Dynamic transitions and bifurcations for a class of axisymmetric geophysical fluid flow*

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Abstract. In this article, we aim to classify the dynamic transitions and bifurcations for a family of axisymmetric geophysical fluid problems of a generic fourth-second order structure. A transition theorem is established by reducing the governing partial differential equations to a complex-valued ordinary differential equation, derived by employing approximate invariant manifolds. We develop an algorithm for the numerical determination of the transition/bifurcation types. Finally we apply the transition theorem and algorithm to examine the baroclinic instability in a two–layer quasi-geostrophic system in an annular channel and with different bathymetry profiles. Our numerical results show that with concave bathymetry the transition (bifurcation) is always continuous (supercritical Hopf bifurcation), whereas for convex bathymetry a jump transition (subcritical Hopf bifurcation) may occur in the basic azimuthal currents that rotate in the same direction.

Key words. Dynamic transitions; Baroclinic flows; Quasi-geostrophic models; Topographic effects; Axially symmetric problems.

AMS subject classifications. 76D99, 76E20, 37L15, 37L99

1. Introduction.

Dynamic transitions are the transitions from one state of a system to another state, which are ubiquitous in geophysical fluid dynamics. The physical state of a system is described by the qualitative behaviour or topological structure of solutions to the governing equations. The phenomena of dynamic transitions and bifurcations are usually studied by the classical bifurcation theory, see for instance [19, 26] and references therein. In some applications the local bifurcation theory is insufficient to classify the transitions especially when there is no bifurcation of solutions involved. In recent years, Ma and Wang [36] develop another paradigm-phase transition dynamics to probe the types of transitions and the structure of transition states. Under the principle of exchange of stabilities (PES) they establish that dynamic transitions of all dissipative systems can be classified into three categories: continuous (attractor bifurcation), jump, and mixed, cf. Section 2 for the precise definition. Roughly speaking, a continuous transition means that the basic state bifurcates to a local attractor; a jump transition says that a system will jump to another state discontinuously, and a random transition indicates that both continuous and jump transitions are possible depending on the initial perturbation. It should be pointed out that the theory of phase transition would recover the results from classical bifurcation theory in applications where the latter is applicable. The theory has been successfully applied in the study of many

*Submitted to the editors DATE.

Funding: The work of D. Han was supported by the National Science Foundation grant DMS-1912715. The work of M. Hernandez was supported in part by the National Science Foundation (NSF) grant DMS-1515024, and by the Office of Naval Research (ONR) grant N00014-15-1-2662. The work of Q. Wang was supported by the National Science Foundation of China (NSFC) grant No. 11901408.

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instability problems in fluid dynamics, for instance, the instabilities and transitions of quasi-geostrophic channel flows [14, 27, 32], tropical atmospheric circulations [21, 28], Rayleigh-Benard convection [33, 47, 46, 17], instabilities associated with magnetic fluid flows [52, 31], and many transition problems with random effects [10, 11, 12, 30, 20].

Many phase transition phenomena in geophysical fluid dynamics exhibit the formation of periodic and quasi-periodic patterns of fluid flows. For instance, the transitions associated with the baroclinic instability and barotropic instability are closely linked to the large-scale ocean surface circulation [2, 4, 24, 40, 49, 50, 16] and the large-scale blocking pattern formation [8, 29, 45] in atmospheric dynamics. Dynamic transitions are also observed during the onset of convection instability in thermal circulation [33, 47, 46, 22, 35, 34, 41]. Although the governing equations in these transition problems are different, their linear parts have a common structure of a fourth-second order coupling. For instance, the two-layer QG (geostrophic-quasi) equations [43, 37] with viscous effects and governing the baroclinic instability in the two-layer flows over an annular channel with a bathymetry η_b [44], given by

$$
\begin{aligned}
&\partial_t p_1 + Re^{-1}\Delta^2 u_1 = -1/r\left(-\partial_\phi u_1 \partial_r + \partial_r u_1 \partial_\phi\right) p_1,\\
&\partial_t p_2 + Re^{-1}\Delta^2 u_2 = -1/r\left(-\partial_\phi u_2 \partial_r + \partial_r u_2 \partial_\phi\right)\left(p_2 - F_2\eta_b(r)\right),\\
&p_1 = \left[-\Delta u_1 + F_1(u_1 - u_2)\right], \quad p_2 = -\Delta u_2 + F_2(u_2 - u_1),
\end{aligned} \tag{1.1}
$$

and the equivalent barotropic dimensionless equation [1]

$$
\partial_t\left(\Delta\psi - S\psi\right) + \partial_x\psi\partial_y\Delta\psi - \partial_y\psi\partial_x\Delta\psi + Re^{-1}\partial_x\psi = Re^{-1}\Delta^2\psi, \tag{1.2}
$$

which is used to modeling the evolution of the western boundary currents on a horizontal periodic channel $(x, y) \in \mathbb{R}/(2\pi\mathbb{Z}) \times (y_1, y_2)$, where F_1, F_2, S and Re are dimensionless parameters. The main objective of this article is to examine and classify the dynamic transitions and bifurcations associated with a family of geophysical fluid flows. In particular, we would like to determine if the transition from one flow regime to another is continuous or jump or even mixed from the perspective of the dynamic transition, in the sense that the state of the flow may gradually or suddenly change from one configuration to another.

To capture as many transition problems as possible, including these problems in a horizontal periodic channel or an annular channel, we consider in an abstract setting the following axisymmetric nonlinear system with an underlying fourth-second order linear structure

$$
\partial_t \mathcal{A}u = \mathcal{L}_\lambda u + \mathcal{G}(u, u), \tag{1.3}
$$

where $(r, \phi) \in (R_1, R_2) \times \mathbb{R}/(2\pi\mathbb{Z}) \to u(r, \phi) \in \mathbb{R}^d$ is the state function, $R_2 > R_1$, λ is a real control parameter, ∂_t represents differentiation with respect to time, $\mathcal{A}$ and $\mathcal{L}_\lambda$ are linear differential operators given, respectively, by

$$
\begin{aligned}
&(\mathcal{A}u)^i(r, \phi) := \sum_{j=1}^{d} \sum_{|\alpha| \le N_i - 2} q^i_{j\alpha}(r) D^\alpha u^j(r, \phi),\\
&(\mathcal{L}_\lambda u)^i(r, \phi) := \sum_{j=1}^{d} \sum_{|\alpha| \le N_i} p^i_{j\alpha}(r, \lambda) D^\alpha u^j(r, \phi), \qquad 1 \le i \le d,
\end{aligned} \tag{1.4}
$$

with $|\alpha| = \alpha_1 + \alpha_2$, $\alpha = (\alpha_1, \alpha_2)$, $D^\alpha = \partial r^{\alpha_1} \partial \phi^{\alpha_2}$ and $N_i \in \{2, 4\}$, and $\mathcal{G}$ is the nonlinear differential operators given by

$$\mathcal{G}(u,v)^i(r,\phi) := \sum_{j,k=1}^{d} \sum_{\substack{|\alpha| \le N_i - 1 \\ |\beta| \le N_i - 1 \\ |\alpha| \ne |\beta| \\ |\alpha| + |\beta| \le N_i}} g^i_{jk,\alpha\beta}(r) D^\alpha u^j(r,\phi) D^\beta v^k(r,\phi), \quad 1 \le i \le d. \tag{1.5}$$

For those coefficient functions $q^i_{j\alpha}(r)$, $p^i_{j\alpha}(r)$ and $g^i_{jk,\alpha\beta}(r)$, we suppose that all of them are bounded and smooth functions defined on $[R_1, R_2]$, satisfying

$$p^i_{j\alpha}(r,\lambda) = 0, \quad j \ne i, \quad |\alpha| = N_i, \tag{1.6}$$

$$\begin{aligned} &\min_{r \in [R_1,R_2]} \left| q^i_{i\alpha}(r) \right| > 0 \quad \text{for} \quad |\alpha| = N_i - 2, \\ &\min_{r \in [R_1,R_2]} \left| p^i_{i\alpha}(r,\lambda) \right| > 0 \quad \text{for} \quad |\alpha| = N_i, \end{aligned} \tag{1.7}$$

and required that they could guarantee the invertibility of $\mathcal{A}$ and the highest part of $\mathcal{L}_\lambda$.

We note that the nonlinear system (1.3) not only contains the Eqs. (1.1) and (1.2), but also covers a broad class of equations arising in geophysical fluid dynamics, such as the l-layer QG equations governing barotropic and baroclinic instability [7, 15, 3, 23, 51, 5, 28, 18] and viscosity instability and others [2, 4, 24, 40, 49, 50, 42] where $d = l$, $M_1, ..., M_d = 2$, $N_1, ..., N_d = 4$, the Boussinesq equations for thermal circulation [33, 47, 46, 22, 35, 34, 41] with temperature and/or salinity in a stream function formulation ($M_1 = 2$, $M_2 = M_3 = 0$, $N_1 = 4$, $N_2 = N_3 = 2$), and other variants of the Boussinesq equations [35, 41, 38] for different problems in atmospheric and oceanic sciences, fluid systems and magnetic fluid systems.

The system (1.3) is not complete without boundary conditions. For this we consider the domain $\mathcal{D} = (R_1, R_2) \times (\mathbb{R}/2\pi\mathbb{Z})$, and, for each $j = 1, ..., d$, the boundary conditions imposed on the component u^j are of the form

$$\begin{aligned} &\sum_{|\alpha| \le N_j - 1} b^L_{l,j,\alpha} D^\alpha u^j(R_1) = 0, \quad l = 1, \cdots, D_j, \\ &\sum_{|\alpha| \le N_j - 1} b^R_{l,j,\alpha} D^\alpha u^j(R_2) = 0, \quad l = 1, \cdots, D_j, \end{aligned} \tag{1.8}$$

where $D_j = 2$ if the j^{th} equation is of fourth order ($N_j = 4$), and $D_j = 1$ if it is of second order ($N_j = 2$). Note also that implicit in the above formulation are the periodic boundary conditions

$$u^j(r, \phi + 2k\pi) = u^j(r,\phi), \quad r \in (R_1, R_2), \quad k \in \mathbb{Z}.$$

In a nutshell, the application of the phase transition theory contains three main steps. First, the principle of exchange of stability (PES) condition must be verified. Next, the corresponding reduced equations must be obtained. Finally, the transition number must be computed, and its sign (or that of its real part) determines the type of transition from one state to another.

For our problem (1.3), the first step amounts to solve the generalized eigenvalue problem

$$\beta(\lambda)\mathcal{A}u = \mathcal{L}_\lambda u. \tag{1.9}$$

Except in the case where the coefficients of the differential operators $\mathcal{A}$ and $\mathcal{L}$ are constants, see [33, 47, 46, 22, 35, 34, 41], it is challenging, if not impossible, to solve (1.9) analytically. Thus, in this article, we shall adapt the spectral method [48] for solving single one dimensional second/fourth order differential equations to solve the general eigenvalue problem (1.9). For the second step, the reduced equations corresponding to our problem (1.3) are obtained by approximating the center manifold, leading to a system of the form

$$\frac{dz}{dt} = \beta_{1,m}z + \tau z|z|^2 + \tau_1 z|z|^4 + o(|z|^5), \quad z \in \mathbb{C},$$

where the coefficients τ (τ_1) of the third (fourth)-order term above are determined by an algebraic expression involving the first eigenvector and the nonlinear part in (1.3). These coefficients have to be computed numerically except for the case where non-linear interactions between eigenvectors can be expressed as finite combinations of these. Here $\beta_{1,m}$ is the eigenvalue of (1.9) with the largest real part, satisfying PES condition (3.3) or (3.4).

Throughout this article, we call τ and τ_1 as the first transition number and second transition number for the nonlinear system (1.3), respectively. The main transition theorem can be summarized as follows: 1) If the real part of τ is negative (positive), then the transition of the nonlinear system (1.3) from zero is continuous (jump), which corresponds to a supercritical (subcritical) Hopf bifurcation; 2) When $\tau = 0$, if the real part of τ_1 is negative (positive), then the transition type of the nonlinear system (1.3) from zero is continuous (jump), which corresponds to a supercritical (subcritical) Hopf bifurcation.

For these results to be of practical use, in the present work we also provide an algorithm which is utilized to determine the sign of τ and τ_1. We then apply the transition theorem and the numerical algorithm to study the dynamic transitions associated with the baroclinic instability in two layer basic flows over an annular channel and with different bathymetry profiles. Our numerical experiments show that for a concave bathymetry, the transition of the two-layer system is always continuous, whereas for a convex bathymetry, a jump transition may occur in the basic azimuthal currents that rotate in the same direction. Related to our results are those in [3, 39, 51] which study the effect of the ratio of the bathymetry slope to the mean isopycnal slop on the baroclinic instability based on different non-dissipative models. A relatively thorough numerical simulation is performed in [25] for characterizing the topographic regime using the Eady model. To the best of our knowledge, our results here provide the first complete classification of the transitions and bifurcations associated with baroclinic instability in the two-layer QG model of different bathymetry profiles.

The rest of the article is organized as follows. In Section 1 we recall the definition of dynamic transitions and the classification following [36]. In Section 2, we establish the reduced equations and transition theorem for (1.3), subject to the boundary conditions (1.8), under the assumption that the required PES condition holds. The supplemented algorithm used to estimate the type of transition is given in Section 3. In Section 4, as an application of the algorithm and general transition theorem, we study the effect of the curved bottom slopes on the dynamic transition associated to baroclinic instability arising in two layer currents.

2. Preliminary. In this section we recall the mathematical definition of dynamic transitions and the classification from chapter 2 of [36]. To this end, let Y be a Hilbert space, we consider the following abstract evolution equation defined on Y

$$\frac{dv}{dt} = L_\lambda v + G(v), \quad v \in Y, \tag{2.1}$$

where $\{L_\lambda\}$ represents a family of linear operators parameterized by a scalar control parameter λ, $G(u)$ accounts for the nonlinear terms. We make the following assumptions.

Assumptions about the operator: L_λ takes the form of

$$L_\lambda = -A + B_\lambda, \tag{2.2}$$

where A is a linear homeomorphism from X_1 to Y, which is also a sectorial operator on X_1, i.e. $D(A) = X_1 \subset Y$ and X_1 is compactly and densely embedded in Y, and B_λ is a parameterized family of bounded linear compact operators continuously depending on λ, defined on the interpolated space $X_\gamma = D(A^\gamma)$ for some $\gamma \in [0, 1]$. Note that $L_\lambda : X_1 \to Y$ is a closed operator, and the spectrum of L_λ consists only of isolated eigenvalues with finite algebraic multiplicities.

Assumptions about the nonlinear operator G: It is assumed that $G : X_\alpha \to Y$ is continuous for some fixed $\alpha \in [0, 1)$. Furthermore, we assume that $G(0) = 0$, and the tangent map of G at 0 is assumed to be the null map, i.e., $DG(0) = 0$.

Definition 2.1. *We call the system (2.1) undergoes a dynamic transition from $(v, \lambda) = (0, \lambda_0)$ if the following two conditions hold true*

(1) *For $\lambda < \lambda_0$, 0 is locally asymptotically stable for (2.1).*
(2) *For $\lambda > \lambda_0$, there exists a neighborhood U of $v = 0$ independent of λ_0 such that for every $w \in U \setminus (\Gamma_\lambda)$, the solution $v(t, w)$ of (2.1) with initial value w satisfies*

$$\lim_{t\to+\infty} \sup \|v(t, w)\| \geq \delta(\lambda) > 0, \quad \lim_{\lambda\to\lambda_0} \delta(\lambda) \geq 0,$$

where Γ_λ is the stable manifold of $v = 0$ with codim $\Gamma_\lambda \geq 1$ for $\lambda > \lambda_0$.

The following lemma established in [36] (Theorem 2.1.3) provides a relatively complete description of the phase transitions associated to dissipative dynamical systems (2.1).

Lemma 2.2. *Let the (generalized) eigenvalues (counting multiplicity) of L_λ be given by $\{\beta_i(\lambda) | i = 1, 2, \cdots, n-1, n, \cdots\}$ ordered by decreasing real part. If the PES condition*

$$\Re\beta_i(\lambda) \begin{cases} < 0, & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0, & \lambda > \lambda_0 \end{cases}, \quad 1 \leq i \leq m \tag{2.3}$$

$$\Re\beta_j(\lambda_0) < 0, \quad j \geq m+1$$

holds, then the system (2.1) always undergoes a dynamic transition from $(v, \lambda) = (0, \lambda_0)$, and there is a neighborhood U of $v = 0$ such that the transition is one of the following three types:

(1) ***Continuous transition****: There exists an open and dense set $U_\lambda \subset U$ such that for any $w \in U_\lambda$, the solution $v(t, w)$ of (2.1) satisfies*

$$\lim_{\lambda\to\lambda_0} \lim_{t\to+\infty} \sup |v(t, w)| = 0.$$

(2) ***Jump transition****: For every $\lambda_0 < \lambda < \lambda_0 + \epsilon$ with some $\epsilon > 0$, there exists an open and dense subset $U_\lambda \subset U$ such that for any $w \in U_\lambda$, the solution $v(t, w)$ of (2.1) satisfies*

$$\lim_{t\to+\infty} \sup |v(t, w)| \geq \delta > 0,$$

where $\delta > 0$ is independent of λ.

(3) ***Mixed transition****: For every $\lambda_0 < \lambda < \lambda_0 + \epsilon$ for some $\epsilon > 0$, U can be decomposed into two open sets U^1_λ and U^2_λ: $\overline{U} = \overline{U^1_\lambda} \cup \overline{U^2_\lambda}$, $\overline{U^1_\lambda} \cap \overline{U^2_\lambda} = \emptyset$ such that*

$$\lim_{\lambda\to\lambda_0} \lim_{t\to+\infty} \sup |v(t, w)| = 0, \quad \forall\, w \in U^1_\lambda,$$
$$\lim_{t\to+\infty} \sup |v(t, w)| \geq \delta > 0, \quad \forall\, w \in U^2_\lambda,$$

where U^1_λ and U^2_λ are called metastable domains.

According to Definition 2.1 and Lemma 2.2, one can see that dynamic transitions in dissipative system are different than bifurcations. It should be pointed out that in most applications the theory of phase transition would recover the results from classical bifurcation theory in applications where the latter is applicable. For many dissipative systems in geophysical fluid dynamics, $\mathcal{A}$ is invertible . Here, we assume that $\mathcal{A}$ has a bounded inverse. Note if we denote $L_\lambda = \mathcal{A}^{-1}\mathcal{L}_\lambda, G = \mathcal{A}^{-1}\mathcal{G}$, then the nonlinear system (1.3) can be rewritten as the standard form (2.1), which satisfy all assumptions aforementioned. In the following, we will establish a dynamic transition theorem for the axially symmetric nonlinear system (1.3) based on the dynamic transition classification scheme in Lemma 2.2.

3. Dynamic transitions in axially symmetric problems.

The first step needed to determine the transition type of (1.3) is to characterize the spectral properties of the linear problem associated with (1.3). More precisely, we need to study the generalized eigenvalue problem (1.9) subject to the boundary conditions (1.8).

Our standing assumption is that all the coefficients in these equations are independent of the periodic variable ϕ, which justifies the use of separation of variables so that, in looking for eigenfunctions for the linearized equations (1.9), one can, without loss of generality, assume that $u^j(r, \phi) = e^{\mathrm{i}m\phi}U^j(r)$. Then, for a given m, the eigenvalue problem (1.9) with form of

$$\beta(\lambda)\mathcal{A}e^{\mathrm{i}m\phi}U^j(r) = \mathcal{L}_\lambda e^{\mathrm{i}m\phi}U^j(r)$$

becomes

$$\beta_m(\lambda)\mathcal{A}_m U^j = \mathcal{L}_m(\lambda)U^j, \quad m \in \mathbb{Z} \tag{3.1}$$

where now

$$(\mathcal{A}_m U)^i(r) = \sum_{\substack{1\le j\le d\\ |\alpha|\le N_i-2}} (\mathrm{i}m)^{\alpha_2} q^i_{j\alpha}(r)\partial_r^{\alpha_1}U^j,$$

$$(\mathcal{L}_m(\lambda)U)^i(r) = \sum_{\substack{1\le j\le d\\ |\alpha|\le N_i}} (\mathrm{i}m)^{\alpha_2} p^i_{j\alpha}(r,\lambda)\partial_r^{\alpha_1}U^j.$$

In general, U can be complex valued, and we do not impose any normalization condition at this point. This particular issue will be taken care of by means of the dual problem introduced below. The corresponding boundary conditions need to be modified to take into account this special angular dependence, and it is easy to see that, for a given m, they take the form

$$\begin{aligned} &\sum_{|\alpha|\le N_j-1} b^L_{l,j,\alpha}(\mathrm{i}m)^{\alpha_2}\partial_r^{\alpha_1}U^j(r_1) = 0, \quad l=1,...,D_j,\\ &\sum_{|\alpha|\le N_j-1} b^R_{l,j,\alpha}(\mathrm{i}m)^{\alpha_2}\partial_r^{\alpha_1}U^j(r_2) = 0, \quad l=1,...,D_j. \end{aligned} \tag{3.2}$$

3.1. PES condition. Note that the real coefficients of the operators $\mathcal{A}$ and $\mathcal{L}_\lambda$ imply that $\overline{\beta_m(\lambda)}$ is also an eigenvalue of the eigenvalue problem (1.9), the corresponding eigenfunction reads $e^{-\mathrm{i}m\phi}\overline{U^j(r)}$, and $\overline{\beta_m(\lambda)}$ and $\overline{U^j(r)}$ solve

$$\overline{\beta_m(\lambda)}\mathcal{A}_{-m}\overline{U}^j = \mathcal{L}_{-m}(\lambda)\overline{U}^j$$

derived by taking the complex-conjugate of (3.1). Thus, we have $\beta_{-m}(\lambda) = \overline{\beta_m(\lambda)}$. If $\beta_m(\lambda)$ complex, then the eigenfunctions respectively corresponding to $\beta_m(\lambda)$ and $\beta_{-m}(\lambda)$ are complex-conjugate each other. Particularly, if $\beta_m(\lambda)$ is real, then $\beta_{-m}(\lambda) = \beta_m(\lambda)$, and both the real part and imaginary part of $e^{\mathrm{i}m\phi}U^j(r)$ are eigenfunctions of the eigenvalue problem (1.9), i.e., its geometric multiplicity is 2.

We denote the solutions of (3.1) by $\{(\beta_{m,k}(\lambda), U_{m,k}(\lambda))\}_{k\in\mathbb{N}}$, where the eigenvalues, counting multiplicity, are ordered by decreasing real part, that is,

$$\Re\beta_{m,1}(\lambda) \ge \Re\beta_{m,2}(\lambda) \ge \cdots \to -\infty.$$

Here we assume that there exists certain λ_0 and a neighborhood Λ of λ_0, such that there exists a unique $\bar{m}\in\mathbb{Z}^+$, with the property that for $\lambda\in\Lambda$

$$\begin{cases} \Re\beta_{\bar{m},1}(\lambda) = \Re\overline{\beta_{-\bar{m},1}(\lambda)} \begin{cases} >0 & \text{if } \lambda>\lambda_0,\\ =0 & \text{if } \lambda=\lambda_0,\\ <0 & \text{if } \lambda<\lambda_0, \end{cases}\\ \Re\beta_{m,k}(\lambda)<0, \quad (m,k)\ne(\bar{m},1), \quad (m,k)\ne(-\bar{m},1), \end{cases} \tag{3.3}$$

if the leading eigenvalue of the problem (1.9) is complex, or

$$\begin{cases} \beta_{\bar{m},1}(\lambda) = \beta_{-\bar{m},1}(\lambda) \begin{cases} >0 & \text{if } \lambda>\lambda_0,\\ =0 & \text{if } \lambda=\lambda_0,\\ <0 & \text{if } \lambda<\lambda_0, \end{cases}\\ \Re\beta_{m,k}(\lambda)<0, \quad (m,k)\ne(\bar{m},1), \quad (m,k)\ne(-\bar{m},1) \end{cases} \tag{3.4}$$

if the leading eigenvalue of the problem (1.9) is real.

If the leading eigenvalue is complex, and the PES condition (3.3) holds, then the center-unstable space H_c is spanned by the real part and imaginary part of the first eigenvectors, which is of dimension two and can be expressed as

$$H_c = \left\{ u(r,\phi) = z e^{\mathrm{i}\bar{m}\phi} U_{\bar{m},1}(r) + \bar{z} e^{-\mathrm{i}\bar{m}\phi} \overline{U_{\bar{m},1}(r)} :\ z \in \mathbb{C} \right\}. \tag{3.5}$$

If the leading eigenvalue is real, the PES condition (3.4) infers that its geometric multiplicity is equal to algebraic multiplicity, which is 2. Because both the real part and imaginary part of $e^{\mathrm{i}m\phi}U^j(r)$ are eigenfunctions of the eigenvalue problem (1.9) in this case. Hence, the center-unstable space H_c is also give by (3.5) for the case of real leading eigenvalue.

Regarding the non-linear interactions in (1.4), we let

$$\boxed{\mathcal{G}^i_{m,m'}(U,V) = \sum_{j,k=1}^{d} \sum_{\substack{|\alpha|\le N_i-1\\ |\beta|\le N_i-1\\ |\alpha|\neq|\beta| \\ |\alpha|+|\beta|\le N_i}} (\mathrm{i}m)^{\alpha_2}(\mathrm{i}m')^{\beta_2} g^i_{jk,\alpha\beta}(r)\partial_r^{\alpha_1}U^j(r)\partial_r^{\beta_1}V^k(r)}. \tag{3.6}$$

In most applications the non-linear interactions involve exactly one derivative with respect to the angular variable in each term. In our context, this amounts to require that

$$g^i_{jk\alpha\beta} \equiv 0 \quad \text{whenever} \quad \alpha_2 + \beta_2 \neq 1. \tag{3.7}$$

Thus we obtain the algebraic constraint

$$\mathcal{G}(e^{\mathrm{i}m\phi}U, e^{\mathrm{i}m'\phi}V) = e^{\mathrm{i}(m+m')\phi}\mathcal{G}_{m,m'}(U,V).$$

It is also convenient to introduce the dual generalized eigenvalue problem

$$\overline{\beta_m}(\lambda)\mathcal{A}^*_m U^* = \mathcal{L}^*_m(\lambda)U^* \tag{3.8}$$

where now

$$(\mathcal{A}^*_m U^*)^i(r) = \sum_{\substack{1\le j\le d\\ |\alpha|\le N_i-2}} (-1)^{\alpha_1}(\mathrm{i}m)^{\alpha_2}\partial_r^{\alpha_1}\left(q^i_{j\alpha}(r)U^{*j}\right),$$

$$(\mathcal{L}^*_m(\lambda) U^*)^i(r) = \sum_{\substack{1\le j\le d\\ |\alpha|\le N_i}} (-1)^{\alpha_1}(\mathrm{i}m)^{\alpha_2}\partial_r^{\alpha_1}\left(p^i_{j\alpha}(r,\lambda)U^{*j}\right).$$

In most applications of interest, the eigenvalues of (3.1) turn out to be semisimple. Because of this, we can assume that upon appropriately normalizing U^*, we have the orthogonality conditions

$$\langle \mathcal{A}_m U_{m,k}, U^*_{m,k'}\rangle = \sum_{i=1}^{d}\int_{R_1}^{R_2} (\mathcal{A}_m U_{m,k})^i(r)\overline{(U^*_{m,k'})^i(r)}dr = \delta_{k,k'},$$

where $U_{m,k}$ ($U^*_{m,k}$) is the eigenvector (adjoint eigenvector) corresponding to $\beta_{m,k}$.

Thus $U^*_{\bar{m},1}$ provides us with a practical way to implement the Riesz projection, $\Pi_{\mathfrak{c}}$, onto H_c (or, more precisely, onto its image under $\mathcal{A}_m$. Note that, as we shall later see in Sec. 3.5, the specifics of this dual problem are not relevant from a numerical point of view, since in the implementation of the spectral Galerkin method the dual problem can be solved by computing the left eigenvectors of the matrices representing the operators involved in (3.1).

3.2. Center manifold reduction. We next turn to the approximation problem of the center-unstable invariant manifold function $h : H_c \to H_s := (H_c)^\perp$ such that $h(0) = 0$ and $Dh(0) = 0$, where H_c is given in (3.5), H_s is the function spaces generated by the stable eigenvectors associated with the eigenvalue problem (1.9) in the vicinity of $\lambda = \lambda_0$. Since the non-linear interactions in the case at hand are quadratic, the leading order approximation of h is the bilinear form

$$h_2(\xi) = \frac{1}{2} D^2 h(0)(\xi, \xi), \ \ \xi \in H_c. \tag{3.9}$$

This term is determined by usage of the backward-forward procedure introduced in [12]; see also [10, Sec. 3.2]. This procedure relies on the pullback characterization of approximations to (local) invariant manifolds as identified in [12, Chap. 4], that we apply in the deterministic context of this article.

To do so, we first let ξ in H_c be parametrized by $z \in \mathbb{C}$ as in (3.5). Then we solve the linearized equations backwards in time with initial datum ξ at $t = 0$, and denote the corresponding solution by $u^{(1)}$. Next, we solve the linearized equations forward in time by using $\Pi_{\mathfrak{s}}\mathcal{G}(u^{(1)}, u^{(1)})$ as a source term, and impose that the corresponding solution vanishes as $t \to -\infty$, where $\Pi_{\mathfrak{s}}$ is the projection onto H_s.

More precisely, we solve for all $T > 0$, the following *backward-forward system* associated with the evolution equation (1.3):

$$\frac{\mathrm{d}\mathcal{A}_{\mathfrak{c}} u^{(1)}}{\mathrm{d}t} = \mathcal{L}^{\mathfrak{c}}_\lambda u^{(1)}, \quad u^{(1)}(0) = \xi, \quad t \in [-T, 0], \tag{3.10a}$$

$$\frac{\mathrm{d}\mathcal{A}_{\mathfrak{s}} u^{(2)}}{\mathrm{d}t} = \mathcal{L}^{\mathfrak{s}}_\lambda u^{(2)} + \Pi_{\mathfrak{s}}\mathcal{G}(u^{(1)}(s), u^{(1)}(s)), \quad u^{(2)}(-T) = 0, \quad t \in [-T, 0], \tag{3.10b}$$

where $\mathcal{L}^{\mathfrak{c}}_\lambda := \Pi_{\mathfrak{c}}\mathcal{L}_\lambda$ (resp. $\mathcal{A}_{\mathfrak{c}} := \Pi_{\mathfrak{c}}\mathcal{A}$), and $\mathcal{L}^{\mathfrak{s}}_\lambda := \Pi_{\mathfrak{s}} L_\lambda$ (resp. $\mathcal{A}_{\mathfrak{s}} := \Pi_{\mathfrak{s}}\mathcal{A}$), in which $\Pi_{\mathfrak{c}}$ is the projection onto H_c. Note that Eq. (3.10a) being a finite dimensional linear ordinary differential equation (ODE), the solution $u^{(1)}$ exists for all negative times. The existence and uniqueness of $u^{(2)}$ solving (3.10b) can be justified by making use of the fact that $\mathcal{L}^{\mathfrak{s}}_\lambda$ is a negative definite operator on the preimage under $\mathcal{A}_{\mathfrak{s}}$ of set $\Pi_{\mathfrak{s}}\mathcal{G}(H_c, H_c)$.

In the system above, the initial value ξ (in $H_{\mathfrak{c}}$) of $u^{(1)}$ is prescribed at $t = 0$, and the initial value of $u^{(2)}$ at $t = -T$. The solution of this system is obtained by using a two-step *backward-forward integration procedure* — where Eq. (3.10a) is integrated first backward and Eq. (3.10b) is then integrated forward — made possible due to the partial coupling present in (3.10) where $u^{(1)}$ forces (via the nonlinear terms in $\Pi_{\mathfrak{s}}\mathcal{G}$) the evolution equation of $u^{(2)}$ but not reciprocally. Due to this forcing introduced by $u^{(1)}$ which emanates (backward) from ξ, the solution $u^{(2)}$ depends thus naturally on ξ. For that reason, we will emphasize this dependence as $u^{(2)}[\xi]$ hereafter.

Adopting the language of non-autonomous dynamical systems [6, 13], the proof of [12, Prop. 4.1] can be adapted to infer that the leading approximation $h_2(\xi)$ (defined in (3.9)) is in fact obtained as the *pullback limit* of the $u^{(2)}$-component of the solution to the system (3.10), *i.e.*,

$$\boxed{h_2(\xi) = \lim_{T\to+\infty} u^{(2)}[\xi](-T, 0).} \tag{3.11}$$

Note that in (3.11), the dependence on two time arguments for $u^{(2)}[\xi]$ is made apparent to emphasize the two-time description employed for the proper description of the non-autonomous dynamics inherent to (3.10b); see e.g. [9].

We turn now to the explicit determination of $u^{(2)}[\xi]$ in the case of H_c given by (3.5). First, let us remark that due to the simple form of H_c and (3.1), Eq. (3.10a) can be solved explicitly and thus

$$u^{(1)}(t, r, \phi) = ze^{\beta_{\bar{m},1}t}e^{\mathrm{i}\bar{m}\phi}U_{\bar{m},1}(r) + \bar{z}e^{\beta_{-\bar{m},1}t}e^{-\mathrm{i}\bar{m}\phi}U_{-\bar{m},1}(r), \quad t \le 0. \tag{3.12}$$

Using this expression into the right-hand side of Eq. (3.10b), and the following ansatz for its solution $u^{(2)}(t, \cdot, \cdot)$ $(t \le 0)$,

$$u^{(2)}(t, r, \phi) = z^2 u^{(2)}_{20}(t, r)e^{2\mathrm{i}\bar{m}\phi} + |z|^2 u^{(2)}_{11}(t, r) + \bar{z}^2 u^{(2)}_{02}(t, r)e^{-2\mathrm{i}\bar{m}\phi}, \tag{3.13}$$

we observe that $u^{(2)}_{20}, u^{(2)}_{11}$ and $u^{(2)}_{02}$ solve the following auxiliary (uncoupled) system of PDEs

$$\begin{aligned}
&\frac{d}{dt}\mathcal{A}_{2\bar{m}}u^{(2)}_{20} = \mathcal{L}_{2\bar{m}}(\lambda)u^{(2)}_{20} + e^{2\beta_{\bar{m},1}t}\mathcal{G}_{\bar{m},\bar{m}}(U_{\bar{m},1}, U_{\bar{m},1}),\\
&\frac{d}{dt}\mathcal{A}_0 u^{(2)}_{11} = \mathcal{L}_{2\bar{m}}(\lambda)u^{(2)}_{11} + e^{2\Re\beta_{\bar{m},1}t}\mathcal{G}_{\bar{m},-\bar{m}}(U_{\bar{m},1}, U_{-\bar{m},1})\\
&\qquad\qquad + e^{2\Re\beta_{\bar{m},1}t}\mathcal{G}_{-\bar{m},\bar{m}}(U_{-\bar{m},1}, U_{\bar{m},1}),\\
&\frac{d}{dt}\mathcal{A}_{-2\bar{m}}u^{(2)}_{02} = \mathcal{L}_{-2\bar{m}}(\lambda)u^{(2)}_{02} + e^{2\beta_{-\bar{m},1}t}\mathcal{G}_{-\bar{m},-\bar{m}}(U_{-\bar{m},1}, U_{-\bar{m},1}).
\end{aligned} \tag{3.14}$$

The solutions of the diagonal system (3.14) vanishing at $t = -\infty$ are given by

$$\begin{aligned}
u^{(2)}_{20}(t, r) &= e^{2\beta_{\bar{m},1}t}\varphi_{20}(r),\\
u^{(2)}_{11}(t, r) &= e^{2\Re\beta_{\bar{m},1}t}\varphi_{11}(r),\\
u^{(2)}_{02}(t, r) &= e^{2\beta_{-\bar{m},1}t}\varphi_{02}(r),
\end{aligned} \tag{3.15}$$

where φ_{20}, φ_{11} and φ_{02} solve the system of PDEs

$$\begin{aligned}
&2\beta_{\bar{m},1}\mathcal{A}_{2\bar{m}}\varphi_{20} - \mathcal{L}_{2\bar{m}}(\lambda)\varphi_{20} = \mathcal{G}_{\bar{m},\bar{m}}(U_{\bar{m},1}, U_{\bar{m},1}),\\
&2\Re\beta_{\bar{m},1}\mathcal{A}_0\varphi_{11} - \mathcal{L}_0(\lambda)\varphi_{11} = \mathcal{G}_{\bar{m},-\bar{m}}(U_{\bar{m},1}, U_{-\bar{m},1})\\
&\qquad\qquad + \mathcal{G}_{-\bar{m},\bar{m}}(U_{-\bar{m},1}, U_{\bar{m},1}),\\
&2\beta_{-\bar{m},1}\mathcal{A}_{-2\bar{m}}\varphi_{02} - \mathcal{L}_{-2\bar{m}}(\lambda)\varphi_{02} = \mathcal{G}_{-\bar{m},-\bar{m}}(U_{-\bar{m},1}, U_{-\bar{m},1}),
\end{aligned} \tag{3.16}$$

supplemented with the boundary conditions

$$\begin{aligned}\sum_{|\alpha|\le N_j-1} b^L_{l,j,\alpha}(\mathrm{i}2\bar m)^{\alpha_2}\partial_r^{\alpha_1}\varphi^j_{20}(R_1)=0,\\ \sum_{|\alpha|\le N_j-1} b^R_{l,j,\alpha}(\mathrm{i}2\bar m)^{\alpha_2}\partial_r^{\alpha_1}\varphi^j_{20}(R_2)=0,\end{aligned}\tag{3.17}$$

$$\begin{aligned}\sum_{|\alpha|\le N_j-1} b^L_{l,j,\alpha}\partial_r^{\alpha}\varphi^j_{11}(R_1)=0,\quad l=1,...,D_j,\\ \sum_{|\alpha|\le N_j-1} b^R_{l,j,\alpha}\partial_r^{\alpha}\varphi^j_{11}(R_2)=0,\quad l=1,...,D_j,\end{aligned}\tag{3.18}$$

$$\begin{aligned}\sum_{|\alpha|\le N_j-1} b^L_{l,j,\alpha}(-\mathrm{i}2\bar m)^{\alpha_2}\partial_r^{\alpha_1}\varphi^j_{02}(R_1)=0,\\ \sum_{|\alpha|\le N_j-1} b^R_{l,j,\alpha}(-\mathrm{i}2\bar m)^{\alpha_2}\partial_r^{\alpha_1}\varphi^j_{02}(R_2)=0.\end{aligned}\tag{3.19}$$

3.3. Transition theorem. The reduced equation is then obtained by setting $u=\xi+h_2(\xi)$ in Eq. (1.3) with h_2 as given by (3.11) and with $u^{(2)}$ obtained as described above, and by projecting onto the subspace

$$\mathrm{span}\{\mathcal{A}\xi,\ \xi\in H_{\mathfrak{c}}\}.\tag{3.20}$$

Thus, using the expression of $u^{(2)}$ given by (3.13)-(3.15) with φ_{20}, φ_{11} and φ_{02} solving (3.16), and projecting onto $e^{\mathrm{i}a_{\bar m}\phi}U^*_{\bar m,1}$, we get the following reduced equation

$$\begin{aligned}\frac{dz}{dt}=&\beta_{\bar m,1}z+z|z|^2\langle\mathcal{G}_{\bar m,0}(U_{\bar m,1},\varphi_{11})+\mathcal{G}_{0,\bar m}(\varphi_{11},U_{\bar m,1}),U^*_{\bar m,1}\rangle\\ &+z|z|^2\langle\mathcal{G}_{-\bar m,2\bar m}(U_{-\bar m,1},\varphi_{20})+\mathcal{G}_{2\bar m,-\bar m}(\varphi_{20},U_{-\bar m,1}),U^*_{\bar m,1}\rangle+o(|z|^3).\end{aligned}\tag{3.21}$$

We have thus proved the following Lemma.

Lemma 3.1. *The stability and transition of zero solution to (1.3) with any sufficiently small initial condition and in the vicinity of the critical control parameter $\lambda=\lambda_0$ can be reduced to these of the zero solution to the equation taking the simple form*

$$\frac{dz}{dt}=\beta_{\bar m,1}z+\tau z|z|^2+o(|z|^3)\tag{3.22}$$

where

$$\begin{aligned}\tau=&\langle\mathcal{G}_{\bar m,0}(U_{\bar m,1},\varphi_{11})+\mathcal{G}_{0,\bar m}(\varphi_{11},U_{\bar m,1})\\ &+\mathcal{G}_{-\bar m,2\bar m}(U_{-\bar m,1},\varphi_{20}),U^*_{\bar m,1}\rangle\\ &+\langle\mathcal{G}_{2\bar m,-\bar m}(\varphi_{20},U_{-\bar m,1}),U^*_{\bar m,1}\rangle.\end{aligned}\tag{3.23}$$

From the above lemma the transition type can be determined by the sign of $\Re\tau$. More precisely, we have the following dynamic transition theorem:

Theorem 3.2. *Let τ be given by (3.23), for (1.3) at $\lambda = \lambda_0$ and subjected to the boundary conditions (1.8), then the following assertions hold true:*

(1) *If $\Re\tau < 0$ and $\beta_{\bar{m},1} \notin \mathbb{R}$, then the system undergoes a continuous transition. As a result, a stable periodic orbit bifurcates on $\lambda > \lambda_0$, whose expression is approximately given by*

$$u(t,r,\phi) = a_{\bar{m}}(\tau)\left(\cos(\Im\beta_{\bar{m},1}t + \bar{m}\phi)\Re U_{\bar{m},1}(r) - \sin(\Im\beta_{\bar{m},1}t + \bar{m}\phi)\Im U_{\bar{m},1}(r)\right) + O(\Re\beta_{\bar{m},1}), \tag{3.24}$$

where

$$a_{\bar{m}}(\tau) = 2\left(\frac{\Re\beta_{\bar{m},1}}{|\Re\tau|}\right)^{1/2}. \tag{3.25}$$

(2) *if $\Re\tau < 0$ and $\Im\beta_{\bar{m},1} = 0$, then the system undergoes a continuous transition. As a result, an S^1 attractor bifurcates, which contains at least two pairs of stationary solutions approximately given by*

$$u_1(r,\phi) = \pm a_{\bar{m}}(\tau)\cos(a_{\bar{m}}\phi)U_{\bar{m},1}(r) + O(|\Re\beta_{\bar{m},1}|),$$
$$u_2(r,\phi) = \pm a_{\bar{m}}(\tau)\sin(a_{\bar{m}}\phi)U_{\bar{m},1}(r) + O(|\Re\beta_{\bar{m},1}|),$$

(3) *If $\Re\tau > 0$, the transition of the system at $\lambda = \lambda_0$ is jump.*

Proof. In broad terms, the transition type for (1.3) is uniquely determined by the stability of the zero solution of the reduced system obtained by projecting the equations onto its center manifold. By the lemma Theorem 3.1, the claim on the transition type then follows directly by noting that

$$\frac{d}{dt}|z|^2 = 2|z|^2(\Re\beta_{\bar{m},1} + \Re\tau|z|^2) + o(|z|^4)$$

so that, at the critical value $\Re\beta_{\bar{m},1} = 0$, the origin is a stable solution provided $\Re\tau < 0$, and it is unstable if $\Re\tau > 0$. In the former case, there exists an open U_λ containing the zero point in the vicinity of $\lambda = \lambda_0$ such that for any $z_0 \in U_\lambda$, the solution $z(t,z_0)$ of (3.22) satisfies

$$\lim_{\lambda\to\lambda_0}\lim_{t\to+\infty}\sup|z(t,z_0)| = 0.$$

By the (2.2), it means that the dynamic transition is continuous type. For the latter case, due to the instability of zero solution for each $\lambda \in [\lambda_0, +\infty)$, for every $\lambda_0 < \lambda < \lambda_0 + \epsilon$ for some $\epsilon > 0$, there exists an open U_λ containing the zero point such that for any $z_0 \in U_\lambda$, the solution $z(t,z_0)$ of (3.22) satisfies

$$\lim_{\lambda\to\lambda_0}\lim_{t\to+\infty}\sup|z(t,z_0)| > 0,$$

which implies the transition is jump (catastrophic). Thus, when $\Re\tau < 0$ and $\Re\beta_{\bar{m},1} > 0$, it deduces from the lemma Theorem 3.1 that the bifurcated solution is a periodic orbit, approximately given by

$$z(t) = \left(\frac{\Re\beta_{\bar{m},1}}{|\Re\tau|}\right)^{1/2}\exp\left(i\Im\beta_{\bar{m},1}t\right) + O(\Re\beta_{\bar{m},1}).$$

If $\Im\beta_{\bar m,1} = 0$, then the bifurcated solution is a fixed point, which can in fact be considered as a special case of the above formula. Noting that, by substituting the above in the definition of $H_{\mathfrak{c}}$ we get, in terms of the original formulation, a bifurcated solution of the form

$$\begin{aligned} u(t,r,\phi) =& 2\left(\Re\beta_{\bar m,1}/|\Re\tau|\right)^{1/2}\cos(\Im\beta_{\bar m,1}t+\bar m\phi)\Re u_{\bar m,1}(r)\\ &- 2\left(\Re\beta_{\bar m,1}/|\Re\tau|\right)^{1/2}\sin(\Im\beta_{\bar m,1}t+\bar m\phi)\Im u_{\bar m,1}(r) + O(\Re\beta_{\bar m,1}),\end{aligned}$$

which proves the theorem. ∎

3.4. Higher order approximations at the transition. Although the construction just described is usually enough to determine the transition type of a given system, it is sometimes necessary to use a higher order approximation of the relevant (locally) invariant manifolds. For instance, this is the case when $\Re\tau$ vanishes identically, so the transition type is undetermined, or when one is interested in further properties of the bifurcated solution very close to the criticality. In any case, by following the logic of the previous derivation, one can easily obtain, at least in theory, higher order approximations of all the involved quantities.

Suppose we seek an approximation of the invariant manifold of the form

$$h(\xi) = h_2(\xi) + h_3(\xi) + o(|\xi|^3), \qquad \xi(r,\phi) = 2\Re\left(ze^{\mathrm{i}\bar m\phi}U_{\bar m,1}(r)\right),$$

where $|h_j(\xi)| = O(|\xi|^j)$, $j = 2, 3$. By using the exact same procedure previously described, we find that h_2 is given by

$$h_2(\xi)(r,\phi) = z^2e^{2\mathrm{i}\bar m\phi}\varphi_{20}(r) + |z|^2\varphi_{11}(r) + \bar z^2e^{-2\mathrm{i}\bar m\phi}\varphi_{02}(r)$$

with φ_{20}, φ_{11} and φ_{02} solving (3.16).

Similarly, using the ansatz

$$\begin{aligned} h_3(\xi)(r,\phi) = & z^3e^{3\mathrm{i}\bar m\phi}\varphi_{30}(r) + z|z|^2e^{\mathrm{i}\bar m\phi}\varphi_{21}(r)\\ &+ \bar z|z|^2e^{-\mathrm{i}\bar m\phi}\varphi_{12}(r) + \bar z^3e^{-3\mathrm{i}\bar m\phi}\varphi_{03}(r)\end{aligned}$$

we get, upon performing calculations similar to those in Section 2.2, that the coefficients φ_{ij}, $i + j = 3$, are uniquely determined by the equations

$$\begin{aligned} &3\beta_{\bar m,1}\mathcal{A}_{3\bar m}\varphi_{30} - \mathcal{L}_{3\bar m}(\lambda)\varphi_{30} = \mathcal{G}_{\bar m,2\bar m}(U_{\bar m,1},\varphi_{20}) + \mathcal{G}_{2\bar m,\bar m}(\varphi_{20},U_{\bar m,1})\\ &(2\Re\beta_{\bar m,1} + \beta_{\bar m,1})\mathcal{A}_{\bar m}\varphi_{21} - \mathcal{L}_{\bar m}(\lambda)\varphi_{21} = \mathcal{G}_{-\bar m,2\bar m}(U_{-\bar m,1},\varphi_{20})\\ &+ \mathcal{G}_{2\bar m,-\bar m}(\varphi_{20},U_{-\bar m,1}) + \mathcal{G}_{\bar m,0}(U_{\bar m,1},\varphi_{11}) + \mathcal{G}_{0,\bar m}(\varphi_{11},U_{\bar m,1})\\ &(2\Re\beta_{\bar m,1} + \overline{\beta_{\bar m,1}})\mathcal{A}_{-\bar m}\varphi_{12} - \mathcal{L}_{-\bar m}(\lambda)\varphi_{12} = \mathcal{G}_{\bar m,-2\bar m}(U_{\bar m,1},\varphi_{02})\\ &+ \mathcal{G}_{-2\bar m,\bar m}(\varphi_{02},U_{\bar m,1}) + \mathcal{G}_{-\bar m,0}(U_{-\bar m,1},\varphi_{11}) + \mathcal{G}_{0,-\bar m}(\varphi_{11},U_{-\bar m,1})\\ &3\overline{\beta_{\bar m,1}}\mathcal{A}_{-3\bar m}\varphi_{03} - \mathcal{L}_{-3m}(\lambda)\varphi_{03} = \mathcal{G}_{-\bar m,-2\bar m}(U_{-\bar m,1},\varphi_{02}) + \mathcal{G}_{-2\bar m,-\bar m}(\varphi_{02},U_{-\bar m,1})\end{aligned}$$

subjected to the boundary conditions

$$\begin{aligned}
&\sum_{|\alpha|\le N_j-1} b^L_{l,j,\alpha}(\mathrm{i}3\bar m)^{\alpha_2}\partial_r^{\alpha_1}\varphi^j_{30}(R_1)=\sum_{|\alpha|\le N_j-1} b^L_{l,j,\alpha}(\mathrm{i}\bar m)^{\alpha_2}\partial_r^{\alpha_1}\varphi^j_{21}(R_1)=0,\\
&\sum_{|\alpha|\le N_j-1} b^L_{l,j,\alpha}(-\mathrm{i}\bar m)^{\alpha_2}\partial_r^{\alpha_1}\varphi^j_{12}(R_1)=\sum_{|\alpha|\le N_j-1} b^L_{l,j,\alpha}(-3\mathrm{i}\bar m)^{\alpha_2}\partial_r^{\alpha_1}\varphi^j_{03}(R_1)=0,\\
&\sum_{|\alpha|\le N_j-1} b^R_{l,j,\alpha}(3\mathrm{i}\bar m)^{\alpha_2}\partial_r^{\alpha_1}\varphi^j_{30}(R_2)=\sum_{|\alpha|\le N_j-1} b^R_{l,j,\alpha}(\mathrm{i}\bar m)^{\alpha_2}\partial_r^{\alpha_1}\varphi^j_{21}(R_2)=0,\\
&\sum_{|\alpha|\le N_j-1} b^R_{l,j,\alpha}(-\mathrm{i}\bar m)^{\alpha_2}\partial_r^{\alpha_1}\varphi^j_{12}(R_2)=\sum_{|\alpha|\le N_j-1} b^R_{l,j,\alpha}(-3\mathrm{i}\bar m)^{\alpha_2}\partial_r^{\alpha_1}\varphi^j_{03}(R_2)=0.
\end{aligned}$$

The reduced equations then admit the approximation

$$\frac{dz}{dt}=\beta_{\bar m,1}z+\tau z|z|^2+\tau_1 z|z|^4+o(|z|^5),$$

where, τ is given by (3.23), and the second transition number τ_1 is given by

$$\begin{aligned}
\tau_1 =&\langle \mathcal{G}_{-2\bar m,3\bar m}(\varphi_{02},\varphi_{30})+\mathcal{G}_{3\bar m,-2\bar m}(\varphi_{30},\varphi_{02}),U^*_{\bar m,1}\rangle\\
&+\langle \mathcal{G}_{0,\bar m}(\varphi_{11},\varphi_{21})+\mathcal{G}_{0,\bar m}(\varphi_{11},\varphi_{21}),U^*_{\bar m,1}\rangle\\
&+\langle \mathcal{G}_{2\bar m,-\bar m}(\varphi_{20},\varphi_{12})+\mathcal{G}_{-\bar m,2\bar m}(\varphi_{12},\varphi_{20}),U^*_{\bar m,1}\rangle.
\end{aligned}\tag{3.26}$$

We thus obtain the following lemma:

Lemma 3.3. *The stability and transition of zero solution to* (1.3) *in the vicinity of the critical control parameter $\lambda=\lambda_0$ and with any sufficiently small initial condition can be reduced to these of the zero solution to the equation taking the simple form*

$$\frac{dz}{dt}=\beta_{\bar m,1}z+\tau z|z|^2+\tau_1 z|z|^4+o(|z|^5).$$

Similarly, we have the following transition theorem:

Theorem 3.4. *In the case of $\Re\tau=0$, let τ_1 be given by* (3.26), *for* (1.3) *at $\lambda=\lambda_0$ and subject to the boundary conditions* (3.2), *the following assertions hold true:*

(1) *If $\Re\tau_1<0$ and $\beta_{\bar m,1}\notin\mathbb{R}$, then the system undergoes a continuous transition. As a result, a stable periodic orbit bifurcates on $\lambda>\lambda_0$, whose expression is approximately given by*

$$\begin{aligned}
u(t,r,\phi)=&b_{\bar m}(\tau)\left(\cos(\Im\beta_{\bar m,1}t+\bar m\phi)\Re U_{\bar m,1}(r)-\sin(\Im\beta_{\bar m,1}t+\bar m\phi)\Im U_{\bar m,1}(r)\right)\\
&+O(\Re\beta_{\bar m,1}),\quad b_{\bar m}(\tau)=2\Big(\Re\beta_{\bar m,1}/|\Re\tau|\Big)^{1/4}.
\end{aligned}\tag{3.27}$$

(12) *If $\Re\tau_1<0$ and $\Im\beta_{\bar m,1}=0$, then the system undergoes a continuous transition. As a result, an S^1 attractor bifurcates on $\lambda>\lambda_0$, which contains at least two pairs of stationary solutions approximately given by*

$$\begin{aligned}
u_1(r,\phi)&=\pm b_{\bar m}(\tau)\cos(a_{\bar m}\phi)U_{\bar m,1}(r)+O(|\Re\beta_{\bar m,1}|),\\
u_2(r,\phi)&=\pm b_{\bar m}(\tau)\sin(a_{\bar m}\phi)U_{\bar m,1}(r)+O(|\Re\beta_{\bar m,1}|),
\end{aligned}$$

(3) *The transition is of jump type if* $\Re\tau_1 > 0$.

Proof. The proof of the theorem is similar to that of Theorem 3.2 and is thus omitted. ∎

3.5. Numerical determination of the transition number τ. In view of the method just described, it is clear that in order to find the value of the transition number τ, it is enough to solve a series of linear problems, which are in turn determined by the operators $\mathcal{A}_m$ and $\mathcal{L}_m$. In order to achieve this, our approach consists of using a spectral method to encode both the action of these operators and the boundary conditions (3.2).

More precisely, given a family of orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ (such as Legendre polynomials or Chebyshev polynomials) and a sufficiently large positive integer N, our aim is to approximate the j^{th} component of our target function u by using a basis $\{\phi_n^j\}_{n=0}^{N-N_j}$ of the form

$$\phi_n^j = d_n^j \left(P_n + \sum_{k=1}^{N_j} c_{n,k}^j P_{n+k} \right), \qquad n = 0, ..., N - N_j,$$

where, for each n and j, the coefficients $\{c_{n,k}^j\}_{k=1}^{N_j}$ need to be chosen so that ϕ_n^j satisfies (3.2), and the positive constants d_n^j are chosen by fixing an appropriate normalization for the j^{th} component of system. For problems with constant or polynomial coefficients, and with a reasonably sized N (say, $N \sim 100$), one can simply take $d_n^j \equiv 1$. In general, however, these constants should be chosen so as to exploit the properties of the particular problem at hand.

If the j^{th} equation is of fourth order ($N_j = 4$), the coefficients $c_{n,k}^j$ are determined, for each $n = 0, ..., N - 4$, by the equations

$$\sum_{k=1}^{4} \left(\sum_{|\alpha|\le 3} b_{1,j,\alpha}^L (im)^{\alpha_2} P_{n+k}^{(\alpha_1)}(-1) \right) c_{n,k}^j = - \sum_{|\alpha|\le 3} b_{1,j,\alpha}^L (im)^{\alpha_2} P_n^{(\alpha_1)}(-1),$$

$$\sum_{k=1}^{4} \left(\sum_{|\alpha|\le 3} b_{2,j,\alpha}^L (im)^{\alpha_2} P_{n+k}^{(\alpha_1)}(-1) \right) c_{n,k}^j = - \sum_{|\alpha|\le 3} b_{2,j,\alpha}^L (im)^{\alpha_2} P_n^{(\alpha_1)}(-1),$$

$$\sum_{k=1}^{4} \left(\sum_{|\alpha|\le 3} b_{1,j,\alpha}^R (im)^{\alpha_2} P_{n+k}^{(\alpha_1)}(1) \right) c_{n,k}^j = - \sum_{|\alpha|\le 3} b_{1,j,\alpha}^R (im)^{\alpha_2} P_n^{(\alpha_1)}(1),$$

$$\sum_{k=1}^{4} \left(\sum_{|\alpha|\le 3} b_{2,j,\alpha}^R (im)^{\alpha_2} P_{n+k}^{(\alpha_1)}(1) \right) c_{n,k}^j = - \sum_{|\alpha|\le 3} b_{1,j,\alpha}^R (im)^{\alpha_2} P_n^{(\alpha_1)}(1).$$

Similarly, if the j^{th} equation is of second order ($N_j = 2$), the coefficients $c_{n,k}^j$ are determined, for each $n = 0, ..., N-2$, by the equations

$$\sum_{k=1}^{2}\left(\sum_{|\alpha|\le 1} b^L_{1,j,\alpha}(im)^{\alpha_2}P^{(\alpha_1)}_{n+k}(-1)\right)c^j_{n,k} = -\sum_{|\alpha|\le 1} b^L_{1,j,\alpha}(im)^{\alpha_2}P^{(\alpha_1)}_{n}(-1),$$
$$\sum_{k=1}^{2}\left(\sum_{|\alpha|\le 1} b^R_{1,j,\alpha}(im)^{\alpha_2}P^{(\alpha_1)}_{n+k}(1)\right)c^j_{n,k} = -\sum_{|\alpha|\le 1} b^R_{1,j,\alpha}(im)^{\alpha_2}P^{(\alpha_1)}_{n}(1).$$

Then, letting $\{e_i\}_{i=1}^d$ be the canonical basis of $\mathbb{R}^d$, we let A_m, M_m and S_m be the matrices encoding the values

$$\begin{aligned}
&(A_m)^{i,j}_{p,q} = \langle \mathcal{A}_m(\phi^j_q e_j), \phi^i_p e_i\rangle,\\
&(S_m)^{i,j}_{p,q} = \langle \mathcal{L}_m(\lambda)(\phi^j_q e_j), \phi^i_p e_i\rangle,\\
&(M_m)^{i,j}_{p,q} = \langle \phi^j_q e_j, \phi^i_p e_i\rangle,\\
&i,j = 1,...,d, \quad p = 0,...,N-N_i,\ q = 0,...,N-N_j.
\end{aligned}$$

Then we decompose

$$u = \sum_{j=1}^{d}\sum_{q=0}^{N-N_j} \hat{u}^j_q \phi^j_q e_j, \quad f = \sum_{j=1}^{d}\sum_{q=0}^{N-N_j} \hat{f}^j_q \phi^j_q e_j$$

so that the equation $\alpha\mathcal{A}_m u = \mathcal{L}_m(\lambda)u + f$ has the solution

$$\hat{u} = (\alpha A_m - S_m)^{-1} M_m \hat{f}.$$

The form of the matrices A_m and S_m depends on the particular equation under study, but the mass matrix M_m can be found using only properties of the chosen orthogonal polynomials:

$$M^{i,j}_{p,q} = \langle \phi^i_p, \phi^j_q\rangle = d^i_p d^j_q \langle P_p + \sum_{n=1}^{N_i} c^i_{p,n}P_{p+n}, L_q + \sum_{m=1}^{N_j} c^j_{q,m}L_{q+m}\rangle$$

Making use of the basis $\{\phi^j_n\}_{n=0}^{N-N_j}$ or $\{P_n\}_{n=0}^N$, f^j can be expressed in form of

$$f^j = \sum_{r=0}^{N} \bar{f}^j_r P_r \quad \text{or} \quad f^j = \sum_{q=0}^{N-N_j} \hat{f}^j_q \phi^j_q,$$

where $\hat{f}^j_q$ can be determined from $\bar{f}^j_r$ by using the following forward transforms

$$\hat{f}^i_p = \delta^{ij} d^i_p \left(|P_p|^2 \bar{f}^j_p + \sum_{n=1}^{N_j} c^i_{p,n}|P_{p+n}|^2 \bar{f}^j_{p+n}\right), \quad 1 \le i,j \le d, \tag{3.28}$$

Algorithm 3.1 Find the transition number τ in $\mathbb{C}$, given by (3.23).

1. Given N, for each $m \in \mathbb{N}$, find the first left eigenvector $U^*_{m,1}$ and the first right eigenvector $U_{m,1}$ of the matrix $A_m^{-1}S_m$, i.e.,

$$\beta_{m,1}U_{m,1} = A_m^{-1}S_mU_{m,1}, \quad \overline{\beta}_{m,1}U^*_{m,1} = U^*_{m,1}A_m^{-1}S_m,$$

2. Normalize the first left eigenvector $U^*_{m,1}$ so that

$$\langle \mathcal{A}_mU_{m,1}, U^*_{m,1}\rangle = 1.$$

3. Find the backward transforms $u_{m,1}$ and $u_{-m,1}$ of $U_{m,1}$ and $U_{-m,1}$, respectively, and find the forward transforms G_{2m} and G_0 of

$$\mathcal{G}_{m,m}(u_{m,1}, u_{m,1}) \quad \text{and} \quad \mathcal{G}_{m,-m}(u_{m,1}, u_{-m,1}) + \mathcal{G}_{-m,m}(u_{-m,1}, u_{m,1}).$$

by the formula (3.28) and (3.29).
4. Solve

$$\Phi_{20} = (2\beta_{m,1}A_{2m} - S_{2m})^{-1}G_{2m}, \quad \Phi_{11} = (2\Re\beta_{m,1}A_0 - S_0)^{-1}G_0$$

5. Find the backward transforms φ_{20} and φ_{11} of Φ_{20} and Φ_{11}, respectively, and compute

$$\zeta = \mathcal{G}_{m,0}(u_{m,1}, \varphi_{11}) + \mathcal{G}_{0,m}(\varphi_{11}, u_{m,1}) + \mathcal{G}_{-m,2m}(u_{-m,1}, \varphi_{20}) + \mathcal{G}_{2m,-m}(\varphi_{20}, u_{-m,1})$$

6. Find the forward transform Z of ζ and compute

$$\tau = \langle Z, U^*_{m,1}\rangle.$$

---

and $\bar{f}^j_r$ can be determined from $\hat{f}^j_q$ by the following backward transforms

$$\bar{f}^j_r = \delta^{ij}\left(d^i_r|P_r|^2\hat{f}^i_r + d^i_{r-n}\sum_{n=1}^{N_i} c^i_{r-n,n}|P_{r-n}|^2\hat{f}^i_{r-n}\right), \quad 1 \le i, j \le d. \tag{3.29}$$

The procedure for finding the transition number τ given by (3.23), as described in the previous section, can now be summarized below. Note that in Steps 2 and 6 of Algorithm 3.1, we use the cannonical inner product of $(\mathbb{R}^N)^d$. This choice corresponds, in physical space, to the L^2-inner product. An implementation of all the routines required to run the Algorithm 3.1 using Matlab can be found in the website https://github.com/marcoher/spectral4ptd.

4. Application to a two layer QG model. In this section we apply the transition theorem (Theorem 3.2) and the numerical algorithm 3.1 to investigate the dynamic transitions associated with the baroclinic instability in the two-layer basic flows over an annular channel (see Fig. 1). The governing equations for this problem is the Eq. (1.1), where the viscous effects are not taken into account in [51].

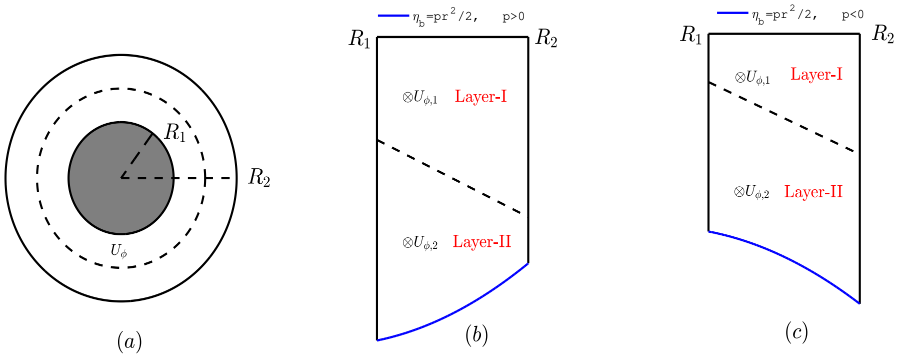

Figure 1. *Schematic representation of the domain, bathymetry and mean circulation.* **Panel (a):** *Top-down view of the annular channel, with dashed lines representing isobaths or mean streamlines and shaded region excluded from the domain.* **Panel (b)-(c):** *Cross-sectional view of the mean flow configuration. The bathymetry, $\eta_b(r)$, is shown here, corresponding to the azimuthal flow with solid body rotation.*

The basic currents $(\Psi_1(r), \Psi_2(r))$ independent of the variable ϕ flowing over the annular channel is determined by the following equations

$$\Delta^2\Psi_i = 0, \quad i = 1, 2, \tag{4.1}$$

which has infinite number of solutions. Here, we are interested in the dynamic transition arising in four different profiles, given by

$$\Psi_j(r)^{(k)} = \Omega_j\Phi^{(k)}(r), \quad j = 1, 2, \quad k = 1, 2, 3, 4, \tag{4.2}$$

where Ω_j are the constant angular velocities of the flow in each layer, and

$$\Psi^{(1)}(r) = \frac{1}{2}r^2, \quad R_1 \le r \le R_2, \tag{4.3a}$$

$$\Psi^{(2)}(r) = \frac{1}{2}\left[\left(1 + \frac{2\ln(R_2/R_1)}{R_2^2/R_1^2 - 1}\right)r^2 - 2R_1^2\ln(r)\right], \tag{4.3b}$$

$$\Psi^{(3)}(r) = \frac{1}{2}\left(r^2 + \frac{1}{2}\frac{R_2^2\ln(R_2/R_1)r^2\ln(r)}{R_2^2\ln(R_2) - R_1^2\ln(R_1)} - \frac{R_2^2\ln(r)}{2}\right), \tag{4.3c}$$

$$\Psi^{(4)}(r) = \frac{1}{2}\frac{\left(R_2^2 - R_1^2\right)\left(r^2\ln(r) - (R_2+R_1)^2\ln(r)/4\right)}{R_2^2\ln(R_2) - R_1^2\ln(R_1) - (R_2+R_1)^2/4\ln(R_2/R_1)}, \tag{4.3d}$$

whose derivatives – the basic azimuthal velocity component $U_\phi = (U_{\phi,1}, U_{\phi,2})$ – are shown in Fig. 2. Note that these basic solutions have been chosen so that the basic velocity profile

$$U_j^{(k)} = (U_{j,r}^{(k)}, U_{j,\phi}^{(k)}) = (0, \partial_r\Psi_j^{(k)}), \ \ j = 1, 2; \ \ k = 1, 2, 3, 4, \tag{4.4}$$

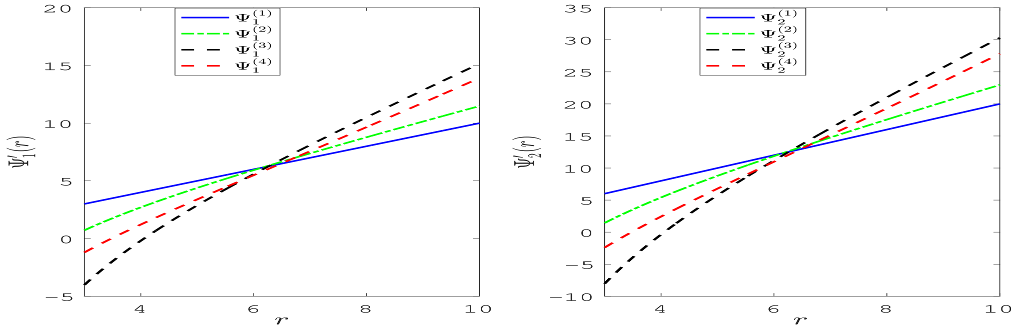

Figure 2. *Angular component of the basic velocity profiles with rotiational speed $\Omega_j = j(j = 1, 2)$ and internal/external radii $R_1 = 3$ and $R_2 = 10$, respectively.*

has, in all cases, the same average along the radial direction as would be given by solid body rotation. That is,

$$\frac{1}{R_2 - R_1}\int_{R_1}^{R_2} U_{j,\phi}^{(k)}(r)dr = \Omega_j \frac{R_2 + R_1}{2}, \qquad j = 1, 2. \tag{4.5}$$

Please also note that only is the stability of the first profile $\Psi^{(1)}(r)$ considered in [51] and references therein in the case of neglecting dissipation term $\Delta^2 u$ in (1.1). These researches only focus on the effect of the value of p on the baroclinic instability, while the effect of which on the transition type has not been considered.

4.1. Eigenvalue Problem and Principle of Exchange of Stability. We let $\lambda = Re$ in the Eq. (1.3), the corresponding operators $\mathcal{A}$, $\mathcal{L}_{Re}$ and $\mathcal{G}$ defined in (1.4) are specified as, respectively,

$$\begin{aligned} \mathcal{A}u &= \begin{pmatrix} -\Delta u_1 + F_1(u_1 - u_2) \\ -\Delta u_2 + F_2(u_2 - u_1) \end{pmatrix}, \\ \mathcal{L}_{Re}u &= -\frac{1}{Re}\Delta^2 u - \frac{1}{r}(\partial_r Q \partial_\phi + \partial_r \Psi \partial_\phi \mathcal{A})u, \\ \mathcal{G}(u, v) &= -\left[-\frac{1}{r}\partial_\phi u \partial_r + \frac{1}{r}\partial_r u \partial_\phi\right]\mathcal{A}v, \end{aligned} \tag{4.6}$$

where

$$Q = \begin{pmatrix} \Delta\Psi_1 - F_1(\Psi_1 - \Psi_2) \\ \Delta\Psi_2 - F_2(\Psi_2 - \Psi_1 - \eta_b) \end{pmatrix} \tag{4.7}$$

and η_b corresponds to a parabolic bathymetry given by

$$\eta_b(r) = pr^2/2. \tag{4.8}$$

Then, the perturbation equations for the basic currents is given by the nonlinear system (1.4), which are supplemented by the following boundary conditions

$$
\begin{aligned}
u_1\Big|_{r=R_1,R_2} &= \left(\partial_{rr}u_1 + \frac{1}{r}\partial_r u_1\right)\Big|_{r=R_1,R_2} = 0,\\
u_2\Big|_{r=R_1,R_2} &= \left(\partial_{rr}u_2 + \frac{1}{r}\partial_r u_2\right)\Big|_{r=R_1,R_2} = 0,
\end{aligned}
\tag{4.9}
$$

which fit (1.8) as well.

Correspondingly, the eigenvalue problem (1.9) reads

$$
\mathcal{L}_{Re}u = \beta\mathcal{A}u,
\tag{4.10}
$$

with the boundary condition (4.9). Thus, for m in $\mathbb{Z}$, denoting an azimuthal wavenumber, we let $u_j(r,\phi) = v_j(r)\exp(im\phi)$, then the eigenvalue problem (3.1) for the two layer model is

$$
\mathcal{L}_m(Re)v = \beta\mathcal{A}_m v,
\tag{4.11}
$$

where

$$
\mathcal{A}_m v = \begin{pmatrix} -\Delta_m v_1 + F_1(v_1 - v_2) \\ -\Delta_m v_2 + F_2(v_2 - v_1) \end{pmatrix},
\tag{4.12a}
$$

$$
\begin{aligned}
\mathcal{L}_m(Re)v = &\begin{pmatrix} -\frac{1}{Re}\Delta_m^2 v_1 - \frac{im}{r}\partial_r Q_1 v_1 \\ -\frac{1}{Re}\Delta_m^2 v_2 - \frac{im}{r}\partial_r Q_2 v_2 \end{pmatrix}\\
&- \frac{im}{r}\begin{pmatrix} \partial_r\Psi_1\left[-\Delta_m v_1 + F_1(v_1 - v_2)\right] \\ \partial_r\Psi_2\left[-\Delta_m v_2 + F_2(v_2 - v_1)\right] \end{pmatrix}.
\end{aligned}
\tag{4.12b}
$$

Here Δ_m is the second order elliptic operator given by

$$
\Delta_m = \partial_{rr} + \frac{1}{r}\partial_r - \frac{m^2}{r^2}.
$$

Besides, the bilinear operators $\mathcal{G}_{m,m'}$ associated with the two layer problem is uniquely determined by (4.6) and the identities

$$
\boxed{\mathcal{G}(e^{im\cdot}u, e^{im'\cdot}v) = e^{i(m+m')\cdot}\mathcal{G}_{m,m'}(u,v),}
\tag{4.13}
$$

for all radial functions u, v.

For any m in $\mathbb{Z}$, the solutions of the generalized eigenvalue problem (4.11) are denoted hereafter by

$$
\{(\beta_{m,k}, v_{m,k})\}_{k\in\mathbb{N}},
\tag{4.14}
$$

and we assume, without loss of generality, that they are arranged such that

$$\Re\beta_{m,1} \geq \Re\beta_{m,2} \geq \Re\beta_{m,3} \geq \cdots \to -\infty.$$

From a numerical point of view, once Ψ_1, Ψ_2 and η_b in (4.8) have been specified, one can directly study (4.11) and determine the values of these parameters $F_1, F_2, R_1, R_2, \Omega_1, \Omega_2$ and Re for which the corresponding PES condition for the two layer problems holds based on the 3.1 provided in the previous section. As a matter of fact, given the number of free parameters in the system, it is not possible to give a comprehensive description of all the possible transition types that may occur in this problem. Instead, we follow [51] and fix the parameter values $R_1 = 3$, $R_2 = 10$, $F_1 = F_2 = 0.5$, $\Omega_1 = 1$ and $\Omega_2 = 2$. We are then left with two free parameters, the Reynolds number Re and the bathymetry slope p, and then verify the corresponding PES condition. For this purpose, and using the periodicity in the angular direction, it suffices to find, for each value of p and wavenumber m, the value of the Reynolds number that makes the leading eigenfunction unstable. In other words, we look for the modal critical Reynolds numbers, which is the family of curves $Re = Re^*_m(p)$ determined by the equation

$$\Re\beta_{m,1}(p, Re) = 0, \qquad m = 1, 2.$$

The curves $Re = Re^*_m(p)$ corresponding to the case of solid body rotation, shown in Fig. 3. It has been observed that, the results are qualitatively similar when any of the other profiles in (4.3) are used (not shown). The critical Reynolds number is then obtained by finding, for each p, the first value of Re at which linear instability occurs. That is, we seek for a curve $Re = Re^*(p)$ such that for each given p the PES holds exactly at $Re^*(p)$. It is easy to see that such curve is in fact given by $Re^*(p) = \min_m Re^*_m(p)$, shown in Fig. 4. It is noteworthy that $Re^*(p)$ is piecewise smooth, with some cusp points at which $Re^*_m(p) = Re^*_{m+1}(p)$. Except these cusp points, the PES condition

$$\begin{cases} \Re\beta_{\bar{m},1}(Re) \begin{cases} > 0 & \text{if } Re > Re^*, \\ = 0 & \text{if } Re = Re^*, \\ < 0 & \text{if } Re < Re^*, \end{cases} \\ \Re\beta_{m,k}(Re^*) < 0 \qquad \forall (m,k) \neq (\bar{m}, 1), \end{cases}$$
$$Re^* := Re^*_{\bar{m}}(p) = \min_m Re^*_m(p), \tag{4.15}$$

holds true. At these cusp points, there are two pair of simple complex conjugate eigenvalues becoming critical, which is a degenerate case and is not within the scope of the present work.

4.2. Types of dynamic transition and bifurcation. Utilizing the procedure described in the previous section, the transition number τ given by (3.23) for the two layer problems can be found by implementing the 3.1 and setting the values of Re and m to coincide with $Re^*(p)$ and $\bar{m}$ as discussed above. In this manner, we compute all values of $\tau(p)$ except at these cusp points of p where degeneracies occur. Hence, the transition number $\tau(p)$ is a discontinuous function, whose graph is shown in Fig. 5.

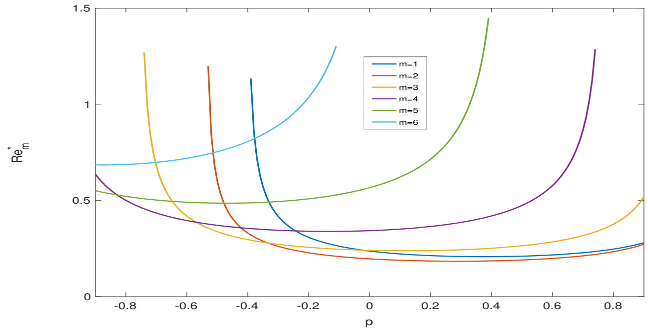

Figure 3. *The modal critical Reynolds numbers,* $Re = Re^*_m(p)$ *(solutions of* $\Re\beta_{m,1}(p, Re) = 0$*), corresponding to the case of solid body rotation, where* $R_1 = 3$*,* $R_2 = 10$*,* $F_1 = F_2 = 0.5$*,* $\Omega_1 = 1$ *and* $\Omega_2 = 2$*.*

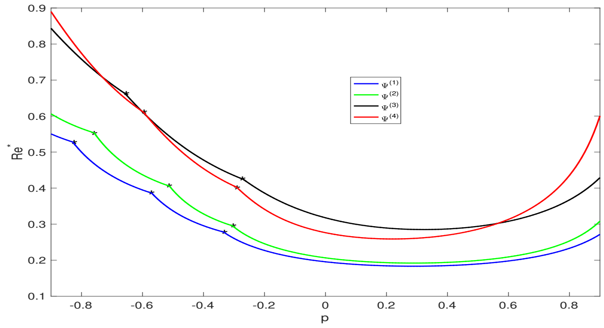

Figure 4. *The curves of neutral stability,* $Re = Re^*(p)$ *(solutions of* $\max_m \Re\beta_{m,1}(p, Re) = 0$*), for all four different basic rotation profiles, the points with asterisk are the cusp points at which* $Re^*_m(p) = Re^*_{m+1}(p)$*, where* $R_1 = 3$*,* $R_2 = 10$*,* $F_1 = F_2 = 0.5$*,* $\Omega_1 = 1$ *and* $\Omega_2 = 2$*.*

Recalling the basic profiles given in (4.3), the numerical results reported in Fig. 5 show that for the profiles $\Psi^{(1)}(r)$ and $\Psi^{(2)}(r)$, the dynamic transitions that occur are continuous for most values of p except some that, when crossed, lead to jump transitions. On the other hand, such jump transitions never occur for the basic profiles $\Psi^{(3)}(r)$ and $\Psi^{(4)}(r)$, and only continuous transitions take place as p varies over the prescribed range. Therefore, the slope of the bathymetry has an evident effect on the type of transition associated with these basic profiles which have uniform rotation direction. For profiles with non uniform rotation direction, the transition type is same for both concave bathymetry and convex bathymetry, i.e., the transition is continuous type.

4.3. New states. From Theorem 3.2, we know that the new states bifurcating from the basic profiles (4.2) can be uniformly given by

$$
\begin{aligned}
\psi =& 2\left(\Re\beta_{\bar m,1}/|\Re\tau|\right)^{1/2}\cos(\Im\beta_{\bar m,1}t + \bar m\phi)\Re u_{\bar m,1}(r) \\
&- 2\left(\Re\beta_{\bar m,1}/|\Re\tau|\right)^{1/2}\sin(\Im\beta_{\bar m,1}t + \bar m\phi)\Im u_{\bar m,1}(r) + O(\Re\beta_{\bar m,1}).
\end{aligned}
\tag{4.16}
$$

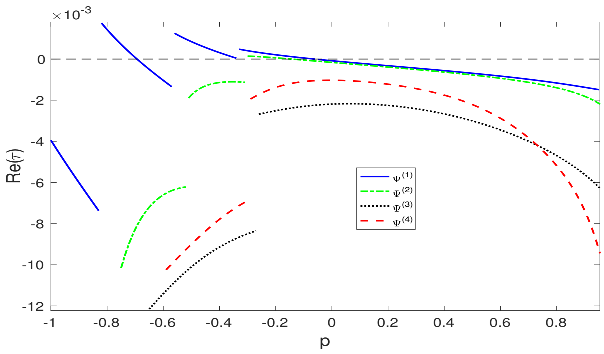

Figure 5. *The transition number for all four different basic rotation profiles with $R_1 = 3$, $R_2 = 10$, $F_1 = F_2 = 0.5$, $\Omega_1 = 1$ and $\Omega_2 = 2$.*

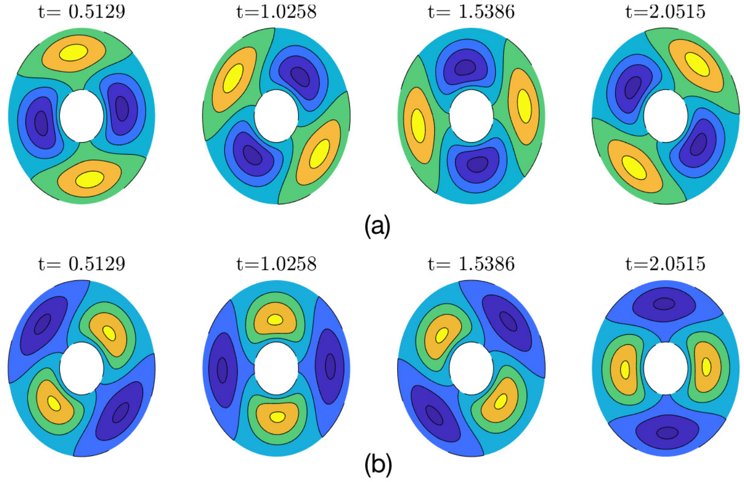

Figure 6. *The evolution of the stable periodic solution* ((a) upper layer, (b) lower layer) bifurcating from the profile with solid body rotation with $p = 0.1$, $m = 2$, $Re = 0.2182 > Re^* = 0.1882$, $R_1 = 3$, $R_2 = 10$, $F_1 = F_2 = 0.5$, $\Omega_1 = 1$ and $\Omega_2 = 2$ at time $T/4-(a), T/2-(b), 3T/4-(c)$ and $T-(d)$, where T is the minimal period.

For $0 < \Re\beta_{\bar{m},1} \ll 1$, the leading order of (4.16) provides a good approximation of the new states that appear after a dynamic transition takes place. In general, the new observed state should be given by the original steady-state (4.2) plus the bifurcated solution (4.17), i.e.,

$$\begin{aligned}\psi(r,\phi) =&\Psi^k(r) + 2\left(\Re\beta_{\bar{m},1}/|\Re\tau|\right)^{1/2}\cos(\Im\beta_{\bar{m},1}t + \bar{m}\phi)\Re u_{\bar{m},1}(r)\\ &- 2\left(\Re\beta_{\bar{m},1}/|\Re\tau|\right)^{1/2}\sin(\Im\beta_{\bar{m},1}t + \bar{m}\phi)\Im u_{\bar{m},1}(r) + O(\Re\beta_{\bar{m},1}), \quad k = 1,2,3,4.\end{aligned} \tag{4.17}$$

For illustration, we show the bifurcated solution and the new observed state corresponding to the case of solid body rotation ($k = 1$) as follows in Fig. 6 and Fig. 7, respectively.

4.4. Concluding remarks. Our research shows that there exist transitions in these basic current flows, parallel to the bathymetric isobaths if the turbulent Reynolds number $Re =$

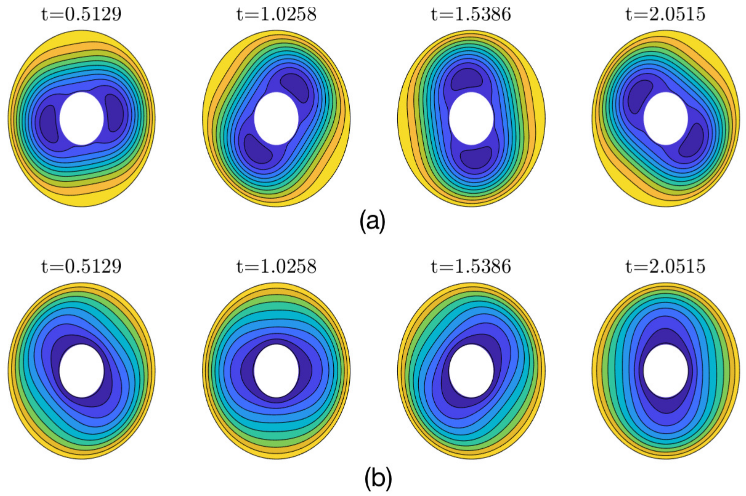

Figure 7. *The evolution of the observed periodic solution (upper layer-a, lower layer-b) corresponding to the profile with solid body rotation with $p = 0.1$, $m = 2$, $Re = 0.2182 > Re^* = 0.1882$, $R_1 = 3$, $R_2 = 10$, $F_1 = F_2 = 0.5$, $\Omega_1 = 1$ and $\Omega_2 = 2$ at time $T/4-(a), T/2-(b), 3T/4-(c)$ and $T-(d)$, where T is the minimal period.*

LU/ν is greater than a critical value Re^*. For those profiles corresponding to basic currents without a uniform rotation direction, the type of the transition is continuous for both concave bathymetry and convex bathymetry. On the other hand, for those profiles with the same rotation direction, the type of transition can be catastrophic for relatively convex bathymetry. Therefore, the bathymetry has an apparent effect on the type of transition in the flow in an annular channel.

Since the bottom profile is of the form

$$\eta_b = pr^2/2, \quad p \in [-1, 1],$$

the absolute value of p determines the slope of the bathymetry. From Fig.4, we see that the critical Reynolds number $Re^*(p)$ with $|p| > 0$ is clearly larger than that with $p = 0$. Physically, this means that the two-layer basic flow in an annular domain with a concave or convex bathymetry lose their stability more easily than those with a flatter one.

From Fig. 5, we can infer that the configuration of the two-layer flow always changes continuously from a basic stationary profile to a periodically evolving one when the control parameter Re crosses the corresponding threshold, provided the basic currents rotate in the same direction. However, for basic profiles where the upper and lower layer currents rotate in the opposite direction, the system may jump to another state even if the basic solution is linearly stable. As a result, the structure of the two-layer flow always changes suddenly, thus leading to a sudden change in the physical states involved in the two-layer flow.

Acknowledgments. The authors would also like to thank Professors Shouhong Wang and Mickael Chekroun for their insightful comments and discussions, and thank two anonymous referees for helpful suggestions improving the readability of the paper.

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