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Guts, volume and skein modules of 3-manifolds

by

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Abstract. We consider hyperbolic links that admit alternating projections on surfaces in compact, irreducible 3-manifolds. We show that, under some mild hypotheses, the volume of the complement of such a link is bounded below in terms of a Kauffman bracket function defined on link diagrams on the surface.

In the case that the 3-manifold is a thickened surface, this Kauffman bracket function leads to a Jones-type polynomial that is an isotopy invariant of links. We show that coefficients of this polynomial provide 2-sided linear bounds on the volume of hyperbolic alternating links in the thickened surface. As a corollary of the proof of this result, we deduce that the twist number of a reduced, twist-reduced, alternating link projection with checkerboard disk regions is an invariant of the link.

1. Introduction. The goal of the paper is to show that, under mild hypotheses, the volume of a hyperbolic link in a compact, irreducible 3-manifold M that admits an alternating projection on a closed surface $F \subset M$ is bounded below in terms of a Kauffman bracket function defined on link diagrams on F. For $M = F \times [-1, 1]$, this function leads to a Jones polynomial link invariant, and coefficients of it provide 2-sided linear bounds on the volume of hyperbolic alternating links. As a corollary, we deduce that the twist number of a reduced, twist-reduced, alternating link projection with checkerboard disk regions is an invariant of the link in $F \times [-1, 1]$. Our results generalize work of Dasbach and Lin [9] and Futer, Kalfagianni and Purcell [13, 14] who obtained similar results for families of links in S^3.

Let M be an irreducible compact 3-manifold with or without boundary. A link L admits a projection on an orientable embedded surface F in M if $L \subset F \times [-1, 1] \subset M$, and it is projected via the obvious projection

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$\pi : F \times [-1,1] \to F = F \times \{0\}$. Given a connected surface F in M, we define a Kauffman bracket function and from this we construct a polynomial

$$J_0(\pi(L)) = \langle \pi(L) \rangle_0 = a_m t^m + a_{m-1} t^{m-1} + \cdots + b_{n+1} t^{n+1} + b_n t^n$$

in $\mathbb{Z}[t^{1/4}, t^{-1/4}]$. See Section 2 for the precise definition.

The polynomial $J_0(\pi(L))$ depends on the topology of F, the projection $\pi(L)$ and a priori on the topology of the complement $M \backslash N(F)$. In the special case that $M = F \times [-1,1]$, it is an isotopy invariant of L in M, but we do not expect that it is an isotopy invariant of L in a general M. Nevertheless, as our results below show, if $\pi(L)$ is alternating on F, and under mild additional hypotheses, $J_0(\pi(L))$ encodes intrinsic geometric information on the link complement $M \setminus L$.

Our first result is the following theorem, where the terms reduced and twist-reduced are defined in Section 3.

THEOREM 1.1. *Let M be an irreducible, compact 3-manifold with empty or incompressible boundary. Let $F \subset M$ be an incompressible, closed, orientable surface such that $M \setminus N(F)$ is atoroidal and ∂-anannular. Suppose that a link L admits a reduced alternating projection $\pi(L) \subset F$ that is checkerboard colorable, twist-reduced and with all the regions of $F \setminus \pi(L)$ disks. Then L is hyperbolic and*

$$\mathrm{vol}(M \setminus L) \geq v_8 \max\{|a_{m-1}| - |a_m|, |b_{n+1}| - |b_n|\} - \tfrac{1}{2}\chi(\partial M),$$

where $a_{m-1}, a_m, b_{n+1}, b_n$ are the first two and the last two coefficients of the polynomial $J_0(\pi(L))$, and $v_8 = 3.66386\ldots$ is the volume of a regular ideal octahedron.

Given an alternating link projection $\pi(L)$ as in the statement of Theorem 1.1, let S_A, S_B denote the two checkerboard surfaces corresponding to $\pi(L)$. Also let M_A and M_B denote the manifolds obtained by cutting $M \setminus L$ along S_A and S_B, respectively. By [20], S_A, S_B are essential in $M \setminus L$, and by Jaco–Shalen–Johannson theory M_A and M_B contain hyperbolic submanifolds called the *guts* of S_A and S_B, respectively. We show that the Euler characteristics of these guts and the twist number $t_F(\pi(L))$ can be calculated from $J_0(\pi(L))$.

THEOREM 1.2. *Let M, F and $\pi(L)$ be as in the statement of Theorem 1.1 and let $t_F(\pi(L))$ denote the twist number of $\pi(L)$. Also let $\mathrm{guts}(M_A)$ and $\mathrm{guts}(M_B)$ denote the guts of S_A and S_B, respectively. Then*

(1) $\chi(\mathrm{guts}(M_A)) = |a_m| - |a_{m-1}| + \frac{1}{2}\chi(\partial M)$,
(2) $\chi(\mathrm{guts}(M_B)) = |b_n| - |b_{n+1}| + \frac{1}{2}\chi(\partial M)$,
(3) $t_F(\pi(L)) = |a_{m-1}| + |b_{n+1}| - |a_m| - |b_m| + \chi(F)$.

Theorem 1.1 follows by combining Theorem 1.2 with a result of Agol, Storm and Thurston [2] asserting that the negative Euler characteristic of the guts of an essential surface in a hyperbolic 3-manifold M linearly bounds the volume of M from below.

For $M = S^3$ and $F = S^2$, the polynomial J_0 is the classical Jones polynomial. In the case that $M = F \times I$, one can use the structure of the Kauffman skein module of M to see that $J_0(\pi(L))$ is also an isotopy invariant of the link L. In fact, for every link in M one obtains a finite collection of Jones-type polynomial invariants that have been used to settle open questions about the topology of alternating links in thickened surfaces [1, 4]. As a corollary of Theorem 1.2 we obtain the following.

Corollary 1.3. *Let L be a link in $F \times [-1, 1]$ that admits a checkerboard colorable, reduced alternating projection $\pi(L) \subset F$ that is twist-reduced and has all its regions disks. Then any two such projections of L have the same twist number. That is, $t_F(\pi(L))$ is an isotopy invariant of L.*

Note that Theorem 1.2 also implies that the quantities $\chi(\mathrm{guts}(M_B))$ and $\chi(\mathrm{guts}(M_B))$ are invariants of L in $M = F \times [-1, 1]$. For reduced, twist-reduced alternating diagrams on a 2-sphere in S^3, invariance of the twist number is a consequence of the Tait flyping conjecture [28]. The corresponding conjecture for links in thickened surfaces is currently open. A second proof of the twist number invariance for alternating links in S^3 follows from the work of Dasbach and Lin [9, 10] that relates this twist number to the Jones polynomial. Our approach generalizes their approach.

Several families of hyperbolic links in S^3, including alternating ones, satisfy a "coarse volume conjecture": coefficients of the Jones and colored Jones polynomials provide two-sided linear bounds of the volume of the link complement [9, 13–17, 25]. The next theorem provides a similar result for alternating links in thickened surfaces and there is a similar result for links with alternating projections on Heegaard tori in lens spaces (see Corollary 4.2).

Theorem 1.4. *Suppose that $\pi(L)$ is a reduced alternating projection on $F = F \times \{0\}$ in $M = F \times [-1, 1]$, that is twist-reduced, checkerboard colorable, and with all the regions of $F \setminus \pi(L)$ disks. Then the interior of $M \setminus L$ admits a hyperbolic structure. If $F = T^2$, then*

$$\frac{v_8}{2} \cdot \beta_L \leq \mathrm{vol}(M \setminus L) < 10 v_4 \cdot \beta_L,$$

and if F has genus at least two, then

$$\frac{v_8}{2} \cdot (\beta_L - 2\chi(F)) \leq \mathrm{vol}(M \setminus L) < 6 v_8 \cdot (\beta_L + \chi(F)).$$

Here $\beta_L := |a_{m-1}| + |b_{n+1}| - |a_m| - |b_n|$ is obtained from the Jones polynomial invariant J_0 of L, and $v_4 = 1.01494\ldots$ is the volume of a regular ideal tetrahedron.

A key idea in the proof of Theorem 1.2 is to relate the coefficients of $J_0(\pi(L))$ to the topology of the checkerboard graphs of any projection

$\pi(L) \subset F$. This idea is reminiscent of techniques that were used to study the Jones polynomial of adequate and alternating links in S^3 [8, 13, 26, 27]. Under a graph-theoretic condition, which we call geometric adequacy, we show that the first two and the last two coefficients of $J_0(\pi(L)$ can be calculated from the checkerboard graphs. The checkerboard graphs of reduced, alternating projections turn out to satisfy these graph-theoretic conditions. On the other hand, these graphs form spines of the checkerboard surfaces. We use the work of Howie and Purcell [20] in a crucial way to show that the graph-theoretic combinatorics that determines the coefficients a_{m-1} and b_{n+1} is exactly the one dictating the calculation of the Euler characteristics of the guts of the checkerboard surfaces.

There exist open conjectures predicting that the volume of hyperbolic 3-manifolds is determined by certain asymptotics of quantum invariants [5, 7, 11, 22]. For links in S^3 these invariants include the Jones polynomial and its generalizations. The relations of skein-theoretic invariants and volume via guts of surfaces established in [13, 14] and in Theorems 1.4 and 1.2 are robust and seem independent of these conjectures.

The paper is organized as follows: In Section 2, we consider projections of links on a surface and we define the polynomial $J_0(\pi(L))$. Then, we explain how known results on the structure of Kauffman skein modules imply that $J_0(\pi(L))$ is an invariant of isotopy for links in $F \times [-1, 1]$. Finally, we restrict ourselves to projections $\pi(L) \subset F$ that have disk regions. For such projections we define the notion to geometric adequacy, under which we obtain formulae for the coefficients $|a_{m-1}|, |b_{n+1}|, |a_m|, |b_n|$ (Theorem 2.6). We also compare geometric adequacy with other notions of adequacy that have recently appeared in the literature [4].

In Section 3 we consider alternating projections $\pi(L) \subset F \subset M$ that are checkerboard colorable and have disk regions. We define a notion of diagram reducibility that generalizes the corresponding notion for link diagrams on a 2-sphere in S^3, and interplays nicely with the complexity of "edge representativity" considered in [20] and with geometric adequacy (Proposition 3.5 and Lemma 3.8). This interplay allows us to relate our work in Section 2 to work of Howie and Purcell on weakly generalized alternating links. The main result in this section is Theorem 3.13 that is a generalized version of Theorem 1.2. The more general version replaces the hypothesis that F is incompressible in M with a hypothesis of "high representativity". We also prove Corollary 1.3.

In Section 4 we prove Theorem 1.1 and we derive Theorem 1.4.

2. Skein polynomials and geometric adequacy

2.1. Bracket polynomials for link diagrams on surfaces. Let M be a 3-manifold with or without boundary and let F be an orientable surface in M. Given a link $L \subset F \times [-1,1] \subset M$ we can consider its image under the projection $\pi : F \times [-1,1] \to F = F \times \{0\}$. Throughout the paper we will refer to $\pi(L) \subset F$ as a *link projection* or a *link diagram.*

Given a crossing of a link diagram $D = \pi(L)$, we define the A and B resolutions of the crossing as indicated in Figure 1.

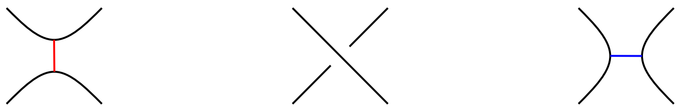
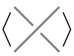
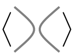
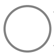
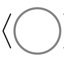


Fig. 1. The A resolution (left) and the B resolution (right) of a crossing.

Let $\mathcal{D}(F)$ denote the set of all (unoriented) link diagrams on F, taken up to isotopy on F. Also let X_F denote the set of all collections of disjoint simple closed curves (a.k.a. multi-curves) on F. We define a *Kauffman bracket*

$$\langle\ \ \rangle : \mathcal{D}(F) \to \mathbb{Z}[A, A^{-1}, (A^2 - A^{-2})^{-1}]X_F$$

by the following skein relations:

(1) $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$,
(2) $\langle L \sqcup$  $\rangle = (-A^2 - A^{-2})\langle L\rangle$,
(3) $\langle$  $\rangle = 1$.

As we are working on a surface that may have essential curves, we also require that the unknots in the above relations bound disks on F.

To describe $\langle D\rangle$ as a function in $\mathbb{Z}[A, A^{-1}, (A^2 - A^{-2})^{-1}]X_F$ in more detail, we need some preparation.

Definition 2.1. A *Kauffman state* for a link diagram $D \subset F$ is an assignment of the A or the B resolution for each crossing of D. The result of applying any state to D is a collection of disjoint simple closed curves on F called *state circles.*

Given a state s we use $|s| = |s(D)|$ to denote the number of state circles resulting from D by applying s, and we let $a(s)$ be the number of A resolutions in s, and $b(s)$ the number of B resolutions. Also, we will use s_t and s_{nt} to denote the set of contractible and non-contractible state circles resulting from s, and we will write $|s_t|$ and $|s_{nt}|$ for the cardinalities of these sets.

Finally, we will use s_A to denote the state where all the resolutions are A, and s_B for the state where all the resolutions are B.

Given a link projection $D = \pi(L)$ in $\mathcal{D}(F)$ we define

$$\langle D\rangle_0 = \sum_{\{s \,|\, s_{nt} = \emptyset\}} A^{a(s)-b(s)}(-A^2 - A^{-2})^{|s_t|-1}, \tag{2.1}$$

that is, we sum over all states that when applied to D produce only contractible state circles. Similarly, given a collection X of simple closed disjoint curves on F, none of which is contractible, we define

$$\langle D\rangle_X = \sum_{\{s\,|\,s_{nt}=X\}} A^{a(s)-b(s)}(-A^2-A^{-2})^{|s_t|-1}, \tag{2.2}$$

where $|s_t|$ is the number of contractible curves in $s = s(D)$. Clearly, for a given D, the value of $\langle D\rangle_X$ will be non-zero for only finitely many collections X.

Using the defining skein relations, for any $D \in \mathcal{D}(F)$ we can write in $\mathbb{Z}[A, A^{-1}]$:

$$\langle D\rangle = \langle D\rangle_0 + \sum_{X\in X_F} \langle D\rangle_X X. \tag{2.3}$$

Note that by definition, $\langle D\rangle_0$ is always a polynomial in $\mathbb{Z}[A, A^{-1}]$, and $\langle D\rangle_X$ are not in $\mathbb{Z}[A, A^{-1}]$ if there exist states for which $s_t = 0$, in which case we get the factor $(-A^2 - A^{-2})^{-1}$. Thus, in particular, $(-A^2 - A^{-2})\langle D\rangle_X$ always lies in $\mathbb{Z}[A, A^{-1}]$.

The argument used for S^2 in the 3-sphere works to show that $\langle D\rangle$ is invariant under Reidemeister moves II and III on F and that it changes by a power of A under Reidemeister move I. Then, $\langle D\rangle = \langle D\rangle_0$ is an invariant of framed links and a normalization of it gives the Jones polynomial (see e.g. [26]). We do not expect that $\langle\pi(L)\rangle$ an isotopy invariant of the framed link L in a general 3-manifold M.

A way to generalize the Jones polynomial to links in arbitrary 3-manifolds is to consider skein modules: given a 3-manifold M, let $\mathcal{L}(M)$ denote the set of isotopy classes of framed links in M. The *Kauffman skein module* of a 3-manifold M, denoted by $\mathcal{S}(M)$, is the quotient of the free $\mathbb{Z}[A, A^{-1}]$ module generated by $\mathcal{L}(M)$ by the submodule generated by all relations of the form

-  $-\,A$  $-\,A^{-1}$ ,
- $L \sqcup \bigcirc - (-A^2 - A^{-2})L$,
- $\bigcirc - 1$.

Here the crossing modifications take place in a small 3-ball in M that intersects the link to be modified at a single crossing, and the notation $\bigcirc$ is used to define the isotopy class of the knot that bounds a smooth disk in M. Given a link $L \subset M$, let $\bar{L}$ denote the class of L in M. Now the image of $\bar{L}$ under the map

$$\mathcal{L}(M) \to \mathcal{S}(M)$$

is an isotopy invariant of the framed link L.

Let us now discuss the special case where $M = F \times [-1,1]$, a thickened surface. It is known that $\mathcal{S}(F \times [-1,1])$ is free over $\mathbb{Z}[A, A^{-1}]$ with basis $X_F \cup \{\emptyset\}$, where $\emptyset$ is the empty knot [29]. Now any framed link in $F \times [-1,1]$ can be projected on $F = F \times \{0\}$, and any two link diagrams of F represent the same element in $\mathcal{L}(M)$ if and only if they are related by Reidemeister moves II and III on F. Thus $D = \pi(L)$, and the expression in (2.2) can immediately be viewed as the image of L in $\mathcal{S}(F \times [-1,1])$. It follows that $\langle D \rangle_0, \langle D \rangle_X$ are invariants of L in $F \times [-1,1]$. These invariants were recently considered by Boden, Karimi and Sikora [4] and were used to prove versions of two of the Tait conjectures for alternating links in $F \times [-1,1]$.

We will be concerned with $\langle D \rangle_0$, the sum of elements that appear when, after using the skein relations, we are left with just the empty knot. As pointed out in [4], one can make $\langle D \rangle_0$ an isotopy invariant of oriented non-framed links by considering $(-1)^{w(D)} A^{-3w(D)} \langle D \rangle_0$, where $w(D)$ is the writhe number of D. Setting

$$J_0(t) = ((-1)^{w(D)} A^{-3w(D)} \langle D \rangle_0)|_{t=A^4},$$

we obtain an isotopy invariant of L. Note that our convention differs from [4], as they set $t = A^{-4}$. Let

$$J_0 = a_m t^m + a_{m-1} t^{m-1} + \cdots + b_{n+1} t^{n+1} + b_n t^n,$$

where m and n denote the highest and lowest degrees of J_0, respectively.

Proposition 2.2. *For any link $L \subset M = F \times [-1,1]$, the polynomial $J_0 = J_0(L)$ is an isotopy invariant of L.*

Remark 2.3. We are interested in the absolute values of the coefficients of J_0 which are the same as those of $\langle D \rangle_0$, since as mentioned earlier, they remain unchanged under Reidemeister move I on F. By slightly abusing our setting, when talking about these coefficients, we will feel free to use $J_0(D)$ or $\langle D \rangle_0$ interchangeably.

In general let us start with a 3-manifold M and a connected, closed, orientable surface F embedded in M and a projection of $\pi(L) \subset F$ of a link $L \subset F \times [-1,1] \subset M$. We can define $\langle \pi(L) \rangle_0$ and $J_0(D)$ as above, but in general it is hard to decide when they descend to isotopy link invariants in M: Firstly, understanding the structure of $\mathcal{S}(M)$ is known to be a very hard problem. There is no algorithm for computing $\mathcal{S}(M)$ in general, and these modules have only been explicitly computed for some simple families of 3-manifolds. See [12] and references therein. For our purposes here, we will consider $F \times [-1,1]$ embedded in a 3-manifold M that is closed or has incompressible boundary. The inclusion induces a map

$$\mathcal{S}(F \times [-1,1]) \to \mathcal{S}(M), \tag{2.4}$$

and the image of L in $\mathcal{S}(F \times [-1, 1])$ is easy to calculate as we said above. However, in general very little is known about the structure of the map (2.4). For instance, for closed M, if one works over the field $\mathbb{Q}(A)$, then the skein module of M is finitely generated while the skein module of $F \times [-1, 1]$ is infinitely generated [18]. Thus, in this case, the map (2.4) has a substantial kernel and it is expected that this is the case over $\mathbb{Z}[A, A^{-1}]$ as well. Given a link $L \subset F \times [-1, 1] \subset M$, with $D = \pi(L)$ on F, in general we do not expect that $J_0(\pi(L))$ is an invariant of isotopy of L in M. In the remaining part of the paper, unless working with $M = F \times [-1, 1]$, we will consider J_0 as a function on the set of link projections $\pi(L)$ on F. In this setting, it is rather striking that, as Theorems 1.1 and 4.1 show, $J_0(\pi(L))$ captures intrinsic geometric information of the complement $M \setminus L$.

2.2. Geometrically adequate links. Recall that M is a 3-manifold and $F \subset M$ is an embedded orientable surface, and let $D = \pi(L) \subset F$ be a link projection. Given a Kauffman state s on D we define the *state graph* $G_s = G_s(D)$ as follows: the vertices of G_s correspond to the state circles of s, and the edges correspond to the crossings of $\pi(L)$. Each edge connects the subarcs of the state circles that remain from the splitting of that crossing in s. We will use G_A to denote the graph corresponding to s_A, and G_B to denote the graph corresponding to s_B. From now on we will restrict ourselves to projections D where all the state circles in s_A and in s_B are contractible on F. See Figure 3 below for an example of a link diagram D with this property, where we also show the graphs G_A, G_B.

Definition 2.4. We say that the diagram $D = \pi(L) \subset F$ is *geometrically A-adequate* if $G_A(D)$ has no 1-edge loops and all circles of s_A are contractible. Likewise, we say D is *geometrically B-adequate* if $G_B(D)$ has no 1-edge loops and all circles of s_B are contractible. If D is both geometrically A-adequate and B-adequate, we say it is *geometrically adequate.*

Definition 2.5. With the notation as above, suppose that the diagram $D = \pi(L) \subset F$ is geometrically adequate. Define the *reduced graph of G_A*, written G'_A, to be the graph where, if two edges e_1 and e_2 are adjacent to the same pair of vertices, we remove one of them if $e_1 \cup e_2$ bounds a disk on F. Let e'_A denote the number of edges of G'_A. Similarly define the reduced graph G'_B and denote its number of edges by e'_B.

In S^3 a link diagram $D = D(L)$ on a projection 2-sphere is called *A-adequate* if $G_A(D)$ has no 1-edge loops, and *B-adequate* if $G_B(D)$ has no 1-edge loops. For such diagrams, Futer, Kalfagianni and Purcell [13] have established relations between coefficients of the Kauffman bracket of D and geometric properties and invariants of the link complement $S^3 \setminus L$. In particular, they show that when D represents a hyperbolic link, coefficients of the Kauffman bracket of D provide linear bounds for the volume of the complement of the link.

In this paper, we will generalize these geometric relations to links that admit alternating projections on surfaces in 3-manifolds. A key step for this generalization is Theorem 2.6 below that also holds for projections on a sphere in S^3.

THEOREM 2.6. *Suppose that* $D=\pi(L)\subset F\subset M$ *and let* $a_m, a_{m-1}, b_{n+1}, b_n$ *denote the first two and last two coefficients of* $J_0(D)$.

(1) *If D is geometrically A-adequate, then* $|a_m|=1$ *and* $|a_{m-1}|=e'_A-|s_A|+1$.
(2) *If D is geometrically B-adequate, then* $|b_n| = 1$ *and* $|b_{n+1}| = e'_B-|s_B|+1$.

Theorem 2.6 and its proof should be compared with [9, Proposition 2.1] and the usual calculation of the degree of the Jones polynomial for ordinary alternating links in [26]. We will split the proof into two lemmas. The first one concerns the determination of $|a_m|$ and $|b_n|$.

LEMMA 2.7. *Suppose that* $D = \pi(L) \subset F \subset M$ *is a link diagram and let* a_m, b_n *denote the first and the last coefficients of* $J_0(D)$.

(1) *If D is geometrically A-adequate, then* $|a_m| = 1$.
(2) *If D is geometrically B-adequate, then* $|b_n| = 1$.

Proof. Let $c = c(D)$ denote the number of crossings of D and consider the all-A state, s_A. Then $a(s_A) = c$ and $b(s_A) = 0$. By the definition of $\langle D\rangle_0$ (see (2.1)), the contribution of s_A is

$$A^c(-A^2 - A^{-2})^{|s_A|-1}.$$

The highest degree here, then, is $c + 2|s_A| - 2$, and the coefficient belonging to it is $(-1)^{|s_A|-1} = \pm 1$.

Now we will show that all the other states have degrees less than $c + 2|s_A| - 2$. We can view any state as being obtained from s_A by a finite series of changing an A resolution to a B resolution. We can write this series out as $s_A \to s_1 \to s_2 \to \cdots \to s_k$. We will show that s_{i+1} has degree at most s_i. First, note that $a(s_i) = a(s_{i-1})-1$ and $b(s_i) = b(s_{i-1})+1$. Next, by changing a single resolution, we are doing one of the following:

- We merge two contractible circles to a contractible one (so $|s_i| = |s_{i-1}|-1$).
- We split a contractible circle into two contractible ones (so $|s_i| = |s_{i-1}|+1$).
- We split a contractible circle into two non-contractible ones (so $|s_i| = |s_{i-1}| - 1$).
- We merge two non-contractible circles into a contractible one (so $|s_i| = |s_{i-1}| + 1$).

- We merge a contractible and non-contractible circle into a non-contractible one (so $|s_i| = |s_{i-1}| - 1$).
- We rearrange a non-contractible circle of s_{i-1} to a non-contractible one of s_i (so $|s_i| = |s_{i-1}|$).

In particular, each resolution change will either increase the number of state circles by 1, leave it the same, or decrease it by 1. As a result, the highest degree in the contribution of s_i to $\langle D \rangle_0$ will be less than or equal to that of s_{i-1}.

As the highest degree contribution to $\langle D \rangle_0$ coming from s_A is $c + 2|s_A|$, while the highest degree contribution coming from $s_k \neq s_A$ is $c-2k+2|s_k|-2$, in order for s_k to contribute to the highest degree of $\langle D \rangle_0$, we would need to have

$$c + 2|s_A| - 2 \leq c - 2k + 2|s_k| - 2, \quad \text{i.e.} \quad |s_A| \leq |s_k| - k.$$

This would mean that each state change $s_{i-1} \to s_i$ must increase the number of state circles by 1, limiting what sort of changes we can make from the five possibilities discussed above. However, since all the state circles in s_A are contractible and G_A has no 1-edge loops, the first change $s_A \to s_1$ will merge two contractible circles to one, so $|s_1| = |s_A| - 1$ and the highest degree of s_1 is strictly less than that of s_A. Since this degree cannot increase during the change from s_A to s_k, the contribution of s_k has degree less than the contribution of s_A. So it does not contribute to the highest degree of $\langle D \rangle_0$, and we are done with part (1).

To see part (2), let $D^* \subset F \subset M$ denote the link diagram obtained from D by switching all the crossings of D simultaneously. By the definition we can see that $\langle D^* \rangle_0$ is obtained from $\langle D \rangle_0$ by changing A to A^{-1}. Thus, we have $|b_n(D)| = |a_m(D^*)|$ and the conclusion follows from part (1). ∎

Now we turn to the second lemma, which treats the second and the penultimate coefficients of $J_0(D)$.

Lemma 2.8. *Suppose that $D = \pi(L) \subset F \subset M$ is a link diagram and let a_{m-1}, b_{n+1} denote the second and the penultimate coefficients of $J_0(D)$.*

(1) *If D is geometrically A-adequate, then $|a_m| = 1$ and $|a_{m-1}| = e'_A - |s_A| + 1$.*
(2) *If D is geometrically B-adequate, then $|b_n| = 1$ and $|b_{n+1}| = e'_B - |s_B| + 1$.*

Proof. Let $c = c(D)$ denote the number of crossings of D. We know that the highest degree of $\langle D \rangle_0$ is $c + 2|s_A|$. Then the second highest degree has exponent $c + 2|s_A| - 4$. A contribution to this degree can come either from the second highest degree of s_A, or from the highest degree of a state s_k in which $k \neq 1$ crossings of D are assigned the B resolution.

First, we will deal with s_A. Recall that the part of D_0 coming from s_A is

$$\begin{aligned} A^c(-A^2 - A^{-2})^{|s_A|-1} &= (-1)^{|s_A|-1} A^c (A^2 + A^{-2})^{|s_A|-1} \\ &= (-1)^{|s_A|} A^c \sum_{i=0}^{|s_A|} \binom{|s_A|-1}{i} A^{2|s_A|-2i-2} A^{-2i} \\ &= (-1)^{|s_A|-1} \sum_{i=0}^{|s_A|-1} \binom{|s_A|-1}{i} A^{c+2|s_A|-4i-2}. \end{aligned}$$

In particular, the second highest degree is $c + 2|s_A| - 6$ (when $i = 1$), and the coefficient is $(-1)^{|s_A|-1}(|s_A| - 1)$. Next, we deal with states $s_k \neq s_A$. As in the proof of Lemma 2.7, we can write s_k as a finite series of resolution changes, $s_A \to s_1 \to \cdots \to s_k$. We claim that s_k can only contribute to the degree $c + 2|s_A| - 6$ if all resolution changes happen to parallel edges in G_A (i.e. edges that are adjacent to the same pair of vertices and any two of which encircle a disk on F).

Recall that the highest degree that s_k contributes is $c - 2k + 2|s_k|$. For this to contribute to the second highest degree of $\langle D \rangle_0$, we must have

$$c + 2|s_A| - 6 = c - 2k + 2|s_k| - 2, \quad \text{i.e.} \quad |s_A| = |s_k| - k + 2.$$

This means that, in our series from s_A to s_k, either we must increase the number of state circles in all but exactly two resolution changes, where we do not change the number at all, or we must decrease the number of state circles exactly once, and the other resolution changes must increase it. As in Lemma 2.7, we know that $s_A \to s_1$ must decrease the number of state circles. Hence, for s_k to contribute to the second highest degree the remaining resolution changes must increase the number of state circles.

As s_A has no non-contractible circles, any following state will only have them if we introduce them from a resolution change. Note that we need to change at least two resolutions from s_A to create non-contractible state circles. However, turning these circles into contractible ones again will require merging and this step will decrease the number of state circles. Hence such a sequence cannot contribute to a_{m-1}. Thus, to increase the number of state circles, we must, after $s_A \to s_1$, always split a single contractible circle into two contractible circles. This can only happen in the change $s_{i-1} \to s_i$, however, if there is a 1-edge loop in the state graph $G_{s_{i-1}}$. Such 1-edge loops are created when we merge two state circles. In our series, this can only happen in the first state change, $s_A \to s_1$, and so all other changes must be adjacent to the same two state circles. There are now two cases to consider. Let e_1 be the edge of G_A affected during the change $s_A \to s_1$. Then either

- $s_{i-1} \to s_i$ affects an edge e_i with $e_1 \cup e_i$ bounding a disk on F, or
- $s_{i-1} \to s_i$ affects an edge e_i with $e_1 \cup e_i$ not bounding a disk on F.

The two cases are illustrated in Figure 2. If we are in the second case, we will create two non-contractible circles, each parallel to the curve $e_i \cup e_2$. If we are in the first case, we create two contractible circles, and so are fine. As such, s_k only contributes to the second highest degree if all resolution changes happen to parallel edges that bound a disk on F. We can view this as a single edge of G'_A.

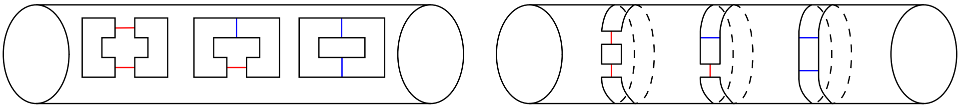


Fig. 2. The two cases of changing two edges of G_A adjacent to the same pair of vertices to edges of G_B.

Each family of parallel edges has several states associated to it: if the family has k parallel edges (and thus k crossings), and we want a state to have $1 \leq j \leq k$ differences from s_A, then there are $\binom{k}{j}$ such states. While the highest degree remains the same, other values do change. If a state s has j changes in resolutions from s_A, then $a(s) = c - j$, $b(s) = j$, and $|s| = |s_A| - 2 + j$. Then the highest degree coefficient such a state contributes is

$$\begin{aligned}(-1)^{|s|} A^{a(s)-b(s)} A^{2|s|-2} &= (-1)^{|s_A|+j-3} A^{c-2j+2|s|-2} \\ &= (-1)^{|s_A|-3} (-1)^j A^{c+2|s_A|-6}.\end{aligned}$$

Summing over all possible states for this family shows that the family contributes the coefficient

$$(-1)^{|s_A|-3} \sum_{j=1}^{k} \binom{k}{j} (-1)^j.$$

Using the binomial theorem, we obtain

$$0 = (1 + (-1))^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j = 1 + \sum_{j=1}^{k} \binom{k}{j} (-1)^j = 1 + (-1)$$

so the coefficient contributed by a single family is just $(-1)^{|s_A|-1}$.

Now, adding the coefficient we get from s_A to all the coefficients we get from edges of G'_A, we deduce that the coefficient is

$$\begin{aligned}|a_{m-1}| &= |(-1)^{|s_A|-2} e'_A + (-1)^{|s_A|-1} (|s_A| - 1)| = |(-1)^{|s_A|-2} (e'_A - |s_A| + 1)| \\ &= e'_A - |s_A| + 1\end{aligned}$$

and we are done with part (1).

To prove (2) we can apply the argument of (1) to the diagram D^* as in the proof of Lemma 2.7. This finishes the proof of the lemma and that of Theorem 2.6. ∎

The notion of geometric adequacy is well suited for the connections of skein invariants with geometry of the link complement that we explore in this paper. Recently, Boden, Karimi and Sikora have considered link diagrams on a surface $F = F \times \{0\}$ inside thickened surfaces $F \times [-1, 1]$, and they also defined notions of A-adequacy and B-adequacy. We now compare their definitions to ours. In the terminology of [4] a link diagram D on F is called *A-adequate* if for any state s that differs from s_A by a single resolution we have the following: either $|s_t| \leq |s_A|$ or the number of non-contractible state circles in s is different than that in s_A. One defines D being *B-adequate* in a similar way.

LEMMA 2.9. *Suppose that $D = \pi(L) \subset F \subset M$ is a link diagram such that all the state circles in s_A are contractible on F. Then, if D is geometrically A-adequate, it is also A-adequate.*

Proof. To show that D is A-adequate, we must show that, for any state s adjacent to s_A, either $|s_t| \leq |s_A|$ or s and s_A have a different set of non-contractible loops. There are two ways a state change $s_A \to s$ can increase $|s_t|$: either we split a contractible state circle into two such circles, or we merge two essential circles in s_A into a contractible circle. By assumption, as s_A has only contractible circles, we must split a single state circle into two. However, in order to split a state circle, we must have an edge of G_A connecting that state circle to itself. As G_A, by assumption, has no 1-edge loops, this cannot happen, and so we are done. ∎

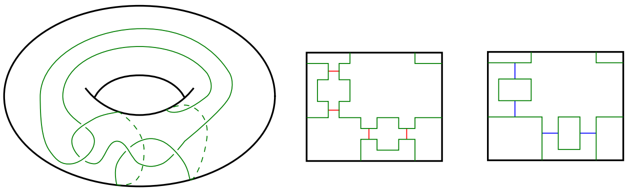


Fig. 3. A link diagram $D = \pi(L)$ (left) together with the graphs G_A (center) and G_B (right). All the state circles in s_A and s_B are contractible on the torus. The diagram D is geometrically B-adequate but not geometrically A-adequate. In fact, all the four edges of G_A are 1-edge loops.

The converse of Lemma 2.9 is not true. The diagram D of Figure 3 is A-adequate in the sense of [4] but is not geometrically A-adequate: indeed, while the state circles in s_A are contractible, each of the four states that are obtained from s_A by a single resolution change contains non-contractible circles. Thus D is A-adequate. However, G_A contains 1-edge loops, hence D is not geometrically A-adequate.

3. Guts of surfaces and Kauffman brackets. In this section we focus on links that admit alternating projections on surfaces in 3-manifolds. We find that under suitable diagrammatic conditions such links are geometrically adequate on the one hand, and on the other hand they fit into the class of weakly generalized alternating links studied in [20]. This will allow us to combine our work in the last section with the geometric techniques of [20] and prove Theorems 1.2 and 1.1.

3.1. Reduced alternating link projections. Suppose M is a compact, orientable, irreducible 3-manifold with empty or incompressible boundary, and F a closed, connected, orientable surface in M. Next we introduce several properties and definitions concerning projections of links $L \subset F \times [-1, 1] \subset M$ on F. Some of these properties are directly quoted from [20] and others are suitably adapted to better fit our purposes.

Definition 3.1. A link diagram $\pi(L) \subset F$ is *prime* if, whenever a disk $D \subset F$ has ∂D intersecting $\pi(L)$ transversely exactly twice, then either

- $F = S^2$, and either $\pi(L) \cap D$ or $\pi(L) \cap (F \setminus D)$ is a single arc, or
- F has positive genus, and $\pi(L) \cap D$ is a single arc.

Definition 3.2. We say that a link diagram $\pi(L) \subset F$ is *reduced alternating* if

(1) each component of L projects to at least one crossing in $\pi(L)$,
(2) $\pi(L)$ is prime and alternating on F, and
(3) for every essential, simple closed curve γ on F that intersects $\pi(L)$ at exactly two points near a crossing, one of the two subarcs of $\pi(L)$ with endpoints on γ contains no crossings of $\pi(L)$.

We note that for the notion in Definition 3.1 the authors of [20] use the term "weakly prime". We also note that Definition 3.2 is different than the definition of reduced diagrams given in [20], in that condition (3) is not required in their definition. For alternating projections of $F = S^2$ in $M = S^3$ the two definitions are equivalent.

Given an alternating link projection $\pi(L) \subset F \subset M$ for each crossing of $\pi(L)$ we can label the four regions around it by the letters A and B in an alternating fashion. This is done so that the two opposite regions of the crossing that are merged during the A splitting are labeled by A. Similarly, the two opposite regions of the crossing that are merged during the B splitting are labeled by B. This way the corners of every region of $F \setminus \pi(L)$ receive the label A or B.

Definition 3.3. With the notation and setting as above, we will say that the link diagram $\pi(L)$ is *checkerboard colorable* if for every region R of $F \setminus \pi(L)$ the letters at all corners of R are the same. Thus every region of a checkerboard colorable diagram is labeled by A or B.

Next we recall two complexity functions for link diagrams $\pi(L) \subset F \subset M$ from [20].

DEFINITION 3.4. The *edge representativity* $e(\pi(L), F)$ is the minimum number of intersections between $\pi(L)$ and any essential curve on F. If there are no essential curves, then we say $e(\pi(L), F) = \infty$.

The *representativity* $r(\pi(L), F)$ is the minimum number of intersections between $\pi(L)$ and the boundary of any compressing disk for F. If there are no compressing disks for F, then we say $r(\pi(L), F) = \infty$.

As an example to clarify the definitions above we discuss the alternating link diagram $\pi(L)$ of Figure 3 viewed on a standard Heegaard torus $F = T^2$ in S^3. We have $e(\pi(L), F) = r(\pi(L), F) = 2$ and the diagram is checkerboard colorable, prime, and all the regions of $T^2 \setminus \pi(L)$ are disks. However, $\pi(L)$ is not reduced in the sense of Definition 3.2. The next proposition shows this phenomenon does not happen when $e(\pi(L), F) > 2$.

PROPOSITION 3.5. *Let $\pi(L) \subset F \subset M$ be an alternating link diagram such that $\pi(L)$ is checkerboard colorable and all the regions of $F \setminus \pi(L)$ are disks. Then $\pi(L)$ is reduced if and only if*

- *$\pi(L)$ is prime,*
- *each component of L projects to at least one crossing in $\pi(L)$, and*
- *the edge representativity satisfies $e(\pi(L), F) > 2$.*

Proof. First, suppose $e(\pi(L), F) > 2$ and $\pi(L)$ is prime. We only need to prove that $\pi(L)$ satisfies part (3) of Definition 3.2. So suppose γ is a simple closed curve on F intersecting $\pi(L)$ exactly twice. Because $e(\pi(L), F) > 2$, we know that γ cannot be essential, and so must bound a disk E on F. If $F = S^2$, then, as $\pi(L)$ is prime, either $\pi(L) \cap E$ or $\pi(L) \cap (F \setminus E)$ is a single arc without any crossings. If $F \neq S^2$, then $\pi(L) \cap E$ is a single arc. In either case, we have one of the subarcs of $\pi(L)$ with endpoints on γ contain no crossings, and so we are done.

Now suppose $\pi(L)$ is reduced alternating. Then we already know that $\pi(L)$ is prime, and each component of L projects to at least one crossing in $\pi(L)$. We need to show $e(\pi(L), F) > 2$. Suppose not. As D is checkerboard colorable, we must have $e(\pi(L), F)$ be an even number. If $e(D, F) = 0$, then there is a region of $F \setminus \pi(L)$ that contains an essential curve. This would mean we have a non-disk region, and so cannot happen.

If $e(\pi(L), F) = 2$, then we can find some essential closed curve γ intersecting $\pi(L)$ exactly twice. As $\pi(L)$ is reduced, this means that one of the two subarcs of $\pi(L)$ with endpoints on γ must contain no crossings. Call this subarc ℓ. We also have γ split into two subarcs, γ_1 and γ_2. There are four cases to consider. First, we could have $\ell \cup \gamma_i$ bound a disk on F for some $i = 1, 2$. We can use this disk to homotope γ off of D, and so make it an essential curve intersecting our knot zero times, a contradiction. Second, we could have ℓ form a single component. If it does, L has a component with no crossing in D, contradicting the assumption that each component of L projects to at least one crossing on F, included in the definition of reduced. Third, we could have $\ell \cup \gamma_i$ essential and parallel to all of γ for some i, say $i = 1$. But then $\ell \cup \gamma_2$ is homotopically trivial, and so we are in the first case, and get a contradiction. Finally, we could have $\ell \cup \gamma_i$ essential and not parallel to γ. As $\pi(L)$ is checkerboard colorable and ℓ contains no crossings, everything to one side of ℓ must be the same color. But then we have a region adjacent to itself across a knot arc, contradicting D being checkerboard colorable. In any of the cases where $e(\pi(L), F) = 2$, we contradict one of our assumptions, and so it cannot happen. So then we are left with $e(\pi(L), F) > 2$, and we are done. ∎

Definition 3.6. Following [20] we say a link diagram $\pi(L) \subset F$ is *weakly generalized alternating* if it is prime, checkerboard colorable, alternating, and the representativity satisfies $r(\pi(L), F) \geq 4$.

We have the following:

Corollary 3.7. *A reduced alternating diagram $\pi(L) \subset F$ that has disk regions and is checkerboard colorable is also weakly generalized alternating.*

Proof. We only need to check that $r(\pi(L), F) \geq 4$. By Proposition 3.5, we have $e(\pi(L), F) > 2$, which, since the diagram is checkerboard colorable with disk regions, implies $e(\pi(L), F) \geq 4$. Since any curve on F bounding a compression disk is also essential, we are done. ∎

Our next lemma together with Corollary 3.7 allows us to relate our work in Section 2 to the work of [20] on weakly generalized alternating links.

Lemma 3.8. *Suppose $\pi(L)$ is an alternating diagram on a projection surface F of genus at least 1 in a 3-manifold M. Suppose that $\pi(L)$ is reduced, checkerboard colorable and all regions of $F \setminus \pi(L)$ are disks. Then $\pi(L)$ is geometrically adequate.*

Proof. First, as $\pi(L)$ is alternating on F and all regions of $F \setminus D$ are disks, we see that s_A and s_B must have only contractible circles.

We need to show that G_A and G_B have no 1-edge loops. The proof is the same for both G_A and G_B, so we will focus on G_A. Suppose $\pi(L)$ is as in the statement of the lemma, but G_A has at least one 1-edge loop, ℓ. Then ℓ connects a state circle to itself, and ℓ crosses $\pi(L)$ exactly twice at a crossing. We may then find some simple arc γ in the state circle connecting the two endpoints of ℓ. But then $\gamma \cup \ell$ is a simple closed curve intersecting $\pi(L)$ exactly twice. By Proposition 3.5 we have $e(\pi(L), F) \geq 4$. Thus $\gamma \cup \ell$ must be contractible on F.

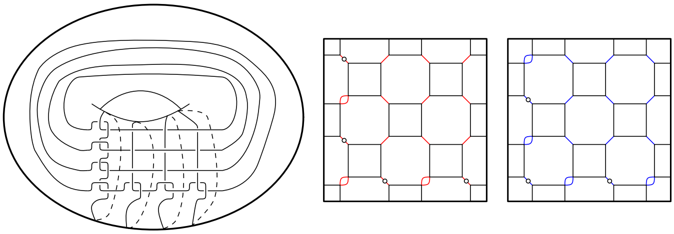


Fig. 4. A reduced, checkerboard colorable, alternating diagram $\pi(L)$ with disk regions, on a torus. In the terminology of [20] it is weakly generalized alternating. Both G_A (left) and G_B (right) have no 1-edge loops, which makes $\pi(L)$ geometrically adequate.

By homotoping $\gamma \cup \ell$ we can get two such simple closed curves, one with the crossing to the left of the curve, and the other with the crossing to the right. One of them would bound a disk such that the existence of the crossing corresponding to ℓ violates the primeness of $\pi(L)$. Thus $\pi(L)$ must be geometrically A-adequate. See Figure 4 for an example. ∎

3.2. Twist number relations and invariance. A *twist region* of an alternating projection $\pi(L) \subset F$ is either a string of bigons of $\pi(L)$ arranged vertex to vertex that is maximal in the sense that no larger string of bigons contains it, or a single crossing adjacent to no bigon.

Definition 3.9. An alternating diagram $\pi(L) \subset F$ is called *twist-reduced* if whenever there is a disk $D \subset F$ such that ∂D intersects $\pi(L)$ exactly four times adjacent to two crossings, then one of the following holds:

- D contains a (possibly empty) sequence of bigons that is part of a larger twist region containing the two crossings, or
- $F \setminus D$ contains a disk D' with $\partial D'$ intersecting $\pi(L)$ four times adjacent to the same two crossings as ∂D, and D' contains a string of bigons that forms a larger twist region containing the original two crossings. See Figure 5.

The *twist number* $t_F(\pi(L))$ of a diagram is the number of twist regions in a twist-reduced diagram.

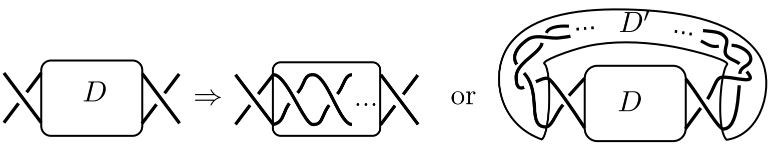


Fig. 5. A twist-reduced diagram. Figure modified from [21].

Lemma 3.10. *Suppose that $\pi(L)$ is a twist-reduced, reduced alternating diagram with twist number $t_F(\pi(L))$ on a projection surface $F \subset M$ of genus at least 1. Suppose that $\pi(L)$ is checkerboard colorable and all regions of*

$F \setminus \pi(L)$ are disks. Then

$$|a_{m-1}| + |b_{n+1}| - 2 = t_F(\pi(L)) - \chi(F),$$

where a_{m-1} and b_{n+1} are the second and the penultimate coefficients of the polynomial $J_0(\pi(L))$.

Proof. By Lemma 3.8, $\pi(L)$ is geometrically adequate; the state graphs G_A and G_B have no 1-edge loops. Suppose that $\pi(L)$ has c crossings. First note that the twist number is

$$t_F(\pi(L)) = c - (c - e'_A) - (c - e'_B) = e'_A + e'_B - c.$$

By definition, crossings that correspond to twist regions of $\pi(L)$ correspond to edges of G_A or G_B that are parallel; every pair bounds a bigon on F. Call a twist region of $\pi(L)$ an *A-twist* (or *B-twist*) *region* if, in G_A (or G_B), all crossings of the twist region are represented by edges such that every pair bounds a bigon on F. Then note that $c - e'_A$ is exactly the number of edges in G_A that are not in G'_A, and so counts the number of crossings that are in an A-twist region except for one for each such twist region (that is represented in G'_A). Likewise, $c - e'_B$ is the number of crossings in B-twist regions minus one for each such twist region. Then $(c - e'_A) + (c - e'_B) = c - t_F(\pi(L))$.

Next, note that

$$|s_A| + |s_B| = c + 2 - 2g(F) = c + \chi(F).$$

Putting these together, along with Lemmas 2.7 and 2.8, we get

$$\begin{aligned} |a_{m-1}| + |b_{n+1}| - 2 &= e'_A + e'_B - |s_A| - |s_B| \\ &= \big(t_F(\pi(L)) + c\big) - (c + \chi(F)) \\ &= t_F(\pi(L)) - \chi(F), \end{aligned}$$

which finishes the proof of the lemma. ∎

It is known that the twist number of a reduced, twist-reduced alternating projection $\pi(L)$ on a 2-sphere in S^3 is an isotopy invariant of L. This has been proven in two ways. Firstly, it follows from work of Dasbach and Lin [9] showing this twist number can be obtained from the Jones polynomial of L. Secondly, it follows from the Tait flyping conjecture proven in Menasco and Thistlethwaite [28], which shows that any two reduced, prime, alternating link diagrams are related by a series of flypes. Following the approach of [9], we have a generalization of twist number invariance for alternating links in thickened surfaces.

Corollary 3.11. *Let L be a link in $F \times [-1, 1]$ admitting a checkerboard colorable, reduced alternating projection $\pi(L) \subset F$ that is twist-reduced and has all its regions disks. Then any two such projections of L have the same twist number. That is, $t_F(\pi(L))$ is an isotopy invariant of L.*

Proof. By Lemma 3.10,

$$|a_{m-1}| + |b_{n+1}| - 2 + \chi(F) = t_F(\pi(L)).$$

Since $|a_{m-1}|, |b_{n+1}|$ are isotopy invariants of L in $F \times [-1,1]$ (Proposition 2.2), the conclusion follows. ∎

The Tait flyping conjecture is unknown for links in thickened surfaces. Hence the second method of deducing invariance of the twist number is not currently available. However, Boden, Karimi, and Sikora were able to show the first two Tait conjectures by proving that, for reduced alternating diagrams in thickened surfaces, the crossing number and the writhe are invariants [4].

In general the twist number of weakly generalized alternating knots is not an invariant. Howie [19] has produced weakly alternating projections of the same knot on a Heegaard torus in S^3 with different twist numbers (the knot 9_{29} is one example). On the other hand no such examples are known for weakly alternating projections on incompressible surfaces. In view of this and Corollary 1.3 we ask the following:

Question 3.12. *Let M be a 3-manifold that is closed or has incompressible boundary, and $F \subset M$ an incompressible surface. Suppose that $\pi(L)$ is a reduced, twist-reduced, checkerboard colorable, alternating diagram on F where all the regions of $F \setminus \pi(L)$ are disks. Is $t_F(\pi(L))$ an invariant of the isotopy type of L in M?*

As Howie's examples take place on the compressible Heegaard torus in S^3, these do not give an answer to this question.

3.3. Guts and Kauffman bracket. Here we will prove Theorem 1.2 stated in the Introduction. In fact we prove a more general result (Theorem 3.13) in which the assumption that F is incompressible ($r(\pi(L), F) = \infty$) is relaxed to $r(\pi(L), F) > 4$.

Suppose that $D = \pi(L)$ is a weakly generalized alternating diagram on a surface $F \subset M$ such that the regions of $F \setminus \pi(L)$ are all disks. The projection gives rise to two spanning surfaces of L, the checkerboard surfaces that we will denote by $S_A = S_A(D)$ and $S_B = S_A(D)$. Our convention will be that S_A is constructed by attaching half-twisted bands to the disks bounded by the state circles $s_A(D)$, where we attach a half-twisted band for each crossing of D, so that the band retracts onto the corresponding edge of the graph G_A and the surface S_A retracts to G_A. Similarly we define S_B that retracts onto G_B. See Figure 6.

By [20, Theorem 3.19] the surfaces S_A and S_B are π_1-essential in the complement of $X = M \setminus L$. Let $M_A = X \setminus\setminus S_A := X \setminus N(S_A)$ and let $M_B = X \setminus\setminus S_B := X \setminus N(S_B)$. Recall also that $a_m, a_{m-1}, b_{n+1}, b_n$ are the first two and the last two coefficients, respectively, in the polynomial $J_0(\pi(L))$.

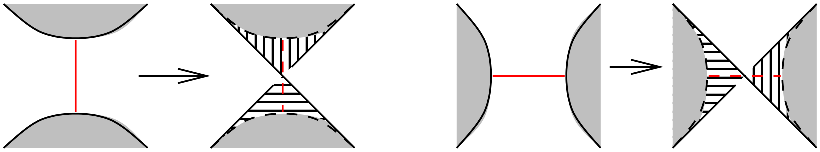


Fig. 6. The construction of S_A and S_B. The red lines (see the pdf file) indicate the edge of S_A and S_B that corresponds to the bands shown.

THEOREM 3.13. *Let M be a 3-manifold that is closed or has incompressible boundary, and $F \subset M$ a projection surface such that that $M \setminus N(F)$ is atoroidal and ∂-anannular. Let $\pi(L)$ be a reduced, alternating diagram on F that is twist-reduced with twist number $t_F(\pi(L))$. Suppose that $\pi(L)$ is checkerboard colorable, all the regions of $F \setminus \pi(L)$ are disks, F has genus at least 1 and $r(\pi(L), F) > 4$. Then:*

(1) $\chi(\mathrm{guts}(M_A)) = 1 - |a_{m-1}| + \frac{1}{2}\chi(\partial M)$,
(2) $\chi(\mathrm{guts}(M_B)) = 1 - |b_{n+1}| + \frac{1}{2}\chi(\partial M)$,
(3) $t_F(\pi(L)) = |a_{m-1}| + |b_{n+1}| - 2 + \chi(F)$.

Let us first explain how to deduce Theorem 1.2: As discussed earlier, if F is incompressible in M we have $r(\pi(L), F) = \infty$. Thus, in particular, $r(\pi(L), F) > 4$ and Theorem 1.2 is a special case of Theorem 3.13.

Proof of Theorem 3.13. First note that by Lemma 3.8, $D = \pi(L)$ is geometrically A-adequate and geometrically B-adequate. We will give the proof for part (1) and M_A. The proof works the same, after swapping S_A and S_B, to give part (2). Finally, (3) follows from Lemma 3.10.

The graph G_A gives a cellular decomposition of the surface F. The number of 0-cells is the number of the vertices of G_A, denoted by $|s_A|$, and the number of 1-cells is the number of edges, $e_A = c(\pi(L))$. The number of 2-cells is the number of complementary regions of G_A, which is the same as the number $|s_B|$ of vertices of G_B. If we consider $\pi(L)$ as a 4-valent graph on F, we can label the components of $F \setminus \pi(L)$ by A or B according to whether they correspond to a vertex of G_A or G_B. We will refer to these as *A-regions* and *B-regions*, respectively. Now let $|s'_B|$ denote the number of non-bigon B-regions and recall that e'_A denotes the number of edges in the reduced graph G'_A. We have

$$\chi(F) = |s_A| - e_A + |s_B| = |s_A| - e'_A + |s'_B|, \tag{3.1}$$

where the second equality follows since, by definition and the fact that D is twist-reduced, the number of edges we remove from G_A to obtain G'_A is exactly the number of bigon B-regions. Equation (3.1) gives

$$\chi(F) - |s'_B| = |s_A| - e'_A. \tag{3.2}$$

Since, as we mentioned above, D is geometrically A-adequate, by Theorem 2.6 we have

$$|s_A| - e'_A = 1 - |a_{m-1}| = |a_m| - |a_{m-1}|. \tag{3.3}$$

By Corollary 3.7, D is weakly generalized alternating. Now we can apply [20, Theorem 6.6] to D to conclude that

$$\chi(\mathrm{guts}(M_A)) = \chi(F) + \tfrac{1}{2}\chi(\partial M) - |s'_B|. \tag{3.4}$$

Now combining (3.4) with (3.2) and (3.3), we get

$$\chi(\mathrm{guts}(M_A)) = 1 - |a_{m-1}| + \tfrac{1}{2}\chi(\partial M),$$

which is part (1) of the theorem.

We will now sketch the proof of (3.4), referring the reader to [20] for precise definitions and details. We do this not only for reasons of completeness, but because it is interesting to see the correspondence between the combinatorics in the calculation of $|a_{m-1}|$ from the proof of Theorem 2.6 and those involved in the calculation of $\chi(\mathrm{guts}(M_A))$. On the one hand, edges that are parallel on G_A (i.e. co-bound a disk on F) do not contribute due to cancelations in the Kauffman state sum expression of $|a_{m-1}|$. On the other hand, strings of parallel edges on G_A correspond to components of I-bundle pieces of the JSJ-decomposition of M_A and they do not contribute to $\chi(\mathrm{guts}(M_A))$.

Set $\tilde{S}_A = \partial N(S_A)$. The *parabolic locus* P is $\partial M_A \cap \partial N(L)$. Considering $\pi(L)$ as a 4-valent graph on F, the authors of [20] define a *chunk decomposition* of M_A into two compact, oriented, irreducible 3-manifolds with boundary, say C_1, C_2, each containing a copy of F as a boundary component (and possibly more boundary components coming from ∂M). The component of ∂C_i that corresponds to F comes equipped with a checkerboard coloring with the regions of $F \setminus \pi(F)$ called *faces*. The chunks are glued together along the B-labeled faces. The decomposition generalizes previously known polyhedral decompositions constructed from alternating and adequate link projections in S^3 (see, for example, [13] and references therein). Even though the chunks are not simply connected, [20] shows that the techniques that were used for polyhedral decompositions generalize and adapt in the setting of chunks.

Recall that $M \setminus N(F)$ is atoroidal and ∂-anannular. By the annulus version of JSJ-decomposition one can cut M_A along a collection of essential annuli that are disjoint from P into I-bundles, Seifert fibered pieces and hyperbolic pieces which are the ones that form the guts. Seifert fibered pieces turn out to be solid tori and as such they do not contribute to the Euler characteristic computation.

Let R be an essential annulus in M_A, disjoint from P, with $\partial R \subset \tilde{S}_A$. Such an annulus R is either *parabolically compressible*, or not, in which case it is called *parabolically incompressible*.

If R is parabolically incompressible, then [20, Lemma 6.9] argues that F must be a 2-sphere, contradicting the assumption of F having genus at least 1.

Suppose now that R is parabolically compressible. This means that there is a disk D with interior disjoint from R, with ∂D meeting R in an essential arc α on R, and with $\partial D \setminus \alpha$ lying on $\tilde{S}_A \cup P$, with α meeting P transversely exactly once. If we do surgery along such a disk, we obtain an *essential product disk*: these are disks meeting P transversely exactly twice, with boundary otherwise on $\tilde{S}_A$. Such disks are known to correspond to I-bundle components of the above-mentioned JSJ-decomposition (see [13, Definition 4.5] or [20, Definition 6.7]).

Now let us look at an essential product disk E caused by surgering R. If it meets S_B, then S_B cuts E into subrectangles $E_1, \dots, E_n$. By looking how such rectangles must sit in the diagram and in the chunk decomposition, one can show that E must be boundary parallel, a contradiction to E being essential.

However, if E does not run through S_B, then ∂E must meet the chunk in two A-faces and two B-faces, and so ∂E must meet P exactly four times. Such an E is parallel to F. However, as D is twist-reduced, this implies ∂E contains a series of B-bigons.

Case 1. First suppose that we do not have B-regions that are bigons. Then $\mathrm{guts}(M_A) = M_A$. Recall that M_A is obtained via C_1 and C_2, where we glue these chunks together along B-labeled faces. Then, as $\chi(C_i) = \frac{1}{2}\chi(\partial C_i)$, we must have

$$\chi(C_1) = \tfrac{1}{2}\chi(F) + \tfrac{1}{2}\chi(\partial M|_{C_1}), \qquad \chi(C_2) = \tfrac{1}{2}\chi(F) + \tfrac{1}{2}\chi(\partial M|_{C_2}).$$

Gluing the chunks together along white faces will add their Euler characteristics together, and subtract one for every B-face we glue along. As there are no white bigons, we glue along $|s_B| = |s'_B|$ such faces, and so

$$\chi(\mathrm{guts}(M_A)) = \chi(F) + \tfrac{1}{2}\chi(\partial M) - |s'_B|.$$

Case 2. If $F \setminus \pi(L)$ has B-regions that are bigons, then each such bigon will form a quad, with two sides on P and two sides on $\tilde{S}_A$. These will give essential product disks and thus I-bundle parts. The existence of I-bundles leads to parabolically compressible annuli and, as mentioned above, to essential product disks. All the essential product disks parabolically compress to the strings of the ones corresponding to bigons (see [13, Figure 4.2]). Surgering along one of these basic essential product disks increases the Euler characteristic of the I-bundle submanifold by 1 and it does not change the guts. After we remove all B-bigons of $\pi(L)$, we have replaced each B twist region by a single crossing and we have eliminated all the I-bundle components. This has modified S_A into a new surface S'_A and $\mathrm{guts}(M_A)$ is the same as the guts of S'_A. But now we have no B-bigon regions left, and the B-regions of the new link projection are exactly the non-bigon B-regions of $\pi(L)$, which are exactly $|s'_B|$. ∎

4. Relations to hyperbolic geometry invariants. Let $D = \pi(L)$ be a reduced, alternating link diagram on a surface $F \subset M$, such that $M \setminus L$ is hyperbolic. In this section we show that the skein-theoretic quantities $|a_{m-1}(\pi(L))|$, $|b_{n+1}(\pi(L))|$ provide bounds on the volume of the complement $M \setminus L$. The relations to volume come from two sources: First, by a result of Agol, Storm and Thurston [2] the negative Euler characteristic of the guts of an essential surface in a hyperbolic 3-manifold M bounds linearly the volume of M from below. We will apply this result to the surfaces S_A, S_B associated to projections of weakly generalized alternating links. Second, by work of Kalfagianni and Purcell [21], if F is a Heegaard torus or M is a thickened surface, the twist number of weakly generalized alternating projections provides two-sided bounds of their volume.

We prove the following theorem which, in particular, implies Theorem 1.1 stated in the Introduction.

THEOREM 4.1. *Let M be a 3-manifold that is closed or has incompressible boundary, and $F \subset M$ a projection surface such that that $M \setminus N(F)$ is atoroidal and ∂-anannular. Let $D = \pi(L)$ be a reduced and twist-reduced alternating diagram on F that is checkerboard colorable and all the regions of $F \setminus D$ are disks. Suppose, moreover, that F has genus at least 1 and $r(D, F) > 4$. Then L is hyperbolic and*

$$\mathrm{vol}(M \setminus L) \geq v_8 \max\{|a_{m-1}|, |b_{n+1}|\} - 1 - \tfrac{1}{2}\chi(\partial M),$$

where a_{m-1}, b_{n+1} are the second and the penultimate coefficients of the polynomial $J_0(\pi(L))$, and $v_8 = 3.66386\ldots$ is the volume of a regular ideal octahedron.

Proof. By Corollary 3.7, D is weakly generalized alternating. Let S_A, S_B denote the checkerboard surfaces of the projection. By [20, Theorem 1.1], S_A, S_B are π_1-essential in $X = M \setminus L$, and X is hyperbolic. By cutting the link complement along S_A and S_B we obtain manifolds $M_A = X \setminus\setminus S_A$ and $M_B = X \setminus\setminus S_B$, respectively. By [2, Theorem 9.1] we have

$$\mathrm{vol}(M \setminus L) \geq -v_8\chi(\mathrm{guts}(M_A)), \qquad \mathrm{vol}(M \setminus L) \geq -v_8\chi(\mathrm{guts}(M_B)).$$

Since we have assumed that $\pi(L)$ is reduced, by Lemma 3.8 it is geometrically A-adequate and B-adequate. By Theorem 3.13 we have

$$\chi(\mathrm{guts}(M_A)) = 1 - |a_{m-1}| + \tfrac{1}{2}\chi(\partial M),$$
$$\chi(\mathrm{guts}(M_B)) = 1 - |b_{n+1}| + \tfrac{1}{2}\chi(\partial M).$$

Thus we obtain

$$\begin{aligned}\operatorname{vol}(M \setminus L) &\geq v_8\big(|a_{m-1}| - 1 - \tfrac{1}{2}\chi(\partial M)\big),\\ \operatorname{vol}(M \setminus L) &\geq v_8\big(|b_{n+1}| - 1 - \tfrac{1}{2}\chi(\partial M)\big),\end{aligned}$$

and the result follows. ∎

To see how Theorem 1.1 follows, note that if F is incompressible the hypothesis $r(D, F) > 4$ is satisfied.

Next we discuss two special cases where the quantity $|a_{m-1}| + |b_{n+1}| - 2$ of Theorem 4.1 also provides upper bounds of the volume. The first result concerns weakly generalized alternating knots on a Heegaard torus.

Corollary 4.2. *Let F be a Heegaard torus in $M = S^3$, or in a lens space $M = L(p, q)$. Let $D = \pi(L)$ be a reduced and twist-reduced alternating diagram on F that is checkerboard colorable and all the regions of $F \setminus D$ are disks. Suppose, moreover, that $r(D, F) > 4$. Then $M \setminus L$ is hyperbolic, and*

$$\frac{v_8}{2} \cdot (|a_{m-1}| + |b_{n+1}| - 2) \leq \operatorname{vol}(M \setminus L) < 10v_4 \cdot (|a_{m-1}| + |b_{n+1}| - 2),$$

where $v_4 = 1.01494\ldots$ is the volume of a regular ideal tetrahedron.

Proof. By Corollary 3.7 the projection $\pi(L)$ is weakly generalized alternating. Hyperbolicity follows from [20, Theorem 1.1]. By [21, Corollary 1.5], which also relies on [20] for the lower bound, we have

$$\frac{v_8}{2} \cdot t_F(\pi(L)) \leq \operatorname{vol}(M \setminus L) < 10v_4 \cdot t_F(\pi(L)),$$

where $t_F(\pi(L))$ is the twist number of $\pi(L)$. By Lemma 3.10, $t_F(\pi(L)) = |a_{m-1}| + |b_{n+1}| - |a_m| - |b_n| = |a_{m-1}| + |b_{n+1}| - 2$ and the result follows. ∎

Our second result is Theorem 1.4, which we now prove.

Proof of Theorem 1.4. By Corollary 3.7 the projection $\pi(L)$ is weakly generalized alternating. Hyperbolicity follows from [20, Theorem 1.1], where for $F \neq T^2$ the hyperbolic structure is chosen so that non-torus boundary components of $M \setminus L$ are totally geodesic. By [21, Theorem 1.4], which relies on [20] for the lower bound, we have

$$\frac{v_8}{2} \cdot t_F(\pi(L)) \leq \operatorname{vol}(Y - K) < 10v_4 \cdot t_F(\pi(L)) \tag{4.1}$$

if $F = T^2$, and

$$\frac{v_8}{2} \cdot (t_F(\pi(L)) - 3\chi(F)) \leq \operatorname{vol}(Y - K) < 6v_8 \cdot t_F(\pi(L))$$

if F has genus greater than one. Thus in both cases the result follows immediately by Lemma 3.10. ∎

Remark 4.3. Theorem 1.4 is the analogue of the "volumish theorem" of [9] for alternating links in thickened surfaces, where the authors rely on the two-sided volume bounds in terms of the twist number of alternating projections given by Lackenby [24].

Remark 4.4. In [6] Champanerkar and Kofman show that if $\pi(L)$ is an alternating projection as in Theorem 1.4, then $\pi(L)$ admits two-sided linear bounds in terms of coefficients of a specialization of the Krushkal polynomial [23]. Then they also combine this with (4.1) to conclude that Krushkal's polynomial also gives two-sided bounds of the volume of alternating links in thickened surfaces. Krushkal has informed the present authors that Andrew Will [30] also obtained a similar result. His approach, however, does not lead to a proof of invariance of $t_F(\pi(L))$.

Remark 4.5. In [3] Bavier shows that if M is closed and $\pi(K)$ is a weakly generalized alternating knot projection, twist-reduced, on a surface $F \subset M$ of genus at least 1, then the twist number $t_F(\pi(K)$ provides two-sided bounds on the cusp volume of $M \setminus K$. We close the section by noting that Lemma 3.10 implies that Theorem 1.1 of [3] and the resulting applications to Dehn filling given therein can also be stated in terms of the skein-theoretic quantity $J_0(\pi(K))$.

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