

Integrability and renormalization under $T\bar{T}$

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Smirnov and Zamolodchikov recently introduced a new class of two-dimensional quantum field theories, defined through a differential change of any existing theory by the determinant of the energy-momentum tensor. From this $T\bar{T}$ flow equation one can find a simple expression for both the energy spectrum and the S -matrix of the $T\bar{T}$ deformed theories. Our goal is to find the renormalized Lagrangian of the $T\bar{T}$ deformed theories. In the context of the $T\bar{T}$ deformation of an integrable theory, the deformed theory is also integrable and, correspondingly, the S -matrix factorizes into two-to-two S -matrices. One may thus hope to be able to extract the renormalized Lagrangian from the S -matrix. We do this explicitly for the $T\bar{T}$ deformation of a free massive scalar, to second order in the deformation parameter. Once one has the renormalized Lagrangian one can, in principle, compute all other observables, such as correlation functions. We briefly discuss this, as well as the relation between the renormalized Lagrangian, the $T\bar{T}$ flow equation, and the S -matrix. We also mention a more general class of integrability-preserving deformations of a free scalar field theory.

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I. INTRODUCTION

The bootstrap is a powerful, nonperturbative, method to study quantum field theory. Rather than starting with a specific theory, one starts with a set of consistency relations for the S -matrix that any theory, or any class of theories, must satisfy. Assuming one succeeds in finding a solution, a question arises: what theory, if any, is this the solution of?

In this paper we study this question in the context of integrable two-dimensional field theories, for a specific type of S -matrix. Two-dimensional integrable quantum field theories (QFTs) can be characterized as theories that have no particle production. One can discuss integrability of the classical theory (no particle production at tree level), as well as integrability of the quantum theory; the former does not always imply the latter. In the simplest example of an integrable theory, the sinh-Gordon model, it so happens that the renormalized Lagrangian takes the same functional form as the classical Lagrangian, and hence quantum integrability follows immediately from classical integrability.

Recently, Smirnov and Zamolodchikov [1] introduced a rich new class of integrable two-dimensional theories. They gave both the S -matrix for these theories, as well as the

classical Lagrangian. Our goal is to find the renormalized Lagrangian. Renormalization of these theories is highly nontrivial. Indeed, from the form of the classical Lagrangian, one would say that these theories are non-renormalizable. If these were standard (nonintegrable) QFTs, this would mean that one lacks predictive power in the UV: when renormalizing, the finite part of counterterms must be fixed experimentally, and for non-renormalizable theories there are an infinite number of counterterms. However, these theories are integrable, and this gives an infinite number of constraints which, one may hope, uniquely fix all counterterms.

The central equation in the analysis of Smirnov and Zamolodchikov [1] is the $T\bar{T}$ flow equation. Starting from any Lagrangian, the $T\bar{T}$ deformation generates a one-parameter family of Lagrangians satisfying the equation,

$$\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda} = -4(T_{zz}^\lambda T_{\bar{z}\bar{z}}^\lambda - (T_{z\bar{z}}^\lambda)^2), \quad (1.1)$$

where $\mathcal{L}(\lambda = 0)$ is the original Lagrangian and $T_{\mu\nu}^\lambda$ are the components of the energy-momentum tensor of the finite λ theory. The composite operator on the right-hand side is defined via point splitting. It is important to note that both sides of the equation are renormalized and UV finite; this equation is not an RG flow equation. Rather, it is an equation describing some particular one-parameter family of theories. From the $T\bar{T}$ flow equation, it is simple to derive a differential equation for the energy spectrum as a function of λ , and to derive the λ dependence of the

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S -matrix. It is, however, not simple to find the renormalized Lagrangian; this will be our goal. Our approach will be to first solve the classical version of (1.1). Starting with any theory, one may solve the classical $T\bar{T}$ flow equation perturbatively in λ , using the classical energy-momentum tensor for the right-hand side. The result will be the classical Lagrangian of the $T\bar{T}$ deformation of the original Lagrangian. This Lagrangian must then be renormalized. We find an unambiguous renormalized Lagrangian by demanding that it gives the correct S -matrix: the one required by the (renormalized) version of (1.1). We will do this explicitly to one-loop order for the $T\bar{T}$ deformation of a free scalar.

The paper is organized as follows. In Sec. II we review the $T\bar{T}$ deformation, as well as some elementary aspects of integrability that pertain to it. In Sec. III we consider the $T\bar{T}$ deformation of a free scalar. In Sec. III A we compute the renormalized Lagrangian to one-loop order for the $T\bar{T}$ deformation of a free massless scalar. In Sec. III B we compute the renormalized Lagrangian to one-loop order for the $T\bar{T}$ deformation of a free massive scalar. An interesting result is that the renormalized Lagrangian is qualitatively different from the classical Lagrangian. In Sec. IV we discuss further aspects of $T\bar{T}$. In Sec. IV A we discuss the relation between the $T\bar{T}$ flow equation and the S -matrix. In Sec. IV B we discuss the relation between the renormalized Lagrangian and the $T\bar{T}$ flow equation. In Sec. IV C we discuss the relation between the renormalized Lagrangian and correlation functions under $T\bar{T}$. In Sec. IV D we show that there is a broad class of deformations of a free theory, going beyond $T\bar{T}$, that are integrable, at least classically. In Sec. V we discuss future directions. In Appendixes A and B we collect some useful integrals.

II. INTEGRABILITY

In this section we review some elementary aspects of integrability of two-dimensional quantum field theories [2], as well as some relevant aspects of $T\bar{T}$.

Consider the two-to-two S -matrix in a field theory with a single particle species, of mass m . The energy and momenta of the particles are parametrized by the rapidity θ_i ,

$$E(\theta_i) = m \cosh \theta_i, \quad p(\theta) = m \sinh \theta_i. \quad (2.1)$$

As a result of two-dimensional kinematics, the ingoing momenta are the same as the outgoing momenta. The two-to-two S -matrix is thus only a function of the rapidity difference, $\theta = \theta_1 - \theta_2$, and is denoted by $S(\theta)$. Correspondingly, the Mandelstam variables, in signature $(+, -)$, are

$$\begin{aligned} s &= (p_1 + p_2)^2 = 2m^2(1 + \cosh \theta), \\ t &= 4m^2 - s, \quad u = 0. \end{aligned} \quad (2.2)$$

The t channel corresponds to $\theta \rightarrow i\pi - \theta$, and the u channel corresponds to $\theta = i\pi$.

The assumption of integrability is that there is no particle production: the 2 to n S -matrix, for $n > 2$ is zero. As a result, unitarity, which ordinarily is the inequality $|S(\theta)|^2 \leq 1$, becomes the equality $|S(\theta)|^2 = 1$. Combined with crossing symmetry, the symmetry of the S -matrix under the interchange of the s and t channels, we have the set of equations,

$$|S(\theta)|^2 = 1 \quad \text{and} \quad S(i\pi - \theta) = S(\theta). \quad (2.3)$$

A solution of these equations is the CDD factor,

$$\mathcal{S}_\alpha(\theta) = \frac{\sinh \theta - i \sin \alpha}{\sinh \theta + i \sin \alpha}, \quad (2.4)$$

for any real α . It is easy to see that this is a solution, since $\sinh(i\pi - \theta) = \sinh \theta$, and taking a ratio as above ensures unitarity. A product of $\mathcal{S}_\alpha(\theta)$ over various α , $\prod_\alpha \mathcal{S}_\alpha(\theta)$, is clearly also a solution. A single factor, $\mathcal{S}_\alpha(\theta)$, is the S -matrix for the sinh-Gordon model, where the parameter α is related to the coupling.

A. $T\bar{T}$ flow equation

An interesting question is which theory, if any, has an S -matrix that is some product of the $\mathcal{S}_\alpha(\theta)$. Progress in this direction was recently made by Smirnov and Zamolodchikov [1]. They consider an alternative basis of solutions of (2.3),

$$S'_s(\theta) = \exp(i\lambda_s m^{2s} \sinh(s\theta)), \quad (2.5)$$

where s is an odd integer, and λ_s is a constant of dimension $-2s$, so that $\lambda_s m^{2s}$ is dimensionless. A result of [1] is the following: let $S(\theta)$ be the S -matrix of some integrable theory. Since the theory is integrable, it has an infinite number of conserved currents, $\bar{\partial} T_{s+1} = \partial \Theta_{s-1}$. Suppose one constructs a one-parameter family of theories, depending on the parameter λ_s , that are deformations of this theory, and have a Lagrangian that solves the differential equation,

$$\frac{\partial \mathcal{L}}{\partial \lambda_s} = -4(T_{s+1}^{\lambda_s} \bar{T}_{s+1}^{\lambda_s} - \Theta_{s-1}^{\lambda_s} \bar{\Theta}_{s-1}^{\lambda_s}), \quad (2.6)$$

where $\lambda_s = 0$ corresponds to the original theory, and $T_{s+1}^{\lambda_s}$ and $\Theta_{s-1}^{\lambda_s}$ are the conserved currents of the deformed theory. Then, [1] argue that the deformed theory is integrable, and moreover, has an S -matrix given by $S(\theta)S'_s(\theta)$.

The simplest case (the $T\bar{T}$ deformation) is that of a deformation with $s = 1$, in which case the currents are simply components of the stress tensor, $T_{\mu\nu}$. Since every theory has a stress tensor, the $T\bar{T}$ deformation can be performed on any theory, integrable or not. The differential equation for the λ dependence of the Lagrangian is therefore,

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -4(T^\lambda \bar{T}^\lambda - (T_{z\bar{z}}^\lambda)^2). \quad (2.7)$$

Moreover, consider the simplest subcase of this, in which one deforms a free massive scalar (so that the S -matrix of the initial theory is the identity),

$$\mathcal{L} = 2\partial\phi\bar{\partial}\phi + V(\phi), \quad V = \frac{1}{2}m^2\phi^2, \quad (2.8)$$

where $\partial \equiv \partial_z$ is the derivative with respect to the holomorphic coordinate $z = x_1 + ix_2$, and similarly $\bar{\partial} \equiv \partial_{\bar{z}}$ is the derivative with respect to the antiholomorphic coordinate $\bar{z} = x_1 - ix_2$. The S -matrix of the $T\bar{T}$ deformation of the free scalar is (2.5) with $s = 1$,

$$S(\theta) = \exp(i\lambda m^2 \sinh \theta). \quad (2.9)$$

The classical Lagrangian for the $T\bar{T}$ deformation of a massless scalar was first found by Cavaglià, Negro, Szécsényi, and Tateo [3]. The classical Lagrangian for the $T\bar{T}$ deformation of a scalar theory of the form (2.8), with an arbitrary potential $V(\phi)$, was found in [4] and takes the form

$$\mathcal{L} = \frac{V}{1 - \lambda V} + \frac{-1 + \sqrt{1 + 8\bar{\lambda}\partial\phi\bar{\partial}\phi}}{2\bar{\lambda}}, \quad \bar{\lambda} = \lambda(1 - \lambda V). \quad (2.10)$$

For $\lambda = 0$, this reduces to the starting Lagrangian (2.8). One can explicitly verify that the Lagrangian (2.10), combined with its classical stress tensor, is a solution of the $T\bar{T}$ flow equation (2.7).

We now make several comments:

- (1) An essential aspect of the $T\bar{T}$ deformation is that one knows both how the theory changes, as described by the flow equation (2.7), and how the S -matrix changes, by picking up the phase factor (2.9). Let us briefly review how this connection can be understood. One may notice that in the massless case, (2.10) is the Nambu-Goto action for a long string in three spacetime dimensions, in static gauge. In work predating the study of $T\bar{T}$ in the form initiated by [1], Dubovsky, Flauger, and Gorbenko [5] found the S -matrix for this Nambu-Goto action to be (2.9); see also [6]. Their computation did not explicitly use the Lagrangian. Rather, they appealed to the knowledge of the energy spectrum of a string, and then applied the thermodynamic Bethe ansatz (TBA) equation, which (for integrable theories) relates the energy spectrum to the S -matrix. More generally, if one takes the S -matrix for any integrable theory and multiplies it by the factor (2.9) [7], then one can turn the TBA equation into a differential equation for the energy spectrum as a function of λ [3]. At the same

time, one can start from the definition of the $T\bar{T}$ deformation (2.7) and compute the energy spectrum, as a function of λ , to find the same equation [1,3,8]. This establishes, for integrable theories, that the $T\bar{T}$ deformation (2.7) is equivalent to multiplying the S -matrix by the factor (2.9). In fact, the $T\bar{T}$ deformation is not unique to integrable theories; one can study the $T\bar{T}$ deformation of any theory. The relation between the flow equation (2.7) and the change in the S -matrix in this general case will be discussed in Sec. IV A.

- (2) In stating that the Lagrangian (2.10) is a solution of the $T\bar{T}$ flow equation (2.7), we treated the flow equation as a classical equation. Consequently, (2.10) is the classical Lagrangian. In the study of Smirnov and Zamolodchikov [1], and in particular in the computation of the energy spectrum and the S -matrix, the $T\bar{T}$ flow equation is treated as a quantum equation, with operators on the right-hand side.¹ In particular, the Lagrangian on the left-hand side of the $T\bar{T}$ flow equation is implicitly taken to be the renormalized Lagrangian. Our goal in this paper is to find the renormalized Lagrangian: to renormalize, perturbatively in λ , the classical Lagrangian (2.10). The counterterms are chosen so that (2.9) is the S -matrix of the renormalized Lagrangian. This is what we show in Sec. III, to one-loop level.
- (3) In finding consistent S -matrices through the bootstrap, it is important to not only impose unitarity and crossing, as was done in Eq. (2.3), but also to ensure that the S -matrix has correct analytic behavior. The S -matrices (2.5) do not have correct analytic behavior—they grow exponentially at large imaginary momenta. Such growth is inconsistent with the behavior of a local quantum field theory. However, rather than just discarding these theories, the approach in much of the $T\bar{T}$ literature is to regard them as quantum field theories coupled to gravity [9–13].² The question of if it is correct to interpret the $T\bar{T}$ deformed theories at finite λ as gravitational theories is orthogonal to the focus of this paper: we will work

¹More precisely, the composite operator on the right-hand side is defined by point-splitting, and the right-hand side is defined up to a total derivative (which is irrelevant for the change in the action). The fact that via point-splitting one gets a finite operator is nontrivial; this was shown in [8], and relies crucially on both two dimensions and $T_{\mu\nu}$ being a conserved current.

²The AdS_3 dual of the $T\bar{T}$ deformed theories is correspondingly an interesting question, first studied in [14]. At large N one expects the $T\bar{T}$ deformation to correspond to a change in boundary conditions [15]. See also [16,17] for AdS studies, [18,19] for related developments on $T\bar{T}$ and string theory in AdS_3 , and [20] for studies of dS. Separately, understanding the Hagedorn behavior and partition function of the $T\bar{T}$ theories is an interesting question [1,3,21–23].

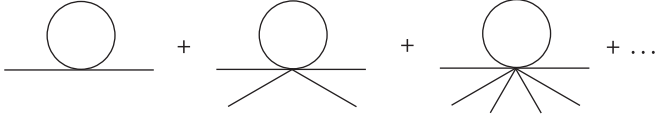


FIG. 1. The only divergent diagrams in the sinh-Gordon model are tadpole diagrams. The combinatorics works out so that their only effect is to renormalize the mass.

perturbatively in λ around $\lambda = 0$, and in this regime these theories are quantum field theories.

We finish this section with a brief review of the one-loop S -matrix in the sinh-Gordon model; this is a good warmup for the one-loop calculation that we will perform in Sec. III.

B. Sinh-Gordon model

As we mentioned, the theory that gives the CDD factor S -matrix in (2.4) is the sinh-Gordon model,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{b^2}(\cosh(b\phi) - 1), \quad \alpha = \frac{\pi b^2}{8\pi + b^2}, \quad (2.11)$$

in which the coupling b is related to the free parameter α in the S -matrix. For imaginary b , this is the sine-Gordon model. In this section we briefly recall some properties of the sinh-Gordon model, as originally discussed in [24].

It is instructive to expand the Lagrangian perturbatively in b ,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{m^2b^2}{4!}\phi^4 + \frac{m^2b^4}{6!}\phi^6 + \dots, \quad (2.12)$$

and to verify, to a few order in b^2 , that (2.4) is in fact the correct S -matrix. An immediate question that arises is that (2.12) is the bare Lagrangian; presumably, we will encounter UV divergences and will need to renormalize the theory. In fact, in two-dimensional theories with nonderivative interactions, all UV divergences are of the trivial tadpole type. In particular, the quartic interaction leads to the divergent tadpole shown on the left side of Fig. 1, which we cancel with a mass counterterm,

$$-\frac{1}{2}\delta m^2\phi^2, \quad \delta m^2 = \frac{1}{2}m^2b^2 \int \frac{d^2p}{(2\pi)^2} G(p), \quad (2.13)$$

where $G(p)$ is the propagator. Similarly, the ϕ^{2n+2} interaction in the Lagrangian requires a counterterm,

$$-\frac{1}{(2n)!}\delta m^2b^{2n-2}\phi^{2n}. \quad (2.14)$$

Summing all these terms, the full set of counterterms at one loop is

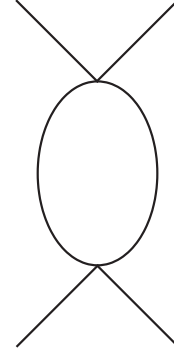


FIG. 2. The bubble diagram contribution to the sinh-Gordon S -matrix.

$$\mathcal{L}_{\text{ctr}} = -\frac{\delta m^2}{b^2}(\cosh(b\phi) - 1). \quad (2.15)$$

Through a redefinition of the mass, the renormalized sinh-Gordon Lagrangian is the same as the classical sinh-Gordon Lagrangian. Thus, the quantum sinh-Gordon model maintains all the integrability properties of the classical theory.

We now turn to the two-to-two S -matrix. For integrable theories, the initial momenta are the same as the final momenta, and so we define the S -matrix $S(\theta)$ by

$$\begin{aligned} \text{out} \langle k_3, k_4 | k_1, k_2 \rangle_{\text{in}} &= S(\theta) (2\pi)^2 2\omega(k_1) 2\omega(k_2) \\ &\times \delta(k_1 - k_4) \delta(k_2 - k_3). \end{aligned} \quad (2.16)$$

With this definition, the S -matrix at zeroth order in the coupling is $S^{(0)}(\theta) = 1$. The S -matrix is related to the scattering amplitude $\mathcal{A}$ through

$$S = \frac{\mathcal{A}}{4m^2 \sinh \theta}. \quad (2.17)$$

At tree level, the amplitude is simply $-im^2b^2$, which gives the S -matrix,

$$S^{(1)}(\theta) = \frac{-ib^2}{4 \sinh \theta}. \quad (2.18)$$

At one loop, the s -channel amplitude, shown in Fig. 2, is

$$\begin{aligned} \mathcal{A}_s &= \frac{-m^4b^4}{2} \int \frac{d^2q}{(2\pi)^2} G(p_1 + p_2 - q) G(q) \\ &= \frac{im^4b^4}{8\pi} \frac{(i\pi - \theta)}{m^2 \sinh \theta}, \end{aligned} \quad (2.19)$$

where the integral is performed in Appendix A, Eq. (A16). Adding also the t and u channels, the one-loop S -matrix is thus

$$S^{(2)}(\theta) = \frac{b^4}{32\pi \sinh \theta} \left(i - \frac{\pi}{\sinh \theta} \right). \quad (2.20)$$

These order b^2 (2.18) and order b^4 (2.20) contributions to the S -matrix agree with the Taylor expansion of the exact S -matrix (2.4). One could in principle proceed to any order.

III. RENORMALIZATION OF $T\bar{T}$ DEFORMED THEORY

We previously wrote the classical Lagrangian for the $T\bar{T}$ deformation of a free massive scalar. In this section we compute the renormalized Lagrangian to order λ^2 , at one-loop level. The classical Lagrangian was given by Eq. (2.10). The first few orders in λ are

$$\mathcal{L} = 2\partial\phi\bar{\partial}\phi + \frac{1}{2}m^2\phi^2 - 4\lambda(\partial\phi\bar{\partial}\phi)^2 + \frac{1}{4}\lambda m^4\phi^4 + 16\lambda^2(\partial\phi\bar{\partial}\phi)^3 + 2\lambda^2m^2(\partial\phi\bar{\partial}\phi)^2\phi^2 + \frac{\lambda^2}{8}m^6\phi^6 + \dots \quad (3.1)$$

The renormalized Lagrangian that we will find is given at the end of the section, see Eq. (3.34).

We will compute the renormalized Lagrangian by taking the classical Lagrangian, using it to compute the S -matrix, and then adding appropriate counterterms so that the S -matrix is given by what the definition of the $T\bar{T}$ deformed theories says it should be, Eq. (2.9). Stated differently, as is standard in quantum field theory, in computing the S -matrix using the classical Lagrangian, we will encounter

UV divergences. We will cancel the divergences through the addition of counterterms. The finite parts of the counterterms, which are usually ambiguous, will be fixed by demanding that integrability be preserved at the quantum level.

For computing the S -matrix, we switch to Lorentzian signature, $x_1 = it$ and $x_2 = x$, so that the holomorphic and antiholomorphic coordinates are, respectively, $z = i\sigma_+$ and $\bar{z} = i\sigma_-$, where $\sigma_{\pm} = t \pm x$. The action (3.1), to order λ^2 , is thus

$$\begin{aligned} -I = - \int dx_1 dx_2 \mathcal{L} = i \int dt dx \left(2\partial_+\phi\partial_-\phi - \frac{1}{2}m^2\phi^2 + 4\lambda(\partial_+\phi\partial_-\phi)^2 - \frac{1}{4}\lambda m^4\phi^4 \right. \\ \left. + 16\lambda^2(\partial_+\phi\partial_-\phi)^3 - 2\lambda^2m^2(\partial_+\phi\partial_-\phi)^2\phi^2 - \frac{\lambda^2}{8}m^6\phi^6 + \dots \right), \end{aligned} \quad (3.2)$$

where we are using the nonstandard convention that ∂_+ denotes $\frac{\partial}{\partial\sigma_+}$. The light-cone momenta are $p_{\pm} = \omega \pm k$. In terms of the rapidity θ_i of particle i , the light-cone momenta are $p_{i,\pm} = me^{\pm\theta_i}$. The Mandelstam variable is $s = p^2$, where $p = p_1 + p_2$, is in light-cone variables, $s = p^2 = p_+p_-$. Finally, the Feynman propagator is

$$\begin{aligned} \langle \phi(x_1)\phi(x_2) \rangle = \int \frac{d^2p}{(2\pi)^2} G(p) e^{-\frac{i}{2}(p_-\sigma_{12,+} + p_+\sigma_{12,-})}, \\ G(p) = \frac{i}{p_+p_- - m^2 + i\epsilon}. \end{aligned} \quad (3.3)$$

In Sec. III A we compute the S -matrix for the special case of zero mass. In Sec. III B we consider the massive theory.

A. Renormalization of $T\bar{T}$ deformation of massless free scalar

In the massless limit, the Lagrangian (2.10) becomes [3]

$$\mathcal{L} = \frac{1}{2\lambda} \left(-1 + \sqrt{1 + 8\lambda\partial\phi\bar{\partial}\phi} \right). \quad (3.4)$$

This is of course the gauge-fixed Nambu-Goto action for a string embedded in three spacetime dimensions, with

embedding coordinates $X^{\mu}(\tau, \sigma)$ given by $X^0 = \tau$, $X^1 = \sigma$, and $X^2 = \sqrt{2\lambda}\phi$.³

To study the S -matrix in the massless limit, we hold s fixed while taking m to zero and θ to infinity. In this limit, the S -matrix (2.9) becomes

$$S = \exp(i\lambda s/2). \quad (3.5)$$

At zero coupling our action is that of a free massless scalar, $\mathcal{L} = 2\partial\phi\bar{\partial}\phi$, with correlation functions, $\langle \phi(z_1)\phi(z_2) \rangle = -\frac{1}{4\pi} \log z_{12}\bar{z}_{12}$. Since the interactions involve only $\partial\phi$ and $\bar{\partial}\phi$, it is convenient to think of our fundamental fields as $\partial\phi$ and $\bar{\partial}\phi$, with propagators given by

³There is an important distinction between this theory and the standard Nambu-Goto theory, defined as the area swept out by a string in $D = 3$ embedding spacetime dimensions. For the Nambu-Goto theory, there is D dimensional Poincare invariance, which must be preserved under quantization. For instance, in the standard quantization of string theory in light-cone gauge, D can be anything; however, the spectrum is only consistent with Poincare symmetry if $D = 26$ (or $D = 3$, which is exceptional [25]). In the context of the $T\bar{T}$ deformation of $D - 2$ free scalars, we have no such restriction, as there was no Poincare symmetry to start with. Physically, the action (3.4) describes small fluctuations around a static long string. The S -matrix for Nambu-Goto was recently studied in [26].

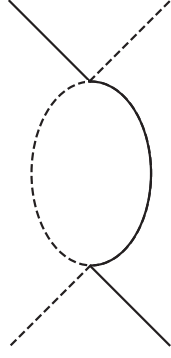


FIG. 3. The one-loop contribution to the S -matrix in the $T\bar{T}$ deformation of a massless free scalar (the gauge-fixed Nambu-Goto theory). Solid lines represent propagators of $\partial\phi$, while dashed lines are propagators of $\bar{\partial}\phi$.

$$\begin{aligned}\langle\partial\phi(z_1)\partial\phi(z_2)\rangle &= -\frac{1}{4\pi}\frac{1}{z_{12}^2}, & \langle\bar{\partial}\phi(z_1)\bar{\partial}\phi(z_2)\rangle &= -\frac{1}{4\pi}\frac{1}{\bar{z}_{12}^2}, \\ \langle\partial\phi(z_1)\bar{\partial}\phi(z_2)\rangle &= \frac{1}{4}\delta^2(x_{12}),\end{aligned}\quad (3.6)$$

where $z_{ij} \equiv z_i - z_j$, and to get the last equation we used that $\bar{\partial}\frac{1}{z} = 2\pi\delta^2(z)$.

Let us now compute the S -matrix, to order λ^2 . The S -matrix is related to the amplitude through $S = \mathcal{A}/2s$. At tree level, the interaction $-4\lambda(\partial\phi\bar{\partial}\phi)^2$ in the Lagrangian gives an amplitude $\mathcal{A} = i\lambda s^2$. Correspondingly, the first order S -matrix is

$$S^{(1)} = i\frac{\lambda}{2}s, \quad (3.7)$$

in agreement with (3.5).

At one-loop level, we must compute the bubble diagram shown in Fig. 3, where each interaction vertex is $-4\lambda(\partial\phi\bar{\partial}\phi)^2$. Since the particles are massless, one of the two incoming particles is left moving, while the other is right moving. This uniquely fixes the contractions between the ingoing and outgoing particles and the $\partial\phi$ and $\bar{\partial}\phi$ that appear in the interaction vertices. Within the loop, along one of the lines we have a propagator for $\partial\phi$, and along the other line a propagator for $\bar{\partial}\phi$.⁴ Thus, the one-loop diagram is, in Euclidean signature,

$$L(p_E^2) = \frac{1}{(4\pi)^2} \int d^2x e^{ip_E \cdot x} \frac{1}{z^2 \bar{z}^2} = \frac{1}{8\pi} \int \frac{dr}{r^3} J_0(|p_E r|), \quad (3.8)$$

where $p = p_1 + p_2$, and the subscript E denotes Euclidean, and $J_0(r)$ is the Bessel function. We first regulate the integral by putting a hard cutoff,

⁴The contribution of the mixed propagator $\langle\partial\phi\bar{\partial}\phi\rangle$ can be ignored, as it leads to a local and divergent contribution to the amplitude.

$r = 1/\Lambda$, then evaluate the integral, and then take $\Lambda \rightarrow \infty$, to get

$$\begin{aligned}L(p_E^2) &= \frac{1}{8\pi} \int_{\Lambda^{-1}}^{\infty} \frac{dr}{r^3} J_0(|p_E r|) \\ &= \frac{\Lambda^2}{16\pi} - \frac{p_E^2}{32\pi} (1 - \gamma + \log 2) + \frac{p_E^2}{64\pi} \log\left(\frac{p_E^2}{\Lambda^2}\right).\end{aligned}\quad (3.9)$$

Converting from Euclidean to Lorentzian signature, using $dt dx = -i dx_1 dx_2$ and $p_E^2 = -p^2$, the s -channel amplitude is thus

$$\begin{aligned}\mathcal{A}_s &= -16\lambda^2 s^2 L(s), \\ L(s) &= -\frac{i\Lambda^2}{16\pi} - \frac{is}{32\pi} (1 - \gamma + \log 2) + \frac{is}{64\pi} \log\left(\frac{-s}{\Lambda^2}\right).\end{aligned}\quad (3.10)$$

Adding the t -channel amplitude (where $t = -s$), and dividing by $2s$ to convert to the S -matrix, the second order S -matrix element is thus

$$S^{(2)} = i\lambda^2 \frac{\Lambda^2}{\pi} s - \frac{\lambda^2}{8} s^2. \quad (3.11)$$

The real piece matches the order λ^2 part of the S -matrix that we want, see Eq. (3.5).⁵ On the other hand, the imaginary and power law divergent piece of $S^{(2)}$ will need to be canceled by a counterterm of the form $(\partial\phi\bar{\partial}\phi)^2$. This corresponds to a fairly trivial renormalization of λ .

1. Quantum effective action

In fact, if instead of the on-shell S -matrix, we study the quantum effective action directly, by integrating out high energy modes, we find a number of interesting counterterms. As we will show, these terms do not contribute to the S -matrix at second order in λ . However, they are necessary in the computation of off-shell quantities, such as correlation functions, and play a role in the calculation of higher order terms in the S -matrix.

Let us introduce an external field configuration $\bar{\phi}$, through a shift of the field, $\phi \rightarrow \phi + \bar{\phi}$. The λ^2 contribution to the effective action is given by the one particle irreducible correlation function of the form

⁵The real piece comes from a $-i\pi$ that comes from the $\log(-s)$ term. Namely, replacing s with $s + i\epsilon$, we have $\log(-s - i\epsilon) = -i\pi + \log(s)$. The $\log(s)$ term gets canceled by the corresponding t -channel contribution.

$$\Gamma_{\text{eff}}^{(2)}(\bar{\phi}) = 8\lambda^2 \int d^2x_1 \int d^2x_2 \langle T\bar{T}(z_1, \bar{z}_1) T\bar{T}(z_2, \bar{z}_2) \rangle_{\text{1PI}}, \quad (3.12)$$

with either $n = 0, 2$, or 4 external legs associated with the background field $\bar{\phi}$. We denote these terms by V_n . Thus, for instance, V_2 is given by

$$V_2 = -\frac{\lambda^2}{\pi^3} \int d^2x_1 \int d^2x_2 \left[\partial\bar{\phi}_1 \partial\bar{\phi}_2 \frac{1}{\bar{z}_{12}^4 \bar{z}_{12}^2} + \text{c.c.} \right], \quad (3.13)$$

where we made use of the correlation functions (3.6), and ϕ_i denotes $\phi(z_i)$. This integral exhibits a UV divergence when the points z_1 and z_2 collide. (In addition, there is a spurious infrared divergence, which is regulated by the decaying external background $\bar{\phi}$.) The UV divergence is cured in the standard way, through the addition of a local counterterm, $I_{\text{c.t.}}$, to the action. In order to obtain the structure of the counterterm, we expand $\bar{\phi}_2$ around z_1 ,

$$V_2 = -\frac{\lambda^2}{3\pi^3} \int d^2x_1 \partial\bar{\phi}_1 \bar{\partial}^3 \partial\bar{\phi}_1 \int d^2x_2 \frac{1}{|z_{12}|^2}, \quad (3.14)$$

and extract the UV divergent part of the x_2 integral. Introducing a sharp UV cutoff Λ yields

$$\int d^2x_2 \frac{1}{|z_{12}|^2} = 2\pi \int_{\Lambda^{-1}} \frac{dr}{r} = 2\pi \log \Lambda + \dots, \quad (3.15)$$

where the ellipsis encode finite contributions. Thus, the divergent part of the two-point vertex takes the form

$$V_2^{\text{div}} = \frac{2\lambda^2}{3\pi^2} \log \Lambda \int d^2x \bar{\partial}\partial\phi(\bar{\partial}\partial)^2\phi, \quad (3.16)$$

where the argument of the logarithm can be made dimensionless by dividing Λ by some IR scale. Similarly, the four-point vertex is given by

$$\begin{aligned} V_4 = 8\lambda^2 \int d^2x_1 \int d^2x_2 & \left[\partial\bar{\phi}_1 \bar{\partial}\bar{\phi}_1 \partial\bar{\phi}_2 \bar{\partial}\bar{\phi}_2 \frac{16}{(4\pi)^2 |z_{12}|^4} \right. \\ & + (\partial\bar{\phi}_1)^2 (\partial\bar{\phi}_2)^2 \frac{2}{(4\pi)^2 \bar{z}_{12}^4} \\ & \left. + (\bar{\partial}\bar{\phi}_1)^2 (\bar{\partial}\bar{\phi}_2)^2 \frac{2}{(4\pi)^2 \bar{z}_{12}^4} \right]. \end{aligned} \quad (3.17)$$

As before, we Taylor expand $\bar{\phi}_2$ around z_1 and integrate over x_2 to get the structure of possible UV divergences. Only one divergent integral, which takes the form (3.15), survives. In the minimal subtraction scheme, we thus end up with the following set of counterterms:

$$\begin{aligned} I_{\text{c.t.}} = V_2^{\text{div}} + V_4^{\text{div}} = \lambda^2 \log \Lambda & \left(\frac{2}{3\pi^2} \int d^2x \bar{\partial}\partial\phi(\bar{\partial}\partial)^2\phi \right. \\ & \left. + \frac{16}{\pi} \int d^2x (\partial\phi\bar{\partial}\phi) \partial\bar{\partial}(\partial\phi\bar{\partial}\phi) \right). \end{aligned} \quad (3.18)$$

In fact, both counterterms are proportional to the equations of motion, $\bar{\partial}\partial\phi = 0$. For the first term this is manifest, while for the second term this requires a few manipulations,

$$\begin{aligned} \partial(\partial\phi\bar{\partial}\phi) \bar{\partial}(\partial\phi\bar{\partial}\phi) & \sim (\partial^2\bar{\partial}\phi)(\partial\phi\bar{\partial}^2\phi) \sim \frac{1}{4} \partial(\partial\phi)^2 \bar{\partial}(\partial\phi)^2 \\ & \sim \frac{1}{4} \bar{\partial}(\partial\phi)^2 \partial(\partial\phi)^2 \sim 0. \end{aligned} \quad (3.19)$$

Here in the first equality we acted with the derivatives inside the parentheses and used the equations of motion. We then regrouped the terms to get the second equality, and then integrated by parts in both directions. The equality with zero then follows by the equations of motion, provided the field is on-shell. The effective action is consistent with our computation of the S -matrix, as these terms do not contribute to the perturbative S -matrix at order λ^2 ; they vanish when contracted with on-shell external particles obeying $\partial\bar{\partial}\phi = 0$.

B. Renormalization of $T\bar{T}$ deformation of massive free scalar

In this section we compute the S -matrix, to order λ^2 , of the $T\bar{T}$ deformation of a free massive scalar. The relevant part of the bare action was given in (3.2).

At tree level, there are two contributions to the amplitude. The first is from the quartic interaction without derivatives, which gives a contribution to the amplitude of $-6i\lambda m^4$. The second is from the quartic interaction with derivatives, which gives a contribution to the amplitude of $2i\lambda m^4(2\cosh^2\theta + 1)$. Converting from the amplitude to the S -matrix via (2.17), the tree level S -matrix is trivially seen to be the order λ piece of (2.9)

$$S^{(1)}(\theta) = i\lambda m^2 \sinh \theta. \quad (3.20)$$

At one loop, there are two contributions to the amplitude. The first comes from the tadpole diagram, and the second comes from the bubble diagram.

1. Quartic tadpole diagrams

From the sextic terms in the Lagrangian, we have one-loop tadpole diagrams that contribute to the S -matrix, see Fig. 4.⁶ From the last three terms in the action (3.2), we get the quartic effective action,

⁶There are also tadpole diagrams that renormalize the mass; these are trivial to remove with mass counterterms, and we simplify notation by taking the mass in the Lagrangian to be the physical mass.

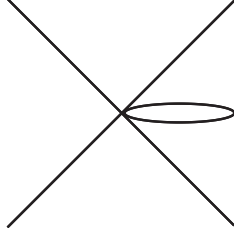


FIG. 4. Tadpole diagram contribution to the S -matrix of the $T\bar{T}$ deformation of a massive free scalar.

$$i\mathcal{L}_{\text{tad}} = \lambda^2 \left(2(\partial_+ \phi \partial_- \phi)^2 (72\langle \partial_+ \phi \partial_- \phi \rangle - m^2 \langle \phi^2 \rangle) - 8m^2 \partial_+ \phi \partial_- \phi \phi^2 \langle \partial_+ \phi \partial_- \phi \rangle - \frac{15}{8} m^6 \phi^4 \langle \phi^2 \rangle \right).$$

The corresponding amplitude is

$$\mathcal{A}_{\text{tad}} = i\lambda^2 m^4 ((2 + \cosh 2\theta)(72\langle \partial_+ \phi \partial_- \phi \rangle - m^2 \langle \phi^2 \rangle) - 16\langle \partial_+ \phi \partial_- \phi \rangle - 45m^2 \langle \phi^2 \rangle). \quad (3.21)$$

Now, using

$$\langle \phi^2 \rangle = \frac{-1}{4\pi} \log \left(\frac{m^2}{\Lambda^2} \right), \quad \langle \partial_+ \phi \partial_- \phi \rangle = \frac{m^2}{4} \langle \phi^2 \rangle - \frac{\Lambda^2}{16\pi}, \quad (3.22)$$

where the first equation is found in Appendix A, Eq. (A6), and the second is immediate after writing $p^2 = p^2 - m^2 + m^2$, we get that the tadpole diagram contribution to the amplitude is

$$\mathcal{A}_{\text{tad}} = \frac{i\lambda^2 m^6}{4\pi} (15 - 17 \cosh 2\theta) \log \left(\frac{m^2}{\Lambda^2} \right) - \frac{i\lambda^2 m^4 \Lambda^2}{2\pi} (16 + 9 \cosh 2\theta). \quad (3.23)$$

2. Bubble diagram

We now compute the contribution of the bubble diagram to the S -matrix. Since there are two quartic interactions in the Lagrangian (3.2), one with derivatives and one without, we need to consider the bubble diagram in three separate cases, depending on which of the two is chosen for each of the vertices, see Fig. 5.

a. No derivative.—If neither vertex has derivatives, then the amplitude is the same as the corresponding one for the sinh-Gordon model, computed in Sec. II B, provided that the coupling $(m^2 b^2/4!)$ is replaced by $\frac{1}{4}\lambda m^4$. This gives

$$\mathcal{A}^{(a)} = \frac{9}{2\pi} \lambda^2 m^6 \left(i - \frac{\pi}{\sinh \theta} \right). \quad (3.24)$$

b. One derivative.—Next, we look at the bubble diagram with one vertex having derivatives. We get for the s -channel amplitude

$$\mathcal{A}_s^{(b)} = -48\lambda^2 m^6 \left(\frac{1}{2} e^{-(\theta_1 + \theta_2)} L_{++} + \frac{1}{2} e^{(\theta_1 + \theta_2)} L_{--} + 2 \cosh \theta L_{-+} \right), \quad (3.25)$$

where $L_{\mu\nu}$ is defined by the integral,

$$L_{\mu\nu} = \frac{1}{4} \int \frac{d^2 k}{(2\pi)^2} \frac{k_\mu}{k^2 - m^2} \frac{(p_\nu - k_\nu)}{(p - k)^2 - m^2}. \quad (3.26)$$

The integral $L_{\mu\nu}$ is computed in Appendix A, see (A18). Combined with the trivial relations, $p_+^2 e^{-(\theta_1 + \theta_2)} = s$ and $p_-^2 e^{\theta_1 + \theta_2} = s$, we get for the amplitude

$$\mathcal{A}_s^{(b)} = \frac{3i\lambda^2 m^6}{\pi} \left(1 - (2 + \cosh 2\theta) \frac{(i\pi - \theta)}{\sinh \theta} + 2 \cosh \theta \log \frac{m^2}{\Lambda^2} \right). \quad (3.27)$$

Adding to this the amplitude in the t channel, $\theta \rightarrow i\pi - \theta$ [note that $\cosh(i\pi - \theta) = -\cosh \theta$ and $\sinh(i\pi - \theta) = \sinh \theta$], and the amplitude in the u channel, $\theta = i\pi$, we get the amplitude

$$\mathcal{A}^{(b)} = \mathcal{A}_s^{(b)} + \mathcal{A}_t^{(b)} + \mathcal{A}_u^{(b)} = -\frac{3i\lambda^2 m^6}{\pi} \left(2 \log \frac{m^2}{\Lambda^2} + i\pi \left(2 \sinh \theta + \frac{3}{\sinh \theta} \right) \right). \quad (3.28)$$

c. Two derivatives.—Finally, we look at the bubble diagram with both vertices having derivatives. We get for the s -channel amplitude

$$\mathcal{A}_s^{(c)} = 16\lambda^2 m^4 \left[\frac{s^2}{p_+^4} L_{++++} + 8 \cosh \theta \frac{s}{p_+^2} L_{-+++} + (1 + 4 \cosh^2 \theta) L_{+--+} + 4 \cosh^2 \theta L_{++--} \right], \quad (3.29)$$

where we defined the integral,

$$L_{\mu\nu\alpha\beta} = \frac{1}{16} \int \frac{d^2 k}{(2\pi)^2} \frac{k_\mu k_\nu}{k^2 - m^2} \frac{(p_\alpha - k_\alpha)(p_\beta - k_\beta)}{(p - k)^2 - m^2}. \quad (3.30)$$

The integral $L_{\mu\nu\alpha\beta}$ is computed in Appendix A, see (A21).

Using this to get $\mathcal{A}_s^{(c)}$, and then adding the s , t , and u channels, $\mathcal{A}^{(c)} = \mathcal{A}_s^{(c)} + \mathcal{A}_t^{(c)} + \mathcal{A}_u^{(c)}$, gives

$$\mathcal{A}^{(c)} = \frac{i\lambda^2 m^6}{12\pi} \left(\pi i \frac{6(2 + \cosh 2\theta)^2}{\sinh \theta} + 8 + 57 \frac{\Lambda^2}{m^2} + 90 \log \frac{m^2}{\Lambda^2} + 8 \cosh 2\theta \left(-2 + 3 \frac{\Lambda^2}{m^2} + 3 \log \frac{m^2}{\Lambda^2} \right) \right). \quad (3.31)$$

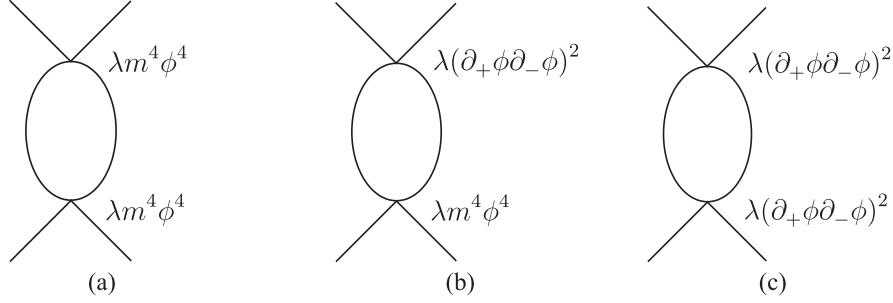


FIG. 5. The contribution of s -channel bubble diagrams to the S -matrix of the $T\bar{T}$ deformation of a massive free scalar. The diagrams are (a) $\mathcal{A}_s^{(a)}$, (b) $\mathcal{A}_s^{(b)}$, (c) $\mathcal{A}_s^{(c)}$.

d. All terms.—The full amplitude coming from the bubble diagram is the sum of these three contributions, $\mathcal{A}_{\text{bub}} = \mathcal{A}^{(a)} + \mathcal{A}^{(b)} + \mathcal{A}^{(c)}$, and is

$$\mathcal{A}_{\text{bub}} = -2\lambda^2 m^6 \sinh^3 \theta + \frac{i\lambda^2 m^6}{12\pi} \left(62 + 57 \frac{\Lambda^2}{m^2} + 18 \log \frac{m^2}{\Lambda^2} \right) + 24 \cosh 2\theta \left(-\frac{2}{3} + \frac{\Lambda^2}{m^2} + \log \frac{m^2}{\Lambda^2} \right). \quad (3.32)$$

3. Renormalized Lagrangian

The total one-loop amplitude is the sum of the contribution $\mathcal{A}_{\text{tad}}$ (3.23) of the tadpole diagrams and the contribution $\mathcal{A}_{\text{bub}}$ (3.32) of the bubble diagrams. Adding these two together, and converting to an S -matrix, we find that the second-order S -matrix is

$$S^{(2)}(\theta) = -\frac{1}{2}\lambda^2 m^4 \sinh^2 \theta - \frac{im^4 \lambda^2}{48\pi \sinh \theta} \left(-46 + 69 \frac{\Lambda^2}{m^2} - 36 \log \frac{m^2}{\Lambda^2} \right) + \left(32 + 60 \frac{\Lambda^2}{m^2} + 54 \log \frac{m^2}{\Lambda^2} \right) \sinh^2 \theta.$$

In fact, as discussed in Appendix A, when evaluating divergent integrals, we discarded finite pieces, so more precisely the order λ^2 piece of the S -matrix is

$$S^{(2)}(\theta) = -\frac{1}{2}\lambda^2 m^4 \sinh^2 \theta - \frac{im^4 \lambda^2}{16\pi \sinh \theta} \left(23 \frac{\Lambda^2}{m^2} - 12 \log \frac{m^2}{\Lambda^2} \right) + \left(20 \frac{\Lambda^2}{m^2} + 18 \log \frac{m^2}{\Lambda^2} \right) \sinh^2 \theta, \quad (3.33)$$

up to finite terms that are of the functional form of those that can be obtained by changing the cutoff Λ in this expression by a finite amount.

Since the linear term in λ in the S -matrix (2.9) is purely imaginary, unitarity completely fixes the real part of the λ^2 term in the S -matrix: it was guaranteed that the real part of $S^{(2)}(\theta)$ would come out to $-\frac{1}{2}\lambda^2 m^4 \sinh^2 \theta$. On the other

hand, unitarity tells us nothing about the imaginary part of $S^{(2)}(\theta)$.

The second order S -matrix that we wanted to get is not (3.33), but rather the order λ^2 piece of (2.9), which contains the same real part but a vanishing imaginary part. We obtained (3.33) as the S -matrix coming from the classical (bare) Lagrangian (3.1). We will now add counterterms to it in order to cancel off the imaginary pieces of (3.33). Our ability to add local counterterms to the Lagrangian in order to get the S -matrix that we want is not a generic property, and is special to the theory being integrable. In particular, generic amplitudes contain logarithms. However, with local counterterms we can only cancel polynomials of the Mandelstam variable s (equivalently, powers of $\sinh \theta$) that may appear in the amplitude; we can never cancel a term involving θ [which involves a logarithm, when expressed in Mandelstam variables, see Appendix A, Eq. (A15)]. Indeed, we found that even though individual contributions to the amplitude had terms involving θ , they canceled from the final amplitude. This is as it should be, since the S -matrix in an integrable theory can only have poles and not branch cuts.

Adding the appropriate counterterms to cancel the imaginary part of the S -matrix, the renormalized Lagrangian is thus

$$\mathcal{L} = 2\partial\phi\bar{\partial}\phi + \frac{1}{2}m^2\phi^2 - 4g(\partial\phi\bar{\partial}\phi)^2 + \frac{1}{4}hm^4\phi^4 + \dots, \quad (3.34)$$

where the renormalized couplings are⁷

⁷The renormalized couplings also have finite pieces; we have not written them, but it is straightforward to include them: in the one-loop integrals that appeared in the computation of the S -matrix, one should simply keep the finite pieces instead of discarding them. If one were to use the renormalized Lagrangian to compute correlation functions, it would be important to have the correct finite pieces. Indeed, our ability to uniquely fix the finite pieces of the counterterms, by matching to the desired S -matrix, is essential. For future calculations, it will be better to use dimensional regularization, rather than the hard cutoff that we have used.

$$g = \lambda + \frac{\lambda^2 m^2}{8\pi} \left(10 \frac{\Lambda^2}{m^2} + 9 \log \frac{m^2}{\Lambda^2} \right), \quad (3.35)$$

$$h = \lambda + \frac{\lambda^2 m^2}{24\pi} \left(7 \frac{\Lambda^2}{m^2} + 39 \log \frac{m^2}{\Lambda^2} \right). \quad (3.36)$$

Already at one-loop level, we see that the renormalized Lagrangian is qualitatively different from the classical Lagrangian. Namely, in the classical Lagrangian (3.1) there is only one scale, λ , and it appears as the coupling of both quartic terms, ϕ^4 as well as $(\partial\phi\bar{\partial}\phi)^2$. In the renormalized Lagrangian, the couplings g and h for these two quartic terms are different. It would be interesting to know what happens at higher orders in λ . In any case, it is clear that quantum integrability gives rise to a more intricate renormalized Lagrangian than the classical Lagrangian coming from classical integrability. This is qualitatively different from what occurs in the sinh-Gordon model where, as we saw in Sec. II B, the renormalized Lagrangian takes the same functional form as the classical Lagrangian.

IV. THE RENORMALIZED LAGRANGIAN, THE $T\bar{T}$ FLOW EQUATION, AND THE S -MATRIX

In Sec. IV A we will start with the $T\bar{T}$ flow equation and derive the S -matrix of the $T\bar{T}$ deformed theory (regardless of if the original theory is integrable or not). In the previous section, we used the S -matrix to get the renormalized Lagrangian for the $T\bar{T}$ deformed theory. In Sec. IV B we discuss general aspects of using the renormalized Lagrangian to verify that the $T\bar{T}$ flow equation is satisfied. This then brings us back full cycle: from the flow equation to the S -matrix to the renormalized Lagrangian and back to the flow equation. In Sec. IV C we discuss how one can use the renormalized Lagrangian to compute correlation functions in the $T\bar{T}$ deformed theory. In Sec. IV D we discuss integrable deformations of a free theory that go beyond the $T\bar{T}$ deformation.

A. From the $T\bar{T}$ flow equation to the S -matrix

In this section we present a derivation for the change in the S -matrix of any theory, integrable or not, under a $T\bar{T}$ deformation.⁸

We consider the n body S -matrix, where for notational simplicity we take all particles to be incoming. Let $S(\theta_i)$ be the S -matrix of some theory. The S -matrix of the $T\bar{T}$ deformed theory will be shown to be

$$S(\theta_i) \exp \left(i\lambda m^2 \frac{1}{2} \sum_{1 \leq i < j \leq n} \sinh(s\theta_{ij}) \right), \quad (4.1)$$

⁸We are grateful to J. Maldacena for suggesting all the key steps in the derivation, as well as very helpful discussions.

where $\theta_{ij} \equiv \theta_i - \theta_j$, and θ_i are assigned to the particles sequentially in order from left to right.⁹ More generally, the change in the S -matrix under a $T_{s+1}\bar{T}_{s+1}$ deformation will be shown to be

$$S(\theta_i) \exp \left(i\lambda m^{2s} \frac{1}{2} \sum_{1 \leq i < j \leq n} \sinh(s\theta_{ij}) \right). \quad (4.2)$$

The currents, written as one-forms, along with the corresponding charges, are

$$\star j = T_{s+1} dz + \Theta_{s-1} d\bar{z}, \quad Q = \int \star j, \quad (4.3)$$

$$\star \tilde{j} = \bar{T}_{s+1} d\bar{z} + \bar{\Theta}_{s-1} dz, \quad \tilde{Q} = \int \star \tilde{j}, \quad (4.4)$$

and the currents are conserved, $d\star j = d\star \tilde{j} = 0$. In terms of these one-forms, the $T_{s+1}\bar{T}_{s+1}$ deformation can be written in a simple form, as the wedge product,

$$(T_{s+1}\bar{T}_{s+1} - \Theta_{s-1}\bar{\Theta}_{s-1}) dz \wedge d\bar{z} = \star j \wedge \star \tilde{j}. \quad (4.5)$$

In order to show that the $T_{s+1}\bar{T}_{s+1}$ deformation, as defined by (2.6), gives rise to the S -matrix (4.2), it is necessary to show that the finite λ theory has the matrix element,

$$\langle \text{out} | \int \star j \wedge \star \tilde{j} | \text{in} \rangle = -\frac{m^{2s}}{8} \left(\sum_{1 \leq i < j \leq n} \sinh(s\theta_{ij}) \right) \langle \text{out} | \text{in} \rangle, \quad (4.6)$$

$\theta_{ij} \equiv \theta_i - \theta_j.$

As we said, we will take the out state to be empty, and the in state to have n particles, each with rapidity θ_i , so that $|\text{in}\rangle = |\theta_1, \dots, \theta_n\rangle$.

It will be convenient to work in Euclidean signature. It is then natural to use the radial coordinate as the time coordinate: we take the contour of integration in (4.3) to be a circle. We let ϕ be the angle around the circle. Furthermore, we define the scalar quantity $\tilde{Q}(\phi)$ to be the integral of the current part way around the circle, up to angle ϕ ,

$$\tilde{Q}(\phi) = \int_0^\phi \star \tilde{j}. \quad (4.7)$$

Notice that $d\tilde{Q}(\phi) = \star \tilde{j}$, and in addition, $\tilde{Q}(\phi = 2\pi) = \tilde{Q}$. Since the current is conserved, we have that $d(\star j \tilde{Q}(\phi)) = \star j d\tilde{Q}(\phi)$. This allows us to rewrite

⁹In the case of two-to-two scattering, taking $\theta_3 = i\pi + \theta_1$ and $\theta_4 = i\pi + \theta_2$ (we added the $i\pi$ to make the particles outgoing) gives back the phase factor discussed before; see Eq. (2.9), given by $\exp(i\lambda m^2 \sinh \theta_{12})$.

$$\int \star j \wedge \star \tilde{j} = \int d(\star j \tilde{Q}(\phi)) = \int_{\partial} \star j \tilde{Q}(\phi), \quad (4.8)$$

where in the last step we applied Stokes's theorem. The boundary ∂ that appears here consists of two disconnected circles: a small circle around the origin (corresponding to a Lorentzian time that is in the far past) and a large circle around the origin (corresponding to a Lorentzian time that is in the far future). Since we have only incoming particles, only the small circle is relevant. The left-hand side of (4.6) is thus

$$\langle \text{out} | \int \star j \wedge \star \tilde{j} | \text{in} \rangle = \langle 0 | \int_{\partial} \star j \tilde{Q}(\phi) | \theta_1, \dots, \theta_n \rangle. \quad (4.9)$$

The integral on the right is an integral over the angle ϕ , around the circle. As we move around the circle, $\star j$ is the current at the angle ϕ , while $\tilde{Q}(\phi)$ defined in (4.7) is the integral of the current from 0 to ϕ . For the moment, suppose we consider a state of a single particle, $|\theta_i\rangle$. It is an eigenstate of the charge operator,

$$Q|\theta_i\rangle = \frac{1}{2} m^s e^{s\theta_i} |\theta_i\rangle, \quad \tilde{Q}|\theta_i\rangle = \frac{1}{2} m^s e^{-s\theta_i} |\theta_i\rangle. \quad (4.10)$$

In the case of $s = 1$, the charges Q and $\tilde{Q}$ are just the holomorphic and antiholomorphic momenta, respectively. We assume that the state of n particles that we have, $|\theta_1, \dots, \theta_n\rangle$, consists of localized particles at separated angles ϕ_i , with the particle at ϕ_i having rapidity θ_i . This state is an eigenstate of the operator $Q(\phi)$,

$$Q(\phi) |\theta_1, \dots, \theta_n\rangle = \frac{1}{2} m^s \sum_{i=1}^k e^{s\theta_i} |\theta_1, \dots, \theta_n\rangle, \quad (4.11)$$

$$\tilde{Q}(\phi) |\theta_1, \dots, \theta_n\rangle = \frac{1}{2} m^s \sum_{i=1}^k e^{-s\theta_i} |\theta_1, \dots, \theta_n\rangle,$$

where k is the maximum index i such that the particle with rapidity θ_i is at an angle ϕ_i for which $\phi_i < \phi$. An illustration is shown in Fig. 6. $Q(\phi)$ is a sequence of step functions, undergoing jumps at ϕ_i . We assume that the value at precisely ϕ_i is given by the average of the value to the left and to the right of ϕ_i ,

$$Q(\phi_i) \equiv \frac{1}{2} (Q(\phi_i - \epsilon) + Q(\phi_i + \epsilon)). \quad (4.12)$$

When we go fully around the circle and get back to $\phi = 0$, we must have that $0 = Q(0) = Q(2\pi)$, when acting on this state, which follows from charge conservation (since we assumed that the out state is empty),

$$\sum_{i=1}^n e^{\pm s\theta_i} = 0. \quad (4.13)$$

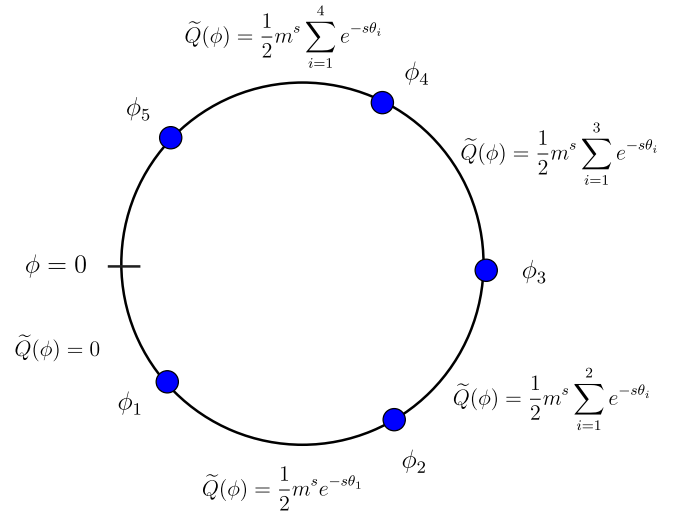


FIG. 6. The circle is a constant time slice, in radial quantization. We have shown a time slice in the far past (small circle) where the particles are well separated, and represented by the blue dots. A particle at angle ϕ_i has rapidity θ_i . $\tilde{Q}(\phi)$ is the integral of the antiholomorphic current around the circle, up to angle ϕ ; it undergoes discrete jumps when passing through the particles.

We may now evaluate (4.14),

$$\begin{aligned} \langle 0 | \int_{\partial} \star j \tilde{Q}(\phi) | \theta_1, \dots, \theta_n \rangle &= \frac{1}{2} m^s \sum_{i=1}^n e^{s\theta_i} \tilde{Q}(\phi_i) = \frac{1}{8} m^{2s} \left(\sum_{1 \leq j < i \leq n} e^{s\theta_{ij}} + \sum_{1 \leq j \leq i \leq n} e^{s\theta_{ij}} \right) \\ &= -\frac{1}{8} m^{2s} \sum_{1 \leq i < j \leq n} \sinh s\theta_{ij}, \end{aligned} \quad (4.14)$$

where in the last step we used that the total charges is zero (4.13), and thus so is the product of the total left charge and total right charge, $0 = \sum_{i,j=1}^n e^{s\theta_{ij}}$. We have thus shown (4.6), and hence (4.2).

Although we took all particles to be incoming, the answer is of course valid if some of the particles are outgoing, which one can achieve by sending θ to $i\pi + \theta$. If one wishes, one can also modify the derivation to make some of the particles outgoing. Letting the index a denote the ingoing particles and the index b denote the outgoing particles, we get a contribution that is like the one we found before, Eq. (4.14), but now with a sum only over the a indices. This came from the small boundary circle. There is also the large boundary circle, as discussed below Eq. (4.8), which gives an additional contribution which is of the form (4.14), but with b indices. Thus, we have

$$-\frac{1}{8} m^{2s} \sum_{a < \bar{a}} \sinh s\theta_{a\bar{a}} - \frac{1}{8} m^{2s} \sum_{b < \bar{b}} \sinh s\theta_{b\bar{b}}. \quad (4.15)$$

Multiplying the two momentum conservation equations, $\sum_a e^{\theta_a} = \sum_b e^{\theta_b}$ and $\sum_b e^{-\theta_b} = \sum_a e^{-\theta_a}$, gives that $\sum_{a,b} \sinh \theta_{ab} = 0$. Hence, (4.15) reproduces (4.14).

The S -matrix “dressing” (4.2), applied to any theory, was first considered in [5,7] (for $s = 1$). This is naturally the S -matrix of a QFT coupled to Jackiw-Teitelboim (JT) gravity [9]. The desired result of this section—that this is also the S -matrix of the $T\bar{T}$ deformed theory—then requires showing that the $T\bar{T}$ deformed theory is the same as the theory coupled to JT gravity [10]. The derivation in this section is more direct, by bypassing JT gravity, and it is also valid for $T_{s+1}\bar{T}_{s+1}$ deformations.

B. From the renormalized Lagrangian to the $T\bar{T}$ flow equation

The $T\bar{T}$ deformed theories are defined by the $T\bar{T}$ flow equation, Eq. (1.1). From the flow equation one can find the S -matrix for the $T\bar{T}$ deformed theories, as we saw in the previous section. Then, from the S -matrix, one may hope to construct the renormalized Lagrangian; we did this explicitly in Sec. III for the $T\bar{T}$ deformation of a free scalar. One thing that remains of interest to verify is that the renormalized Lagrangian satisfies the original $T\bar{T}$ flow equation. In this section, we make some general comments on this. Everything we will say also trivially extends to the higher-spin $T_{s+1}\bar{T}_{s+1}$ deformations described by (2.6).

In our setup, we started with the $T\bar{T}$ flow equation interpreted as a classical equation. The solution of this gave the classical Lagrangian, which was our starting point for constructing the renormalized Lagrangian. It is then no longer obvious how the $T\bar{T}$ flow equation manages to hold as a quantum equation, with the renormalized Lagrangian on the left and renormalized composite operators on the right.

First we note that, as a result of the Ward identity, if one has a conserved current, then the renormalized current can differ from the canonical unrenormalized current only by terms that are themselves conserved. For instance, in the context of ϕ^4 theory in four dimensions, the renormalized stress tensor is, see e.g., [27,28],

$$T^{\mu\nu} = T_0^{\mu\nu} + \eta_0(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\phi_0^2, \quad (4.16)$$

where ϕ_0 is the bare field and $T_0^{\mu\nu}$ is the classical stress tensor,

$$T_0^{\mu\nu} = \partial^\mu\phi_0\partial^\nu\phi_0 - \eta^{\mu\nu}\left(\frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{1}{4!}\lambda_0\phi_0^4\right). \quad (4.17)$$

Notice that the renormalized stress tensor may differ from the classical stress tensor: one must choose a specific η_0 so that $T^{\mu\nu}$ is finite. The improvement term on the right-hand side of (4.16) is itself a conserved quantity, which is why it

appears. There are many other improvement terms; however, these all contain higher derivatives, and as our couplings λ and m are of non-negative dimension, they are excluded on dimensional grounds.

Let us now turn to field theory in two dimensions. In two dimensions, the equation for conservation of the stress tensor is

$$\bar{\partial}T = \partial\Theta. \quad (4.18)$$

We may deform the stress tensor by any quantity that is itself conserved. In particular, we may send $T \rightarrow T - \partial\rho$ and $\Theta \rightarrow \Theta - \bar{\partial}\rho$. Since we need $\Theta = \bar{\Theta}$, we must have $\bar{\partial}\rho = \partial\bar{\rho}$. Provided that this is satisfied, we can choose any ρ , and thereby get a new stress tensor. The renormalized stress tensor must thus take the form

$$T = T_0 - \partial\rho, \quad \Theta = \Theta_0 - \bar{\partial}\rho, \quad (4.19)$$

where T_0 and Θ_0 are the components of the bare stress tensor, and ρ is something that would need to be computed for the specific theory.¹⁰ Since in the $T\bar{T}$ deformed theory the coupling λ has negative dimension, there are an infinite number of things that ρ can potentially be.

In any case, the quantity that we are interested in is the one that appears on the right-hand side of the $T\bar{T}$ flow equation,

$$X = T\bar{T} - \Theta\bar{\Theta}. \quad (4.20)$$

The relation between the bare X_0 and the renormalized X is thus

$$\begin{aligned} X &= (T_0 - \partial\rho)(\bar{T}_0 - \bar{\partial}\bar{\rho}) - (\Theta_0 - \bar{\partial}\rho)^2 \\ &= X_0 + \partial(\bar{Y} + \rho\bar{\partial}\bar{\rho}) + \bar{\partial}(Y - \rho\partial\bar{\rho}), \end{aligned} \quad (4.21)$$

where $Y = \rho\Theta_0 - \bar{\rho}T_0$, and we made use of conservation of the bare stress tensor. Thus, X_0 and X differ by a total derivative, which does not impact the action.

C. From the renormalized Lagrangian to correlation functions

Suppose that we wanted to understand the λ dependence of correlation functions of ϕ for the $T\bar{T}$ deformed theory [29–32]. A straightforward way of proceeding would be to compute the correlators perturbatively in λ . If we were to do this using the bare Lagrangian, then we would expect to

¹⁰An important distinction with the ϕ^4 theory in four dimensions is that, in the $T\bar{T}$ deformed theories, the renormalized Lagrangian will acquire new counterterms that were not present in the classical Lagrangian. So, for the purposes of this paragraph, by “bare stress tensor” we mean a stress tensor computed classically from a Lagrangian, but the Lagrangian is not the classical Lagrangian but rather the classical Lagrangian plus all terms that will be picked up after renormalization.

find UV divergences. We could cancel these by adding counterterms to the Lagrangian; however, since the finite part of the counterterms is ambiguous, it would naively seem that the correlation functions are ambiguous. It is here that the results of Sec. III become important: we fully fixed, to order λ^2 , the renormalized action, by demanding that the corresponding S -matrix be $\exp(i\lambda m^2 \sinh \theta)$. Assuming that we can uniquely fix the renormalized action at arbitrary order in λ , this then gives a way to obtain unique correlation functions. If, instead of correlation functions of ϕ , we are interested in correlation functions of composite operators, then we must, in addition, renormalize the composite operators.

1. Stress-tensor two-point function

In this section we evaluate the two-point function of the stress tensor of the $T\bar{T}$ deformation of a massless scalar, to second order in the coupling λ . This is not a sufficiently

high order in λ for anything interesting to happen since, as discussed in Sec. III A, at second order in λ there are no log divergences. Thus, we expect to be able to get a finite answer by simply evaluating the integrals via analytic continuation, with no need for the addition of counterterms. However, the computation we perform is a necessary warmup for the computation at higher orders in λ .

The free massless scalar has a two-point function $\langle \phi(z_1)\phi(z_2) \rangle = -\frac{1}{4\pi} \log z\bar{z}$ [see Eq. (3.6)]. The stress tensor $T_{\mu\nu}$ has components $T_{zz} \equiv T$, $T_{\bar{z}\bar{z}} \equiv \bar{T}$, and $T_{z\bar{z}}$. For the free scalar, the only nonzero components are $T = (\partial\phi)^2$ and $\bar{T} = (\bar{\partial}\phi)^2$. Trivially, the two-point function is

$$\langle T(z_1)T(z_2) \rangle = \frac{1}{8\pi^2} \frac{1}{z_{12}^4}. \quad (4.22)$$

For the Nambu-Goto theory, from the action (3.4), one finds that the stress tensor is

$$T^\lambda = \frac{(\partial\phi)^2}{\sqrt{1+8\lambda\partial\phi\bar{\partial}\phi}}, \quad \bar{T}^\lambda = \frac{(\bar{\partial}\phi)^2}{\sqrt{1+8\lambda\partial\phi\bar{\partial}\phi}}, \quad T_{z\bar{z}}^\lambda = \frac{\partial\phi\bar{\partial}\phi}{\sqrt{1+8\lambda\partial\phi\bar{\partial}\phi}} - \frac{1}{2}\mathcal{L}. \quad (4.23)$$

In computing the two-point function of the stress tensor, $\langle T_{\mu\nu}^\lambda(z_1)T_{\alpha\beta}^\lambda(z_2) \rangle_\lambda$, perturbatively in λ , we must account for the λ dependence of $T_{\mu\nu}^\lambda$, which we expand in λ , $T_{\mu\nu}^\lambda = T_{\mu\nu} + \lambda T^{(1)} + \dots$, as well as the λ dependence of the action (3.4), which we also expand in λ , $I = I_0 + \lambda I_1 + \dots$. Explicitly, we have

$$\begin{aligned} T^\lambda &= (\partial\phi)^2(1 - 4\lambda\partial\phi\bar{\partial}\phi + \dots), & \bar{T}^\lambda &= (\bar{\partial}\phi)^2(1 - 4\lambda\partial\phi\bar{\partial}\phi + \dots), \\ T_{z\bar{z}}^\lambda &= -2\lambda(\partial\phi\bar{\partial}\phi)^2 + \dots, & I &= \int d^2x(2\partial\phi\bar{\partial}\phi - 4\lambda(\partial\phi\bar{\partial}\phi)^2 + \dots). \end{aligned} \quad (4.24)$$

We now proceed to compute, perturbatively in λ , the two-point function of the stress tensor. At zeroth order in λ , the result is that of the free scalar (4.22), while at first order in λ , all contributions manifestly vanish. Thus, we need to look at the order λ^2 contributions. We start with the simplest components, $\langle T_{z\bar{z}}^\lambda(z_1)T_{z\bar{z}}^\lambda(z_2) \rangle$. Since $T_{z\bar{z}}^\lambda$ starts at order λ , we simply Wick contract to get

$$\langle T_{z\bar{z}}^\lambda(z_1)T_{z\bar{z}}^\lambda(z_2) \rangle_\lambda = \frac{\lambda^2}{16\pi^4} \frac{1}{z_{12}^4 \bar{z}_{12}^4}, \quad (4.25)$$

at order λ^2 . This result was found in [29], and all other components were then obtained from it through use of energy-momentum conservation. However, it is instructive to calculate these directly as well. Proceeding, we next look at $\langle T^\lambda(z_1)\bar{T}^\lambda(z_2) \rangle_\lambda$. The only nonzero contribution is

$$\langle T^\lambda(z_1)\bar{T}^\lambda(z_2) \rangle_\lambda = \frac{\lambda^2}{2} \langle T(z_1)\bar{T}(z_2)I_1I_1 \rangle = \frac{8\lambda^2}{(2\pi)^6} \int d^2x_3 d^2x_4 \frac{1}{z_{13}^2 z_{34}^2 z_{41}^2} \frac{1}{\bar{z}_{23}^2 \bar{z}_{34}^2 \bar{z}_{42}^2} = \frac{\lambda^2}{16\pi^4} \frac{1}{z_{12}^4 \bar{z}_{12}^4}, \quad (4.26)$$

where we used (B12) to evaluate the double integral. Next, we look at $\langle T^\lambda(z_1)T_{z\bar{z}}^\lambda(z_2) \rangle$. The only contribution is

$$\langle T^\lambda(z_1)T_{z\bar{z}}^\lambda(z_2) \rangle_\lambda = 2\lambda \langle (\partial\phi_1)^2 (\partial\phi_2 \bar{\partial}\phi_2)^2 I_1 \rangle = \frac{\lambda^2}{8\pi^5} \frac{1}{z_{12}^2} \int d^2x_3 \frac{1}{z_{13}^2 z_{23}^2 \bar{z}_{23}^4} = -\frac{\lambda^2}{12\pi^4} \frac{1}{z_{12}^5 \bar{z}_{12}^3}, \quad (4.27)$$

where we used (B4) to evaluate the integral.

Finally, we look at $\langle T^\lambda(z_1)T^\lambda(z_2) \rangle_\lambda$. There are three distinct contributions. The first comes from the order λ contribution of T^λ ,

$$\langle T^{(1)}(z_1)T^{(1)}(z_2) \rangle = \frac{6\lambda^2}{(2\pi)^4} \frac{1}{z_{12}^6 \bar{z}_{12}^2}. \quad (4.28)$$

A second contribution comes from the mixing of the order λ part of the stress tensor and the first order in λ part of the action,

$$-\langle T^{(1)}(z_1)T^{(0)}(z_2)I_1 \rangle - (1 \leftrightarrow 2) = -32\lambda^2 \int d^2x_3 \langle (\partial\phi_1)^3 \bar{\partial}\phi_1 (\partial\phi_2)^2 (\partial\phi_3)^2 (\bar{\partial}\phi_3)^2 \rangle = \frac{-12\lambda^2}{(2\pi)^4} \frac{1}{z_{12}^6 \bar{z}_{12}^2}.$$

Here, the only piece that contributes comes from a contactlike term: a contraction $\langle \partial\phi_2 \bar{\partial}\phi_3 \rangle$, which is proportional to a delta function [see Eq. (3.6)]. A third contribution comes from two first-order in λ pieces of the action,

$$\langle T(z_1)T(z_2)I_1I_1 \rangle = \frac{4\lambda^2}{(2\pi)^6} \int d^2x_3 d^2x_4 \frac{1}{\bar{z}_{34}^4} \left(\frac{1}{(z_{13}z_{14}z_{23}z_{24})^2} + \frac{2}{(z_{12}z_{14}z_{23}z_{34})^2} \right) + \frac{6\lambda^2}{(2\pi)^4} \frac{1}{z_{12}^6 \bar{z}_{12}^2},$$

where the second term comes from the contactlike contractions between $\partial\phi$ and $\bar{\partial}\phi$. The integrals in the first term are evaluated using the integrals in Appendix B, giving

$$\int d^2x_3 d^2x_4 \frac{1}{\bar{z}_{34}^4} \frac{1}{(z_{13}z_{14}z_{23}z_{24})^2} = \frac{4\pi^2}{3} \frac{1}{z_{12}^6 \bar{z}_{12}^2}, \quad \int d^2x_3 d^2x_4 \frac{1}{\bar{z}_{34}^4} \frac{1}{(z_{14}z_{23}z_{34})^2} = \frac{\pi^2}{z_{12}^4 \bar{z}_{12}^2}. \quad (4.29)$$

Combining all three contributions, in total we have

$$\langle T^\lambda(z_1)T^\lambda(z_2) \rangle_\lambda = \frac{1}{z_{12}^4} \left(\frac{1}{8\pi^2} + \frac{5\lambda^2}{24\pi^4} \frac{1}{z_{12}^2 \bar{z}_{12}^2} \right). \quad (4.30)$$

This completes the computation of the two-point function of $T_{\mu\nu}$ for the $T\bar{T}$ deformation of a massless scalar. It would be interesting to repeat the computation for the $T\bar{T}$ deformation of a massive scalar. We save this for future work; although we have the renormalized Lagrangian, a necessary ingredient in the computation is, in addition, the renormalized $T_{\mu\nu}$ (for the massless case $T_{\mu\nu}$ does not get renormalized at second order).

D. Further integrable deformations of a free theory

In Sec. IV D 1 we show that there is a broad class of integrable deformations of a free theory. In Sec. IV D 2 we argue that this can be thought of as a consequence of the higher spin $T_{s+1}\bar{T}_{s+1}$ deformations, of which $T\bar{T}$ is a subcase. In Sec. IV D 3 we contrast this with theories with nonderivative interactions, like the sinh-Gordon model, which is the unique integrable theory in its class.

1. $\mathcal{L}=f(\lambda\partial\phi\bar{\partial}\phi)$ is integrable

An important property of $T\bar{T}$ is that the $T\bar{T}$ deformation of an integrable theory is also integrable. An example that we studied in Sec. III A is the $T\bar{T}$ deformation of a free massless scalar, corresponding to a classical action that is the gauge fixed Nambu-Goto action (3.4). In this

section we show that, in fact, any action of the general form

$$\mathcal{L} = f(x), \quad x = \lambda\partial\phi\bar{\partial}\phi, \quad (4.31)$$

where $f(x)$ is any function that is analytic in the vicinity of $x = 0$, is classically integrable.

To show that the theory is integrable, we need to find an infinite number of conserved currents, (T_n, Θ_{n-2}) , which we parametrize as

$$T_n = (\partial\phi)^n t_n(x), \quad \Theta_{n-2} = (\partial\phi)^{n-1} (\bar{\partial}\phi) h_n(x). \quad (4.32)$$

The conservation equation $\bar{\partial}T_n = \partial\Theta_{n-2}$ becomes

$$\begin{aligned} \frac{x}{\lambda} \partial\bar{\partial}\phi [nt_n - h_n + xt'_n - xh'_n] \\ = -(\partial\phi)^2 \bar{\partial}^2 \phi xt'_n + (\bar{\partial}\phi)^2 \partial^2 \phi ((n-1)h_n + xh'_n). \end{aligned} \quad (4.33)$$

We need this to be consistent with the equations of motion, which are given by

$$\partial\bar{\partial}\phi = -\frac{\lambda}{2} \frac{f''(x)}{f'(x) + xf''(x)} ((\partial\phi)^2 \bar{\partial}^2 \phi + (\bar{\partial}\phi)^2 \partial^2 \phi). \quad (4.34)$$

Thus, we need to have that

$$-xt'_n = (n-1)h_n + xh'_n. \quad (4.35)$$

Inserting this into (4.33), we see that consistency with the equations of motion further requires

$$nt_n - h_n + xt'_n - xh'_n = 2 \frac{f'(x) + xf''(x)}{f''(x)}. \quad (4.36)$$

We can turn these two equations, (4.35) and (4.36), into one equation. Solving (4.35) for t_n gives

$$t_n = -h_n + (n-1) \int^x dy \frac{h_n(y)}{y}. \quad (4.37)$$

Inserting this into (4.36) gives an integral equation for h_n ,

$$-xh'_n - nh_n + \frac{1}{2}n(n-1) \int^x dy \frac{h_n(y)}{y} = \frac{f'(x) + xf''(x)}{f''(x)}. \quad (4.38)$$

Given a Lagrangian $\mathcal{L} = f(x)$, this can be explicitly solved to obtain h_n . Therefore, the theory is classically integrable.

2. $T_{s+1}\bar{T}_{s+1}$ deformations of a free massless scalar

In addition to the $T\bar{T}$ deformation, there are also the higher spin $T_{s+1}\bar{T}_{s+1}$ deformations, defined earlier in (2.6).¹¹ These are like the $T\bar{T}$ deformations, but with the higher spin currents, rather than the stress tensor. They too preserve integrability. One may want to view the general integrable theory that we just discussed, $\mathcal{L} = f(x)$, as being formed from a superposition of $T_{s+1}\bar{T}_{s+1}$ deformations of a free massless scalar.

The classical Lagrangian for the $T_{s+1}\bar{T}_{s+1}$ deformation of a free massless scalar is straightforward to find: one solves the $T_{s+1}\bar{T}_{s+1}$ flow equation (2.6) as well as the conservation equation $\bar{\partial}T_n = \partial\Theta_{n-2}$, perturbatively in λ . The resulting Lagrangian is [42]

$$\mathcal{L}_s = \sum_{n=0}^{\infty} c_n \lambda^n (\partial\phi\bar{\partial}\phi)^{n+1}, \quad (4.39)$$

where the coefficients c_n are given recursively through

$$c_n = \sum_{m=0}^{n-1} \frac{2^{n+1}}{(m+1)(m-n)} d_m d_{n-1-m}, \quad (4.40)$$

¹¹Other variants of $T\bar{T}$ include supersymmetric versions [33,34], the $J\bar{T}$ deformation [35], simultaneous $J\bar{T}$ and $T\bar{T}$ deformations [36,37], $T\bar{T}$ in curved space [38], and deformations in quantum mechanics [39]. See also [40,41].

$$d_p = \frac{1}{2p} \sum_{m=1}^p c_m d_{p-m} \times \frac{(1+ms)(4m^2 + 2p(1+p) - m(3+6p+s))}{m-p-1}. \quad (4.41)$$

Explicitly, the first several coefficients are

$$c_0 = 2, \quad c_1 = -4, \quad c_2 = 4(s+1)^2, \quad c_3 = -(7s+3)(s+1)^3. \quad (4.42)$$

For the special case of $s=1$ (the $T\bar{T}$ deformation), this reproduces the Nambu-Goto Lagrangian (3.4) found in [3]. For general s , there does not appear to be a simple closed-form expression for the Lagrangian.

3. Uniqueness of sinh-Gordon

One of the motivations for this section was the well-known property of uniqueness of the sinh-Gordon model. More precisely, within the class of scalar field theories,

$$\mathcal{L} = 2\partial\phi\bar{\partial}\phi + V(\phi), \quad (4.43)$$

with symmetry $\phi \rightarrow -\phi$, the sinh-Gordon model is the unique integrable theory. A simple argument for this is to look at the constraints that the existence of a higher spin conserved current imposes on $V(\phi)$ [43].¹²

The equations of motion are $\partial\bar{\partial}\phi = V'$. Correspondingly, the energy-momentum tensor is conserved,

$$\bar{\partial}T_2 = \partial\Theta_0, \quad T_2 = \frac{1}{2}(\partial\phi)^2, \quad \Theta_0 = V. \quad (4.44)$$

Next, looking at the spin 4 current, it must take the form

$$T_4 = \frac{1}{4}(\partial\phi)^4 + \alpha(\partial^2\phi)^2, \quad (4.45)$$

where α is some parameter. We get Θ_2 by imposing conservation of the current, $\bar{\partial}T_4 = \partial\Theta_2$, which yields

$$\bar{\partial}T_4 = \partial((\partial\phi)^2V) + 2(\partial^2\phi)(\partial\phi)(\alpha V'' - V). \quad (4.46)$$

For this to take the required form, $\bar{\partial}T_4 = \partial\Theta_2$, we must have $(\alpha V'' - V) = 0$, the solution of which is the sinh-Gordon potential.

The reason integrability for Lagrangians of the form $\mathcal{L} = f(\lambda\partial\phi\bar{\partial}\phi)$ is so much less constraining than for Lagrangians of the form (4.43) is clear: for the former,

¹²An alternative is to consider the Taylor expansion of $V(\phi)$ and fix the coefficients by demanding that the tree level 2 to n S-matrix be zero for all $n > 2$ [24,44]. As a practical matter, this is significantly less efficient; see however [45].

unlike the latter, there is a coupling λ of negative dimension, which allows us to have currents (T_n, Θ_{n-2}) for which both components have both holomorphic and antiholomorphic derivatives.

V. DISCUSSION

The perspective in this paper has been the following. Given any quantum field theory, the $T\bar{T}$ flow equation,

$$\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda} = -4(T_{zz}^\lambda T_{\bar{z}\bar{z}}^\lambda - (T_{z\bar{z}}^\lambda)^2), \quad (5.1)$$

gives a one-parameter family of theories, with $\lambda = 0$ corresponding to the original theory. This equation serves as the definition of the $T\bar{T}$ deformed theory. We interpret all quantities in this equation as renormalized and UV finite. From this equation, it is simple to find the λ dependence of certain quantities, such as the energy spectrum and the S -matrix, and (perhaps) difficult to find certain other quantities, such as correlation functions. Our perspective has been that we will use the flow equation (or, more precisely, the S -matrix which follows from it) to find the renormalized Lagrangian of the $T\bar{T}$ deformed theory. Once we have the renormalized Lagrangian we can, in principle, compute all other quantities using standard QFT perturbation theory. Of course, computing with the Lagrangian, or defining the theory through the Lagrangian, is not necessarily optimal. Indeed, for the $T\bar{T}$ theories, the simplicity of the S -matrix and of the flow equation (5.1) suggests that these are ultimately better definitions and starting points for computations. Nevertheless it is, at the very least, conceptually useful to make contact with the Lagrangian.

In Sec. III B we computed, to one-loop order, the renormalized Lagrangian of the $T\bar{T}$ deformation of a free massive scalar. An interesting question is what the Lagrangian looks like to all orders; the simplicity of the S -matrix gives reason to be optimistic that there is a way of writing the Lagrangian so that it too looks simple. Our study has been of the $T\bar{T}$ deformation of an integrable theory. It would be of interest to repeat the calculation for the $T\bar{T}$ deformation of a non-integrable theory. It is not obvious that in this case it is possible to construct a renormalized Lagrangian satisfying the necessary properties.

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APPENDIX A: ONE-LOOP INTEGRALS

In this Appendix we evaluate some one-loop integrals that arise in the computation of the S -matrix of two-dimensional massive theories.

1. Preliminaries

We start with the finite integral,

$$\int \frac{d^2 k}{(2\pi)^2} \frac{1}{(k^2 - m^2)^2} = i \int \frac{d^2 k_E}{(2\pi)^2} \frac{1}{(k_E^2 + m^2)^2} = \frac{i}{4\pi m^2}, \quad (A1)$$

where we Wick rotated to Euclidean signature, $k_0 = ik_0^E$ and $k^2 = -k_E^2$. Next, consider the divergent integral, which we regulate by placing a hard cutoff,

$$\begin{aligned} & \int \frac{d^2 k}{(2\pi)^2} \frac{k^2}{(k^2 - m^2)^2} \\ &= i \int_0^\Lambda \frac{dk_E}{2\pi} \frac{-k_E^3}{(k_E^2 + m^2)^2} \\ &= -\frac{i}{4\pi} \left(-\frac{\Lambda^2}{\Lambda^2 + m^2} + \log \left(\frac{\Lambda^2 + m^2}{m^2} \right) \right). \end{aligned} \quad (A2)$$

The right-hand side is exact; we may simplify it by dropping terms that go to zero as Λ goes to infinity. We will simplify it further, and drop all terms that are finite. We will do the same for all other divergent integrals, keeping only log terms and Λ^2 terms (in fact, there is little point in keeping the Λ^2 terms, but we will do so anyway). Thus, we write the above integral as

$$\int \frac{d^2 k}{(2\pi)^2} \frac{k^2}{(k^2 - m^2)^2} = \frac{i}{4\pi} \log \left(\frac{m^2}{\Lambda^2} \right). \quad (A3)$$

Using Lorentz symmetry we can evaluate the following integral:

$$\int \frac{d^2 k}{(2\pi)^2} \frac{k_\mu k_\nu}{(k^2 - m^2)^2} = \frac{i}{8\pi} \eta_{\mu\nu} \log \left(\frac{m^2}{\Lambda^2} \right). \quad (A4)$$

We will need two additional integrals. The first is

$$\int \frac{d^2k}{(2\pi)^2} \frac{k_\mu k_\nu k_\alpha k_\beta}{(k^2 - m^2)^2} = \frac{i}{32\pi} (\eta_{\mu\nu} \eta_{\alpha\beta} + \eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha}) \times \left(\Lambda^2 + 2m^2 \log \left(\frac{m^2}{\Lambda^2} \right) \right), \quad (\text{A5})$$

and the second is

$$\int \frac{d^2k}{(2\pi)^2} \frac{1}{p^2 - m^2} = \frac{i}{4\pi} \log \left(\frac{m^2}{\Lambda^2} \right). \quad (\text{A6})$$

2. One-loop integrals

We now turn to the one-loop integrals that appear in the calculation of scattering amplitudes. We start with the finite one-loop integral,

$$L_1(p^2) = \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 - m^2} \frac{1}{(p - k)^2 - m^2}. \quad (\text{A7})$$

Introducing Feynman parameters, this becomes

$$L_1(p^2) = \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{1}{[(k - p(1-x))^2 + p^2 x(1-x) - m^2]^2} \quad (\text{A8})$$

$$= \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{1}{[k^2 + p^2 x(1-x) - m^2]^2}, \quad (\text{A9})$$

where in the second equality we shifted the momentum $k \rightarrow k + p(1-x)$. Using (A1) to evaluate the k integral gives

$$L_1(p^2) = \frac{i}{4\pi} \int_0^1 dx \frac{1}{(m^2 - x(1-x)p^2)}. \quad (\text{A10})$$

Through a partial fraction decomposition, one can establish that

$$\int_0^1 dx \frac{1}{A - x(1-x)} = \frac{2}{\sqrt{1-4A}} \log \frac{\sqrt{1-4A} - 1}{\sqrt{1-4A} + 1}. \quad (\text{A11})$$

Thus our integral $L_1(p^2)$ becomes

$$L_1(p^2) = \frac{i}{2\pi p^2 \sqrt{1-4m^2/p^2}} \log \frac{\sqrt{1-4m^2/p^2} - 1}{\sqrt{1-4m^2/p^2} + 1} = \frac{i}{2\pi s \sqrt{-t/s}} \log \frac{\sqrt{-t/s} - 1}{\sqrt{-t/s} + 1}, \quad (\text{A12})$$

where in the second equality we used the Mandelstam variables $s = p^2$ and $t = 4m^2 - p^2$. Let us write this in

terms of θ , which we take to be positive. The Mandelstam variables are

$$s = p^2 = p_+ p_- = 2m^2(1 + \cosh \theta) = 4m^2 \cosh^2 \frac{\theta}{2}, \\ \Rightarrow t = -4m^2 \sinh^2 \frac{\theta}{2}, \quad (\text{A13})$$

and so the expression appearing in $L_1(p^2)$ is

$$\frac{\sqrt{-t/s} - 1}{\sqrt{-t/s} + 1} = -e^{-\theta}, \quad \sqrt{-st} = 2m^2 \sinh \theta. \quad (\text{A14})$$

As a result we have that

$$\frac{1}{\sqrt{-st}} \log \frac{\sqrt{-t/s} - 1}{\sqrt{-t/s} + 1} = \frac{(i\pi - \theta)}{2m^2 \sinh \theta}, \quad (\text{A15})$$

where we used that s is really $s + i\epsilon$, in order to pick the correct sheet. Hence, our integral is

$$L_1(s) = \frac{i}{4\pi m^2} \frac{(i\pi - \theta)}{\sinh \theta}. \quad (\text{A16})$$

Next, we consider the one-loop integral $L_{\mu\nu}$ defined by

$$L_{\mu\nu} = \frac{1}{4} \int \frac{d^2k}{(2\pi)^2} \frac{k_\mu}{k^2 - m^2} \frac{(p_\nu - k_\nu)}{(p - k)^2 - m^2}. \quad (\text{A17})$$

As before, we introduce Feynman parameters, perform the k integral, and then perform the x integral. This gives

$$L_{++} = \frac{i}{16\pi} \frac{p_+^2}{s} \left(-1 + \frac{(i\pi - \theta)}{\sinh \theta} \right), \\ L_{--} = \frac{i}{16\pi} \left(-\log \left(\frac{m^2}{\Lambda^2} \right) + (i\pi - \theta) \frac{\cosh \theta}{\sinh \theta} \right). \quad (\text{A18})$$

In evaluating L_{-+} , to perform the x integral we used

$$\int_0^1 dx \log(A - x(1-x)) \\ = \log A - 2 + \frac{1}{2} \int_0^1 dx \frac{4A - 1}{A - x(1-x)}, \quad (\text{A19})$$

where the integral on the right is just the integral (A11) that appeared earlier. This identity, as well as similar ones, are easily established through integration by parts. Finally, we look at the integral $L_{\mu\nu\alpha\beta}$,

$$L_{\mu\nu\alpha\beta} = \frac{1}{16} \int \frac{d^2k}{(2\pi)^2} \frac{k_\mu k_\nu}{k^2 - m^2} \frac{(p_\alpha - k_\alpha)(p_\beta - k_\beta)}{(p - k)^2 - m^2}. \quad (\text{A20})$$

Evaluating gives

$$\begin{aligned}
L_{++++} &= \frac{i}{64\pi} \frac{p_+^4 m^2}{s^2} \left(-\frac{4}{3} - \frac{\cosh \theta}{3} + \frac{(i\pi - \theta)}{\sinh \theta} \right), \\
L_{-+++} &= \frac{i p_+^2 m^2}{64\pi 3s} \left(1 - 2 \cosh \theta + 3(i\pi - \theta) \frac{\cosh \theta}{\sinh \theta} \right), \\
L_{--++} &= \frac{i m^2}{64\pi} \left(\frac{\Lambda^2}{m^2} - 2 \cosh \theta \log \frac{m^2}{\Lambda^2} + (i\pi - \theta) \frac{\cosh(2\theta)}{\sinh \theta} \right), \\
L_{+--+} &= \frac{i m^2}{64\pi} \left(\frac{\Lambda^2}{m^2} - \frac{1}{3} \cosh \theta + 2 \log \frac{m^2}{\Lambda^2} + \frac{(i\pi - \theta)}{\sinh \theta} \right).
\end{aligned} \tag{A21}$$

APPENDIX B: TWO-DIMENSIONAL MASSLESS INTEGRALS

In this Appendix we record some two-dimensional integrals. In two dimensions, we use the notation $[z]^a$, which denotes

$$\frac{1}{[z]^a} \equiv \frac{1}{z^a \bar{z}^a} = \frac{\bar{z}^{a-\bar{a}}}{|z|^{2a}} = \frac{z^{\bar{a}-a}}{|z|^{2\bar{a}}} = \frac{(-1)^{a-\bar{a}}}{[-z]^a}, \tag{B1}$$

where the difference between a and $\bar{a}$ is an integer, $a - \bar{a} = n$.

1. Two-point integral

The two-point integral is given by

$$\int d^2 x_3 \frac{1}{[z_{13}]^\alpha [z_{32}]^\beta} = \pi (-1)^{\gamma-\bar{\gamma}} C(\alpha) C(\beta) C(\gamma) \frac{1}{[z_{12}]^{\alpha+\beta-1}}, \tag{B2}$$

where $\gamma = 2 - \alpha - \beta$ and $z_{ij} \equiv z_i - z_j$ and

$$C(\alpha) = \frac{\Gamma(1-\bar{\alpha})}{\Gamma(\alpha)} = (-1)^{\alpha-\bar{\alpha}} \frac{\Gamma(1-\alpha)}{\Gamma(\bar{\alpha})}, \tag{B3}$$

where the second equality is a consequence of the difference in α and $\bar{\alpha}$ being an integer. This integral can be easily obtained through Fourier transform for the case of $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$, combined with differentiation for the other cases. A few two-point integrals that we will need in particular are

$$\int d^2 x_3 \frac{1}{z_{13}^2 z_{23}^\beta \bar{z}_{23}^{\beta+2}} = -\pi \frac{\beta}{\beta+1} \frac{1}{z_{12}^{\beta+1} \bar{z}_{12}^{\beta+1}}, \tag{B4}$$

$$\begin{aligned}
\int d^2 x_3 \frac{1}{z_{23}^2 z_{13}^\beta \bar{z}_{13}^\beta} &= -\pi \frac{\beta}{\beta-1} \frac{1}{z_{12}^{\beta+1} \bar{z}_{12}^{\beta-1}}, \\
\int d^2 x_3 \frac{1}{z_{23}^2 z_{13}^\beta \bar{z}_{13}^{\beta+1}} &= -\pi \frac{1}{z_{12}^{\beta+1} \bar{z}_{12}^\beta}.
\end{aligned} \tag{B5}$$

2. Three-point integral

The three-point integral is given by

$$\begin{aligned}
\int d^2 x_4 \frac{1}{[z_{41}]^a [z_{42}]^b [z_{43}]^c} \\
= \frac{1}{[z_{12}]^{\frac{a+b}{2}}} \frac{1}{[z_{13}]^{\frac{a+b+2c-2}{2}}} (A_{a,b,c} f(\chi, \bar{\chi}) + B_{a,b,c} g(\chi, \bar{\chi})),
\end{aligned} \tag{B6}$$

where $\chi = z_{12}/z_{13}$ and

$$A_{a,b,c} = \pi \frac{\Gamma(1-a-b)}{\Gamma(\bar{a}+\bar{b})} \frac{\Gamma(1-c)}{\Gamma(\bar{c})} \frac{\Gamma(\bar{a}+\bar{b}+\bar{c}-1)}{\Gamma(2-a-b-c)}, \tag{B7}$$

$$B_{a,b,c} = \pi \frac{\Gamma(1-a)}{\Gamma(\bar{a})} \frac{\Gamma(1-\bar{b})}{\Gamma(b)} \frac{\Gamma(a+b-1)}{\Gamma(2-\bar{a}-\bar{b})}, \tag{B8}$$

$$\begin{aligned}
f(\chi, \bar{\chi}) &= \chi^{\frac{a+b}{2}} {}_2F_1(b, a+b+c-1, a+b, \chi) \bar{\chi}^{\frac{\bar{a}+\bar{b}}{2}} {}_2F_1 \\
&\times (\bar{b}, \bar{a}+\bar{b}+\bar{c}-1, \bar{a}+\bar{b}, \bar{\chi}),
\end{aligned} \tag{B9}$$

$$\begin{aligned}
g(\chi, \bar{\chi}) &= \chi^{\frac{2-a-b}{2}} {}_2F_1(c, 1-a, 2-a-b, \chi) \bar{\chi}^{\frac{2-\bar{a}-\bar{b}}{2}} {}_2F_1 \\
&\times (\bar{c}, 1-\bar{a}, 2-\bar{a}-\bar{b}, \bar{\chi}).
\end{aligned} \tag{B10}$$

To get this result, we started with the expression for a conformal four-point integral and took one of the points to infinity.¹³ A particular three-point integral that we will need is

$$\int d^2 x_4 \frac{1}{z_{14}^2 z_{24}^2 z_{34}^4} = \frac{2\pi}{3} \frac{1}{z_{12}^3} \left(\frac{1}{z_{23}^3} - \frac{1}{z_{13}^3} \right). \tag{B11}$$

¹³The conformal four-point integral is well known, as it is equivalent to the integral definition of a conformal partial wave, see e.g., [46,47]. An efficient way to evaluate the conformal partial wave integral is by noting that the partial wave is a sum of a conformal block and a shadow block (with coefficients that can be established by taking the $\chi \rightarrow 0$ limit). The two-dimensional conformal block is a product of two one-dimensional conformal blocks, which are trivial to establish. For more details, see for instance [48,49].

3. Double Integral

A particular double integral that we will need is [50]

$$\int d^2x_0 d^2x_1 \frac{[z_{01}]^{\alpha+\beta+\gamma-2}}{[z_0]^\alpha [z_1]^\alpha [1-z_0]^\beta [1-z_1]^\beta} = \pi^2 \cos\left(\frac{\pi}{2}(\alpha - \bar{\alpha} + \beta - \bar{\beta} + \gamma - \bar{\gamma})\right) 2^{\alpha+\bar{\alpha}+\beta+\bar{\beta}+\gamma+\bar{\gamma}-4} \\ \times C(\alpha)C(\beta)C(\gamma)C\left(\frac{3-\alpha-\beta-\gamma}{2}\right)C\left(\frac{2-\alpha-\beta+\gamma}{2}\right)C\left(\frac{2+\alpha-\beta-\gamma}{2}\right)C\left(\frac{2-\alpha+\beta-\gamma}{2}\right),$$

where $C(\alpha)$ was defined in (B3).¹⁴ Taking $\alpha = \bar{\beta} = 2$, $\bar{\alpha} = \beta = 0$, and $\gamma = \bar{\gamma} = -2 + \epsilon$ and sending ϵ to zero gives

$$\int d^2x_3 d^2x_4 \frac{1}{z_{13}^2 z_{34}^2 z_{41}^2} \frac{1}{\bar{z}_{23}^2 \bar{z}_{34}^2 \bar{z}_{42}^2} = \frac{\pi^2}{2} \frac{1}{z_{12}^4 \bar{z}_{12}^4}. \quad (\text{B12})$$

¹⁴An efficient way to evaluate such an integral is to add an auxiliary point, in such a way as to make the integral conformal. The resulting integral, which is similar to ones studied in [51], can then be evaluated through use of the Lorentzian inversion formula [52,53].

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