb{R})$, $\nu_1 \le \nu(\phi) \le \nu_2$ and $|\nu'(s)| \le \tilde{\nu}$ for $s \in \mathbb{R}$, where ν_1, ν_2 and $\tilde{\nu}$ are positive constants and ν_1.

(iii) The permeability Π is isotropic, bounded from above and below, namely, $\Pi = \kappa(x)\mathbb{I}$ with $\mathbb{I}$ being the $d \times d$ identity matrix and $\kappa(x) \in L^\infty(\Omega)$ such that there exist $\kappa_2 > \kappa_1 > 0$, $\kappa_1 \le \kappa(x) \le \kappa_2$ a.e. in Ω.

The main results of this article are summarized in the following two theorems.

Theorem 2.1 (Existence of weak solutions)**.** *Suppose that the assumptions* (i)–(iii) *are satisfied. Then for any* $\boldsymbol{u}_c^0 \in \mathbf{L}^2(\Omega_c)$, $\phi^0 \in H^1(\Omega)$, *and* $T > 0$, *there exists at least one weak solution to the Cahn-Hilliard-Navier-Stokes-Darcy system* (2.1)-(2.16) *in the sense of Definition* 2.1. *Moreover, the following energy inequality holds in the sense of distribution*

$$\frac{d}{dt}E(\boldsymbol{u}_c, \phi) \le -\mathcal{D}(t), \tag{2.32}$$

where E *and* $\mathcal{D}$ *are defined in Eqs.* (2.18) *and* (2.19).

Theorem 2.2 (Weak-strong uniqueness)**.** *The strong solution to the Cahn-Hilliard-Navier-Stokes-Darcy system, if exists such that*

$$\begin{aligned} &\boldsymbol{u}_c \in L^\infty(0,T;\boldsymbol{X}_{c,div}), \boldsymbol{u}_m \in L^\infty\big(0,T;\mathbf{H}^1(\Omega_m)\big), \\ &\phi \in L^\infty\big(0,T;H^2(\Omega)\big) \cap L^2\big(0,T;H^4(\Omega)\big), \end{aligned} \tag{2.33}$$

is unique in the class of the weak solutions in the sense of Definition 2.1.

Remark 2.2. *The energy inequality* (2.32) *can be interpreted as*

$$E(u_c(t), \phi(t)) \le E(u_c^0, \phi^0) - \int_0^t \mathcal{D}(s)ds.$$

Several remarks are in order. First, for the purpose of establishing the weak-strong uniqueness, the regularity assumption (2.33) can be weakened as in the Cahn-Hilliard-Stokes-Darcy model [23]. Second, in the two-dimensional case, uniqueness of weak solutions to the CH-NSD system is beyond immediate reach, in contrast to the single domain case, see [1, 2] for the Cahn-Hilliard-Navier-Stokes system and [21] for the Cahn-Hilliard-Darcy system. This is because the low temporal regularity of the Darcy pressure (cf. Eq. (3.61) in [23]) and the coupling of Navier-Stokes equations and Darcy equations via domain interface boundary condition leads to reduced temporal regularity of $\frac{\partial \boldsymbol{u}_c}{\partial t}$. Finally, we point out that the (finite-time) existence of the strong solution is an outstanding open question for the coupled Cahn-Hilliard-Navier-Stokes-Darcy system. It is also open for the Navier-Stokes-Darcy type system in the case of single phase flow in superposed free flow and porous media. While the spatial regularity can be iteratively improved in individual domains, to gain further temporal regularity one needs to differentiate in time the whole system due to the presence of domain interface boundary conditions. This will be pursued in another work.

3. Existence of weak solutions

In this section, we establish the existence of weak solutions by following the same semi-discretization method as in our earlier work [23] and the classical compactness argument. That is, one constructs an approximate solution which solves an elliptic system resulting from a temporal discretization of the CH-NSD system, obtains a priori estimates of the approximate solution, and finally passes to the limit.

For a large positive integer N, let $\delta = \frac{T}{N}$. The time-discrete scheme reads as follows. Given $(\boldsymbol{u}_c^k, \phi^k) \in \mathbf{L}^2(\Omega_c) \times H^1(\Omega)$, $k = 0, 1, ..., N-1$, find

$$(\boldsymbol{u}_c^{k+1}, p_m^{k+1}, \phi^{k+1}, w^{k+1}) \in \mathbf{X}_{c,div} \times Q_m \times H^1(\Omega) \times H^1(\Omega)$$

such that

$$\begin{aligned}(\phi^{k+1}, \psi) + \delta(\boldsymbol{u}^{k+1} \cdot \nabla \phi^{k+1}, \psi) = \\ (\phi^k, \psi) - \delta(M(\phi^k)\nabla w^{k+1}, \nabla \psi),\ \forall\ \psi \in Y,\end{aligned} \tag{3.1}$$

$$(w^{k+1}, \omega) - \gamma\epsilon(\nabla \phi^{k+1}, \nabla \omega) - \gamma(f(\phi^{k+1}, \phi^k), \omega) = 0,\ \forall\ \omega \in Y, \tag{3.2}$$

$$
\begin{aligned}
&(\boldsymbol{u}_c^{k+1}, \boldsymbol{v}_c)_c + \delta((\boldsymbol{u}_c^{k+1}\cdot\nabla)\boldsymbol{u}_c^{k+1}, \boldsymbol{v}_c)_c = \\
&\quad \delta\left(\frac{\Pi}{\nu(\phi_m^k)} w_m^{k+1}\nabla\phi_m^{k+1}, \nabla q_m\right)_m - \delta\tilde{a}((\boldsymbol{u}_c^{k+1}, p_m^{k+1}), (\boldsymbol{v}_c, q_m)) \\
&\quad + \delta(w_c^{k+1}\nabla\phi_c^{k+1}, \boldsymbol{v}_c)_c + (\boldsymbol{u}_c^k, \boldsymbol{v}_c)_c, \quad \forall \boldsymbol{v}_c \in \boldsymbol{X}_{c,div},\ q_m \in H^1(\Omega_m),
\end{aligned} \tag{3.3}
$$

where

$$
(\boldsymbol{u}_m^{k+1}, \boldsymbol{v}_m)_m = \left(\frac{\Pi}{\nu(\phi_m^k)}(-\nabla p_m^{k+1} + w_m^{k+1}\nabla\phi_m^{k+1}), \boldsymbol{v}_m\right)_m,\ \forall\ \boldsymbol{v}_m \in \mathbf{L}^2(\Omega_m). \tag{3.4}
$$

Here $f(\phi^{k+1}, \phi^k) = \frac{1}{\epsilon^2}((\phi^{k+1})^3 - \phi^k)$ and

$$
\begin{aligned}
\tilde{a}((\boldsymbol{u}_c^{k+1}, p_m^{k+1}), (\boldsymbol{v}_c, q_m)) =& (2\nu(\phi_c^k)\mathbb{D}(\boldsymbol{u}_c^{k+1}), \mathbb{D}(\boldsymbol{v}_c))_c + \left(\frac{\Pi}{\nu(\phi_m^k)}\nabla p_m^{k+1}, \nabla q_m\right)_m \\
&+ \left\langle \frac{\alpha_B \nu(\phi_m^k)}{\sqrt{\mathrm{tr}(\Pi)}} P_\tau \boldsymbol{u}_c^{k+1}, P_\tau \boldsymbol{v}_c \right\rangle_\Gamma - \langle \boldsymbol{u}_c^{k+1}\cdot\mathbf{n}_c, q_m\rangle_\Gamma \\
&+ \left\langle p_m^{k+1} - \frac{1}{2}(\boldsymbol{u}_c^{k+1}\cdot\boldsymbol{u}_c^{k+1}), \boldsymbol{v}_c\cdot\mathbf{n}_c \right\rangle_\Gamma.
\end{aligned} \tag{3.5}
$$

We note that $F(a) - F(b) \leq f(a,b)(a-b)$ thanks to the monotonicity of the cubic function a^3.

Before showing the existence of solutions to the elliptic system (3.1)–(3.4), we note that the following lemma is proved in [23].

Lemma 3.1. *Assume that* $(\boldsymbol{u}_c^{k+1}, p_m^{k+1}, \phi^{k+1}, w^{k+1}) \in \mathbf{X}_{c,div} \times Q_m \times H^3(\Omega) \times H^1(\Omega)$ *is a solution to the system* (3.1)–(3.4)*. Then*

$$
\boldsymbol{u}_m^{k+1} \in \boldsymbol{X}_{m,div}, \quad \boldsymbol{u}_m \cdot \boldsymbol{n}_c = \boldsymbol{u}_c \cdot \boldsymbol{n}_c \ \text{in}\ \boldsymbol{H}^{\frac{1}{2}}(\Gamma), \tag{3.6}
$$

$$
\int_\Omega \phi^{k+1} dx = \int_\Omega \phi^k dx. \tag{3.7}
$$

3.1. Existence of weak solutions to the time-discrete scheme

For the sake of simplicity, we will omit the superscript $k+1$ for the unknown variables in the following subsection. We follow the idea in [41] for showing the existence of solutions to the elliptic system (3.1)–(3.3). That is, we apply Leray-Schauder principle to Eq. (3.1) viewed as a nonlinear equation of w in which the solution operators $\phi(w)$ and $(\mathbf{u}_c, p_m)(w)$ are properly defined via Eqs. (3.2) and (3.3), respectively.

Concerning the solvability of the chemical potential equation, i.e.

$$\gamma\epsilon(\nabla\phi, \nabla\psi) + \gamma(f(\phi, \phi^k), \psi) = (w, \psi), \quad \forall\psi \in H^1(\Omega), \tag{3.8}$$

the following result is essentially proved in [22].

Lemma 3.2. *Let $\phi^k \in H^1(\Omega)$. For a given function $w \in H^1(\Omega)$, there is a unique solution $\phi \in H^3(\Omega)$ to the problem (3.8). Moreover, the solution operator $\phi(w) : H^1(\Omega) \longmapsto H^3(\Omega)$ is bounded and continuous in the strong topology.*

The equation (3.3) can be written as

$$\begin{aligned}
&(\boldsymbol{u}_c, \boldsymbol{v}) + \delta((\boldsymbol{u}_c \cdot \nabla)\boldsymbol{u}_c, \boldsymbol{v})_c + \delta\tilde{a}((\boldsymbol{u}_c, p_m), (\boldsymbol{v}, q))_c = \\
&\delta\Big(\frac{\Pi}{\nu(\phi_m^k)}\boldsymbol{f}_m, \nabla q\Big)_m + \delta(\boldsymbol{f}_c, \boldsymbol{v})_c + (\boldsymbol{u}_c^k, \boldsymbol{v})_c, \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{c,div},\ q \in H^1(\Omega_m).
\end{aligned} \tag{3.9}$$

From Lemma 3.2 we know that ϕ is the unique solution of the equation (3.8) for a given $w \in H^1(\Omega)$. So we can define the source terms $\boldsymbol{f}_c = w_c\nabla\phi_c$ and $\boldsymbol{f}_m = w_m\nabla\phi_m$, where $\boldsymbol{f}_c$ and $\boldsymbol{f}_m$ are viewed as functions of w_c and w_m, respectively. To establish the well-posedness of (3.9), we define an equivalent norm on the space $\boldsymbol{W} = \boldsymbol{X}_{c,div} \times Q_m$:

$$\|(\boldsymbol{u}_c, p_m)\|_{\boldsymbol{W}} = \|\boldsymbol{u}_c\|_{L^2}^2 + \|\mathbb{D}(\boldsymbol{u}_c)\|_{L^2}^2 + \|\nabla p_m\|_{L^2}^2 + \|P_\tau \boldsymbol{u}_c\|_{L^2(\Gamma)}^2 \tag{3.10}$$

Lemma 3.3. *For given $\boldsymbol{u}_c^k \in \mathbf{L}^2(\Omega_c)$ and $w \in H^1(\Omega)$ the problem (3.9) admits a solution $(\boldsymbol{u}_c, p_m) \in \boldsymbol{X}_{c,div} \times Q_m$. Moreover, if δ is sufficiently small, the solution operator $w \in H^1(\Omega) \to (\boldsymbol{u}_c, p_m) \in \boldsymbol{X}_{c,div} \times \dot{H}^1(\Omega_m)$ is completely continuous.*

Proof. We employ the Galerkin method for showing existence of solutions. Since the spaces $\boldsymbol{X}_{c,div}$ and Q_m are separable Hilbert spaces, there exists a sequence $\{(\boldsymbol{a}_i, b_i)\}_{i=1}^{+\infty} \in \boldsymbol{X}_{c,div} \times Q_m$. For a fixed $n \geq 1$, let $\boldsymbol{X}_{c,div}^{(n)} = span\{\boldsymbol{a}_i, i = 1, ..., n\} \subset X_{c,div}$ and $Q_m^{(n)} = span\{b_i, i = 1, ..., n\} \subset Q_m$, and denote $\boldsymbol{W}^{(n)} = \boldsymbol{X}_{c,div}^{(n)} \times Q_m^{(n)}$. Then a Galerkin approximation to the problem (3.9) is to find

$(\boldsymbol{u}_{c,n}, p_{m,n}) \in \boldsymbol{W}^{(n)}$ such that

$$
\begin{aligned}
&(\boldsymbol{u}_{c,n}, \boldsymbol{a}_i)_c + \delta((\boldsymbol{u}_{c,n} \cdot \nabla)\boldsymbol{u}_{c,n}, \boldsymbol{a}_i)_c \\
&= \delta\Big(\frac{\Pi}{\nu(\phi_m^k)}\boldsymbol{f}_m, \nabla b_i\Big)_m - \delta\tilde{a}\big((\boldsymbol{u}_{c,n}, p_{m,n}), (\boldsymbol{a}_i, b_i)\big) \\
&\quad + \delta(\boldsymbol{f}_c, \boldsymbol{a}_i)_c + (\boldsymbol{u}_c^k, \boldsymbol{a}_i)_c, \quad \forall(\boldsymbol{a}_i, b_i) \in \boldsymbol{W}^{(n)}.
\end{aligned} \tag{3.11}
$$

Eqs. (3.11) are a nonlinear system in a finite dimensional Hilbert space. We show the existence of a solution to (3.11) by the Brouwer fixed point theorem in finite dimension, cf. [42] (Lemma 1.4, pp. 110). Since $\boldsymbol{W}^{(n)}$ is a finite dimensional Hilbert space, we introduce the mapping: $\mathcal{F}_n : \boldsymbol{W}^{(n)} \to \boldsymbol{W}^{(n)}$ defined by

$$
\begin{aligned}
&[\mathcal{F}_n(\boldsymbol{u}, p), (\boldsymbol{v}_c, q_m)] = (\boldsymbol{u}, \boldsymbol{v}_c)_c + \delta\big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}_c\big)_c \\
&\quad + \delta\tilde{a}\big((\boldsymbol{u}, p), (\boldsymbol{v}_c, q_m)\big) - \delta\Big(\frac{\Pi}{\nu(\phi_m^k)}\boldsymbol{f}_m, \nabla q_m\Big)_m \\
&\quad - \delta(\boldsymbol{f}_c, \boldsymbol{v}_c)_c - (\boldsymbol{u}_c^k, \boldsymbol{v}_c)_c, \qquad \forall\ (\boldsymbol{v}_c, q_m) \in \boldsymbol{W}^{(n)}.
\end{aligned} \tag{3.12}
$$

It is clear that $\mathcal{F}_n$ is continuous. Next, we recall the definition of $\tilde{a}$ in (3.5), perform integration by parts and calculate

$$
\begin{aligned}
&[\mathcal{F}_n(\boldsymbol{v}_c, q_m), (\boldsymbol{v}_c, q_m)] = (\boldsymbol{v}_c, \boldsymbol{v}_c)_c + \delta\big((\boldsymbol{v}_c \cdot \nabla)\boldsymbol{v}_c, \boldsymbol{v}_c\big)_c + \delta\tilde{a}((\boldsymbol{v}_c, q_m), (\boldsymbol{v}_c, q_m)) \\
&\quad - \delta\left(\frac{\Pi}{\nu(\phi_m^k)}\boldsymbol{f}_m, \nabla q_m\right)_m - \delta(\boldsymbol{f}_c, \boldsymbol{v}_c) - (\boldsymbol{u}_c^k, \boldsymbol{v}_c)_c \\
&\geq \|\boldsymbol{v}_c\|_{L^2}^2 + \delta C\left[\|\mathbb{D}(\boldsymbol{v}_c)\|_{L^2}^2 + \|\nabla q_m\|_{L^2}^2 + \|P_\tau \boldsymbol{v}_c\|_{L^2(\Gamma)}^2\right] \\
&\quad - \delta\|\boldsymbol{f}_c\|_{L^2}\|\boldsymbol{v}_c\|_{L^2} - C\delta\|\boldsymbol{f}_m\|_{L^2}\|\nabla q_m\|_{L^2} - \|\boldsymbol{u}_c^k\|_{L^2}\|\boldsymbol{v}_c\|_{L^2} \\
&> \frac{1}{2}\|\boldsymbol{v}_c\|_{L^2}^2 + \delta C\left[\|\mathbb{D}(\boldsymbol{v}_c)\|_{L^2}^2 + \|\nabla q_m\|_{L^2}^2 + \|P_\tau \boldsymbol{v}_c\|_{L^2(\Gamma)}^2\right] \\
&\quad - C\left[\|\boldsymbol{f}_c\|_{L^2}^2 + \|\boldsymbol{f}_m\|_{L^2}^2 + \|\boldsymbol{u}_c^k\|_{L^2}^2\right],
\end{aligned} \tag{3.13}
$$

where $((\boldsymbol{v}_c \cdot \nabla)\boldsymbol{v}_c, \boldsymbol{v}_c)_c + \big\langle -\frac{1}{2}(\boldsymbol{v}_c \cdot \boldsymbol{v}_c), \boldsymbol{v}_c \cdot \mathbf{n}_c\big\rangle_\Gamma = 0$ by integration by parts. It follows that $[\mathcal{F}_n(\boldsymbol{v}_c, q_m), (\boldsymbol{v}_c, q_m)] > 0$ as long as $\|(\boldsymbol{v}_c, q_m)\|_{\boldsymbol{W}}$ is sufficiently large. Hence there exists a solution $(\boldsymbol{u}_{c,n}, p_{m,n})$ to the Eqs. (3.11).

Now we derive some a priori estimates of $(\boldsymbol{u}_{c,n}, p_{m,n})$. By performing integration by parts, one notes the identity

$$\begin{aligned}&\big((\boldsymbol{u}_{c,n}\cdot\nabla)\boldsymbol{u}_{c,n},\boldsymbol{v}_c\big)_c-\Big\langle\frac{1}{2}(\boldsymbol{u}_{c,n}\cdot\boldsymbol{u}_{c,n}),\boldsymbol{v}_c\cdot\mathbf{n}_c\Big\rangle_\Gamma\\&=\big((\boldsymbol{u}_{c,n}\cdot\nabla)\boldsymbol{u}_{c,n},\boldsymbol{v}_c\big)_c-\big((\boldsymbol{v}_c\cdot\nabla)\boldsymbol{u}_{c,n},\boldsymbol{u}_{c,n}\big)_c.\end{aligned}\tag{3.14}$$

Choosing $\boldsymbol{a}_i=\boldsymbol{u}_{c,n}$, $b_i=p_{m,n}$ in (3.11) yields

$$\begin{aligned}&(\boldsymbol{u}_{c,n},\boldsymbol{u}_{c,n})_c+\delta\big((\boldsymbol{u}_{c,n}\cdot\nabla)\boldsymbol{u}_{c,n},\boldsymbol{u}_{c,n}\big)_c+\delta a((\boldsymbol{u}_{c,n},p_{m,n}),(\boldsymbol{u}_{c,n},p_{m,n}))\\&=\delta\Big(\frac{\Pi}{\nu(\phi_m^k)}\boldsymbol{f}_m,\nabla p_{m,n}\Big)_m+\delta(\boldsymbol{f}_c,\boldsymbol{u}_{c,n})_c+(\boldsymbol{u}_c^k,\boldsymbol{u}_{c,n})_c.\end{aligned}\tag{3.15}$$

By the identity (3.14), the nonlinear term in (3.15) vanishes, i.e.

$$\big((\boldsymbol{u}_{c,n}\cdot\nabla)\boldsymbol{u}_{c,n},\boldsymbol{u}_{c,n}\big)_c-\Big\langle\frac{1}{2}(\boldsymbol{u}_{c,n}\cdot\boldsymbol{u}_{c,n}),\boldsymbol{u}_{c,n}\cdot\mathbf{n}_c\Big\rangle_\Gamma=0.$$

Eq. (3.15) implies

$$\|(\boldsymbol{u}_{c,n},p_{m,n})\|_{\boldsymbol{W}}\leq C\big(\|f_m\|_{L^2}+\|f_c\|_{L^2}+\|\boldsymbol{u}_c^k\|_{L^2}\big).\tag{3.16}$$

Since $\boldsymbol{X}_{c,div}\times Q_m$ is a reflexive Hilbert space, there exists a subsequence still denoted by $\{(\boldsymbol{u}_{c,n},p_{m,n})\}_{n\in\mathbb{N}}$ and a pair $(\boldsymbol{u}_c,p_m)\in\boldsymbol{X}_{c,div}\times Q_m$ such that

$$\boldsymbol{u}_{c,n}\longrightarrow\boldsymbol{u}_c\quad\textit{weakly in}\quad\boldsymbol{X}_{c,div},\tag{3.17}$$

$$\boldsymbol{u}_{c,n}\longrightarrow\boldsymbol{u}_c\quad\textit{strongly in}\quad\mathbf{L}^4(\Omega_c),\tag{3.18}$$

$$p_{m,n}\longrightarrow p_m\quad\textit{weakly in}\quad Q_m,\tag{3.19}$$

$$p_{m,n}\longrightarrow p_m\quad\textit{strongly in}\quad L^2(\Omega_m).\tag{3.20}$$

To pass to the limit in the nonlinear term, one notes that

$$\begin{aligned}&\big((\boldsymbol{u}_{c,n}\cdot\nabla)\boldsymbol{u}_{c,n},\boldsymbol{v}_c\big)_c-\big((\boldsymbol{u}_c\cdot\nabla)\boldsymbol{u}_c,\boldsymbol{v}_c\big)_c\\&=\big((\boldsymbol{u}_{c,n}-\boldsymbol{u}_c)\cdot\nabla\boldsymbol{u}_{c,n},\boldsymbol{v}_c\big)_c-\big((\boldsymbol{u}_c\cdot\nabla)(\boldsymbol{u}_c-\boldsymbol{u}_{c,n}),\boldsymbol{v}_c\big)_c.\end{aligned}$$

By the identity (3.14), and the convergence (3.17), (3.18), one concludes that

$$\begin{aligned}&\big((\boldsymbol{u}_{c,n}\cdot\nabla)\boldsymbol{u}_{c,n},\boldsymbol{v}_c\big)_c-\Big\langle\frac{1}{2}(\boldsymbol{u}_{c,n}\cdot\boldsymbol{u}_{c,n}),\boldsymbol{v}_c\cdot\mathbf{n}_c\Big\rangle_\Gamma\longrightarrow\\&\big((\boldsymbol{u}_c\cdot\nabla)\boldsymbol{u}_c,\boldsymbol{v}_c\big)_c-\Big\langle\frac{1}{2}(\boldsymbol{u}_c\cdot\boldsymbol{u}_c),\boldsymbol{v}_c\cdot\mathbf{n}_c\Big\rangle_\Gamma.\end{aligned}$$

Then passing to the limit in (3.11) with $n \to \infty$ we find that

$$\begin{aligned}&(\boldsymbol{u}_c, \boldsymbol{v}_c)_c + \delta((\boldsymbol{u}_c \cdot \nabla)\boldsymbol{u}_c, \boldsymbol{v}_c))_c + \delta a((\boldsymbol{u}_c, p_m), (\boldsymbol{v}_c, q_m)) \\ &= \delta\Big(\frac{\Pi}{\nu(\phi_m^k)}\boldsymbol{f}_m, \nabla q_m\Big)_m + \delta(\boldsymbol{f}_c, \boldsymbol{v}_c))_c + (\boldsymbol{u}_c^k, \boldsymbol{v}_c))_c,\end{aligned} \tag{3.21}$$

where $\boldsymbol{v}_c$ is linear combination of $\boldsymbol{a}_1, ..., \boldsymbol{a}_n, ...$, and q_m is linear combination of $b_1, ..., b_n, ...$ Since these combination are dense in $\boldsymbol{X}_{c,div}$ and Q_m, (3.21) hold for any $\boldsymbol{v}_c \in \boldsymbol{X}_{c,div}, q_m \in Q_m$ by a continuity argument. Hence $(\boldsymbol{u}_c, p_m)$ is a solution to Eqs. (3.9).

Given $w \in H^1(\Omega)$, let $\phi \in H^3(\Omega)$ be the unique solution to Eq. (3.8) according to Lemma 3.2. We show that the mapping $w \in H^1(\Omega) \to (\boldsymbol{u}_c, p_m) \in \boldsymbol{W}$ via Eqs. (3.9) is completely continuous.

Suppose $(\boldsymbol{u}_c^i, p_m^i), i = 1, 2$ are two solutions corresponding to $\mathbf{f}^i, i = 1, 2$ respectively. Define

$$\boldsymbol{u}_c^e = \boldsymbol{u}_c^1 - \boldsymbol{u}_c^2; \quad p_m^e = p_m^1 - p_m^2; \quad \mathbf{f}^e = \mathbf{f}^1 - \mathbf{f}^2.$$

One obtains

$$\begin{aligned}&(\boldsymbol{u}_c^e, \boldsymbol{v}_c)_c + \delta\Big(2\nu(\phi_c^k)\mathbb{D}(\boldsymbol{u}_c^e), \mathbb{D}(\boldsymbol{v}_c)\Big)_c + \delta\left(\frac{\Pi}{\nu(\phi_m^k)}\nabla p_m^e, \nabla q_m\right)_m + \delta\left\langle\frac{\alpha_B\nu(\phi_c^k)}{\sqrt{\mathrm{tr}(\Pi)}}P_\tau\boldsymbol{u}_c^e, P_\tau\boldsymbol{v}_c\right\rangle_\Gamma \\ &= \delta\langle\boldsymbol{u}_c^e \cdot \mathbf{n}_c, q_m\rangle_\Gamma - \delta\,\langle p_m^e, \boldsymbol{v}_c \cdot \mathbf{n}_c\rangle_\Gamma + \delta\Big(\frac{\Pi}{\nu(\phi_m^k)}\boldsymbol{f}_m^e, \nabla q\Big)_m + \delta(\boldsymbol{f}_c^e, \boldsymbol{v}_c)_c - \delta\big(\boldsymbol{u}_c^e \cdot \nabla\boldsymbol{u}_c^1, \boldsymbol{v}_c\big)_c \\ &\quad - \delta\big(\boldsymbol{u}_c^2 \cdot \nabla\boldsymbol{u}_c^e, \boldsymbol{v}_c\big)_c + \delta\big(\boldsymbol{v}_c \cdot \nabla\boldsymbol{u}_c^e, \boldsymbol{u}_c^2\big)_c + \delta\big(\boldsymbol{v}_c \cdot \nabla\boldsymbol{u}_c^1, \boldsymbol{u}_c^e\big)_c, \quad \forall \boldsymbol{v}_c \in \boldsymbol{X}_{c,div},\ q \in H^1(\Omega_m),\end{aligned} \tag{3.22}$$

where one has applied the identity (3.14) in treating the nonlinear terms. Taking $(\boldsymbol{v}_c, q) = (\boldsymbol{u}_c^e, p_m^e)$ in Eqs. (3.22), and noting that

$$\begin{aligned}\delta|\big(\boldsymbol{u}_c^2 \cdot \nabla\boldsymbol{u}_c^e, \boldsymbol{u}_c^e\big)_c + \big(\boldsymbol{u}_c^e \cdot \nabla\boldsymbol{u}_c^e, \boldsymbol{u}_c^2\big)_c| &\le \delta||\boldsymbol{u}_c^2||_{L^4}||\nabla\boldsymbol{u}_c^e||_{L^2}||\boldsymbol{u}_c^e||_{L^4} \\ &\le \delta||\boldsymbol{u}_c^2||_{L^4}||\boldsymbol{u}_c^e||_{L^2}^{\frac{1}{4}}||\nabla\boldsymbol{u}_c^e||_{L^2}^{\frac{7}{4}} \\ &\le C\delta||\boldsymbol{u}_c^e||_{L^2}^2 + \chi\delta||\mathbb{D}(\boldsymbol{u}_c^e)||_{L^2}^2,\end{aligned}$$

one derives for sufficiently small δ and χ that

$$||(\boldsymbol{u}_c^e, p_m^e)||_{\mathbf{W}} \le C||\mathbf{f}^e||_{L^2}. \tag{3.23}$$

Hence the solution depends continuously on $\mathbf{f}$ in the strong topology. On the other hand, the solution operator $w \in H^1(\Omega) \to \phi \in H^3(\Omega)$ is continuous by Lemma 3.2. Since the embedding $H^3 \hookrightarrow C^1$ is compact, it follows that the mapping $w \in H^1(\Omega) \to \mathbf{f} = w\nabla\phi \in L^2(\Omega)$ is completely continuous. Thus the solution operators $(\boldsymbol{u}_c, p_m) : w \in H^1(\Omega) \to (\boldsymbol{u}_c, p_m) \in \boldsymbol{X}_{c,div} \times \dot{H}^1(\Omega_m)$ is completely continuous. This completes the proof. □

The following lemma is obvious.

Lemma 3.4. *For given $f \in (H^1(\Omega)^{'})$, there exist a unique solution w to the problem*

$$(\omega, \psi) + \delta(M(\phi^k)\nabla\omega, \nabla\psi) = \langle f, \psi\rangle \textit{ for } \ \forall\psi \in Y. \tag{3.24}$$

In addition, the solution is bounded and depends continuously on the data f.

With the help of Lemmas 3.2–3.4,we finally prove the existence of solutions to the problem (3.1)-(3.3) by the Leray-Schauder principle. For convenience, we rewrite Eq. (3.1) as

$$(\phi, \psi) + \delta(\boldsymbol{u} \cdot \nabla\phi, \psi) + \delta(M(\phi^k)\nabla w, \nabla\psi) = (\phi^k, \psi), \ \forall \ \psi \in Y, \tag{3.25}$$

and view it as an equation for w. We have

Lemma 3.5. *Under the assumptions* (i)-(iii) and suppose $\phi^k \in H^1(\Omega)$. There exists at least one weak solution $(\phi, w, \boldsymbol{u}_c, p_m)$ to the problem (3.1)-(3.3) such that

$$\phi \in H^3(\Omega), \quad w \in H^1(\Omega), \quad \boldsymbol{u}_c \in \boldsymbol{X}_{c,div}, \quad p_m \in X_m.$$

Moreover, there holds the discrete energy law

$$\begin{aligned}
&\frac{1}{2\delta}\|\boldsymbol{u}_c\|_{L^2}^2 + \frac{\gamma\epsilon}{2\delta}\|\nabla\phi\|_{L^2}^2 + \frac{\gamma}{\delta}F(\phi) + \left\|\sqrt{2\nu(\phi_c^k)}\mathbb{D}(\boldsymbol{u}_c)\right\|_{L^2}^2 \\
&+ \frac{1}{2\delta}\|\boldsymbol{u}_c - \boldsymbol{u}_c^k\|_{L^2}^2 + \left\|\sqrt{\Pi^{-1}\nu(\phi_m^k)}\boldsymbol{u}_m\right\|_{L^2} + \left\|\sqrt{M(\phi^k)}\nabla w\right\|_{L^2}^2 \\
&+ \left\langle \frac{\alpha_B \nu(\phi_m^k)}{\sqrt{tr(\Pi)}} P_\tau \boldsymbol{u}_c, P_\tau \boldsymbol{u}_c \right\rangle + \frac{\gamma\epsilon}{2\delta}\|\nabla(\phi - \phi^k)\|_{L^2}^2 \\
&\leq \frac{1}{2\delta}\|\boldsymbol{u}_c^k\|_{L^2}^2 + \frac{\gamma\epsilon}{2\delta}\|\nabla\phi^k\|_{L^2}^2 + \frac{\gamma}{\delta}(F(\phi^k), 1) = \frac{1}{\delta}E(\boldsymbol{u}_c^k, \phi^k).
\end{aligned} \tag{3.26}$$

Proof. Here we apply the Leray-Schauder principle. One defines an operator $T : H^1(\Omega) \longrightarrow H^1(\Omega)$ as follows. Given $w \in H^1(\Omega)$, one solves (3.2)-(3.3) according to Lemma 3.1 and Lemma 3.2. Then one introduces

$$f(w) := w - (\phi(w) - \phi^k) - \delta \boldsymbol{u}(w) \cdot \nabla \phi(w). \tag{3.27}$$

Finally, one defines $T(w)$ as the unique solution to equation (3.24) with the source function $f(w)$. Since the solution operators $w \in H^1(\Omega) \longrightarrow \{\phi, \boldsymbol{u}_c, p_m\} \in C^1(\bar{\Omega}) \times \boldsymbol{X}_{c,div} \times H^1(\Omega_m)$ is completely continuous by Lemma 3.2 and Lemma 3.3, the mapping $w \in H^1(\Omega) \longrightarrow f(w) \in L^2(\Omega)$ is completely continuous. Thus by Lemma 3.4 the operator $T : w \in H^1(\Omega) \longrightarrow T(w) \in H^1(\Omega)$ is completely continuous, and hence compact since $H^1(\Omega)$ is a Hilbert space. To apply the Leray-Shauder principle [43], one needs to show that the set

$$\{w \in H^1(\Omega), w = \lambda T(w),\ for\ some\ \lambda \in (0, 1]\}$$

is bounded. Suppose $w = \lambda T(w)$ for some $\lambda \in (0, 1]$. By the definition of T and the linearity of the Eq. (3.24), w satisfies the following equation

$$(w, \psi) + (M(\phi^k)\nabla w, \nabla \psi) = \lambda(f(w), \psi), \quad \forall\ \psi \in H^1(\Omega). \tag{3.28}$$

By taking $\psi = w$ in (3.28), we have

$$(1 - \lambda)\|w\|_{L^2}^2 + \lambda(\phi - \phi^k, w) + \lambda\delta(\boldsymbol{u} \cdot \nabla\phi, w) = -\left\|\sqrt{M(\phi^k)}\nabla w\right\|_{L^2}^2. \tag{3.29}$$

Setting $\omega = \lambda(\phi - \phi^k)$ in (3.2) and $\boldsymbol{v} = \lambda \boldsymbol{u}_c,\ q = \lambda p_m$ in (3.3), performing integration by parts and adding the results together, we have

$$\begin{aligned}
&\lambda E(\boldsymbol{u}_c, \phi) + \lambda\delta \left\|\sqrt{2\nu(\phi_c^k)}\mathbb{D}(\boldsymbol{u}_c)\right\|_{L^2}^2 + \lambda\delta \left\|\sqrt{\Pi^{-1}\nu(\phi_m)}\boldsymbol{u}_m\right\|_{L^2}^2 \\
&+ \frac{\lambda\delta\alpha_B}{\sqrt{tr(\Pi)}} \left\|\sqrt{\nu(\phi_m^k)}P_\tau \boldsymbol{u}_c\right\|_{L^2}^2 + \left\|\sqrt{M(\phi^k)}\nabla w\right\|_{L^2}^2 \\
&+ (1 - \lambda)\|w\|_{L^2}^2 + \frac{\gamma\epsilon\lambda}{2}\|\nabla(\phi - \phi^k)\|_{L^2}^2 + \frac{\lambda}{2}\|\boldsymbol{u}_c - \boldsymbol{u}_c^k\|_{L^2}^2 \leq \lambda E(\boldsymbol{u}_c^k, \phi^k).
\end{aligned} \tag{3.30}$$

It follows immediately that

$$E(\boldsymbol{u}_c, \phi) \leq E(\boldsymbol{u}_c^k, \phi^k), \quad \|\nabla w\|_{L^2}^2 \leq CE(\boldsymbol{u}_c^k, \phi^k).$$

Setting $\omega = \frac{1}{|\Omega|}$ in (3.2), we also have

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega w dx =& \frac{\gamma}{|\Omega|\epsilon}\int_\Omega (\phi^3 - \phi^k) dx \\ \leq& \frac{\gamma}{|\Omega|\epsilon}[\int_\Omega |\phi|^3 dx + \int_\Omega |\phi^k| dx] \\ \leq& \frac{\gamma}{|\Omega|\epsilon}[\|\phi\|_{L^6}^3 + \|\phi^k\|_{L^2}^2]|\Omega|^{\frac{1}{2}} \\ =& \frac{\gamma}{|\Omega|^{\frac{1}{2}}\epsilon}[\|\phi\|_{H^1}^3 + \|\phi^k\|_{L^2}^2] \leq C(\Omega, \epsilon, \gamma, E(\boldsymbol{u}_c^0, \phi^0)).\end{aligned} \tag{3.31}$$

Hence by the Poincaré inequality one concludes that $\|w\|_{H^1}^2 \leq C$. Thus Leray-Schauder principle implies that there exists a fixed point $w = T(w)$, which solves (3.1)-(3.3). The energy law (3.26) follows from (3.30) with $\lambda = 1$. This completes the proof. □

3.2. Construction of the approximation solution and passage to the limit

Recall that $\delta = \frac{T}{N}$ for $T > 0$ and a positive integer N, and that $t_k = k\delta$, $k = 0, 1 \cdots N$. Suppose $(\boldsymbol{u}_c^{k+1}, p_m^{k+1}, \phi^{k+1}, w^{k+1}) \in \mathbf{X}_{c,div} \times Q_m \times H^1(\Omega) \times H^1(\Omega)$ is a solution to the time-discrete system (3.1)–(3.3) according to Lemma 3.5. We define the approximate solutions to Eqs. (2.27)–(2.30) as follows

$$\begin{aligned}&\phi^\delta := \frac{t_{k+1} - t}{\delta}\phi^k + \frac{t - t_k}{\delta}\phi^{k+1}, \\ &\boldsymbol{u}_c^\delta := \frac{t_{k+1} - t}{\delta}\boldsymbol{u}_c^k + \frac{t - t_k}{\delta}\boldsymbol{u}_c^{k+1}, \\ &\hat{\boldsymbol{u}}_m^\delta := -\frac{\Pi}{\nu(\phi_m)}(\nabla p_m^{k+1} - w^{k+1}\nabla\phi_m^{k+1}), \;\; \text{for } t \in [t_k, t_{k+1}). \\ &\hat{p}_m^\delta := p_m^{k+1}, \quad \hat{\phi}^\delta := \phi^{k+1}, \quad \hat{\boldsymbol{u}}_c^\delta := \boldsymbol{u}_c^{k+1}, \\ &\hat{w}^\delta := w^{k+1}, \quad \tilde{\phi}^\delta := \phi^k,\end{aligned}$$

With these definitions, one deduces the following equations, cf. (3.1)-(3.3):

$$(\frac{d\phi^\delta}{dt}, \psi) + (\hat{\boldsymbol{u}}^\delta \cdot \nabla\hat{\phi}^\delta, \psi) + (M(\tilde{\phi}^\delta)\nabla\hat{w}^\delta, \nabla\psi) = 0, \quad \forall\, \psi \in Y, \tag{3.32}$$

$$(\hat{w}^\delta, \omega) - \gamma\epsilon(\nabla\hat{\phi}^\delta, \nabla\omega) - \gamma(f(\hat{\phi}^\delta, \tilde{\phi}^\delta), \omega) = 0, \quad \forall\, \omega \in Y, \tag{3.33}$$

$$(\hat{\boldsymbol{u}}_m^\delta, \boldsymbol{v}_m)_m = \left(\frac{\Pi}{\nu(\tilde{\phi}_m^\delta)}(-\nabla\hat{p}_m^\delta + \hat{w}_m^\delta\nabla\hat{\phi}_m^\delta), \boldsymbol{v}_m\right), \;\forall\, \boldsymbol{v}_m \in \mathbf{L}^2(\Omega_m)_m, \tag{3.34}$$

and the equation

$$
\begin{aligned}
&\langle\frac{d\boldsymbol{u}_c^\delta}{dt},\boldsymbol{v}_c\rangle_c+((\hat{\boldsymbol{u}}_c^\delta\cdot\nabla)\hat{\boldsymbol{u}}_c^\delta,\boldsymbol{v}_c)_c+\Big(\nu(\tilde{\phi}_c^\delta)\mathbb{D}(\hat{\boldsymbol{u}}_c^\delta),\mathbb{D}(\boldsymbol{v}_c)\Big)_c\\
&+\Big(\frac{\Pi}{\nu(\tilde{\phi}_m^\delta)}\nabla\hat{p}_m^\delta,\nabla q_m\Big)_m+\Big\langle\frac{\alpha_B}{\sqrt{tr(\Pi)}}\nu(\tilde{\phi}_m^\delta)P_\tau\hat{\boldsymbol{u}}_c^\delta,P_\tau\boldsymbol{v}_c\Big\rangle_\Gamma\\
&-\langle\hat{\boldsymbol{u}}_c^\delta\cdot\mathbf{n}_c,q_m\rangle_\Gamma+\Big\langle\hat{p}_m^\delta-\frac{1}{2}(\hat{\boldsymbol{u}}_c^\delta\cdot\hat{\boldsymbol{u}}_c^\delta),\boldsymbol{v}_c\cdot\mathbf{n}_c\Big\rangle_\Gamma\\
&=\Big(\frac{\Pi}{\nu(\tilde{\phi}_m^\delta)}\hat{w}_m^\delta\nabla\hat{\phi}_m^\delta,\nabla q_m\Big)_m+(\hat{w}_c^\delta\nabla\hat{\phi}_c^\delta,\boldsymbol{v}_c)_c,\quad\forall\boldsymbol{v}_c\in X_{c,div},\ q_m\in Q_m,
\end{aligned}
\tag{3.35}
$$

with initial conditions

$$
\phi^\delta|_{t=0}=\phi^0,\quad \boldsymbol{u}_c^\delta|_{t=0}=\boldsymbol{u}_c^0. \tag{3.36}
$$

As in [23], we also interpolate the discrete-in-time energy and dissipation function introducing

$$
E^\delta(t)=\frac{t_{k+1}-t}{\delta}E(\boldsymbol{u}_c^k,\phi^k)+\frac{t-t_k}{\delta}E(\boldsymbol{u}_c^{k+1},\phi^k),\qquad for\ t\in[t_k,t_{k+1}] \tag{3.37}
$$

$$
\begin{aligned}
\mathcal{D}^\delta(t)=&\left\|\sqrt{M(\phi^k)}\nabla w^{k+1}\right\|_{L^2}^2+\left\|\sqrt{2\nu(\phi_c^k)}\mathbb{D}(\boldsymbol{u}_c^{k+1})\right\|_{L^2}^2\\
&+\left\|\sqrt{\nu(\phi_m^k)\Pi^{-1}}\boldsymbol{u}_m{}^{k+1}\right\|_{L^2}^2+\frac{\alpha_B}{\sqrt{\mathrm{tr}(\Pi)}}\left\|\sqrt{\nu(\phi_c^k)}P_\tau\boldsymbol{u}_c^{k+1}\right\|_{L^2}^2.
\end{aligned}
\tag{3.38}
$$

The time-discrete energy law translates to

$$
\frac{d}{dt}E^\delta(t)\le-\mathcal{D}^\delta(t). \tag{3.39}
$$

Integrating (3.39) from 0 to T one immediately derives the following estimates

$$
\|\hat{\boldsymbol{u}}_c^\delta\|_{L^\infty(0,T;L^2(\Omega_c))}+\|\hat{\phi}^\delta\|_{L^\infty(0,T;H^1(\Omega))}\le C, \tag{3.40}
$$

$$
\|\nabla\hat{\boldsymbol{u}}_c^\delta\|_{L^2(0,T;L^2(\Omega_c))}+\|P_\tau\hat{\boldsymbol{u}}_c^\delta\|_{L^2(0,T;L^2(\Gamma))}\le C, \tag{3.41}
$$

$$
\|\hat{\boldsymbol{u}}_m^\delta\|_{L^2(0,T;L^2(\Omega_m))}+\|\nabla\hat{w}^\delta\|_{L^2(0,T;L^2(\Omega))}\le C, \tag{3.42}
$$

where the constant C depends on $E(\boldsymbol{u}_c^0,\phi^0)$. Based on these estimates and Eqs. (3.32)–(3.35) the following estimates can be further inferred.

Lemma 3.6. *Let* $\{\hat{\boldsymbol{u}}_c^\delta, \hat{p}_m^\delta, \hat{\boldsymbol{u}}_m^\delta, \hat{\phi}^\delta, \hat{w}^\delta\}$ *be satisfying Eqs.* (3.32)-(3.35). *The following estimates hold*

$$\|\hat{w}^\delta\|_{L^2(0,T;H^1(\Omega))} \leq C_T, \tag{3.43}$$

$$\|\hat{\phi}^\delta\|_{L^2(0,T;H^3(\Omega))} \leq C_T, \tag{3.44}$$

$$\|\nabla \hat{p}_m^\delta\|_{L^{\frac{8}{5}}(0,T;L^2(\Omega_m))} \leq C_T, \tag{3.45}$$

$$\|\partial_t \phi^\delta\|_{L^{\frac{8}{5}}(0,T;(H^1(\Omega))')} \leq C_T, \tag{3.46}$$

$$\|\partial_t \boldsymbol{u}_c^\delta\|_{L^{\frac{4}{3}}(0,T;(H^1(\Omega_c))')} \leq C_T, \tag{3.47}$$

Proof. The estimates (3.43)–(3.44) are derived exactly the same as in [23]. We briefly outline the arguments here for completeness. By the estimates (3.31) and (3.42) one obtains (3.43) as a result of Poincaré's inequality. Then inequality (3.44) follows from Eq. (3.33) and elliptic regularity. Next, by Hölder's inequality, the interpolation inequality [44, 45] and Sobolev inequality, we have

$$\begin{aligned}\|\hat{w}^\delta \nabla \hat{\phi}^\delta\|_{L^2} \leq \|\nabla \hat{\phi}^\delta\|_{L^3} \|\hat{w}^\delta\|_{L^6} &\leq C \|\nabla \hat{\phi}^\delta\|_{L^2}^{\frac{3}{4}} \|\nabla \hat{\phi}^\delta\|_{H^2}^{\frac{1}{4}} \|\hat{w}^\delta\|_{L^6} \\ &\leq C \|\nabla \hat{\phi}^\delta\|_{L^2}^{\frac{3}{4}} \|\nabla \hat{\phi}^\delta\|_{H^2}^{\frac{1}{4}} \|\hat{w}^\delta\|_{H^1}.\end{aligned} \tag{3.48}$$

Since $\|\nabla \hat{\phi}^\delta\|_{H^2}^{\frac{1}{4}} \|\hat{w}^\delta\|_{H^1} \in L^{\frac{8}{5}}(0,T)$ by Hölder's inequality, one derives that

$$\|\hat{w}^\delta \nabla \hat{\phi}^\delta\|_{L^{\frac{8}{5}}(0,T;L^2(\Omega))} \leq C_T. \tag{3.49}$$

The inequality (3.45) follows immediately from Eqs. (3.35) with $\boldsymbol{v}_c = 0$ and $q_m = \hat{p}_m$. Likewise, one has

$$|(\hat{\boldsymbol{u}}^\delta \cdot \nabla \hat{\phi}^\delta, \psi)| \leq \|\hat{\boldsymbol{u}}^\delta\|_{L^2} \|\nabla \hat{\phi}^\delta\|_{L^3} \|\psi\|_{L^6} \leq \|\hat{\boldsymbol{u}}^\delta\|_{L^2} \|\nabla \hat{\phi}^\delta\|_{L^2}^{\frac{3}{4}} \|\nabla \hat{\phi}^\delta\|_{H^2}^{\frac{1}{4}} \|\psi\|_{H^1}.$$

Hence the inequality (3.46) follows from Eq. (3.32), the estimates (3.40), (3.42) and (3.44).

By the identity (3.14) and the interpolation inequality, one has

$$\begin{aligned}&\left| \left((\hat{\boldsymbol{u}}_c^\delta \cdot \nabla) \hat{\boldsymbol{u}}_c^\delta, \boldsymbol{v}_c \right)_c - \left\langle \frac{1}{2} (\hat{\boldsymbol{u}}_c^\delta \cdot \hat{\boldsymbol{u}}_c^\delta), \boldsymbol{v}_c \cdot \mathbf{n}_c \right\rangle_\Gamma \right| \\ &= \left| \left((\hat{\boldsymbol{u}}_c^\delta \cdot \nabla) \hat{\boldsymbol{u}}_c^\delta, \boldsymbol{v}_c \right)_c - \left((\boldsymbol{v}_c \cdot \nabla) \hat{\boldsymbol{u}}_c^\delta, \hat{\boldsymbol{u}}_c^\delta \right)_c \right| \\ &\leq 2 \|\boldsymbol{v}_c\|_{L^6} \|\hat{\boldsymbol{u}}_c^\delta\|_{L^3} \|\nabla \hat{\boldsymbol{u}}_c^\delta\|_{L^2} \leq C \|\boldsymbol{v}_c\|_{H^1} \|\hat{\boldsymbol{u}}_c^\delta\|_{L^2}^{\frac{1}{2}} \|\nabla \hat{\boldsymbol{u}}_c^\delta\|_{L^2}^{\frac{3}{2}}\end{aligned}$$

It then follows from Eq. (3.35), the trace inequality, the inequality (3.48) and Korn's inequality that

$$
\begin{aligned}
|\langle \frac{d\boldsymbol{u}_c^\delta}{dt}, \boldsymbol{v}_c\rangle| \leq & C\bigg(\|\hat{\boldsymbol{u}}_c^\delta\|_{L^2}^{\frac{1}{2}} \|\nabla \hat{\boldsymbol{u}}_c^\delta\|_{L^2}^{\frac{3}{2}} + \|\mathbb{D}(\hat{\boldsymbol{u}}_c^\delta)\|_{L^2} \\
& + \|\nabla \hat{p}_m^\delta\|_{L^2} + \|\nabla\hat{\phi}^\delta\|_{L^2}^{\frac{3}{4}} \|\nabla\hat{\phi}^\delta\|_{H^2}^{\frac{1}{4}} \|\hat{w}^\delta\|_{H^1} \bigg) \|\boldsymbol{v}_c\|_{H^1}
\end{aligned} \tag{3.50}
$$

Since the right hand side of (3.50) is in $L^{\frac{4}{3}}(0,T)$, the estimate (3.47) is thus proved. This completes the proof of the lemma. □

We are now ready to pass to the limit and prove the main Theorem 2.1.

Proof. The estimates in (3.40)-(3.45) imply the existence of

$$
\begin{aligned}
& \boldsymbol{u}_c \in L^\infty(0,T;\mathbf{H}(\mathrm{div};\Omega_c)) \cap L^2(0,T;\boldsymbol{X}_{c,div}), \\
& \boldsymbol{u}_m \in L^2(0,T;\boldsymbol{X}_{m,div}), \quad p_m \in L^{8/5}(0,T;Q_m), \\
& \phi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)), \quad w \in L^2(0,T;H^1(\Omega)), \\
& \frac{\partial\phi}{\partial t} \in L^{\frac{8}{5}}\big(0,T;(H^1(\Omega))'\big), \quad \frac{\partial \boldsymbol{u}_c}{\partial t} \in L^{\frac{4}{3}}\big(0,T;\mathbf{X}'_{c,div}\big),
\end{aligned}
$$

such that the following convergence (of subsequences) holds as $\delta \to 0$

$$\hat{\boldsymbol{u}}_c^\delta \longrightarrow \boldsymbol{u}_c \quad \textit{weakly} * \textit{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega_c)), \tag{3.51}$$

$$\textit{weakly in } L^2(0,T;\boldsymbol{H}^1(\Omega_c)), \tag{3.52}$$

$$p_m^\delta \longrightarrow p_m \qquad \textit{weakly in } L^{8/5}(0,T;Q_m), \tag{3.53}$$

$$\hat{\boldsymbol{u}}_m^\delta \longrightarrow \boldsymbol{u}_m \quad \textit{weakly in } L^2(0,T;\boldsymbol{L}^2(\Omega_m)), \tag{3.54}$$

$$\hat{\phi}^\delta \longrightarrow \phi \quad \textit{weakly} \ * \ \textit{in } L^\infty(0,T;H^1(\Omega_m)), \tag{3.55}$$

$$\textit{weakly in } L^2(0,T;H^3(\Omega)), \tag{3.56}$$

$$\hat{w}^\delta \longrightarrow w^\epsilon \qquad \textit{weakly in } L^2(0,T;H^1(\Omega)). \tag{3.57}$$

By the definition of $\hat{\boldsymbol{u}}_c^\delta$ and $\boldsymbol{u}_c^\delta$, $\hat{\phi}^\delta$, $\tilde{\phi}^\delta$ and ϕ^δ, we also have

$$\|\hat{\boldsymbol{u}}_c^\delta - \boldsymbol{u}_c^\delta\|_{L^2(L^2)}^2 = \frac{\delta}{3}\sum_{k=0}^{N-1} \|\boldsymbol{u}_c^{k+1} - \boldsymbol{u}_c^k\|_{L^2}^2 \leq C\delta \to 0 \ as \ \delta \to 0. \tag{3.58}$$

$$\|\nabla(\hat{\phi}^\delta - \phi^\delta)\|_{L^2(L^2)}^2 = \frac{\delta}{3}\sum_{k=0}^{N-1} \|\nabla(\phi^{k+1} - \phi^k)\|_{L^2}^2 \leq C\delta \to 0 \ as \ \delta \to 0. \tag{3.59}$$

Since $\int_\Omega \hat{\phi}^\delta dx = \int_\Omega \tilde{\phi}^\delta dx$ by Lemma 3.1, Poincaré's inequality gives

$$\|\hat{\phi}^\delta - \phi^\delta\|_{L^2(0,T;H^1(\Omega))} \to 0 \ as\ \delta \to 0.$$

Likewise, one has $\|\hat{\phi}^\delta - \tilde{\phi}^\delta\|_{L^2(0,T;H^1(\Omega))} \to 0\ as\ \delta \to 0$. Thus the sequences $\{\hat{\boldsymbol{u}}_c^\delta\}$ and $\{\boldsymbol{u}_c^\delta\}$, if convergent, converge to the same limit. So do the sequences $\{\tilde{\phi}^\delta\}$, $\{\phi^\delta\}$ and $\hat{\phi}^\delta$. On the other hand, from the definition of $\boldsymbol{u}_c^\delta$, $\tilde{\phi}^\delta$ and ϕ^δ, as well as the estimates (3.40)-(3.45), we infer they also satisfy estimates analogous to (3.40)-(3.45). Hence the convergence (3.51)-(3.52) and (3.55)-(3.56) holds for $\boldsymbol{u}_c^\delta$ and ϕ^δ, respectively.

Since $\phi^\delta \in L^\infty\big(0,T;H^1(\Omega)\big) \cap L^2\big(0,T;H^3(\Omega)\big)$ and $\frac{\partial \phi^\delta}{\partial t} \in L^{\frac{8}{5}}\big(0,T;(H^1(\Omega))'\big)$, the Aubin-Lions-Simon lemma (cf. [46] Corollary 4) yields

$$\begin{aligned}&\frac{\partial \phi}{\partial t} \in L^{\frac{8}{5}}\big(0,T;(H^1(\Omega))'\big), \quad \phi^\delta \longrightarrow \phi \\ &strongly\ in\ L^2\big(0,T;H^2(\Omega)\big) \cap C\big(0,T;L^2(\Omega)\big).\end{aligned} \tag{3.60}$$

Due to the fact that $\phi \in L^\infty\big(0,T;H^1(\Omega)\big)$ and $\frac{\partial \phi}{\partial t} \in L^{\frac{8}{5}}\big(0,T;(H^1(\Omega))'\big)$, it follows that (cf. [42] pp. 178)

$$\phi \in C_w\big(0,T;H^1(\Omega)\big), \tag{3.61}$$

that is

$$\lim_{t \to t_0} \big(\phi(t),\varphi\big)_{H^1} = \big(\phi(t_0),\varphi\big)_{H^1}, \forall t_0 \in [0,T] \text{ and } \varphi \in H^1(\Omega).$$

Similarly, one has that

$$\begin{aligned}&\frac{\partial \boldsymbol{u}_c}{\partial t} \in L^{\frac{4}{3}}\big(0,T;\mathbf{X}'_{c,div}\big), \quad \boldsymbol{u}_c^\delta \longrightarrow \boldsymbol{u}_c \quad strongly\ in \\ &L^2\big(0,T;\boldsymbol{H}^{1-\beta}(\Omega_c)\big) \cap C\big(0,T;\boldsymbol{H}^\alpha(\Omega_c)\big), \beta \in (0,\frac{1}{2}), \alpha \in (-1,0),\end{aligned} \tag{3.62}$$

and that

$$\boldsymbol{u}_c \in C_w\big(0,T;\boldsymbol{L}^2(\Omega_c)\big). \tag{3.63}$$

Because

$$\|(\hat{\phi}^\delta)^3 - \phi^3\|_{L^2} \leq C\|\hat{\phi}^\delta - \phi\|_{L^2}(\|\hat{\phi}^\delta\|^2_{L^\infty} + \|\phi\|^2_{L^\infty})$$

$$\leq C\|\hat{\phi}^\delta - \phi\|_{L^2}\big(\|\hat{\phi}^\delta\|_{H^1}^{\frac{3}{2}}\|\hat{\phi}^\delta\|_{H^3}^{\frac{1}{2}} + \|\phi\|_{H^1}^{\frac{3}{2}}\|\phi\|_{H^3}^{\frac{1}{2}}\big),$$

by the strong convergence (3.60), one readily derives that

$$f\big(\hat{\phi}^\delta, \tilde{\phi}^\delta\big) \longrightarrow f(\phi) \quad \textit{strongly in } L^4\big(0,T;L^2(\Omega)\big).$$

Likewise, the weak strong convergence implies that

$$\hat{w}^\delta \nabla \hat{\phi}^\delta \longrightarrow w\nabla\phi, \quad (\hat{\boldsymbol{u}}_c^\delta \cdot \nabla)\hat{\boldsymbol{u}}_c^\delta \longrightarrow (\boldsymbol{u}_c \cdot \nabla)\boldsymbol{u}_c$$

in the sense of distributions. For the nonlinear interface term, one has

$$\begin{aligned}
&\left\langle \frac{1}{2}(\hat{\boldsymbol{u}}_c^\delta \cdot \hat{\boldsymbol{u}}_c^\delta), \boldsymbol{v}_c \cdot \mathbf{n}_c \right\rangle_\Gamma = -\big((\boldsymbol{v}_c \cdot \nabla)\hat{\boldsymbol{u}}_c^\delta, \hat{\boldsymbol{u}}_c^\delta\big)_c \longrightarrow \\
&-\big((\boldsymbol{v}_c \cdot \nabla)\boldsymbol{u}_c, \boldsymbol{u}_c\big)_c = \left\langle \frac{1}{2}(\boldsymbol{u}_c \cdot \boldsymbol{u}_c), \boldsymbol{v}_c \cdot \mathbf{n}_c \right\rangle_\Gamma, \text{ in } \mathcal{D}(0,T), \forall \boldsymbol{v}_c \in X_{c,div}.
\end{aligned}$$

In addition by assumptions (1)-(2), one also has

$$\begin{aligned}
&\nu(\tilde{\phi}^\delta) \longrightarrow \nu(\phi) \quad \textit{strongly in } C\big(0,T;L^2(\Omega)\big), \\
&M(\tilde{\phi}^\delta) \longrightarrow M(\phi) \quad \textit{strongly in } C\big(0,T;L^2(\Omega)\big).
\end{aligned}$$

These convergence results allow us pass to the limit in Eqs. (3.32)-(3.35), first in the sense of distributions, then to corresponding function spaces by continuity. Specifically, one has for $h(t) \in \mathcal{D}(0,T)$

$$\begin{aligned}
&\int_0^T \left[\left\langle \frac{d\phi}{dt}, \psi \right\rangle + (\boldsymbol{u}\cdot\nabla\phi, \psi) + (M(\phi)\nabla w, \nabla\psi)\right] h(t)dt = 0, \ \forall\ \psi \in Y, \\
&\int_0^T \left[(w,\omega) - \gamma\epsilon(\nabla\phi, \nabla\omega) - \gamma(f(\phi), \omega)\right] h(t)dt = 0, \ \forall\ \omega \in Y, \\
&\int_0^T \left[\left\langle \frac{d\boldsymbol{u}_c}{dt}, \boldsymbol{v}_c \right\rangle_c + ((\boldsymbol{u}_c\cdot\nabla)\boldsymbol{u}_c, \boldsymbol{v}_c)_c + \big(\nu(\phi_c)\mathbb{D}(\boldsymbol{u}_c), \mathbb{D}(\boldsymbol{v}_c)\big)_c\right. \\
&+\Big(\frac{\Pi}{\nu(\phi_m)}\nabla p_m, \nabla q\Big)_m + \left\langle \frac{\alpha\nu(\phi_m)}{\sqrt{tr(\Pi)}} P_\tau \boldsymbol{u}_c, P_\tau \boldsymbol{v}_c \right\rangle_\Gamma - \langle \boldsymbol{u}_c \cdot \mathbf{n}_c, q\rangle \\
&+\left\langle p_m - \frac{1}{2}|\boldsymbol{u}_c|^2, \boldsymbol{v}_c \cdot \mathbf{n}_c \right\rangle - \Big(\frac{\Pi}{\nu(\phi_m)} w_m \nabla\phi_m, \nabla q\Big)_m \\
&\left.-(w_c\nabla\phi_c, \boldsymbol{v}_c)_c\right] h(t)dt = 0, \quad \forall \boldsymbol{v}_c \in X_{c,div}, \ q \in H^1(\Omega_m), \\
&\int_0^T \left[(\hat{\boldsymbol{u}}_m^\delta, \boldsymbol{v}_m)_m - \left(\frac{\Pi}{\nu(\tilde{\phi}_m^\delta)}(-\nabla \hat{p}_m^\delta + \hat{w}_m^\delta \nabla \hat{\phi}_m^\delta), \boldsymbol{v}_m\right)_m\right] h(t)dt = 0,
\end{aligned}$$

$\forall\ \boldsymbol{v}_m \in \mathbf{L}^2(\Omega_m)$, This shows that $(\boldsymbol{u}_c, \boldsymbol{u}_m, p_m, \phi, w)$ almost everywhere in time satisfies the Eqs. (2.27)–(2.30). Furthermore, in light of the initial conditions (3.36), the strong convergence (3.60) and (3.62), and by the weak continuity in time (3.61) and (3.63), one infers that

$$\phi|_{t=0} = \phi^0, \quad \boldsymbol{u}_c|_{t=0} = \boldsymbol{u}_c^0.$$

Finally, we show that weak solutions satisfy the energy inequality (2.32). The argument is entirely the same as in [23]. We reproduce it here for completeness. Multiplying the inequality (3.39) by $h(t)$ for $h \in C^1(0,T)$ with $h \geq 0, h(T) = 0$ and integrating, one derives

$$E(\boldsymbol{u}_c^0, \phi^0)h(0) + \int_0^t E^\delta(s)h'(s)ds \geq \int_0^t \mathcal{D}^\delta(s)h(s)ds.$$

By the strong convergence (3.60), (3.62), the weak convergence (3.52), (3.54), (3.57), one passes to the limit to obtain

$$E(\boldsymbol{u}_c^0, \phi^0) + \int_0^t E(s)h'(s)ds \geq \liminf_{\delta\to 0} \int_0^t \mathcal{D}^\delta(s)h(s)ds \geq \int_0^t \mathcal{D}(s)h(s)ds, \tag{3.64}$$

where the last inequality follows from the lower semi-continuity of norms and the almost everywhere convergence of ν and M. The energy inequality (2.32) is thus established. This completes the proof of Theorem 2.1. □

4. Weak-strong uniqueness

In this section we prove the weak-strong uniqueness (Theorem 2.2). We largely follow the lines of proof from [23] for the Cahn-Hilliard-Stokes-Darcy system. Special care is paid to the treatment of the nonlinear advection term in the Navier-Stokes equation and the nonlinear Lions interface boundary conditions.

Proof. Suppose $(\tilde{\boldsymbol{u}}_c, p_m, \tilde{\boldsymbol{u}}_m, \tilde{\phi}, \tilde{w})$ is a strong solution to the Cahn-Hilliard-Navier-Stokes-Darcy system such that

$$\tilde{\boldsymbol{u}}_c \in L^\infty(0,T;\boldsymbol{X}_{c,div}), \quad \tilde{\boldsymbol{u}}_m \in L^\infty\big(0,T;\mathbf{H}^1(\Omega_m)\big),$$

$$\tilde{\phi} \in L^\infty\big(0,T;H^2(\Omega)\big) \cap L^2\big(0,T;H^4(\Omega)\big),$$
$$\nabla p_m \in L^\infty(0,T;L^2(\Omega_m)).$$

It follows from Eqs. (2.4) and (2.8) that $\tilde{w} \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^2(\Omega)\big)$, hence $\tilde{w} \in L^4\big(0,T;H^1(\Omega)\big)$ by interpolation. It then follows from the equations that

$$\frac{\partial \tilde{\boldsymbol{u}}_c}{\partial t} \in L^\infty(0,T;\boldsymbol{X}'_{c,div}), \quad \frac{\partial \tilde{\phi}}{\partial t} \in L^4(0,T;(H^1(\Omega))').$$

Owing to the regularity, one can use $(\tilde{w}, \tilde{\boldsymbol{u}}_c, \tilde{\boldsymbol{u}}_m)$ as test functions, which gives the energy equality (2.17). That is

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|\tilde{\boldsymbol{u}}_c\|_{L^2}^2 + \frac{\gamma\epsilon}{2}\|\nabla\tilde{\phi}\|_{L^2}^2 + \gamma(F(\tilde{\phi}),1)\right\} \\
&= -\|\sqrt{M}\nabla\tilde{w}\|_{L^2}^2 - \|\sqrt{2\nu}\mathbb{D}(\tilde{\boldsymbol{u}}_c)\|_{L^2}^2 \\
&\quad - \|\sqrt{\nu\Pi^{-1}}\tilde{\boldsymbol{u}}_m\|_{L^2}^2 - \frac{\alpha_B}{\sqrt{\mathrm{tr}(\Pi)}}\|\sqrt{\nu}P_\tau\tilde{\boldsymbol{u}}_c\|_{L^2}^2.
\end{aligned} \tag{4.1}$$

For the weak solution $(\boldsymbol{u}_c, p_m, \boldsymbol{u}_m, \phi, w)$ in the sense of Definition 2.1, the energy inequality (2.32) holds (Theorem 2.1), i.e.

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|\boldsymbol{u}_c\|_{L^2}^2 + \frac{\gamma\epsilon}{2}\|\nabla\phi\|_{L^2}^2 + \gamma(F(\phi),1)\right\} \\
&\leq -\|\sqrt{M}\nabla w\|_{L^2}^2 - \|\sqrt{2\nu}\mathbb{D}(\boldsymbol{u}_c)\|_{L^2}^2 \\
&\quad - \|\sqrt{\nu\Pi^{-1}}\boldsymbol{u}_m\|_{L^2}^2 - \frac{\alpha_B}{\sqrt{\mathrm{tr}(\Pi)}}\|\sqrt{\nu}P_\tau\boldsymbol{u}_c\|_{L^2}^2.
\end{aligned} \tag{4.2}$$

Since $\tilde{\boldsymbol{u}}_c \in L^\infty(0,T;\boldsymbol{X}_{c,div})$, for almost all $t \in (0,T)$ it permits to use $\boldsymbol{v}_c = \tilde{\boldsymbol{u}}_c$ and $q_m = 0$ as test functions in Eq. (2.29). Meanwhile one multiplies by $\boldsymbol{u}_c$ the strong form (2.5) for $\tilde{\boldsymbol{u}}_c$, and performs integration by parts. Adding together the resultants gives

$$\begin{aligned}
&\left(\frac{\partial \boldsymbol{u}_c}{\partial t}, \tilde{\boldsymbol{u}}_c\right)_c + \left(\frac{\partial \tilde{\boldsymbol{u}}_c}{\partial t}, \boldsymbol{u}_c\right)_c + \Big(2\nu(\phi_c)\mathbb{D}(\boldsymbol{u}_c), \mathbb{D}(\tilde{\boldsymbol{u}}_c)\Big)_c + \Big(2\nu(\tilde{\phi}_c)\mathbb{D}(\tilde{\boldsymbol{u}}_c), \mathbb{D}(\boldsymbol{u}_c)\Big)_c \\
&\quad + \left\langle \frac{\alpha_B\nu(\phi_c)}{\sqrt{\mathrm{tr}(\Pi)}}P_\tau\boldsymbol{u}_c, P_\tau\tilde{\boldsymbol{u}}_c \right\rangle_\Gamma + \left\langle \frac{\alpha_B\nu(\tilde{\phi}_c)}{\sqrt{\mathrm{tr}(\Pi)}}P_\tau\tilde{\boldsymbol{u}}_c, P_\tau\boldsymbol{u}_c \right\rangle_\Gamma \\
&= -\Big(\left(\boldsymbol{u}_c\cdot\nabla\right)\boldsymbol{u}_c, \tilde{\boldsymbol{u}}_c\Big)_c - \Big(\left(\tilde{\boldsymbol{u}}_c\cdot\nabla\right)\tilde{\boldsymbol{u}}_c, \boldsymbol{u}_c\Big)_c + \Big(w_c\nabla\phi_c, \tilde{\boldsymbol{u}}_c\Big)_c + \Big(\tilde{w}_c\nabla\tilde{\phi}_c, \boldsymbol{u}_c\Big)_c
\end{aligned}$$

$$
\begin{aligned}
&-\left\langle p_m-\frac{1}{2}(\boldsymbol{u}_c\cdot\boldsymbol{u}_c),\tilde{\boldsymbol{u}}_c\cdot\mathbf{n}_c\right\rangle_\Gamma-\left\langle \tilde{p}_m-\frac{1}{2}(\tilde{\boldsymbol{u}}_c\cdot\tilde{\boldsymbol{u}}_c),\boldsymbol{u}_c\cdot\mathbf{n}_c\right\rangle_\Gamma\\
=&-\Big(\left(\boldsymbol{u}_c\cdot\nabla\right)\boldsymbol{u}_c,\tilde{\boldsymbol{u}}_c\Big)_c-\Big(\left(\tilde{\boldsymbol{u}}_c\cdot\nabla\right)\tilde{\boldsymbol{u}}_c,\boldsymbol{u}_c\Big)_c+\Big(w_c\nabla\phi_c,\tilde{\boldsymbol{u}}_c\Big)_c\\
&+\Big(\tilde{w}_c\nabla\tilde{\phi}_c,\boldsymbol{u}_c\Big)_c-\langle p_m,\tilde{\boldsymbol{u}}_c\cdot\mathbf{n}_c\rangle_\Gamma-\langle\tilde{p}_m,\boldsymbol{u}_c\cdot\mathbf{n}_c\rangle_\Gamma\\
&+\Big(\left(\tilde{\boldsymbol{u}}_c\cdot\nabla\right)\boldsymbol{u}_c,\boldsymbol{u}_c\Big)_c+\Big(\left(\boldsymbol{u}_c\cdot\nabla\right)\tilde{\boldsymbol{u}}_c,\tilde{\boldsymbol{u}}_c\Big)_c\\
=&-\Big((\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c)\cdot\nabla(\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c),\tilde{\boldsymbol{u}}_c\Big)_c+\Big(\tilde{\boldsymbol{u}}_c\cdot\nabla(\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c),(\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c)\Big)_c\\
&+\Big(w_c\nabla\phi_c,\tilde{\boldsymbol{u}}_c\Big)_c+\Big(\tilde{w}_c\nabla\tilde{\phi}_c,\boldsymbol{u}_c\Big)_c-\langle p_m,\tilde{\boldsymbol{u}}_c\cdot\mathbf{n}_c\rangle_\Gamma-\langle\tilde{p}_m,\boldsymbol{u}_c\cdot\mathbf{n}_c\rangle_\Gamma. \qquad (4.3)
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&\big(\Pi^{-1}\nu(\phi_m)\boldsymbol{u}_m,\tilde{\boldsymbol{u}}_m\big)_m+\Big(\Pi^{-1}\nu(\tilde{\phi}_m)\tilde{\boldsymbol{u}}_m,\boldsymbol{u}_m\Big)_m \qquad (4.4)\\
&=\langle p_m,\tilde{\boldsymbol{u}}_m\cdot\mathbf{n}_c\rangle_\Gamma+\langle\tilde{p}_m,\boldsymbol{u}_m\cdot\mathbf{n}_c\rangle_\Gamma+\Big(w_m\nabla\phi_m,\tilde{\boldsymbol{u}}_m\Big)_m+\Big(\tilde{w}_m\nabla\tilde{\phi}_m,\boldsymbol{u}_m\Big)_m\\
&=\langle p_m,\tilde{\boldsymbol{u}}_c\cdot\mathbf{n}_c\rangle_\Gamma+\langle\tilde{p}_m,\boldsymbol{u}_c\cdot\mathbf{n}_c\rangle_\Gamma+\Big(w_m\nabla\phi_m,\tilde{\boldsymbol{u}}_m\Big)_m+\Big(\tilde{w}_m\nabla\tilde{\phi}_m,\boldsymbol{u}_m\Big)_m,
\end{aligned}
$$

where we have utilized the fact that $\mathbf{u}_m\cdot\mathbf{n}_c=\mathbf{u}_c\cdot\mathbf{n}_c$ on Γ (likewise for $\tilde{\mathbf{u}}_m$ and $\tilde{\mathbf{u}}_c$). Take summation of Eqs. (4.3) and (4.4)

$$
\begin{aligned}
&\left\langle\frac{\partial\boldsymbol{u}_c}{\partial t},\tilde{\boldsymbol{u}}_c\right\rangle_c+\left\langle\frac{\partial\tilde{\boldsymbol{u}}_c}{\partial t},\boldsymbol{u}_c\right\rangle_c+\Big(2\nu(\phi_c)\mathbb{D}(\boldsymbol{u}_c),\mathbb{D}(\tilde{\boldsymbol{u}}_c)\Big)_c+\Big(2\nu(\tilde{\phi}_c)\mathbb{D}(\tilde{\boldsymbol{u}}_c),\mathbb{D}(\boldsymbol{u}_c)\Big)_c\\
&+\left\langle\frac{\alpha_B\nu(\phi_c)}{\sqrt{\operatorname{tr}(\Pi)}}P_\tau\boldsymbol{u}_c,P_\tau\tilde{\boldsymbol{u}}_c\right\rangle_\Gamma+\left\langle\frac{\alpha_B\nu(\tilde{\phi}_c)}{\sqrt{\operatorname{tr}(\Pi)}}P_\tau\tilde{\boldsymbol{u}}_c,P_\tau\boldsymbol{u}_c\right\rangle_\Gamma\\
&+\big(\nu(\phi_m)\Pi^{-1}\boldsymbol{u}_m,\tilde{\boldsymbol{u}}_m\big)_m+\Big(\nu(\tilde{\phi}_m)\Pi^{-1}\tilde{\boldsymbol{u}}_m,\boldsymbol{u}_m\Big)_m\\
=&-\Big((\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c)\cdot\nabla(\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c),\tilde{\boldsymbol{u}}_c\Big)_c+\Big(\tilde{\boldsymbol{u}}_c\cdot\nabla(\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c),(\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c)\Big)_c \qquad (4.5)\\
&+\Big(w_c\nabla\phi_c,\tilde{\boldsymbol{u}}_c\Big)_c+\Big(\tilde{w}_c\nabla\tilde{\phi}_c,\boldsymbol{u}_c\Big)_c+\Big(w_m\nabla\phi_m,\tilde{\boldsymbol{u}}_m\Big)_m+\Big(\tilde{w}_m\nabla\tilde{\phi}_m,\boldsymbol{u}_m\Big)_m.
\end{aligned}
$$

To deal with the Cahn-Hilliard equations, one notes that $\tilde{\phi}\in L^4\big(0,T;H^3(\Omega)\big)$, since by the Gagliardo-Nirenberg inequality

$$
||\nabla\Delta\tilde{\phi}||_{L^2}\leq C||\Delta\tilde{\phi}||_{L^2}^{\frac{1}{2}}||\Delta\tilde{\phi}||_{H^2}^{\frac{1}{2}}.
$$

Hence in view of Eq. (2.26), for almost every $t\in(0,T)$, $(-\gamma\epsilon\Delta\tilde{\phi})$ can be used as a test function in the weak form (2.27). One obtains

$$
\gamma\epsilon\left\langle\frac{\partial\nabla\phi}{\partial t},\nabla\tilde{\phi}\right\rangle_c+\gamma\epsilon\left\langle\frac{\partial\nabla\tilde{\phi}}{\partial t},\nabla\phi\right\rangle_c+\big(M(\phi)\nabla w,\nabla\tilde{w}\big)+\big(M(\tilde{\phi})\nabla\tilde{w},\nabla w\big)
$$

$$
\begin{aligned}
&= \gamma\big(M(\phi)\nabla w, \nabla f(\tilde{\phi})\big) + \gamma\big(M(\tilde{\phi})\nabla\tilde{w}, \nabla f(\phi)\big) + \gamma\big(\boldsymbol{u}\cdot\nabla\phi, f(\tilde{\phi})\big) \\
&\quad + \gamma\big(\tilde{\boldsymbol{u}}\cdot\nabla\tilde{\phi}, f(\phi)\big) - \big(\boldsymbol{u}\cdot\nabla\phi, \tilde{w}\big) - \big(\tilde{\boldsymbol{u}}\cdot\nabla\tilde{\phi}, w\big). \qquad (4.6)
\end{aligned}
$$

Now one adds together Eqs. (4.1) and (4.2), then subtracts Eqs. (4.5) and (4.6)

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|\tilde{\boldsymbol{u}}_c - \boldsymbol{u}_c\|_{L^2}^2 + \frac{\gamma\epsilon}{2}\|\nabla\tilde{\phi} - \nabla\phi\|_{L^2}^2\right\} + \frac{d}{dt}\left\{\gamma\big(F(\tilde{\phi}),1\big) + \gamma\big(F(\phi),1\big)\right\} \\
&\quad + \int_{\Omega_c} 2\nu(\phi_c)\big|\mathbb{D}(\tilde{\boldsymbol{u}}_c) - \mathbb{D}(\boldsymbol{u}_c)\big|^2 dx + \int_\Omega M(\phi)|\nabla\tilde{w} - \nabla w|^2 dx \\
&\quad + \int_{\Omega_m} \nu(\phi_m)\Pi^{-1}\big|\tilde{\boldsymbol{u}}_m - \boldsymbol{u}_m\big|^2 dx + \frac{\alpha_B}{\sqrt{\operatorname{tr}(\Pi)}}\int_\Gamma \nu(\phi_c)|P_\tau\tilde{\boldsymbol{u}}_c - P_\tau\boldsymbol{u}_c|^2 ds \\
&\leq \Big((\boldsymbol{u}_c - \tilde{\boldsymbol{u}}_c)\cdot\nabla(\boldsymbol{u}_c - \tilde{\boldsymbol{u}}_c), \tilde{\boldsymbol{u}}_c\Big)_c - \Big(\tilde{\boldsymbol{u}}_c\cdot\nabla(\boldsymbol{u}_c - \tilde{\boldsymbol{u}}_c), (\boldsymbol{u}_c - \tilde{\boldsymbol{u}}_c)\Big)_c \\
&\quad - \Big(2[\nu(\phi_c) - \nu(\tilde{\phi}_c)]\mathbb{D}(\tilde{\boldsymbol{u}}_c), \big(\mathbb{D}(\boldsymbol{u}_c) - \mathbb{D}(\tilde{\boldsymbol{u}}_c)\big)\Big)_c - \Big([M(\phi) - M(\tilde{\phi})]\nabla\tilde{w}, (\nabla w - \nabla\tilde{w})\Big) \\
&\quad - \Big((\nu(\phi_m) - \nu(\tilde{\phi}_m))\Pi^{-1}\tilde{\boldsymbol{u}}_m, (\boldsymbol{u}_m - \tilde{\boldsymbol{u}}_m)\Big)_m - \frac{\alpha_B}{\sqrt{\operatorname{tr}(\Pi)}}\Big\langle[\nu(\phi_c) - \nu(\tilde{\phi}_c)]P_\tau\tilde{\boldsymbol{u}}_c, (P_\tau\boldsymbol{u}_c - P_\tau\tilde{\boldsymbol{u}}_c)\Big\rangle_\Gamma \\
&\quad - \Big((w - \tilde{w})\tilde{\boldsymbol{u}}, (\nabla\phi - \nabla\tilde{\phi})\Big) - \Big(\tilde{w}(\tilde{\boldsymbol{u}} - \boldsymbol{u}), (\nabla\phi - \nabla\tilde{\phi})\Big) - \gamma\big(M(\phi)\nabla w, \nabla f(\tilde{\phi})\big) \\
&\quad - \gamma\big(M(\tilde{\phi})\nabla\tilde{w}, \nabla f(\phi)\big) - \gamma\big(\boldsymbol{u}\cdot\nabla\phi, f(\tilde{\phi})\big) - \gamma\big(\tilde{\boldsymbol{u}}\cdot\nabla\tilde{\phi}, f(\phi)\big). \qquad (4.7)
\end{aligned}
$$

Since $\phi \in L^\infty\big(0,T;H^1(\Omega)\big) \cap L^2\big(0,T;H^3(\Omega)\big)$, it follows that $\phi \in L^8\big(0,T;L^\infty(\Omega)\big)$ by interpolation. Now

$$
||\nabla f(\phi)||_{L^2} \leq C(||\phi||_{L^\infty}^2 + 1)||\nabla\phi||_{L^2},
$$

which implies that $f(\phi) \in L^4(0,T;H^1(\Omega))$. Hence $f(\phi)$ can be used as a test function in Eq. (2.27). Owing to the monotonicity of ϕ^3, one has for a.e. $t \in (0,T)$

$$
\begin{aligned}
&\frac{d}{dt}\left\{\gamma\big(F(\tilde{\phi}),1\big) + \gamma\big(F(\phi),1\big)\right\} = \gamma\left\langle\frac{\partial\phi}{\partial t}, f(\phi)\right\rangle + \gamma\left\langle\frac{\partial\tilde{\phi}}{\partial t}, f(\tilde{\phi})\right\rangle \\
&= -\gamma\Big(M(\phi)\nabla w, \nabla f(\phi)\Big) - \gamma\Big(M(\tilde{\phi})\nabla\tilde{w}, \nabla f(\tilde{\phi})\Big).
\end{aligned}
$$

Eq. (4.7) can then be written as

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|\tilde{\boldsymbol{u}}_c - \boldsymbol{u}_c\|_{L^2}^2 + \frac{\gamma\epsilon}{2}\|\nabla\tilde{\phi} - \nabla\phi\|_{L^2}^2\right\} \\
&\quad + \int_{\Omega_c} 2\nu(\phi_c)\big|\mathbb{D}(\tilde{\boldsymbol{u}}_c) - \mathbb{D}(\boldsymbol{u}_c)\big|^2 dx + \int_\Omega M(\phi)|\nabla\tilde{w} - \nabla w|^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\Omega_m}\nu(\phi_m)\Pi^{-1}\big|\tilde{\boldsymbol{u}}_m-\boldsymbol{u}_m\big|^2dx+\frac{\alpha_B}{\sqrt{\mathrm{tr}(\Pi)}}\int_\Gamma\nu(\phi_c)|P_\tau\tilde{\boldsymbol{u}}_c-P_\tau\boldsymbol{u}_c|^2ds\\
\le&\Big((\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c)\cdot\nabla(\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c),\tilde{\boldsymbol{u}}_c\Big)_c-\Big(\tilde{\boldsymbol{u}}_c\cdot\nabla(\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c),(\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c)\Big)_c\\
&-\Big(2[\nu(\phi_c)-\nu(\tilde{\phi}_c)]\mathbb{D}(\tilde{\boldsymbol{u}}_c),\big(\mathbb{D}(\boldsymbol{u}_c)-\mathbb{D}(\tilde{\boldsymbol{u}}_c)\big)\Big)_c\\
&-\Big([M(\phi)-M(\tilde{\phi})]\nabla\tilde{w},(\nabla w-\nabla\tilde{w})\Big) \qquad (4.8)\\
&-\Big((\nu(\phi_m)-\nu(\tilde{\phi}_m))\Pi^{-1}\tilde{\boldsymbol{u}}_m,(\boldsymbol{u}_m-\tilde{\boldsymbol{u}}_m)\Big)_m\\
&-\left\langle\frac{\alpha_B}{\sqrt{\mathrm{tr}(\Pi)}}[\nu(\phi_c)-\nu(\tilde{\phi}_c)]P_\tau\tilde{\boldsymbol{u}}_c,(P_\tau\boldsymbol{u}_c-P_\tau\tilde{\boldsymbol{u}}_c)\right\rangle_\Gamma\\
&-\Big((w-\tilde{w})\tilde{\boldsymbol{u}},(\nabla\phi-\nabla\tilde{\phi})\Big)-\Big(\tilde{w}(\tilde{\boldsymbol{u}}-\boldsymbol{u}),(\nabla\phi-\nabla\tilde{\phi})\Big)\\
&+\gamma\Big([M(\tilde{\phi})-M(\phi)]\nabla\tilde{w},\nabla[f(\tilde{\phi})-f(\phi)]\Big)+\gamma\Big(M(\phi)\nabla[\tilde{w}-w],\nabla[f(\tilde{\phi})-f(\phi)]\Big)\\
&-\gamma\Big((\boldsymbol{u}-\tilde{\boldsymbol{u}})\cdot\nabla(\phi-\tilde{\phi}),f(\tilde{\phi})\Big)-\gamma\Big(\tilde{\boldsymbol{u}}\cdot\nabla(\phi-\tilde{\phi}),f(\tilde{\phi})-f(\phi)\Big)\\
:=&\sum_{j=1}^{8}I_j,
\end{aligned}
$$

where each I_j corresponds to the jth line on the right hand side of the inequality (4.8).

The term I_1 is estimated as follows

$$
\begin{aligned}
|I_1|&\le C||\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c||_{L^4}||\nabla(\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c)||_{L^2}||\tilde{\boldsymbol{u}}_c||_{L^4}\\
&\le||\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c||_{L^2}^{\frac14}||\nabla(\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c)||_{L^2}^{\frac74}||\tilde{\boldsymbol{u}}_c||_{L^4}\\
&\le\chi\int_{\Omega_c}2\nu(\phi_c)\big|\mathbb{D}(\tilde{\boldsymbol{u}}_c)-\mathbb{D}(\boldsymbol{u}_c)\big|^2dx+C||\boldsymbol{u}_c-\tilde{\boldsymbol{u}}_c||_{L^2}^2||\tilde{\boldsymbol{u}}_c||_{L^4}^8,
\end{aligned}
\qquad (4.9)
$$

where χ is a constant to be determined later, and one has utilized the interpolation, as well as the lower and upper bounds of the viscosity ν.

Recall that $\phi-\tilde{\phi}$ is of mean zero. To estimate the rest of the terms, we notice that the Gagliardo-Nirenberg inequality and the Poincaré inequality imply

$$
||\phi-\tilde{\phi}||_{L^\infty}\le C\big(||\nabla\Delta(\phi-\tilde{\phi})||_{L^2}^{\frac14}||\nabla(\phi-\tilde{\phi})||_{L^2}^{\frac34}+||\nabla(\phi-\tilde{\phi})||_{L^2}\big). \qquad (4.10)
$$

It follows from the definition of the chemical potential, cf. Eq. (2.28), that

$$
\gamma\epsilon||\nabla\Delta(\phi-\tilde{\phi})||_{L^2}\le||\nabla(w-\tilde{w})||_{L^2}+\gamma||\nabla\big(f(\phi)-f(\tilde{\phi})\big)||_{L^2}.
$$

Since

$$\nabla(\phi^3 - \tilde{\phi}^3) = 3\tilde{\phi}^2\nabla(\phi - \tilde{\phi}) + 3(\phi^2 - \tilde{\phi}^2)\nabla\phi,$$

one has

$$\begin{aligned}
&||\nabla\big(f(\phi) - f(\tilde{\phi})\big)||_{L^2} \\
&\leq C||\nabla\big(\phi - \tilde{\phi}\big)||_{L^2} + ||\tilde{\phi}||^2_{L^\infty}||\nabla\big(\phi - \tilde{\phi}\big)||_{L^2} \\
&\quad + ||\phi + \tilde{\phi}||_{L^\infty}||\phi - \tilde{\phi}||_{L^\infty}||\nabla\phi||_{L^2} \\
&\leq C||\nabla\big(\phi - \tilde{\phi}\big)||_{L^2} + C(1 + ||\phi||_{L^\infty}) \\
&\quad \times \left(||\nabla\Delta(\phi - \tilde{\phi})||^{\frac{1}{4}}_{L^2}||\nabla(\phi - \tilde{\phi})||^{\frac{3}{4}}_{L^2} + ||\nabla(\phi - \tilde{\phi})||_{L^2}\right) \\
&\leq \frac{1}{2}||\nabla\Delta(\phi - \tilde{\phi})||_{L^2} + C\big(1 + ||\phi||^{\frac{4}{3}}_{L^\infty}\big)||\nabla\big(\phi - \tilde{\phi}\big)||_{L^2}.
\end{aligned} \tag{4.11}$$

Hence

$$||\nabla\Delta(\phi - \tilde{\phi})||_{L^2} \leq 2||\nabla(w - \tilde{w})||_{L^2} + C\big(1 + ||\phi||^{\frac{4}{3}}_{L^\infty}\big)||\nabla\big(\phi - \tilde{\phi}\big)||_{L^2}. \tag{4.12}$$

We estimate $I_j, j = 2\cdots 8$ as follows. By the Lipschitz continuity of ν, one obtains

$$\begin{aligned}
|I_2| &\leq C||\phi_c - \tilde{\phi}_c||_{L^\infty}||\mathbb{D}(\tilde{\boldsymbol{u}}_c)||_{L^2}||\mathbb{D}(\tilde{\boldsymbol{u}}_c) - \mathbb{D}(\boldsymbol{u}_c)||_{L^2} \\
&\leq C\left(||\nabla\Delta(\phi - \tilde{\phi})||^{\frac{1}{4}}_{L^2}||\nabla(\phi - \tilde{\phi})||^{\frac{3}{4}}_{L^2} + ||\nabla(\phi - \tilde{\phi})||_{L^2}\right) \\
&\quad \times ||\mathbb{D}(\tilde{\boldsymbol{u}}_c)||_{L^2}||\mathbb{D}(\tilde{\boldsymbol{u}}_c) - \mathbb{D}(\boldsymbol{u}_c)||_{L^2} \\
&\leq \chi_1||\nabla\Delta(\phi - \tilde{\phi})||^2_{L^2} + \chi\int_{\Omega_c} 2\nu(\phi_c)\big|\mathbb{D}(\tilde{\boldsymbol{u}}_c) - \mathbb{D}(\boldsymbol{u}_c)\big|^2 dx \\
&\quad + C\left(||\mathbb{D}(\tilde{\boldsymbol{u}}_c)||^{\frac{8}{3}}_{L^2} + ||\mathbb{D}(\tilde{\boldsymbol{u}}_c)||^2_{L^2}\right)||\nabla(\phi - \tilde{\phi})||^2_{L^2} \\
&\leq \chi_1\int_\Omega M(\phi)|\nabla\tilde{w} - \nabla w|^2 dx + \chi\int_{\Omega_c} 2\nu(\phi_c)\big|\mathbb{D}(\tilde{\boldsymbol{u}}_c) - \mathbb{D}(\boldsymbol{u}_c)\big|^2 dx \\
&\quad + C\left(1 + ||\phi||^{\frac{8}{3}}_{L^\infty}\right)||\nabla\left(\phi - \tilde{\phi}\right)||^2_{L^2}.
\end{aligned} \tag{4.13}$$

Likewise, one has

$$\begin{aligned}
|I_3| &\leq C||\phi_c - \tilde{\phi}_c||_{L^\infty}||\nabla\tilde{w}||_{L^2}||\nabla(\tilde{w} - w)||_{L^2} \\
&\leq \chi\int_\Omega M(\phi)|\nabla\tilde{w} - \nabla w|^2 dx \\
&\quad + C\left(1 + ||\phi||^{\frac{8}{3}}_{L^\infty} + ||\nabla\tilde{w}||^{\frac{8}{3}}_{L^2}\right)||\nabla\left(\phi - \tilde{\phi}\right)||^2_{L^2};
\end{aligned} \tag{4.14}$$

$$|I_4| \leq \chi \int_{\Omega_m} \Pi^{-1}\nu(\phi_m)\big|\tilde{\boldsymbol{u}}_m - \boldsymbol{u}_m\big|^2 dx \\ + C\big(1 + ||\phi||_{L^\infty}^{\frac{8}{3}} + ||\tilde{\boldsymbol{u}}_m||_{L^2}^{\frac{8}{3}}\big)||\nabla\big(\phi - \tilde{\phi}\big)||_{L^2}^2; \tag{4.15}$$

By the trace theorem and Sobolev imbedding, i.e., $H^1(\Omega) \hookrightarrow H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^4(\Gamma)$, one has

$$\begin{aligned} |I_5| &\leq \frac{\alpha_B}{\sqrt{\mathrm{tr}(\Pi)}}||\nu(\phi) - \nu(\tilde{\phi})||_{L^4(\Gamma)}||P_\tau \tilde{\mathbf{u}}_c||_{L^4(\Gamma)}||P_\tau \tilde{\boldsymbol{u}}_c - P_\tau \boldsymbol{u}_c||_{L^2(\Gamma)} \\ &\leq C\frac{\alpha_B}{\sqrt{\mathrm{tr}(\Pi)}}||\phi - \tilde{\phi}||_{H^1(\Omega)}||\tilde{\mathbf{u}}_c||_{H^1(\Omega_c)}||P_\tau \tilde{\boldsymbol{u}}_c - P_\tau \boldsymbol{u}_c||_{L^2(\Gamma)} \\ &\leq \chi\frac{\alpha_B}{\sqrt{\mathrm{tr}(\Pi)}}\int_\Gamma \nu(\phi_c)|P_\tau \tilde{\boldsymbol{u}}_c - P_\tau \boldsymbol{u}_c|^2 ds + C||\nabla\big(\phi - \tilde{\phi}\big)||_{L^2}^2; \end{aligned} \tag{4.16}$$

Upon performing integration by parts, one derives the estimate for I_6 analogous to the one for I_2

$$\begin{aligned} |I_6| &= \Big|\Big((\phi - \tilde{\phi})\nabla(w - \tilde{w}), \tilde{\boldsymbol{u}}\Big) + \Big((\phi - \tilde{\phi})(\tilde{\boldsymbol{u}} - \boldsymbol{u}), \nabla\tilde{w}\Big)\Big| \\ &\leq \chi\int_{\Omega_m} \Pi^{-1}\nu(\phi_m)\big|\tilde{\boldsymbol{u}}_m - \boldsymbol{u}_m\big|^2 dx \\ &+ \chi\int_\Omega M(\phi)|\nabla\tilde{w} - \nabla w|^2 dx + C||\tilde{\boldsymbol{u}}_c - \boldsymbol{u}_c||_{L^2}^2 \\ &+ C\big(1 + ||\phi||_{L^\infty}^{\frac{8}{3}} + ||\nabla\tilde{w}||_{L^2}^{\frac{8}{3}} + ||\tilde{\boldsymbol{u}}_m||_{L^2}^{\frac{8}{3}}\big)||\nabla\big(\phi - \tilde{\phi}\big)||_{L^2}^2; \end{aligned} \tag{4.17}$$

For I_7, by (4.11), (4.10) and (4.12), one has

$$\begin{aligned} |I_7| &\leq C||\phi - \tilde{\phi}||_{L^\infty}||\nabla\tilde{w}||_{L^2}||\nabla\big(f(\phi) - f(\tilde{\phi})\big)||_{L^2} \\ &\quad + C||\nabla(\tilde{w} - w)||_{L^2}||\nabla\big(f(\phi) - f(\tilde{\phi})\big)||_{L^2} \\ &\leq C||\phi - \tilde{\phi}||_{L^\infty}||\nabla(\phi - \tilde{\phi})||_{L^2}||\nabla\tilde{w}||_{L^2} \\ &\quad + C||\nabla\tilde{w}||_{L^2}(1 + ||\phi||_{L^\infty})||\phi - \tilde{\phi}||_{L^\infty}^2 \\ &\quad + C||\nabla(\tilde{w} - w)||_{L^2}||\nabla(\phi - \tilde{\phi})||_{L^2} \\ &\quad + C(1 + ||\phi||_{L^\infty})||\nabla(\tilde{w} - w)||_{L^2}||\phi - \tilde{\phi}||_{L^\infty} \\ &\leq \chi_1\int_\Omega M(\phi)|\nabla\tilde{w} - \nabla w|^2 dx \\ &\quad + C(1 + ||\nabla\tilde{w}||_{L^2}^2 + ||\phi||_{L^\infty}^2 + ||\phi||_{L^\infty}^{\frac{8}{3}})||\nabla(\phi - \tilde{\phi})||_{L^2}^2 \\ &\quad + C\big(||\nabla\tilde{w}||_{L^2}(1 + ||\phi||_{L^\infty}) + (1 + ||\phi||_{L^\infty}^2)\big) \\ &\quad \times \big(||\nabla\Delta(\phi - \tilde{\phi})||_{L^2}^{\frac{1}{2}}||\nabla(\phi - \tilde{\phi})||_{L^2}^{\frac{3}{2}} + ||\nabla(\phi - \tilde{\phi})||_{L^2}^2\big) \end{aligned} \tag{4.18}$$

$$\leq \chi \int_\Omega M(\phi)|\nabla \tilde{w} - \nabla w|^2 dx$$
$$+ C(1 + ||\nabla \tilde{w}||_{L^2}^2 + ||\nabla \tilde{w}||_{L^2}^{\frac{8}{3}} + ||\phi||_{L^\infty}^2 + ||\phi||_{L^\infty}^{\frac{8}{3}})||\nabla(\phi - \tilde{\phi})||_{L^2}^2,$$

where we have used inequality (4.12). Finally,

$$\begin{aligned}
|I_8| &\leq ||f(\tilde{\phi})||_{L^\infty} ||\boldsymbol{u} - \tilde{\boldsymbol{u}}||_{L^2} ||\nabla(\phi - \tilde{\phi})||_{L^2} + C||\tilde{\boldsymbol{u}}||_{L^2} ||\nabla(\phi - \tilde{\phi})||_{L^2} ||f(\phi) - f(\tilde{\phi})||_{L^\infty} \\
&\leq \chi \int_{\Omega_m} \Pi^{-1}\nu(\phi_m) \big|\tilde{\boldsymbol{u}}_m - \boldsymbol{u}_m\big|^2 dx + C||\boldsymbol{u}_c - \tilde{\boldsymbol{u}}_c||_{L^2}^2 + C||\nabla(\phi - \tilde{\phi})||_{L^2}^2 \\
&\quad + C(1 + ||\phi||_{L^\infty}^2)||\nabla(\phi - \tilde{\phi})||_{L^2} ||\phi - \tilde{\phi}||_{L^\infty} \qquad (4.19) \\
&\leq \chi \int_{\Omega_m} \Pi^{-1}\nu(\phi_m) \big|\tilde{\boldsymbol{u}}_m - \boldsymbol{u}_m\big|^2 dx + C(1 + ||\phi||_{L^\infty}^2)||\nabla(\phi - \tilde{\phi})||_{L^2}^2 \\
&\quad + C(1 + ||\phi||_{L^\infty}^2)||\nabla(\phi - \tilde{\phi})||_{L^2}^{\frac{7}{4}} ||\nabla\Delta(\phi - \tilde{\phi})||_{L^2}^{\frac{1}{4}} + C||\boldsymbol{u}_c - \tilde{\boldsymbol{u}}_c||_{L^2}^2 \\
&\leq \chi \int_{\Omega_m} \Pi^{-1}\nu(\phi_m) \big|\tilde{\boldsymbol{u}}_m - \boldsymbol{u}_m\big|^2 dx + \chi \int_\Omega M(\phi)|\nabla \tilde{w} - \nabla w|^2 dx \\
&\quad + C(1 + ||\phi||_{L^\infty}^2 + ||\phi||_{L^\infty}^{\frac{16}{7}} + ||\phi||_{L^\infty}^{\frac{8}{3}})||\nabla(\phi - \tilde{\phi})||_{L^2}^2 + C||\boldsymbol{u}_c - \tilde{\boldsymbol{u}}_c||_{L^2}^2.
\end{aligned}$$

Collecting the inequalities (4.9), and (4.13)–(4.19), choosing sufficiently small χ, one derives from (4.8) that

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|\tilde{\boldsymbol{u}}_c - \boldsymbol{u}_c\|_{L^2}^2 + \frac{\gamma\epsilon}{2}\|\nabla\tilde{\phi} - \nabla\phi\|_{L^2}^2\right\} + \int_{\Omega_c} 2\nu(\phi_c)\big|\mathbb{D}(\tilde{\boldsymbol{u}}_c) - \mathbb{D}(\boldsymbol{u}_c)\big|^2 dx \\
&\quad + \int_\Omega M(\phi)|\nabla \tilde{w} - \nabla w|^2 dx + \int_{\Omega_m} \Pi^{-1}\nu(\phi_m)\big|\tilde{\boldsymbol{u}}_m - \boldsymbol{u}_m\big|^2 dx \\
&\quad + \frac{\alpha_B}{\sqrt{\mathrm{tr}(\Pi)}} \int_\Gamma \nu(\phi_c)|P_\tau \tilde{\boldsymbol{u}}_c - P_\tau \boldsymbol{u}_c|^2 ds \\
&\leq C||\boldsymbol{u}_c - \tilde{\boldsymbol{u}}_c||_{L^2}^2 + Ch(t)||\nabla(\phi - \tilde{\phi})||_{L^2}^2,
\end{aligned}$$

with $h(t) := 1 + ||\nabla\tilde{w}||_{L^2}^{\frac{8}{3}} + ||\phi||_{L^\infty}^{\frac{8}{3}} + ||\tilde{\boldsymbol{u}}_m||_{L^2}^{\frac{8}{3}}$. Noting that $\phi \in L^8\big(0,T;L^\infty(\Omega)\big)$ and $\tilde{w} \in L^4\big(0,T;H^1(\Omega)\big)$, it follows that $h \in L^1(0,T)$. Gronwall's inequality and Poincare's inequality then imply that

$$\boldsymbol{u}_c = \tilde{\boldsymbol{u}}_c, \quad \phi = \tilde{\phi}, \quad \boldsymbol{u}_m = \tilde{\boldsymbol{u}}_m.$$

This completes the proof of Theorem 2.2. □

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Appendix

We present a derivation of the Cahn-Hilliard-Navier-Stokes-Darcy model studied in this article based on the Onsager's variational principle. We derive the irreversible part of the dynamics mainly based on the Onsager's extremum principle [47, 48]. See [49, 50, 51] for the applications of Onsager's variational principle to multiphase flows.

The free flow in Ω_c is assumed to satisfy the following conservation (momentum and mass, resp.) equations

$$\rho_0\Big(\frac{\partial \boldsymbol{u}_c}{\partial t}+(\boldsymbol{u}_c\cdot\nabla)\boldsymbol{u}_c\Big)-\nabla\cdot\mathbf{S}+\nabla p_c=\mathbf{F}_c, \tag{4.20}$$

$$\nabla\cdot\boldsymbol{u}_c=0, \tag{4.21}$$

$$\frac{\partial\phi_c}{\partial t}+\boldsymbol{u}_c\cdot\nabla\phi_c=-\nabla\cdot\mathbf{J}_c, \tag{4.22}$$

with $\mathbf{S}$ a symmetric tensor, $\mathbf{F}_c$ the force density, $\mathbf{J}_c$ the diffusive flux, to be determined. The total energy of the free flow is

$$E_c=\int_{\Omega_c}\frac{\rho_0}{2}|\boldsymbol{u}_c|^2dx+\gamma\int_{\Omega_c}F(\phi_c)+\frac{\epsilon}{2}|\nabla\phi_c|^2dx, \tag{4.23}$$

where the first term is the total kinetic energy, and the second term represents the total free energy associated with the free flow. As in our work [22], we identify the dissipation in Ω_c as

$$\Phi_c=\int_{\Omega_c}\frac{|\mathbf{J}_c|^2}{2M(\phi_c)}+\frac{|\mathbf{S}|^2}{4\nu(\phi_c)}dx+\int_\Gamma\frac{\alpha_B\nu(\phi_c)}{2\sqrt{tr(\Pi)}}|P_\tau\boldsymbol{u}_c|^2dS, \tag{4.24}$$

where the first term is due to chemical diffusion, the second term is due to viscosity, and the last term is because of friction as a result of fluid particles slipping along the domain interface Γ. The friction mechanism along the domain interface is motivated by the study of single phase flow in superposed free flow and porous media, cf [22] and references therein.

Likewise in Ω_m, we postulate the two-phase flow in porous media satisfies the following conservation of mass

$$\frac{\partial \phi_m}{\partial t} + \boldsymbol{u}_m \cdot \nabla \phi_m = -\nabla \cdot \mathbf{J}_m. \tag{4.25}$$

The fluid equations will be derived through the variational procedure. The total energy and dissipation in porous media are as follows

$$\begin{aligned} E_m &= \gamma \int_{\Omega_m} F(\phi_m) + \frac{\epsilon}{2}|\nabla \phi_m|^2 dx, \\ \Phi_m &= \int_{\Omega_m} \frac{|\mathbf{J}_m|^2}{2M(\phi_m)} + \frac{\nu(\phi_m)}{2\rho_0 g}\Pi^{-1}|\boldsymbol{u}_m|^2 dx, \end{aligned}$$

where the second term in Φ_m represents the Darcy damping in porous media.

Before we derive the forms of $\mathbf{S}, \mathbf{F}_c, \mathbf{J}_c$ and $\mathbf{F}_m, \mathbf{J}_m$, we prescribe boundary conditions. On Γ_c, the no-slip no penetration boundary condition $\boldsymbol{u}_c = 0$ is imposed for velocity, and no chemical flux condition $\mathbf{J}_c \cdot \mathbf{n}_c = 0$ is imposed. Similarly, one imposes $\boldsymbol{u}_m \cdot \mathbf{n}_m = 0$ and $\mathbf{J}_m \cdot \mathbf{n}_c = 0$ on Γ_m. On Γ, for conservation of mass one naturally imposes the following continuity interface boundary conditions

$$\boldsymbol{u}_c \cdot \mathbf{n}_c = \boldsymbol{u}_m \cdot \mathbf{n}_c, \quad \phi_c = \phi_m, \quad \mathbf{J}_c \cdot \mathbf{n}_c = \mathbf{J}_m \cdot \mathbf{n}_m. \tag{4.26}$$

One calculates the rate of change of the total energy E_c, by Eqs. (4.20) and (4.22), and by performing integration by parts

$$\begin{aligned} \frac{d}{dt}E_c = &\int_{\Omega_c} \rho_0 \boldsymbol{u}_c \cdot \frac{\partial \boldsymbol{u}_c}{\partial t} dx + \int_{\Omega_c} w_c \frac{\partial \phi_c}{\partial t} dx \\ &+ \gamma\epsilon \int_{\Gamma_c} \nabla \phi_c \cdot \mathbf{n}_c \frac{\partial \phi_c}{\partial t} dS + \gamma\epsilon \int_{\Gamma} \nabla \phi_c \cdot \mathbf{n}_c \frac{\partial \phi_c}{\partial t} dS \\ = &\int_{\Omega_c} \Big[-\rho_0 (\boldsymbol{u}_c \cdot \nabla)\boldsymbol{u}_c + \nabla \cdot \mathbf{S} - \nabla p_c + \mathbf{F}_c \Big] \cdot \boldsymbol{u}_c dx \end{aligned}$$

$$
\begin{aligned}
&- \int_{\Omega_c} w_c \Big(\boldsymbol{u}_c \cdot \nabla \phi_c + \nabla \cdot \mathbf{J}_c \Big) dx \\
&+ \gamma\epsilon \int_{\Gamma_c} \nabla \phi_c \cdot \mathbf{n}_c \frac{\partial \phi_c}{\partial t} dS + \gamma\epsilon \int_{\Gamma} \nabla \phi_c \cdot \mathbf{n}_c \frac{\partial \phi_c}{\partial t} dS \\
= &- \int_{\Omega_c} \mathbf{S} : \mathbb{D}(\boldsymbol{u}_c) dx + \int_{\Omega_c} \mathbf{F}_c \cdot \boldsymbol{u}_c dx \\
&+ \int_{\Gamma} \Big[\mathbf{n}_c \cdot (\mathbf{S}\mathbf{n}_c) - p_c - \rho_0 \frac{|\boldsymbol{u}_c|^2}{2} \Big] \boldsymbol{u}_c \cdot \mathbf{n}_c dS \\
&+ \int_{\Gamma} P_\tau (\mathbf{S}\mathbf{n}_c) \cdot P_\tau \boldsymbol{u}_c dS + \int_{\Omega_c} \mathbf{J}_c \cdot \nabla w_c - w_c \boldsymbol{u}_c \cdot \nabla \phi_c dx \\
&+ \gamma\epsilon \int_{\Gamma_c} \nabla \phi_c \cdot \mathbf{n}_c \frac{\partial \phi_c}{\partial t} dS + \int_{\Gamma} \gamma\epsilon \nabla \phi_c \cdot \mathbf{n}_c \frac{\partial \phi_c}{\partial t} - w_c \mathbf{J}_c \cdot \mathbf{n}_c dS,
\end{aligned}
\tag{4.27}
$$

where $w_c := \gamma[f(\phi_c) - \epsilon \Delta \phi_c]$ is the chemical potential and the symmetry of the tensor $\mathbf{S}$ has been utilized. In a similar fashion, one calculates the rate of change of total energy in porous media

$$
\begin{aligned}
\frac{d}{dt} E_m = &\int_{\Omega_c} \mathbf{J}_m \cdot \nabla w_m - w_m \boldsymbol{u}_m \cdot \nabla \phi_m dx + \gamma\epsilon \int_{\Gamma_m} \nabla \phi_m \cdot \mathbf{n}_m \frac{\partial \phi_m}{\partial t} dS \\
&- \int_{\Gamma} \gamma\epsilon \nabla \phi_m \cdot \mathbf{n}_c \frac{\partial \phi_m}{\partial t} + w_m \mathbf{J}_c \cdot \mathbf{n}_c dS.
\end{aligned}
\tag{4.28}
$$

If gravity force (matched densities) is the only external force applied in free flow, one identifies the rate of change of the mechanical work with, cf. [51]

$$
\frac{dW}{dt} = \int_{\Omega_c} \mathbf{F}_c \cdot \boldsymbol{u}_c - w_c \nabla \phi_c \cdot \boldsymbol{u}_c = 0,
$$

which leads to the choice

$$
\mathbf{F}_c = w_c \nabla \phi_c. \tag{4.29}
$$

To derive the irreversible part of the dynamics, we resort to Onsager's variational principle which theorizes that the configuration is to minimize

$$
\frac{d}{dt} E_c + \frac{d}{dt} E_m + \Phi_c + \Phi_m - \int_{\Omega_m} p_m \nabla \cdot \boldsymbol{u}_m dx, \tag{4.30}
$$

with respect to rate functions

$$
\boldsymbol{u}_m, \quad p_m, \quad \mathbf{S}, \quad \{\mathbf{J}_c, \mathbf{J}_m\}, \quad \frac{\partial \phi_c}{\partial t}|_{\Gamma_c}, \quad \frac{\partial \phi_m}{\partial t}|_{\Gamma_m}, \quad \frac{\partial \phi_c}{\partial t}|_{\Gamma}, \tag{4.31}
$$

which results in a dissipative dynamic system such that

$$\frac{d}{dt}E_c + \frac{d}{dt}E_m = -2(\Phi_c + \Phi_m). \tag{4.32}$$

The variational procedure gives

$$\frac{\nu(\phi_m)}{\rho_0 g}\Pi^{-1}\boldsymbol{u}_m = -\nabla p_m + w_m\nabla\phi_m, \quad \nabla\cdot\boldsymbol{u}_m = 0, \tag{4.33}$$

$$\mathbf{J}_c = -M(\phi_c)\nabla w_c, \quad \mathbf{J}_m = -M(\phi_m)\nabla w_m, \quad \mathbf{S} = 2\nu(\phi_c)\mathbb{D}(\boldsymbol{u}_c), \tag{4.34}$$

$$\begin{aligned}&\nabla\phi_c\cdot\mathbf{n}_c|_{\Gamma_c} = \nabla\phi_m\cdot\mathbf{n}_m|_{\Gamma_m} = 0,\\ &\nabla\phi_c\cdot\mathbf{n}_c|_{\Gamma} = \nabla\phi_m\cdot\mathbf{n}_c|_{\Gamma}, \quad w_c|_{\Gamma} = w_m|_{\Gamma}.\end{aligned} \tag{4.35}$$

One recognizes that Eqs.(4.33) are the Darcy's law with surface tension effect, cf. [52, 12].

With the help of Eqs. (4.29), (4.33)–(4.35), one may write

$$\begin{aligned}\frac{d}{dt}E_c + \frac{d}{dt}E_m = &-\int_{\Omega_c} M(\phi_c)|\nabla w_c|^2 + 2\nu(\phi_c)\mathbb{D}(\boldsymbol{u}_c):\mathbb{D}(\boldsymbol{u}_c)dx\\ &-\int_{\Omega_m} M(\phi_m)|\nabla w_m|^2 dx + \frac{\nu(\phi_m)}{\rho_0 g}\Pi^{-1}|\boldsymbol{u}_m|^2 dx\\ &+\int_{\Gamma}\Big[\mathbf{n}_c\cdot(2\nu\mathbb{D}(\boldsymbol{u}_c)\mathbf{n}_c) - p_c - \rho_0\frac{|\boldsymbol{u}_c|^2}{2} + p_m\Big]\boldsymbol{u}_c\cdot\mathbf{n}_c dS\\ &+\int_{\Gamma} P_\tau(2\nu\mathbb{D}(\boldsymbol{u}_c)\mathbf{n}_c)\cdot P_\tau\boldsymbol{u}_c dS.\end{aligned} \tag{4.36}$$

Comparing Eq. (4.36) to Eq. (4.32) implied by the Onsager extremum principle, one obtains that

$$-\mathbf{n}_c\cdot(\mathbb{T}(\boldsymbol{u}_c, p_c)\cdot\mathbf{n}_c) + \frac{1}{2}(\boldsymbol{u}_c\cdot\boldsymbol{u}_c) = p_m, \text{ on } \Gamma, \tag{4.37}$$

$$-\boldsymbol{\tau}_j\cdot(\mathbb{T}(\boldsymbol{u}_c, p_c)\cdot\mathbf{n}_c) = \frac{\alpha_B\nu(\phi_c)}{\sqrt{\mathrm{tr}(\Pi)}}\boldsymbol{\tau}_j\cdot\boldsymbol{u}_c, \quad j = 1, 2, \text{ on } \Gamma. \tag{4.38}$$

This completes the derivation of the model.

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