Renormalization-group behavior of ϕ^3 theories in $d = 6$ dimensions

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We investigate possible renormalization-group fixed points at nonzero coupling in ϕ^3 theories in six spacetime dimensions, using beta functions calculated to the four-loop level. We analyze three theories of this type, with (a) a one-component scalar, (b) a scalar transforming as the fundamental representation of a global SU(N) symmetry group, and (c) a scalar transforming as a bi-adjoint representation of a global SU(N) $\otimes$ SU(N) symmetry. We do not find robust evidence for such fixed points in theories (a) or (b). Theory (c) has the special feature that the one-loop term in the beta function is zero; implications of this are discussed.

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I. INTRODUCTION

A topic of fundamental importance in quantum field theory is the renormalization-group (RG) behavior of a scalar field theory in d spacetime dimensions. Here we investigate RG behavior and possible RG fixed points of three scalar field theories with cubic scalar self-interactions in $d = 6$ spacetime dimensions. These are denoted generically as ϕ^3_6 theories and are defined by the path integral

$$Z = \int \prod_x [d\phi(x)] e^{iS}, \tag{1.1}$$

where $S = \int d^6x \mathcal{L}$ and $\mathcal{L}$ is the Lagrangian density. For the theory with a (real) one-component scalar,

$$\mathcal{L}_1 = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m_0^2 \phi^2 - \frac{g}{3!}\phi^3. \tag{1.2}$$

For the theory with a (complex) scalar field ϕ^i, $i = 1, \ldots, N$, transforming according to the fundamental representation of a global SU(N) theory,

$$\mathcal{L}_2 = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m_0^2 \phi^\dagger \phi - \frac{g}{3!} d_{ijk}(\phi^i \phi^j \phi^k + \text{H.c.}), \tag{1.3}$$

where d_{ijk} is the totally symmetric rank-3 tensor for SU(N), and sums over repeated indices are understood. Since $d_{ijk} = 0$ for SU(2), N is restricted to the range $N \geq 3$ for this theory. We will discuss the theory with a bi-adjoint scalar below. The ϕ^3_6 theories are renormalizable, with a dimensionless coupling, g. These theories are invariant under the redefinition (suppressing possible indices on ϕ)

$$\phi \to -\phi, \qquad g \to -g. \tag{1.4}$$

Because of this invariance, one can, without loss of generality, take g to be non-negative, and we shall do so henceforth. For technical simplicity, we also take $m_0 = 0$. As is well known, because of the cubic scalar self-interaction, the energy of the theory is not bounded below. Nevertheless, cubic scalar theories have long been used to provide simple examples of perturbative calculations in quantum field theory. They have also been used in statistical mechanics to model the Yang-Lee edge singularity [1] and percolation [2]. A recent general analysis is [3]. The application to statistical mechanics naturally makes use of a $d = 6 - \epsilon$ expansion to obtain estimates of critical exponents; see also [4]. Here we restrict ourselves to $d = 6$. Recently, ϕ^3_6 theories were used for a test of the a theorem [5].

Quantum loop corrections lead to a dependence of the physical coupling $g = g(\mu)$ on the Euclidean energy/momentum scale μ at which this coupling is measured. The dependence of $g(\mu)$ on μ is described by the RG beta function of the theory,

$$\beta_g = \frac{dg}{d \ln \mu}. \tag{1.5}$$

Because the n-loop integrals involve nth powers of the quantity

$$\frac{g^2 S_6}{(2\pi)^6} = \frac{g^2}{2^6 \pi^3} \tag{1.6}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the unit sphere $\|x\| = 1$ for a vector $x \in \mathbb{R}^d$, it is convenient to define the variable

$$\bar{g} \equiv \frac{g}{8\pi^{3/2}} \tag{1.7}$$

and the corresponding beta function $\beta_{\bar{g}} = d\bar{g}/d\ln\mu$. This beta function has the series expansion

$$\beta_{\bar{g}} = \bar{g} \sum_{n=1}^{\infty} b_n a^n, \tag{1.8}$$

where

$$a \equiv (\bar{g})^2 \tag{1.9}$$

and b_n is the n-loop coefficient. The n-loop ($n\ell$) approximation to $\beta_{\bar{g}}$, denoted $\beta_{\bar{g},n\ell}$, is obtained by replacing $n = \infty$ by finite n in the summand in Eq. (1.8). Because of the prefactor $\bar{g} = \sqrt{a}$ in Eq. (1.8), the beta function $\beta_{\bar{g}}$ always vanishes at the origin in coupling-constant space, $a = 0$. Physically, this just means that in a free theory, there is no running coupling since the coupling is zero. The one-loop and two-loop coefficients in Eq. (1.8) are independent of the scheme used for regularization and renormalization, while the b_n with $n \geq 3$ are scheme dependent [6,7]. The coefficients b_1 and b_2 for the one-component ϕ_6^3 theory were calculated in [8], while b_3 was calculated in [9] (in the $\overline{\rm MS}$ scheme [10]). It was observed early on [8] that the one-component ϕ_6^3 theory is asymptotically free, i.e., $g(\mu) \to 0$ in the ultraviolet (UV) limit, $\mu \to \infty$. For this theory, and for a ϕ_6^3 theory with a general global symmetry group G, the beta function was calculated up to four-loop order, inclusive, in Refs. [3,11]. Recently, Ref. [12] presented a four-loop calculation of the beta function for a ϕ_6^3 theory with a scalar transforming as a bi-adjoint representation of a direct product $G_1 \otimes G_2$ global symmetry group (see also [13]).

An important question is whether, for the region of $\bar{g}$ where a perturbative calculation of the beta function is reliable, the beta function of this theory exhibits evidence for a zero away from the origin, at a physical, positive, value of a. If the theory is asymptotically free, this would be an infrared (IR) fixed point of the renormalization group (IRFP), denoted $a_{\rm IR}$, while if the theory is infrared free, this would be an UV fixed point of the RG (UVFP), denoted $a_{\rm UV}$. In the UV-free case, one thus considers the RG evolution of the theory from the deep UV. If the theory exhibits an IRFP, then as the reference momentum scale μ decreases from large values, $a = a(\mu)$ increases and approaches $a_{\rm IR}$ from below as $\mu \to 0$. In the IR-free case, one envisages starting the RG evolution from the IR; if the theory exhibits an UVFP, then as $\mu \to \infty$, $a(\mu)$ approaches $a_{\rm UV}$ from below.

In this paper we carry out an analysis of zeros of the respective beta functions for three types of ϕ^3 theories in $d = 6$ spacetime dimensions, namely those with a real, one-component scalar, a scalar transforming according to the fundamental representation of SU(N), and a scalar transforming according to a bi-adjoint representation of SU(N) $\otimes$ SU(N). The organization of the paper is as follows. In Sec. II we discuss some relevant methodology. In Secs. III–V we present our results for the three ϕ_6^3 theories under consideration. Our conclusions are given in Sec. VI.

II. METHODOLOGY

In this section we briefly discuss some methodology that is relevant for our study of the beta functions and their zeros in ϕ_6^3 theories. One carries out this study using the beta function calculated (perturbatively) to a given finite n-loop order. The maximal loop order to which one can carry out this study in a scheme-independent manner is the two-loop order. Thus, if b_1 and b_2 are nonzero, then a necessary condition for a theory to exhibit a physical zero of the beta function at a nonzero value of a is that b_1 and b_2 must have opposite signs. If the n-loop beta function has more than one zero on the positive real a axis, we denote the one nearest to the origin as $a_{{\rm IR},n\ell}$ or $a_{{\rm UV},n\ell}$ in the two respective cases of an asymptotically free or infrared free theory. An additional necessary condition for the n-loop beta function to exhibit robust evidence for an IR or UV zero at the respective values $a_{{\rm IR},n\ell}$ and $a_{{\rm UV},n\ell}$ is that the beta functions calculated to $(n+1)$-loop order should also exhibit a zero, and the fractional difference between the n-loop and $(n+1)$-loop values should be small.

Before proceeding, for perspective, it is useful to mention two examples where these conditions are satisfied. The first example is a non-Abelian gauge theory in $d = 4$ spacetime dimensions with gauge group G containing N_f massless fermions transforming according to a given representation R of G. This theory is asymptotically free for N_f less than an upper (u) bound, $N_u = 11C_A/(4T_f)$ [14], where $C_A = C_2(Adj)$ is the quadratic Casimir invariant for the adjoint representation and $T_f = T(R)$ is the trace invariant [15,16]. There is a range of values of N_f less than N_u where the two-loop beta function of this theory has an IR zero [17,18] at a value $a_{{\rm IR},2\ell}$ that goes to zero as N_f (formally generalized to nonzero real values) approaches N_u from below. For N_f less than, but close to N_u, the IR theory is weakly coupled, and one expects that the values of

this IR zero and of physical quantities such as anomalous dimensions of gauge-invariant operators calculated to finite order at the IRFP are reasonably stable with respect to the inclusion of higher-loop terms in the beta function. This has been shown explicitly and quantitatively up to the four-loop [19,20] and five-loop order [21,22]. As N_f decreases below the region near N_u, the IR theory becomes more strongly coupled and higher-order terms in perturbative series expansions become more important. Although the value of the IR zero, $\alpha_{\mathrm{IR},n\ell}$, at n-loop order is scheme dependent if $n \geq 3$, scheme-independent series expansions as power series in the variable $\Delta_f = N_u - N_f$ have been used to obtain scheme-independent calculations of physical quantities such as anomalous dimensions [22–26]. The resultant values of these anomalous dimensions have been compared with lattice simulations [22,24]. (For reviews of lattice measurements, see, e.g., [27,28].) This stability of physical results for N_f slightly below N_u in such non-Abelian gauge theories is the sort of necessary behavior that one would require to certify the existence of an IRFP of the renormalization group in an asymptotically free ϕ_6^3 theory.

An example of a reliably calculated UVFP of the renormalization group in an IR-free theory is provided by an exact solution of the O(N) nonlinear σ model in the $N \to \infty$ limit, in $d = 2 + \epsilon$ dimensions [29], where ϵ is small. This UVFP was calculated nonperturbatively by means of a summation of an infinite number of Feynman diagrams in this $N \to \infty$ limit.

As noted above, for our analysis of the beta functions of the various ϕ_6^3 theories, we shall use the b_n coefficients with $1 \leq n \leq 4$, with b_3 and b_4 calculated in the $\overline{\mathrm{MS}}$ scheme [10], from Refs. [3,9,12]. Effects of scheme transformations on beta function coefficients were calculated in [30,31] (see also [32–34].)

III. ONE-COMPONENT ϕ FIELD

In this section we consider the ϕ_6^3 theory with a one-component (real) scalar field, with the Lagrangian density (1.2). The one-loop and two-loop coefficients in $\beta_{\bar{g}}$ are [8]

$$b_1 = -\frac{3}{4} \tag{3.1}$$

and

$$b_2 = -\frac{5^3}{2^4 \cdot 3^2} = -0.8680556. \tag{3.2}$$

The fact that b_1 is negative means that this theory is asymptotically free. Since these coefficients have the same sign, the beta function has no IR zero at the maximal scheme-independent level, namely the two-loop level.

In the $\overline{\mathrm{MS}}$ scheme, the three-loop coefficient is [9]

$$b_3 = -\frac{5}{2^3}\left(\frac{6617}{2^5 \cdot 3^4} + \zeta_3\right) = -2.3468199. \tag{3.3}$$

In Eq. (3.3) and similar equations we show the simple factorizations of denominators. Although the numerator of b_2 happens to have a simple factorization, most numerator numbers do not; for example, $6617 = 13 \cdot 509$. The four-loop coefficient, again in the $\overline{\mathrm{MS}}$ scheme, is [3,11]

$$b_4 = \frac{3404365}{2^{10} \cdot 3^6} + \frac{4891\zeta_3}{2^5 \cdot 3^3} - \frac{15\zeta_4}{2^5} - \frac{5\zeta_5}{3} = 9.129607, \tag{3.4}$$

where $\zeta_s = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. (Here we could substitute $\zeta_4 = \pi^4/90$, but we leave the ζ_4 term in its abstract form.) Since the b_n with $n \geq 3$ are scheme dependent, so are the zeros of the n-loop beta function for $n \geq 3$. Nevertheless, one may check the zeros of the three-loop beta function away from the origin. These are the solutions of the quadratic equation $b_1 + b_2 a + b_3 a^2 = 0$. We find these solutions are a complex-conjugate pair and hence are unphysical. This analysis at the two-loop and three-loop level provides strong evidence against the existence of an IR zero in the beta function. At the four-loop level, the zeros of the beta function away from the origin, which are the solutions to the cubic equation $b_1 + b_2 a + b_3 a^2 + b_4 a^3 = 0$, are comprised of a complex-conjugate pair and the value $a = 0.622134$. Because the one real positive root was not present at either the maximal scheme-independent two-loop level or at the three-loop level, we do not consider it as robust evidence for an IR zero of the beta function.

IV. SU(N) THEORY WITH SCALAR IN FUNDAMENTAL REPRESENTATION

In this section we investigate the beta function of the ϕ_6^3 theory where ϕ transforms as the fundamental representation of a (global) SU(N) symmetry group. As noted above, we consider N in the range $N \geq 3$ since d_{ijk} vanishes for SU(2), resulting in a free theory. The beta function has been calculated to four-loop order for this theory in [3,11]. The two scheme-independent coefficients are

$$b_1 = -\frac{(N^2 - 20)}{4N} \tag{4.1}$$

and

$$b_2 = -\frac{(5N^4 - 496N^2 + 5360)}{2^4 \cdot 3^2 N^2}. \tag{4.2}$$

The one-loop coefficient b_1 is positive for small N and passes through zero to negative values as N (formally generalized from integral values $N \geq 3$ to positive real values [16]) increases through the value

$$N_{b1z} = 2\sqrt{5} = 4.4721360, \tag{4.3}$$

where this and other floating-point numbers are quoted to the indicated accuracy, and the subscript $b1z$ stands for "b_1 zero." Thus, this theory is IR-free for $N < N_{b1z}$ and UV-free (i.e., asymptotically free) for $N > N_{b1z}$. The physical reason for the change in the sign of b_1 and the resultant change in the renormalization-group behavior as N increases through the value $2\sqrt{5}$ can be traced to the individual contributions to b_1 from one-loop two-point and three-point Feynman diagrams. This is evident from the expression for b_1 in terms of group invariants, namely [12]

$$b_1 = \frac{1}{4}(T_2 - 4T_3), \tag{4.4}$$

where T_2 and T_3 are defined by the traces

$$d^{ii_1i_2}d^{ji_1i_2} = T_2\delta^{ij} \tag{4.5}$$

and

$$d^{ii_1i_2}d^{ji_1i_3}d^{ki_2i_3} = T_3 d^{ijk}. \tag{4.6}$$

The traces T_2 and T_3 occur in the one-loop corrections to the two-point and three-point functions. For SU(N) [35]

$$T_2 = \frac{N^2 - 4}{N} \tag{4.7}$$

and

$$T_3 = \frac{N^2 - 12}{2N}. \tag{4.8}$$

The interplay of both of these types of corrections determines b_1.

The coefficient b_2 is negative for $N < N_{b2z,-}$, positive in the interval $N_{b2z,-} < N < N_{b2z,+}$, and negative for $N > N_{b2z,+}$, where

$$N_{b2z,\pm} = \frac{2}{\sqrt{5}}\sqrt{62 \pm 3\sqrt{241}}. \tag{4.9}$$

Numerically,

$$N_{b2z,-} = 3.5131155, \qquad N_{b2z,+} = 9.319765. \tag{4.10}$$

Consequently, this theory thus has four different regimes of RG behavior, depending on the value of N (again, formally generalized to positive real values):

TABLE I. Regimes of different behavior of ϕ_6^3 theory with SU(N) global symmetry, as a function of N (formally generalized from positive integer to positive real values [16]).

N	b_1	b_2	Properties
$3 \leq N_{b2z,-}$, i.e. $3 \leq N < 3.513$	+	−	IR-free with $a_{\mathrm{UV},2\ell}$
$N_{b2z,-} < N < N_{b1z}$, i.e. $3.513 < N < 4.472$	+	+	IR-free, no $a_{\mathrm{UV},2\ell}$
$N_{b1z} < N < N_{b2z,+}$, i.e. $4.472 < N < 9.320$	−	+	UV-free with $a_{\mathrm{IR},2\ell}$
$N > N_{b2z,+}$, i.e. $N > 9.320$	−	−	UV-free, no $a_{\mathrm{IR},2\ell}$

(1) $3 \leq N < 3.513$: IR-free, with a UV zero of the beta function $\beta_{\bar{g},2\ell}$.
(2) $3.513 < N < 4.472$: IR-free, with no UV zero of $\beta_{\bar{g},2\ell}$.
(3) $4.472 < N < 9.320$: UV-free, with an IR zero of $\beta_{\bar{g},2\ell}$.
(4) $N > 9.320$: UV-free, with no IR zero of $\beta_{\bar{g},2\ell}$.

These properties are summarized in Table I. The respective real intervals in N contain the physical integer values (i) $N = 3$; (ii) $N = 4$; (iii) $5 \leq N \leq 9$; and (iv) $N \geq 10$.

We first consider the asymptotically free regime, defined by the inequality $N > N_{b1z}$. For N in the interval $N_{b1z} < N < N_{b2z,+}$, i.e., $4.472 < N < 9.320$, the two-loop beta function $\beta_{\bar{g},2\ell}$ has an IR zero at $a = -b_1/b_2 \equiv a_{\mathrm{IR},2\ell}$, where

$$a_{\mathrm{IR},2\ell} = \frac{36N(N^2 - 20)}{-5N^4 + 496N^2 - 5360}. \tag{4.11}$$

In Table II we list values of $a_{\mathrm{IR},2\ell}$ for integer values of N in the interval $N_{b1z} < N < N_{b2z,+}$. The calculation leading to this IR zero is expected to be most reliable toward the lower

TABLE II. Values of zeros of the n-loop beta function, $\beta_{\bar{g},n\ell}$ away from the origin, in the variable $a = (\bar{g})^2$, for $2 \leq n \leq 4$, as a function of N, in the SU(N) ϕ_6^3 theory with a scalar field transforming as the fundamental representation of SU(N). The notation "u" denotes unphysical zeros (which are comprised of complex-conjugate pairs here). If $N = 4$ or $N \geq 10$, the theory has no scheme-independent zero of the beta function, and hence these cases are not tabulated. See text for further details.

N	2-loop	3-loop	4-loop
3	0.913	u	u, −0.676
5	0.230	u	u, 0.495
6	0.574	u	u, 0.224
7	1.053	u	u, 0.190
8	2.146	u	u, 0.175
9	9.828	u	u, 0.166

end of this interval, where $a_{\mathrm{IR},2\ell}$ is small, and to become less reliable toward the upper end of the interval, where $a_{\mathrm{IR},2\ell}$ grows to larger values.

To investigate how stable this IR zero of the two-loop beta function $\beta_{\bar{g},2\ell}$ is to the inclusion of higher-order terms, we examine the three-loop and four-loop beta functions, $\beta_{\bar{g},3\ell}$ and $\beta_{\bar{g},4\ell}$. For this purpose, we make use of the expressions for b_3 and b_4, as calculated in the $\overline{\mathrm{MS}}$ scheme, from Refs. [3,11],

$$b_3 = \frac{1}{2^8 \cdot 3^4 N^3}[-211N^6 + (27132 - 62208\zeta_3)N^4 + (-1220688 + 20736\zeta_3)N^2 + 9272896 + 4396032\zeta_3] \tag{4.12}$$

and

$$\begin{aligned} b_4 = \frac{1}{2^{10} \cdot 3^6 N^4}[&(327893 + 870048\zeta_3 - 1321920\zeta_5)N^8 + (-8142840 - 14427072\zeta_3 - 559872\zeta_4 + 31570560\zeta_5)N^6 \\ &+ (112740480 + 155416320\zeta_3 + 11384064\zeta_4 - 421770240\zeta_5)N^4 \\ &+ (-1264882304 - 1477343232\zeta_3 + 35831808\zeta_4 + 1950842880\zeta_5)N^2 \\ &+ 5761837824 + 7029669888\zeta_3 - 791285760\zeta_4 + 995328000\zeta_5]. \end{aligned} \tag{4.13}$$

As before, the zeros of $\beta_{\bar{g},3\ell}$ away from the origin are the solutions of the equation $b_1 + b_2 a + b_3 a^2 = 0$. In the interval $N_{b1z} < N < N_{b2z,+}$ under consideration here, the discriminant $b_2^2 - 4b_1 b_3$ (which is a quartic function in the variable N^2) is positive for a very small interval $N_{b1z} < N < N_{dz}$, but passes through zero to negative values as N increases through the value

$$N_{dz} = 4.497050 \tag{4.14}$$

(where the subscript dz stands for "discriminant zero"). Hence, except for the very small interval $4.472 < N < 4.497$, the zeros of $\beta_{\bar{g},3\ell}$ (away from the origin) are comprised of a complex-conjugate pair of a values and are thus unphysical. This is indicated in Table II. These results show that although the two-loop beta function exhibits an IR zero in this interval $4.472 < N < 9.320$, it is not stable to the inclusion of higher-order perturbative corrections. We also calculate the zeros of the four-loop beta function, given as the roots of the equation $b_1 + b_2 a + b_3 a^2 + b_4 a^3 = 0$. The results are listed in Table II. As is evident, they consist of a real value and an unphysical complex-conjugate pair of roots. The fact that the real value is not at all close to $a_{\mathrm{IR},2\ell}$ provides further evidence against a robust IR zero of the beta function.

In the interval $N > N_{b2z,+}$, i.e., $N > 9.320$, this SU(N) ϕ_6^3 theory does not have an IR zero at the maximal scheme-independent level of two loops.

Finally, we consider the interval $N < N_{b1z}$, i.e., $3 \le N < 4.472$, where the theory is IR-free. As discussed above, in the subinterval $N_{b2z,-} < N < N_{b1z}$, i.e., $3.513 < N < 4.472$, there is no UV zero in the two-loop beta function, $\beta_{\bar{g},2\ell}$. In the subinterval $3 \le N < 3.513$, including the physical integral value $N = 3$, $\beta_{\bar{g},2\ell}$ does have a UV zero, denoted $a_{\mathrm{UV},2\ell}$, which is given by the right-hand side of Eq. (4.11). This two-loop zero, $a_{\mathrm{UV},2\ell}$, has the value $1188/1301 = 0.913144$ for $N = 3$. However, in order for this to be considered as a reliable UV zero of the beta function, it is necessary that the value should be reasonably stable when one includes higher-order terms in the beta function. We find that this is not the case. At the three-loop level, the zeros of $\beta_{\bar{g},3\ell}$ away from the origin for this $N = 3$ case consist of an unphysical complex-conjugate pair of values of a. At the four-loop level, the three roots of the equation $b_1 + b_2 a + b_3 a^2 + b_4 a^3 = 0$ are comprised of a negative value and a complex-conjugate pair, all of which are unphysical. We list these results in Table II. Consequently, although the beta function of this SU(3) ϕ_6^3 theory does exhibit a UV zero at the two-loop level, it does not satisfy the requirement of being stable to higher-loop corrections.

The large-N limit of this theory is also of interest. We define the rescaled coupling

$$\xi \equiv \bar{g} N \tag{4.15}$$

and the corresponding beta function

$$\beta_\xi = \frac{d\xi}{d \ln \mu}. \tag{4.16}$$

This beta function has the series expansion

$$\beta_\xi = \xi \sum_{n=1}^{\infty} \hat{b}_n \hat{a}^n \tag{4.17}$$

where

$$\hat{b}_n = \lim_{N\to\infty} \frac{b_n}{N^n} \tag{4.18}$$

and

$$\hat{a} = \xi^2. \tag{4.19}$$

From the expressions for the b_n, $1 \le n \le 4$ we have

$$\hat{b}_1 = -\frac{1}{4}, \tag{4.20}$$

$$\hat{b}_2 = -\frac{5}{2^4 \cdot 3^2} = -(3.472222 \times 10^{-2}), \tag{4.21}$$

$$\hat{b}_3 = -\frac{211}{2^8 \cdot 3^4} = -(1.01755 \times 10^{-2}), \tag{4.22}$$

and

$$\hat{b}_4 = \frac{1}{2^4 \cdot 3}\left(\frac{327893}{2^6 \cdot 3^5} + \frac{1007\zeta_3}{2 \cdot 3^2} - 85\zeta_5\right) = 4.02503 \times 10^{-3}. \tag{4.23}$$

In the large-N limit, this theory has no IRFP at the maximal, two-loop scheme-independent level, since $\hat{b}_1$ and $\hat{b}_2$ have the same sign.

V. SU(N) ⊗ SU(N) THEORY WITH BI-ADJOINT SCALAR

The condition that a theory is UV-free or IR-free is that for small (physical) values of the coupling near the origin, its beta function is negative or positive, respectively. If, as is usually the case, the one-loop coefficient, b_1, is nonzero, this is equivalent to the condition that b_1 is negative or positive, respectively. In a theory where b_1 depends on a parameter, such as the SU(N) ϕ_6^3 theory, one may formally choose this parameter so that $b_1 = 0$ and then examine the sign of b_2. For example, in the case of the SU(N) ϕ_6^3 theory, if one formally generalizes N from the physical range of integers $N \ge 3$ to non-negative real numbers and sets $N = N_{b1z} = 2\sqrt{5}$, this renders $b_1 = 0$ in Eq. (4.1). Substituting this value of N into b_2, one obtains $b_2 = 8/9 > 0$, so that the theory is IR-free. Of course, this is just a formal result, since it depends on setting N to a noninteger value.

Recently, Ref. [12] reported a physical example of a theory with an identically zero one-loop term, i.e., $b_1 = 0$. This is a ϕ_6^3 theory with a bi-adjoint (BA) scalar, i.e., a scalar transforming according to the representation (Adj, Adj) of a direct-product global symmetry group $G_1 \otimes G_2$. For our purposes, it will suffice to consider the diagonal case where the symmetry group is $G \otimes G$ and, furthermore, to take $G = \mathrm{SU}(N)$, with $N \ge 2$. One motivation for studying this ϕ_6^3 theory with a bi-adjoint scalar is its connection with recent work modeling on-shell gravity as a double copy of a Yang-Mills gauge theory [36] (see [12] for more details). We denote the scalar as $\phi^{a_1 a_2}$, where here $1 \le a_1, a_2 \le N^2 - 1$. The Lagrangian density for this theory is

$$\mathcal{L}_3 = \frac{1}{2}(\partial_\mu \phi^{a_1 a_2})(\partial^\mu \phi^{a_1 a_2}) - \frac{g}{3!} f^{a_1 b_1 c_1} f^{a_2 b_2 c_2} \phi^{a_1 a_2} \phi^{b_1 b_2} \phi^{c_1 c_2}, \tag{5.1}$$

where f^{abc} are the structure constants of the Lie algebra of SU(N) [15].

The two-loop coefficient of the beta function in this theory is [12]

$$b_2^{(BA)} = -\frac{5N^4}{2^6 \cdot 3^2}. \tag{5.2}$$

Thus, this theory is asymptotically free. For its study of this theory, Ref. [12] also calculated the three-loop and four-loop terms in the beta function (in the $\overline{\mathrm{MS}}$ scheme). These are

$$b_3^{(BA)} = \frac{N^2}{2^8 \cdot 3^4}(55N^4 + 7776N^2 + 139968\zeta_3) \tag{5.3}$$

and

$$b_4^{(BA)} = \frac{N^4}{2^{14} \cdot 3^6}[(-298081 - 825120\zeta_3 + 1244160\zeta_5)N^4 + (-7091712 - 22394880\zeta_3 + 33592320\zeta_5)N^2 - 214990848\zeta_3 + 268738560\zeta_5]. \tag{5.4}$$

Because the $b_n^{(BA)}$ are scheme dependent for $n \ge 3$, it is not possible to give a scheme-independent answer to the question of whether the (perturbatively computed) beta function has an IR zero in this theory. With b_3 calculated in the $\overline{\mathrm{MS}}$ scheme, the theory has an IR zero in the three-loop beta function, at $a = -b_3^{(BA)}/b_2^{(BA)}$, i.e.,

$$a_{\mathrm{IR},3\ell,BA} = \frac{180N^2}{55N^4 + 7776N^2 + 139968}. \tag{5.5}$$

We list values of $a_{\mathrm{IR},3\ell,BA}$ in Table III for an illustrative range of values of N.

We now investigate the effect of including the next-higher-order term, namely, the four-loop term, in the beta function. The condition that the four-loop beta function vanishes for a away from the origin is the equation $b_2^{(BA)} + b_3^{(BA)} a + b_4^{(BA)} a^2 = 0$. Using the expressions for $b_3^{(BA)}$ and $b_4^{(BA)}$ in Eqs. (5.3) and (5.4), we find that one solution for $a_{\mathrm{IR},4\ell,BA}$ is quite close to the three-loop value,

TABLE III. Values of IR zeros $a_{\mathrm{IR},n\ell,BA}$ of the n-loop beta function, $\beta_{\bar{g},n\ell}$ away from the origin, in the variable $a = (\bar{g})^2$, for $2 \le n \le 4$, as a function of N, in the SU(N) ⊗ SU(N) ϕ_6^3 theory with a scalar field transforming as a bi-adjoint representation. See text for further details.

N	$a_{\mathrm{IR},3\ell,BA}$	$a_{\mathrm{IR},4\ell,BA}$	$\delta_{\mathrm{IR},3,4}$
2	3.596e-3	3.585e-3	−2.916e-3
3	6.675e-3	6.598e-3	−1.160e-2
4	0.9389e-2	0.9136e-2	−2.690e-2
5	1.133e-2	1.081e-2	−4.609e-2
6	1.247e-2	1.166e-2	−6.556e-2
7	1.295e-2	1.187e-2	−8.281e-2
8	1.293e-2	1.168e-2	−9.670e-2
9	1.258e-2	1.112e-2	−0.1070
10	1.203e-2	1.066e-2	−0.1142

$a_{\mathrm{IR},3\ell,BA}$. We list this solution in Table III together with the corresponding fractional difference $\delta_{\mathrm{IR},3,4}$, where

$$\delta_{\mathrm{IR},n,n+1} \equiv \frac{a_{\mathrm{IR},(n+1)\ell,BA} - a_{\mathrm{IR},n\ell,BA}}{a_{\mathrm{IR},n\ell,BA}}. \tag{5.6}$$

This agreement of the three-loop and four-loop values of the IR zero of beta in the $\overline{\mathrm{MS}}$ scheme was noted for $N = 3$ in [12], and here it is extended to other values of N. We have tested the robustness of this candidate IR zero by applying a scheme transformation from [30] that produces a simplified beta function with a vanishing three-loop coefficient, $(b_3^{(BA)})' = 0$ in the transformed (primed) scheme, and we find that $a'_{\mathrm{IR},4\ell,BA}$ is very close to $a_{\mathrm{IR},4\ell,BA}$ for general N.

We may also consider the large-N limit of this theory. For this purpose we define the variable

$$\eta \equiv \bar{g}N^2 \tag{5.7}$$

and the corresponding beta function

$$\beta_\eta = \frac{d\eta}{d\ln\mu}. \tag{5.8}$$

This has the series expansion

$$\beta_\eta = \eta \sum_{n=1}^{\infty} \tilde{b}_n^{(BA)} \eta^n, \tag{5.9}$$

where

$$\tilde{b}_n^{(BA)} = \lim_{N\to\infty} \frac{b_n^{(BA)}}{N^{2n}}. \tag{5.10}$$

We have $\tilde{b}_1^{(BA)} = 0$, and

$$\tilde{b}_2^{(BA)} = -\frac{5}{2^6 \cdot 3^2} = 0.868056 \times 10^{-2}, \tag{5.11}$$

$$\tilde{b}_3^{(BA)} = \frac{55}{2^8 \cdot 3^4} = 2.65239 \times 10^{-3}, \tag{5.12}$$

and

$$\tilde{b}_4^{(BA)} = \frac{1}{2^{14} \cdot 3^6}(-298081 - 825120\zeta_3 + 1244160\zeta_5) = 1.52248 \times 10^{-5}. \tag{5.13}$$

The IR zeros of the rescaled three-loop and four-loop beta functions are, respectively,

$$\eta_{\mathrm{IR},3\ell,BA} = \frac{36}{11} = 3.272727 \tag{5.14}$$

and

$$\eta_{\mathrm{IR},4\ell,BA} = \frac{144[-110 + (5(-295661 - 825120\zeta_3 + 1244160\zeta_5))^{1/2}]}{-298081 - 825120\zeta_3 + 1244160\zeta_5} = 3.2134506. \tag{5.15}$$

The fractional difference between these is reasonably small:

$$\frac{\eta_{\mathrm{IR},4\ell,BA} - \eta_{\mathrm{IR},3\ell,BA}}{\eta_{\mathrm{IR},3\ell,BA}} = -(1.8112 \times 10^{-2}). \tag{5.16}$$

This small fractional difference can be understood as a consequence of the fact that $\tilde{b}_4^{(BA)}$ is much smaller than $\tilde{b}_3^{(BA)}$. These results are consistent with the inference that in the $N \to \infty$ limit, this theory has an IR zero in the beta function. However, one must treat this inference with considerable caution, since it involves scheme-dependent beta function terms in an essential way.

VI. CONCLUSIONS

In this work we have investigated whether the beta functions for three ϕ_6^3 theories exhibit robust evidence for zeros away from the origin. The one-component theory is asymptotically free, and has an IR zero at the two-loop level. However, we find that it is not stable to the inclusion

of three-loop and four-loop terms in the beta function, and hence we conclude that there is not persuasive evidence for a robust IR zero in this theory. For the ϕ_6^3 theory with a scalar transforming according to the fundamental representation of a (global) SU(N) symmetry group with $N \geq 3$, we find four different types of renormalization-group behavior, depending on the value of N. In particular, for N (generalized from positive integers to positive real numbers) in the interval $4.47 < N < 9.32$, the theory is asymptotically free and has an IR zero in the two-loop beta function, but we find that it is not stable to the inclusion of higher-loop terms. For N in the interval $3 \leq N < 3.51$, the theory is IR-free and has a UV zero in the two-loop beta function, but we again find that this is not stable to the inclusion of higher-loop terms. In the two other intervals, namely $3.51 < N < 4.47$ and $N > 9.32$, the one-loop and two-loop terms in the beta function have the same sign, so the beta function has no physical zero away from the origin in coupling-constant space. The third ϕ_6^3 theory that we consider features a scalar transforming as a bi-adjoint representation of a global SU(N) $\otimes$ SU(N) symmetry with $N \geq 2$. This theory has the property that the one-loop term in the beta function vanishes and the two-loop term is negative, so the theory is asymptotically free. For this theory, the question of whether the higher-loop beta function has an IR zero cannot be answered in a scheme-independent way, and hence results must be treated with the requisite caution. Nevertheless, we do find that in the $\overline{\text{MS}}$ scheme, the three-loop and four-loop calculations yield values of an IR zero in reasonable agreement with each other.

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