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Weighted estimates for the Bergman projection on the Hartogs triangle ☆

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We apply modern techniques of dyadic harmonic analysis to obtain sharp estimates for the Bergman projection in weighted Bergman spaces. Our main theorem focuses on the Bergman projection on the Hartogs triangle. The estimates of the operator norm are in terms of a Bekollé-Bonami type constant. As an application of the results obtained, we give, for example, an upper bound for the L^p norm of the Bergman projection on the generalized Hartogs triangle $\mathbb{H}_{m/n}$ in $\mathbb{C}^2$.

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1. Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. Let $L^2(\Omega)$ denote the space of square-integrable functions with respect to the Lebesgue measure dV on Ω. Let $A^2(\Omega)$ denote the subspace of square-integrable holomorphic functions. The Bergman projection P is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$. Associated with P, there is a unique function K_Ω on $\Omega \times \Omega$ such that for any $f \in L^2(\Omega)$:

$$P(f)(z) = \int_\Omega K_\Omega(z;\bar{w})f(w)dV(w). \tag{1.1}$$

Let P^+ denote the positive Bergman projection defined by:

$$P^+(f)(z) := \int_\Omega |K_\Omega(z;\bar{w})|f(w)dV(w). \tag{1.2}$$

A question of importance in analytic function theory and harmonic analysis is to understand the boundedness of P or P^+ on the space $L^p(\Omega, \mu dV)$, where μ is some non-negative locally integrable function on Ω.

For the unweighted case ($\mu \equiv 1$), the L^p boundedness for the Bergman projection has been studied in various settings. On a wide class of domains, the Bergman projection is L^p regularity for all $1 < p < \infty$. See for instance [16,27,21,22,26,22–24,7,15,3]. In all these results, the domain needs to satisfy certain boundary conditions. On some other domains, the projection has only a finite range of mapping regularity. See for example [30,6,13,14,9]. One important example is the Hartogs triangle $\mathbb{H}$. In [6], Chakrabarti and Zeytuncu showed that the Bergman projection on the Hartogs triangle is L^p-regular if and only if $\frac{4}{3} < p < 4$.

Less is known about the situation when the weight $\mu \not\equiv 1$, and results and progress depend upon the domains being studied. For the case of the unit ball in $\mathbb{C}^n$, the boundedness of P and P^+ in the weighted L^p space was studied by Bekollé and Bonami in [5] and [4]. Let T_z denote the Carleson tent over z in the unit ball $\mathbb{B}_n$ defined as below:

- $T_z := \left\{ w \in \mathbb{B}_n : \left| 1 - \bar{w}\frac{z}{|z|} \right| < 1 - |z| \right\}$ for $z \neq 0$, and
- $T_z := \mathbb{B}_n$ for $z = 0$.

Then the result of Bekollé and Bonami can be stated as follows:

Theorem 1.1. *(Bekollé-Bonami) Let the weight $\mu(w)$ be a positive, locally integrable function on the unit ball $\mathbb{B}_n$. Let $1 < p < \infty$. Then the following conditions are equivalent:*

(1) $P : L^p(\mathbb{B}_n, \mu) \mapsto L^p(\mathbb{B}_n, \mu)$ *is bounded.*
(2) $P^+ : L^p(\mathbb{B}_n, \mu) \mapsto L^p(\mathbb{B}_n, \mu)$ *is bounded.*

(3) *The Bekollé-Bonami constant*

$$B_p(\mu) := \sup_{z\in\mathbb{B}_n} \frac{\int_{T_z}\mu(w)dV(w)}{\int_{T_z}dV(w)}\left(\frac{\int_{T_z}\mu^{-\frac{1}{p-1}}(w)dV(w)}{\int_{T_z}dV(w)}\right)^{p-1}$$

is finite.

Motivated by recent developments on the A_2-Conjecture [18] for singular integrals in the setting of Muckenhoupt weighted L^p spaces, people have made progress on the dependence of the operator norm $\|P\|_{L^p(\mathbb{B}_n,\mu)}$ on $B_p(\mu)$. In [28], Pott and Reguera gave a weighted L^p estimate for the Bergman projection on the upper half plane. Their estimates are in terms of the Bekollé-Bonami constant and the upper bound estimate is sharp. Later, Rahm, Tchoundja, and Wick [29] generalized the results of Pott and Reguera to the unit ball case and also obtained estimates for the Berezin transform.

The purpose of this paper is to establish sharp weighted inequalities for the Bergman projection on the Hartogs triangle $\mathbb{H}$. The Hartogs triangle is a bounded pseudoconvex domain defined by $\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$. The boundary $\mathbf{b}\mathbb{H}$ of $\mathbb{H}$ has a serious singularity at the origin, where $\mathbf{b}\mathbb{H}$ cannot be represented as a graph of a continuous function. Partially because of this, $\mathbb{H}$ exhibits many interesting phenomena unseen on smooth domains and serves as a source of counterexamples to many conjectures in several complex variables. The closure $\bar{\mathbb{H}}$ does not have a Stein neighborhood basis. The $\bar\partial$ problem on $\mathbb{H}$ is not global regular [8], i.e. there exists a $\bar\partial$-closed $(0,1)$ form $h \in C^\infty_{0,1}(\bar{\mathbb{H}})$ such that no solution u of the equation $\bar\partial u = h$ is in $C^\infty(\bar{\mathbb{H}})$. The Bergman projection on $\mathbb{H}$ has only limited L^p regularity for $p \in (3/4, 4)$ [6]. This makes even the unweighted L^p norm estimate of the projection interesting.

We give a Bekollé-Bonami type constant and obtain weighted L^p-norm estimates for P and P^+. Recall that the Hartogs triangle $\mathbb{H}$ is defined by

$$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}.$$

$\mathbb{H}$ is biholomorphic to the product domain of the disc and the punctured disc. By the biholomorphic transformation formula, the kernel $K_{\mathbb{H}}(z_1, z_2; \bar w_1, \bar w_2)$ has the following form:

$$K_{\mathbb{H}}(z_1, z_2; \bar w_1, \bar w_2) = \frac{1}{\pi^2 z_2\bar w_2(1-\frac{z_1\bar w_1}{z_2\bar w_2})^2(1-z_2\bar w_2)^2}.$$

Detailed computation for $K_{\mathbb{H}}$ is provided in the next section. Given functions of several variables f and g, we use $f \lesssim g$ to denote that $f \le Cg$ for a constant C. If $f \lesssim g$ and $g \lesssim f$, then we say f is comparable to g and write $f \approx g$. For a weight μ and a subset U in a domain Ω, we set $\mu(U) := \int_U \mu dV$ and let $\langle f\rangle_U^{\mu dV}$ denote the average of the function $|f|$ with respect to the measure μdV on the set U:

$$\langle f\rangle_U^{\mu dV} = \frac{\int_U |f(w_1,w_2)|\mu dV}{\mu(U)}. \tag{1.3}$$

The main result obtained in this paper is:

Theorem 1.2. *Let $1 < p < \infty$, and p' denote the Hölder conjugate to p. Let μ be a positive, locally integrable weight on $\mathbb{H}$ of the form*

$$\mu(z_1,z_2) = \mu_1(z_1/z_2)\mu_2(z_2). \tag{1.4}$$

Set $\nu = |z_2|^{-p'}\mu^{\frac{-p'}{p}}$ and $du = |z_2|^{-2}dV$. Then the Bergman projection P is bounded on the weighted function space $L^p(\mathbb{H},\mu dV)$ if and only if $[\mu,\nu]_p < \infty$.

Moreover, the following quantitative estimate is provided:

$$[\mu,\nu]_p^{\frac{1}{2p}} \lesssim \|P\|_{L^p(\mathbb{H},\mu dV)} \leq \|P^+\|_{L^p(\mathbb{H},\mu dV)} \lesssim ([\mu,\nu]_p^{0,0})^{\frac{1}{p}} + pp'([\mu,\nu]_p^{1,0} + [\mu,\nu]_p^{0,1}) + (pp')^2([\mu,\nu]_p^{1,1})^{\max\{1,\frac{1}{p-1}\}}. \tag{1.5}$$

Here

$$[\mu,\nu]_p = \sup_{z_1,z_2\in\mathbb{D}} \langle \mu|w_2|^{2-p}\rangle_{T'_{z_1,z_2}}^{du}\left(\langle |w_2|^2\nu\rangle_{T'_{z_1,z_2}}^{du}\right)^{p-1}; \tag{1.6}$$

$$[\mu,\nu]_p^{0,0} = \langle |w_2|^{2-p}\mu\rangle_{\mathbb{H}}^{du}\left(\langle \nu|w_2|^2\rangle_{\mathbb{H}}^{du}\right)^{p-1}; \tag{1.7}$$

$$[\mu,\nu]_p^{1,0} = \left(\langle |w_2|^{2-p}\mu_2\rangle_{\mathbb{D}}^{dV}(\langle |w_2|^2\nu_2\rangle_{\mathbb{D}}^{dV})^{p-1}\right)^{\frac{1}{p}}\left(\sup_{\substack{z\in\mathbb{D},\\|z|>1/2}} \langle \mu_1\rangle_{T_z}^{dV}\left(\langle \nu_1\rangle_{T_z}^{dV}\right)^{p-1}\right)^{\max\{1,\frac{1}{p-1}\}}; \tag{1.8}$$

$$[\mu,\nu]_p^{0,1} = \left(\sup_{\substack{z\in\mathbb{D},\\|z|>1/2}} \langle |w_2|^{2-p}\mu_2\rangle_{T_z}^{dV}(\langle |w_2|^2\nu_2\rangle_{T_z}^{dV})^{p-1}\right)^{\max\{1,\frac{1}{p-1}\}}\left(\langle \mu_1\rangle_{\mathbb{D}}^{dV}\left(\langle \nu_1\rangle_{\mathbb{D}}^{dV}\right)^{p-1}\right)^{\frac{1}{p}}; \tag{1.9}$$

$$[\mu,\nu]_p^{1,1} = \sup_{\substack{z_1,z_2\in\mathbb{D}\\|z_1|>1/2,|z_2|>1/2}} \langle \mu|w_2|^{2-p}\rangle_{T'_{z_1,z_2}}^{du}\left(\langle |w_2|^2\nu\rangle_{T'_{z_1,z_2}}^{du}\right)^{p-1}. \tag{1.10}$$

For the definitions of the induced Carleson tents T'_{z_1,z_2} of the Hartogs triangle, see Section 2.

Remark 1.3. The constant $[\mu,\nu]_p$ serves as a natural generalization of the B_p constant for the Hartogs triangle case. It is not hard to see that $[\mu,\nu]_p$ and the upper bound in Theorem 1.2 are qualitatively equivalent, i.e. $[\mu,\nu]_p$ is finite if and only if the sum

$$([\mu,\nu]_p^{0,0})^{\frac{1}{p}} + pp'([\mu,\nu]_p^{1,0} + [\mu,\nu]_p^{0,1}) + (pp')^2([\mu,\nu]_p^{1,1})^{\max\{1,\frac{1}{p-1}\}}$$

is finite. But they are not quantitatively equivalent. More specifically, $[\mu,\nu]_p$ and the upper bound satisfy the following inequalities:

$$\begin{aligned}([\mu,\nu]_p)^{\frac{1}{p}} \lesssim & ([\mu,\nu]_p^{0,0})^{\frac{1}{p}} + pp'([\mu,\nu]_p^{1,0} + [\mu,\nu]_p^{0,1}) + (pp')^2([\mu,\nu]_p^{1,1})^{\max\{1,\frac{1}{p-1}\}} \\ \lesssim & (pp')^2([\mu,\nu]_p)^{\max\{1,\frac{1}{p-1}\}} \end{aligned} \tag{1.11}$$

As one will see in the proof of Theorem 1.2, the products of averages of μ and ν over different tents will have different impacts on the estimate for the weighted norm of the projection P. The constant $[\mu,\nu]_p$ above fails to reflect such a difference, and hence is unable to give the sharp upper bound. This issue did not occur in the upper half plane case [28] since the average over the whole upper half plane is not included in the B_p constant there.

Remark 1.4. In Theorem 1.2, we consider the weight μ of the form as in (1.4) so that the boundedness of the weighted maximal operator in Lemma 2.6 follows by the Fubini's theorem. See Section 5 for further discussion on this assumption. The measure du on $\mathbb{H}$ is induced by the Lebesgue measure on $\mathbb{D}^2$. The weight ν is chosen to be the dual weight of $|z_2|^{-p}\mu$ with respect to the measure du so that a similar argument as in [28] and [29] works for the Hartogs triangle case.

There has been some recent interest in analyzing the L^p regularity properties of the projection via characteristics of the weight. In [10], Chen considered an A_p^+ condition, which is equivalent to the Bekollé-Bonami condition in the upper half plane setting, and obtained the L^p regularity of the weighted Bergman projection with some special weights on the Hartogs triangle. Using the A_p^+ condition, Chen, Krantz, and Yuan [11] obtained the L^p regularity results for the Bergman projections on domains covered by the polydisc through a rational proper holomorphic map. The result of Chakrabarti and Zeytuncu in [6] can be recovered from [10] by showing that the A_p^+ constant of the weight $\mu \equiv 1$ blows up for $p \notin (\frac{4}{3}, 4)$. Similarly, Theorem 1.2 provides another proof for this result.

The approach we employ in this paper is similar to the ones in [28] and [29]. The lower bound estimate follows from a weak-type inequality argument. To obtain the upper bound estimate, we show that P and P^+ are controlled by a positive dyadic operator. Then an analysis on the weighted L^p norm of the dyadic operator yields the desired estimate. Here we use harmonic analysis strategy from [25] and [20]. In particular, we build the dyadic structure on the Hartogs triangle induced by the dyadic structure on the unit disc via the biholomorphism between $\mathbb{H}$ and $\mathbb{D} \times \mathbb{D}^*$. We also use techniques from multi-parameter harmonic analysis to control the induced product structure on the Hartogs triangle. See also the last remark in Section 5. Our upper bound is sharp. In Section 4.1, we provide an example of weights and functions where the sharp bound is attained. As applications of our results, we recover the L^p-regularity results in [6] and

[14] and give upper bound estimates for the L^p-norm of the Bergman projections on the Hartogs triangle $\mathbb{H}$ and the generalized Hartogs triangle $\mathbb{H}_{m/n}$. See Sections 4.2 and 4.4. It is worth noting that the construction of the positive dyadic operator relies on a dyadic structure on the unit disc where the measure of the set in the structure can be used to estimate the Bergman kernel function. Since the dyadic structures on the disc $\mathbb{D}$ and the ball $\mathbb{B}_n$ are well understood, the approach we use in this paper can also be applied to the setting where the domain is related to the unit disc or ball, such as the polydisc, the product of unit balls, and domains that are biholomorphically equivalent to them.

The paper is organized as follows: In Section 2, we introduce a dyadic structure on the unit disc and a corresponding structure on the Hartogs triangle and provide the results that will be used throughout the paper. In Section 3, we present the dyadic operator $Q^+_{m,n,\nu}$ and prove Theorem 1.2. In Section 4, we give a sharp example for our upper bound estimate. We also provide some examples where the upper bound estimates can be explicitly computed. In Section 5, we make several remarks and possible directions for generalization.

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2. Preliminaries

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$. Let $\mathbb{D}^*$ denote the punctured disc $\mathbb{D}\backslash\{0\}$. The Hartogs triangle $\mathbb{H}$ is defined by

$$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}. \tag{2.1}$$

Note that the mapping $(z_1, z_2) \mapsto (\frac{z_1}{z_2}, z_2)$ is a biholomorphism from $\mathbb{H}$ onto $\mathbb{D} \times \mathbb{D}^*$. The biholomorphic transformation formula (see [19]) then implies that

$$\begin{aligned} K_{\mathbb{H}}(z_1, z_2; \bar{w}_1, \bar{w}_2) &= \frac{1}{z_2\bar{w}_2} K_{\mathbb{D}\times\mathbb{D}^*}\left(\frac{z_1}{z_2}, z_2; \frac{\bar{w}_1}{\bar{w}_2}, \bar{w}_2\right) \\ &= \frac{1}{z_2\bar{w}_2} K_{\mathbb{D}\times\mathbb{D}}\left(\frac{z_1}{z_2}, z_2; \frac{\bar{w}_1}{\bar{w}_2}, \bar{w}_2\right) \\ &= \frac{1}{\pi^2 z_2\bar{w}_2(1-\frac{z_1\bar{w}_1}{z_2\bar{w}_2})^2(1-z_2\bar{w}_2)^2}. \end{aligned} \tag{2.2}$$

Hence, the Bergman projection P and the absolute Bergman projection P^+ on the Hartogs triangle can be expressed as follows

$$P(f)(z) = \int_{\mathbb{H}} \frac{f(w)}{\pi^2 z_2\bar{w}_2(1-\frac{z_1\bar{w}_1}{z_2\bar{w}_2})^2(1-z_2\bar{w}_2)^2} dV(w); \tag{2.3}$$

$$P^{+}(f)(z)=\int\limits_{\mathbb{H}}\frac{f(w)}{\pi^2|z_2w_2||1-\frac{z_1\bar{w}_1}{z_2\bar{w}_2}|^2|1-z_2\bar{w}_2|^2}dV(w). \tag{2.4}$$

We next introduce a dyadic structure on the unit disk. A related construction appears in [2]. Let $\mathcal{D}=\{D_j^k\}$ be a dyadic system on the unit circle with

$$D_j^k=\{e^{2\pi i\theta}:(j-1)2^{-k}\leq\theta<j2^{-k}\},\quad\text{for}\quad j=1,\ldots,2^k.$$

Let $d(\cdot,\cdot)$ denote the Bergman metric on the unit disc $\mathbb{D}$. For $z\in\mathbb{D}$, let $B(z,r)$ denote the ball centered at point z with radius r under this metric. Set $r=2^{-1}\ln 2$. For $k\in\mathbb{N}$, let $\mathbb{S}_{kr}$ denote the circle centered at the origin with radius kr in the Bergman metric. Let $\mathcal{P}_{kr}z$ be the radial projection of z onto the sphere $\mathbb{S}_{Nr}$. By the proof of [29, Lemma 9], $\{\mathcal{P}_{k\theta}D_j^k\}$ satisfy the following three properties:

(1) $\mathbb{S}_{kr}=\cup_{j=1}^{2^k}\mathcal{P}_{kr}D_j^k$;
(2) $\mathcal{P}_{kr}D_j^k\cap\mathcal{P}_{kr}D_i^k=\emptyset$ for $i\neq j$;
(3) For $w_j^k=\mathcal{P}_{kr}e^{2\pi i(j-\frac{1}{2})2^{-k}}$, $\mathbb{S}_{kr}\cap B(w_j^k,\lambda)\subseteq\mathcal{P}_{kr}D_j^k\subseteq\mathbb{S}_{kr}\cap B(w_j^k,C\lambda)$.

Define subsets, K_j^k of $\mathbb{D}$ to be:

$$K_1^0:=\{z\in\mathbb{D}:d(0,z)<r\};$$
$$K_j^k:=\{z\in\mathbb{D}:kr\leq d(0,z)<(k+1)r\ \text{ and }\ \mathcal{P}_{kr}z\in\mathcal{P}_{kr}D_j^k\},k\geq 1,j\geq 1.$$

For $k=0$ and $j=1$, set $c_1^0\in K_1^0$ to be the origin. For $k\geq 1$, set $c_j^k\in K_j^k$ to be the point $\mathcal{P}_{(k+\frac{1}{2})r}w_j^k$. For $\alpha=c_j^k$, the set $K_\alpha:=K_j^k$ is referred to as a kube and the point $\alpha=c_j^k$ is the center of the kube. We define a Bergman tree structure $\mathcal{T}:=\{c_j^k\}$ on centers of the kubes. We say that c_i^{k+1} is a child of c_j^k if $\mathcal{P}_{kr}D_i^{k+1}\subseteq\mathcal{P}_{kr}D_j^k$. We say $c_i^m\geq c_j^k$ if $m\geq k$ and $\mathcal{P}_{kr}c_i^m\in\mathcal{P}_{kr}D_j^k$. We define $\hat{K}_\alpha$ to be the dyadic tent under K_α:

$$\hat{K}_\alpha:=\bigcup_{\beta\in\mathcal{T}:\beta\geq\alpha}K_\beta. \tag{2.5}$$

For $z\in\mathbb{D}$, we say the generation $\text{gen}(z)=N$ if $z\in K_j^N$ for some j.

Using shifted dyadic systems $\mathcal{D}_l=\{D_j^k(l)\}$ on the unit circle with

$$D_j^k(l)=\{e^{2\pi i\theta}:(j-1)2^{-k}+l\leq\theta<j2^{-k}+l\},\quad\text{for}\quad j=1,\ldots,2^k\text{ and }l\in\mathbb{R},$$

one can obtain different dyadic structures on $\mathbb{D}$ with their corresponding Bergman trees $\mathcal{T}_l$. Recall the Carleson tent T_z over $z\in\mathbb{D}$:

- $T_z:=\left\{w\in\mathbb{D}:\left|1-\bar{w}\frac{z}{|z|}\right|<1-|z|\right\}$ for $z\neq 0$, and
- $T_z:=\mathbb{D}$ for $z=0$.

For a subset U, we use the notation $|U|$ to denote the Lebesgue measure of U. The following three lemmas relate the Carleson tent T_z to the dyadic tent $\hat{K}_\alpha$ and the Bergman kernel function on $\mathbb{D}$.

Lemma 2.1. *Let $\mathcal{T}$ be a Bergman tree constructed as above. For $\alpha \in \mathcal{T}$,*

$$|T_\alpha| \approx |\hat{K}_\alpha| \approx |K_\alpha| \approx (1 - |\alpha|)^2.$$

Proof. Suppose $\text{gen}(\alpha) = k$. Let R_{kr} denote the Euclidean distance between $\mathbb{S}_{kr}$ and the origin. Then $|\hat{K}_\alpha| = \pi 2^{-k}(1 - R_{kr}^2)$ and $|K_\alpha| = \pi 2^{-k}(R_{(k+1)r}^2 - R_{kr}^2)$. Recall that $r = 2^{-1}\ln 2$. By the definition of the Bergman distance, $1 - R_{kr} \approx e^{-2kr} = 2^{-k}$. Thus $|\hat{K}_\alpha| \approx |K_\alpha| \approx 2^{-2k}$. Since α is the center of the kube K_α, the Bergman distance $d(0, \alpha) = (k + \frac{1}{2})r$. Hence we obtain

$$(1 - |\alpha|)^2 = (1 - R_{(k+\frac{1}{2})r})^2 \approx 2^{-2(k+\frac{1}{2})} \approx |\hat{K}_\alpha| \approx |K_\alpha|.$$

Notice that the Carleson tent T_α is the intersection set of the unit disc $\mathbb{D}$ and the disc centered at the point $\frac{z}{|z|}$ with Euclidean radius $1 - |\alpha|$. A geometric consideration then yields

$$|T_\alpha| \approx (1 - |\alpha|)^2. \quad \square$$

Lemma 2.2 *([29, Lemma 9]). There is a finite collection of Bergman trees $\{\mathcal{T}_l\}_{l=1}^N$ such that for all $\alpha \in \mathbb{D}$, there is a tree $\mathcal{T}$ from the finite collection and an $\beta \in \mathcal{T}$ such that the dyadic tent $\hat{K}_\beta$ contains the tent T_α and $\sigma(\hat{K}_\beta) \approx |T_\alpha|$.*

Lemma 2.3 *([29, Lemma 15]). For $z, w \in \mathbb{D}$, there is a Carleson tent, T_α, containing z and w such that*

$$|T_\alpha| \approx |1 - z\bar{w}|^2 = \pi^{-1}|K_{\mathbb{D}}(z, \bar{w})|^{-1}. \tag{2.6}$$

Lemma 2.4. *For any dyadic tent $\hat{K}_\beta$ with $\beta \in \mathcal{T}_l$ for some l, there exists a Carleson tent T_z such that $\hat{K}_\beta \subseteq T_z$ and $|\hat{K}_\beta| \approx |T_z|$.*

Proof. Given a dyadic tent $\hat{K}_\beta$, we can find a Carleson tent T_z such that $\hat{K}_\beta$ is a largest dyadic tent in T_z. Without loss of generality, we may assume that z is a positive real number. By Lemma 2.1, $|\hat{K}_\alpha| \approx |K_\alpha|$. It suffices to show that the top kube K_β of the tent $\hat{K}_\beta$ satisfies the inequality $|K_\beta| \approx |T_z|$. Since K_β is a largest kube contained in T_z, all of its ancestors are not contained in T_z. Let k be the generation $\text{gen}(\beta)$ of β. Then T_z intersects with at most two of the Borel subsets $\{Q_j^{k-1}\}_{j=1}^{2^{k-1}}$ of $S_{(k-1)\theta}$. Let $R_{(k-1)r}$ denote the Euclidean distance between $\mathbb{S}_{(k-1)r}$ and the origin. The arc length of the set $P_{(k-1)r}D_j^{k-1}$ equals $R_{(k-1)r}2\pi 2^{1-k}$. Thus the arc length of the intersection set $\mathbb{S}_{(k-1)r} \cap T_t$ is less than $2R_{(k-1)r}2\pi 2^{1-k}$. Note that the point z is a positive real number.

T_z is symmetric about the real number axis. Therefore the point $R_{(k-1)r}e^{2\pi i 2^{1-k}}$ is not in T_z, i.e.

$$|1 - R_{(k-1)r}e^{2\pi i 2^{1-k}}| \geq 1 - z.$$

Since $1 - R_{Nt} \approx e^{-2Nt}$ and $|1 - e^{2\pi it}| \approx t$ for $t \in \mathbb{R}$, we have

$$\begin{aligned}|1 - R_{(k-1)r}e^{2\pi i 2^{1-k}}| &\leq |1 - R_{(k-1)r}| + |R_{(k-1)r} - r_{(k-1)\theta}e^{2\pi i 2^{1-k}}| \\ &\approx e^{-2(k-1)r}(1 + 2^{1-k}) = e^{-(k-1)\ln 2}(1 + 2^{1-k}) \approx 2^{-(k-1)}.\end{aligned}$$

Hence $2^{-(k-1)} \gtrsim 1 - z = 1 - |z|$. Lemma 2.1 then implies that $|T_z| \lesssim 2^{-2(k-1)}$. Since $\mathrm{gen}(\beta) = k$, the Bergman distance $d(\beta, 0)$ equals $(k + \frac{1}{2})r$. Recall that $r = 2^{-1}\ln 2$. We have

$$1 - |\beta| \approx e^{-2(k+\frac{1}{2})\theta} = 2^{-(k+\frac{1}{2})}.$$

Applying Lemma 2.1 again yields $|K_\beta| \approx 2^{-2(k+\frac{1}{2})} \gtrsim |T_z|$. By the containment $K_\beta \subseteq T_z$, there holds $|K_\beta| \leq |T_z|$. Combining these inequalities, we conclude that $|K_\beta| \approx |T_z|$ and the proof is complete. □

Combining Lemmas 2.2 and 2.3, we obtain the following estimate for arbitrary $z, w \in \mathbb{D}$:

$$|1 - z\bar{w}|^{-2} \approx |T_\alpha|^{-1} \approx |\hat{K}_\beta|^{-1} \leq \sum_{m=1}^{M} \sum_{\gamma \in \mathcal{T}_m} \frac{1_{\hat{K}_\gamma}(z) 1_{\hat{K}_\gamma}(w)}{|\hat{K}_\gamma|}. \tag{2.7}$$

Here $\{\mathcal{T}_m\}_{m=1}^M$ is the finite collection in Lemma 2.2.

Similarly, on the bidisk, $\mathbb{D}^2$, we have:

$$\begin{aligned}&|1 - z_1\bar{w}_1|^{-2}|1 - z_2\bar{w}_2|^{-2} \\ \approx &|T_{\alpha_1}|^{-1}|T_{\alpha_2}|^{-1} \\ \approx &|\hat{K}_{\beta_1}|^{-1}|\hat{K}_{\beta_2}|^{-1} \\ \leq &\sum_{m,n=1}^{M} \sum_{\gamma \in \mathcal{T}_m, \eta \in \mathcal{T}_n} \frac{1_{\hat{K}_\gamma \times \hat{K}_\eta}(z_1, z_2) 1_{\hat{K}_\gamma \times \hat{K}_\eta}(w_1, w_2)}{|\hat{K}_\gamma \times \hat{K}_\eta|}.\end{aligned} \tag{2.8}$$

Given a tree structure $\mathcal{T}_m \times \mathcal{T}_n$ on $\mathbb{D}^2$ and a dyadic tent $\hat{K}_{\beta_1} \times \hat{K}_{\beta_2}$ we define the induced tree structure $\mathcal{T}'_{m,n}$ and dyadic tent $\hat{K}'_{\beta_1,\beta_2}$ on $\mathbb{H}$ to be:

$$\mathcal{T}'_{m,n} := \left\{(c_1, c_2) \in \mathbb{H} : \left(\frac{c_1}{c_2}, c_2\right) \in \mathcal{T}_m \times \mathcal{T}_n\right\}, \tag{2.9}$$

$$\hat{K}'_{\beta_1,\beta_2} := \left\{(z_1, z_2) \in \mathbb{H} : \left(\frac{z_1}{z_2}, z_2\right) \in \hat{K}_{\beta_1} \times \hat{K}_{\beta_2}\right\}. \tag{2.10}$$

Similarly the induced Carleson tent T'_{z_1,z_2} on $\mathbb{H}$ can be defined by

$$T'_{z_1,z_2} := \{(w_1, w_2) \in \mathbb{H} : \left(\frac{w_1}{w_2}, w_2\right) \in T_{z_1} \times T_{z_2}\}. \tag{2.11}$$

Set $du = |w_2|^{-2}dV$. For a weight μ and a subset $U \subseteq \mathbb{H}$, we set $\mu(U) := \int_U \mu dV$ and let $\langle f\rangle_U^{\mu dV}$ denote the average of the function $|f|$ with respect to the measure μdV on the set U:

$$\langle f\rangle_U^{\mu dV} = \frac{\int_U |f(w_1, w_2)|\mu dV}{\mu(U)}. \tag{2.12}$$

Given weights μ on $\mathbb{H}$ and $\nu = |z_2|^{-p'}\mu^{-p'/p}$, we define the characteristic of two weights μ, ν to be

$$[\mu, \nu]_p := \sup_{z_1,z_2 \in \mathbb{D}} \langle \mu|w_2|^{2-p}\rangle_{T'_{z_1,z_2}}^{du} \left(\langle |w_2|^2\nu\rangle_{T'_{z_1,z_2}}^{du}\right)^{p-1}. \tag{2.13}$$

By Lemmas 2.2 and 2.4, we can replace T'_{z_1,z_2} by $\hat{K}'_{\gamma,\eta}$ to obtain a quantity of comparable size:

$$[\mu, \nu]_p \approx \sup_{1\le m,n\le M} \sup_{(\gamma,\eta)\in\mathcal{T}'_{m,n}} \langle \mu|w_2|^{2-p}\rangle_{\hat{K}'_{\gamma,\eta}}^{du} \left(\langle |w_2|^2\nu\rangle_{\hat{K}'_{\gamma,\eta}}^{du}\right)^{p-1}. \tag{2.14}$$

From now on, we will abuse the notations $[\mu, \nu]_p$ and $[\mu, \nu]_p^{i,j}$ for $i, j = 0, 1$ to represent both the supremum in T'_{z_1,z_2} and the supremum in the corresponding $\hat{K}'_{\gamma,\eta}$ of similar size.

The proof of Theorem 1.2 will use the weighted strong maximal function on $\mathbb{H}$.

Definition 2.5. For a weight μ, and a Bergman tree $\mathcal{T}'_{m,n}$, we define the following maximal function:

$$\mathcal{M}_{\mathcal{T}'_{m,n},\mu} f(w_1, w_2) := \sup_{(\beta_1,\beta_2)\in\mathcal{T}_m\times\mathcal{T}_n} \frac{1_{\hat{K}'_{\beta_1,\beta_2}}(w_1, w_2)}{\mu(\hat{K}'_{\beta_1,\beta_2})} \int_{\hat{K}'_{\beta_1,\beta_2}} |f(z_1, z_2)|\mu(z_1, z_2)dV(z_1, z_2). \tag{2.15}$$

We set $\langle f\rangle_{Q,\mu} := \frac{\int_Q |f|d\mu}{\mu(Q)}$, then we also have:

$$\mathcal{M}_{\mathcal{T}'_{m,n},\mu} f(w_1, w_2) = \sup_{(\beta_1,\beta_2)\in\mathcal{T}_m\times\mathcal{T}_n} 1_{\hat{K}'_{\beta_1,\beta_2}}(w_1, w_2)\langle f\rangle_{\hat{K}'_{\beta_1,\beta_2},\mu}. \tag{2.16}$$

We have the following L^p regularity result for $\mathcal{M}_{\mathcal{T}'_{m,n},\mu}$.

Lemma 2.6. *Let $\mu(z_1, z_2)$ the same as in Theorem 1.2, then $\mathcal{M}_{\mathcal{T}'_{m,n},\mu}$ is bounded on $L^p(\mathbb{H}, \mu)$ for $1 < p \le \infty$. Moreover, $\|\mathcal{M}_{\mathcal{T}'_{m,n},\mu}\|_{L^p(\mathbb{H},\mu)} \lesssim (p/(p-1))^2$ for $1 < p < \infty$.*

Proof. When $p = \infty$, the boundedness of $\mathcal{M}_{\mathcal{T}'_{m,n},\mu}$ is obvious. We turn to the case $1 < p < \infty$. Set $\mu_2'(w_2) := |w_2|^2\mu_2(w_2)$. Using the biholomorphism $h : (w_1, w_2) \mapsto (w_1w_2, w_2)$ from $\mathbb{D} \times \mathbb{D}^*$ onto $\mathbb{H}$, we transform $\mathcal{M}_{\mathcal{T}'_{m,n},\mu}$ into the following maximal function on $\mathbb{D} \times \mathbb{D}^*$:

$$\mathcal{M}_{\mathcal{T}_{m,n},\mu} f(w_1, w_2) := \sup_{(\beta_1,\beta_2)\in\mathcal{T}_m\times\mathcal{T}_n} \frac{1_{\hat{K}_{\beta_1}}(w_1)1_{\hat{K}_{\beta_2}}(w_2)}{\mu_1(\hat{K}_{\beta_1})\mu_2'(\hat{K}_{\beta_2})} \times \int_{\hat{K}_{\beta_1,\beta_2}} |f(z_1, z_2)|\mu_1(z_1)\mu_2'(z_2)dV(z_1, z_2), \tag{2.17}$$

and it suffices to show that $\mathcal{M}_{\mathcal{T}_{m,n},\mu}$ is L^p bounded on $L^p(\mathbb{D} \times \mathbb{D}^*, |w_2|^2\mu \circ h)$ for $1 < p \le \infty$. Defining the following two 1-parameter maximal functions:

$$\mathcal{M}_{\mathcal{T}_m,\mu_1} f(w_1, w_2) := \sup_{\beta_1\in\mathcal{T}_m} \frac{1_{\hat{K}_{\beta_1}}(w_1)}{\mu_1(\hat{K}_{\beta_1})} \int_{\hat{K}_{\beta_1}} |f(z_1, w_2)|\mu_1(z_1)dV(z_1); \tag{2.18}$$

$$\mathcal{M}_{\mathcal{T}_n,\mu_2'} f(w_1, w_2) := \sup_{\beta_2\in\mathcal{T}_n} \frac{1_{\hat{K}_{\beta_2}}(w_2)}{\mu_2'(\hat{K}_{\beta_2})} \int_{\hat{K}_{\beta_2}} |f(w_1, z_2)|\mu_2'(z_2)dV(z_2), \tag{2.19}$$

we obtain that $\mathcal{M}_{\mathcal{T}_{m,n},\mu} f \le \mathcal{M}_{\mathcal{T}_m,\mu_1} \circ \mathcal{M}_{\mathcal{T}_n,\mu_2'} f$. By Fubini's Theorem, it is enough to show that $\mathcal{M}_{\mathcal{T}_m,\mu_1}$ is bounded on $L^p(\mathbb{D}, \mu_1 dV)$ and $\mathcal{M}_{\mathcal{T}_n,\mu_2'}$ is bounded on $L^p(\mathbb{D}, \mu_2' dV)$. Here we show the L^p boundedness of $\mathcal{M}_{\mathcal{T}_m,\mu_1}$. The boundedness of $\mathcal{M}_{\mathcal{T}_n,\mu_2'}$ follows from an analogous argument.

Note that $\mathcal{M}_{\mathcal{T}_m,\mu_1}$ is bounded on $L^\infty(\mathbb{D}, \mu_1)$. By interpolation, the weak-type (1,1) estimate

$$\mu_1(\{z \in \mathbb{D} : \mathcal{M}_{\mathcal{T}_m,\mu_1} f(z) > \lambda\}) \lesssim \frac{\|f\|_{L^1(\mathbb{D},\mu_1)}}{\lambda} \tag{2.20}$$

is sufficient to finish the proof. For a point $w \in \{z \in \mathbb{D} : \mathcal{M}_{\mathcal{T}_m,\mu_1} f(z) > \lambda\}$, there exists a unique maximal tent $\hat{K}_\alpha$ that contains w and satisfies:

$$\frac{1_{\hat{K}_\alpha}(w)}{\mu_1(\hat{K}_\alpha)} \int_{\hat{K}_\alpha} |f(z)|\mu_1(z)dV(z) > \frac{\lambda}{2}. \tag{2.21}$$

Let $\mathcal{A}_\lambda$ be the set of indices of all such maximal tents $\hat{K}_\alpha$. The union of these maximal tents covers the set $\{z \in \mathbb{D} : \mathcal{M}_{\mathcal{T}_m,\mu_1} f(z) > \lambda\}$. Since the tents $\hat{K}_\alpha$ are maximal, they are also pairwise disjoint and hence

$$\mu_1(\{z \in \mathbb{D} : \mathcal{M}_{\mathcal{T}_m,\mu_1} f(z) > \lambda\}) \le \sum_{\alpha\in\mathcal{A}_\lambda} \mu_1(\hat{K}_\alpha)$$

$$\leq \sum_{\alpha \in \mathcal{A}_\lambda} \frac{2}{\lambda} \int_{\hat{K}_\alpha} f(z)\mu_1(z)dV(z) \leq \frac{2\|f\|_{L^1(\mathbb{D},\mu_1)}}{\lambda}.$$

Thus inequality (2.20) holds and $\mathcal{M}_{\mathcal{T}_m,\mu_1}$ is weak-type (1,1). Using a standard argument for the Hardy-Littlewood maximal function, we further have

$$\|\mathcal{M}_{\mathcal{T}_m,\mu_1}\|_{L^p(\mathbb{D}\times\mathbb{D}^*,|w_2|^2\mu\circ h)} \lesssim \frac{p}{p-1}.$$

Since the same inequality holds for $\mathcal{M}_{\mathcal{T}_n,\mu_2'}$,

$$\begin{aligned}\|\mathcal{M}_{\mathcal{T}'_{m,n},\mu}\|_{L^p(\mathbb{H},\mu)} =& \|\mathcal{M}_{\mathcal{T}_{m,n},\mu}\|_{L^p(\mathbb{D}\times\mathbb{D}^*,|w_2|^2\mu\circ h)} \\ \leq& \|\mathcal{M}_{\mathcal{T}_m,\mu_1} \circ \mathcal{M}_{\mathcal{T}_n,\mu_2'}\|_{L^p(\mathbb{D}\times\mathbb{D}^*,|w_2|^2\mu\circ h)} \lesssim \left(\frac{p}{p-1}\right)^2. \quad \square\end{aligned}$$

Finally, we define two operators Q and Q^+. Let p' be the conjugate index of p. We set

$$Q(f)(z_1,z_2) = \int_{\mathbb{H}} \frac{1}{\pi^2 z_2(1-\frac{z_1\bar{w}_1}{z_2\bar{w}_2})^2(1-z_2\bar{w}_2)^2} f(w_1,w_2)dV(w_1,w_2), \tag{2.22}$$

$$Q^+(f)(z_1,z_2) = \int_{\mathbb{H}} \frac{1}{\pi^2 |z_2||1-\frac{z_1\bar{w}_1}{z_2\bar{w}_2}|^2|1-z_2\bar{w}_2|^2} f(w_1,w_2)dV(w_1,w_2). \tag{2.23}$$

It is clear that $P = QM_{1/\bar{w}_2}$ and $P^+ = Q^+M_{1/|w_2|}$. Moreover, the weighted L^p norm of the projection, $\|P^+ : L^p(\mathbb{H},\mu dV) \to L^p(\mathbb{H},\mu dV)\|$, is equal to the weighted norm of Q^+M_ν acting between two different weighted L^p spaces.

Lemma 2.7. *Let μ be a weight on the Hartogs triangle. Set $\nu := \mu^{\frac{-p'}{p}}|w_2|^{-p'}$. Then*

$$\|P : L^p(\mathbb{H},\mu dV) \to L^p(\mathbb{H},\mu dV)\| = \|QM_\nu : L^p(\mathbb{H},\nu dV) \to L^p(\mathbb{H},\mu dV)\|; \tag{2.24}$$

$$\|P^+ : L^p(\mathbb{H},\mu dV) \to L^p(\mathbb{H},\mu dV)\| = \|Q^+M_\nu : L^p(\mathbb{H},\nu dV) \to L^p(\mathbb{H},\mu dV)\|. \tag{2.25}$$

Proof. We show (2.25) here as the proof for (2.24) is similar. Given $f \in L^p(\mathbb{H},\mu)$, we have

$$\int_{\mathbb{H}} |f|^p \mu dV(w_1,w_2) = \int_{\mathbb{H}} \left|\frac{f}{w_2}\right|^p |w_2|^p \mu dV(w_1,w_2) = \int_{\mathbb{H}} \left|M_{\frac{1}{|w_2|}} f\right|^p |w_2|^p \mu dV(w_1,w_2). \tag{2.26}$$

Thus $\|f\|_{L^p(\mu dV)} = \|M_{1/|w_2|}f\|_{L^p(\mu|w_2|^p dV)}$ and

$$\|P^+ : L^p(\mathbb{H},\mu dV) \to L^p(\mathbb{H},\mu dV)\| = \|Q^+ : L^p(\mathbb{H},|w_2|^p\mu dV) \to L^p(\mathbb{H},\mu dV)\|.$$

We claim further that for $f \in L^p(\mathbb{H}, |w_2|^p \mu dV)$, $\|f\|_{L^p(|w_2|^p \mu dV)} = \|M_{1/\nu} f\|_{L^p(\nu dV)}$. Then (2.25) holds. Recall that $\nu := \mu^{\frac{-p'}{p}} |w_2|^{-p'}$. We have

$$\int_{\mathbb{H}} \left|\frac{f}{\nu}\right|^p \nu dV = \int_{\mathbb{H}} |f|^p \nu^{1-p} dV = \int_{\mathbb{H}} |f|^p (\mu^{-\frac{p'}{p}} |w_2|^{-p'})^{1-p} dV = \int_{\mathbb{H}} |f|^p |w_2|^p \mu dV.$$

Hence the claim is shown and the proof is complete. □

3. Proof of Theorem 1.2

It is sufficient to prove that inequality (1.5) holds.

3.1. Proof for the upper bound

For the upper bound inequality

$$\|P^+\|_{L^p(\mathbb{H},\mu dV)} \lesssim ([\mu,\nu]_p^{0,0})^{\frac{1}{p}} + pp'([\mu,\nu]_p^{1,0} + [\mu,\nu]_p^{0,1}) + (pp')^2([\mu,\nu]_p^{1,1})^{\max\{1,\frac{1}{p-1}\}},$$

we first consider the case $p \geq 2$. The case $1 < p < 2$ will follow from a duality argument.

Recall the tree structure $\{\mathcal{T}'_{m,n}\}_{m=1}^M$ and the dyadic tent $\{\hat{K}'_{\beta_1,\beta_2}\}$ from (2.9) and (2.10). Set the measure $du := |w_2|^{-2} dV$. By Lemma 2.2 and the inequality (2.8), there is a finite collection M such that for (z_1, z_2) and (w_1, w_2) in $\mathbb{H}$, there exists $\hat{K}_{\beta_1}$ and $\hat{K}_{\beta_1}$ such that

$$\begin{aligned}\left|1 - \frac{z_1\bar{w}_1}{z_2\bar{w}_2}\right|^{-2} |1 - z_2\bar{w}_2|^{-2} \approx& |\hat{K}_{\beta_1}|^{-1} |\hat{K}_{\beta_2}|^{-1} \\ \leq& \sum_{m,n=1}^{M} \sum_{\gamma\in\mathcal{T}_m,\eta\in\mathcal{T}_n} \frac{1_{\hat{K}_\gamma\times\hat{K}_\eta}(z_1/z_2, z_2) 1_{\hat{K}_\gamma\times\hat{K}_\eta}(w_1/w_2, w_2)}{|\hat{K}_\gamma \times \hat{K}_\eta|} \\ =& \sum_{m,n=1}^{M} \sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}} \frac{1_{\hat{K}'_{\gamma,\eta}}(z_1, z_2) 1_{\hat{K}'_{\gamma,\eta}}(w_1, w_2)}{u(\hat{K}'_{\gamma,\eta})}. \end{aligned} \tag{3.1}$$

Applying this inequality to the operator $Q^+ M_\nu$ yields

$$\begin{aligned} &\left|Q^+ M_\nu f(z_1, z_2)\right| \\ =& \left|\int_{\mathbb{H}} \frac{|z_2|^{-1} M_\nu f(w_1, w_2)}{\pi^2 |1 - \frac{z_1\bar{w}_1}{z_2\bar{w}_2}|^2 |1 - z_2\bar{w}_2|^2} dV(w_1, w_2)\right| \\ \lesssim& \int_{\mathbb{H}} \sum_{m,n=1}^{M} \sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}} \frac{1_{\hat{K}'_{\gamma,\eta}}(z_1, z_2) 1_{\hat{K}'_{\gamma,\eta}}(w_1, w_2) |M_\nu f(w_1, w_2)|}{|z_2| u(\hat{K}'_{\gamma,\eta})} dV(w_1, w_2) \end{aligned}$$

$$= \sum_{m,n=1}^{M} \sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}} \frac{1_{\hat{K}'_{\gamma,\eta}}(z_1,z_2)}{|z_2|} \langle f\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,\eta}}$$

$$= I_{0,0} + I_{0,1} + I_{1,0} + I_{1,1}, \tag{3.2}$$

where

$$I_{0,0} = \sum_{m,n=1}^{M} \frac{1_{\hat{K}'_{0,0}}(z_1,z_2)}{|z_2|} \langle f\nu|w_2|^2\rangle^{du}_{K'_{0,0}} = M^2 \frac{1_{\mathbb{H}}(z_1,z_2)}{|z_2|} \langle f\nu|w_2|^2\rangle^{du}_{\mathbb{H}}; \tag{3.3}$$

$$I_{1,0} = \sum_{m,n=1}^{M} \sum_{(\gamma,0)\in\mathcal{T}'_{m,n}} \frac{1_{\hat{K}'_{\gamma,0}}(z_1,z_2)}{|z_2|} \langle f\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,0}}; \tag{3.4}$$

$$I_{0,1} = \sum_{m,n=1}^{M} \sum_{(0,\eta)\in\mathcal{T}'_{m,n}} \frac{1_{\hat{K}'_{0,\eta}}(z_1,z_2)}{|z_2|} \langle f\nu|w_2|^2\rangle^{du}_{\hat{K}'_{0,\eta}}; \tag{3.5}$$

$$I_{1,1} = \sum_{m,n=1}^{M} \sum_{\substack{(\gamma,\eta)\in\mathcal{T}'_{m,n} \\ \gamma,\eta\neq 0}} 1_{\hat{K}'_{\gamma,\eta}}(z_1,z_2)|z_2|^{-1} \langle f\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,\eta}}. \tag{3.6}$$

Set

$$Q^{0,0}_{m,n,\nu} f(z_1,z_2) := \frac{1_{\mathbb{H}}(z_1,z_2)}{|z_2|} \langle f\nu|w_2|^2\rangle^{du}_{\mathbb{H}};$$

$$Q^{1,0}_{m,n,\nu} f(z_1,z_2) := \sum_{(\gamma,0)\in\mathcal{T}'_{m,n}} 1_{\hat{K}'_{\gamma,0}}(z_1,z_2)|z_2|^{-1} \langle f\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,0}};$$

$$Q^{0,1}_{m,n,\nu} f(z_1,z_2) := \sum_{(0,\eta)\in\mathcal{T}'_{m,n}} 1_{\hat{K}'_{0,\eta}}(z_1,z_2)|z_2|^{-1} \langle f\nu|w_2|^2\rangle^{du}_{\hat{K}'_{0,\eta}};$$

$$Q^{1,1}_{m,n,\nu} f(z_1,z_2) := \sum_{\substack{(\gamma,\eta)\in\mathcal{T}'_{m,n} \\ \gamma,\eta\neq 0}} 1_{\hat{K}'_{\gamma,\eta}}(z_1,z_2)|z_2|^{-1} \langle f\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,\eta}}.$$

Then it suffices to estimate the L^p norm for each $Q^{i,j}_{m,n,\nu}$. The proof given below uses the idea of how to prove the linear bound for sparse operators in the weighted theory of harmonic analysis, see for example [25] and [20].

We first consider $Q^{0,0}_{m,n,\nu}$. For arbitrary $g \in L^{p'}(\mathbb{H},\mu)$,

$$\left\langle Q^{0,0}_{m,n,\nu} f(z_1,z_2), g(z_1,z_2)\mu \right\rangle$$
$$= \int_{\mathbb{H}} Q^{0,0}_{m,n,\nu} f(z_1,z_2) g(z_1,z_2)\mu dV(z_1,z_2)$$
$$= \int_{\mathbb{H}} 1_{\mathbb{H}}(z_1,z_2)|z_2|^{-1} \langle f\nu|w_2|^2\rangle^{du}_{\mathbb{H}} g(z_1,z_2)\mu dV(z_1,z_2)$$

$$
\begin{aligned}
&=(u(\mathbb{H}))^{-1}\int\limits_{\mathbb{H}} f(z_1,z_2)\nu dV(z_1,z_2)\int\limits_{\mathbb{H}} g(z_1,z_2)|z_2|^{-1}\mu dV(z_1,z_2)\\
&\leq (u(\mathbb{H}))^{-1}\left(\int\limits_{\mathbb{H}}\nu dV\right)^{\frac{p-1}{p}}\|f\|_{L^p(\mathbb{H},\nu)}\left(\int\limits_{\mathbb{H}}|z_2|^{-p}\mu dV\right)^{\frac{1}{p}}\|g\|_{L^{p'}(\mathbb{H},\mu)}\\
&=\left(\langle|w_2|^{2-p}\mu\rangle_{\mathbb{H}}^{du}\left(\langle\nu|w_2|^2\rangle_{\mathbb{H}}^{du}\right)^{p-1}\right)^{\frac{1}{p}}\|f\|_{L^p(\mathbb{H},\nu)}\|g\|_{L^{p'}(\mathbb{H},\mu)}.
\end{aligned}
\tag{3.7}
$$

Therefore

$$
\|Q^{0,0}_{m,n,\nu}\|_{L^p(\mathbb{H},\nu)\to L^p(\mathbb{H},\mu)}\leq\left(\langle|w_2|^{2-p}\mu\rangle_{\mathbb{H}}^{du}\left(\langle\nu|w_2|^2\rangle_{\mathbb{H}}^{du}\right)^{p-1}\right)^{\frac{1}{p}}=([\mu,\nu]_p^{0,0})^{\frac{1}{p}}. \tag{3.8}
$$

We turn to $Q^{1,1}_{m,n,\nu}$. For arbitrary $g\in L^{p'}(\mathbb{H},\mu)$,

$$
\begin{aligned}
&\left\langle Q^{1,1}_{m,n,\nu}f(z_1,z_2),g(z_1,z_2)\mu\right\rangle\\
&=\int\limits_{\mathbb{H}} Q^{1,1}_{m,n,\nu}f(z_1,z_2)g(z_1,z_2)\mu dV(z_1,z_2)\\
&=\int\limits_{\mathbb{H}}\sum_{\substack{(\gamma,\eta)\in\mathcal{T}'_{m,n}\\ \gamma,\eta\neq 0}} 1_{\hat{K}'_{\gamma,\eta}}(z_1,z_2)|z_2|^{-1}\langle f\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,\eta}}g(z_1,z_2)\mu dV(z_1,z_2)\\
&=\sum_{\substack{(\gamma,\eta)\in\mathcal{T}'_{m,n}\\ \gamma,\eta\neq 0}}\langle f\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\int\limits_{\hat{K}'_{\gamma,\eta}} g(z_1,z_2)|z_2|^{-1}\mu dV(z_1,z_2)\\
&=\sum_{\substack{(\gamma,\eta)\in\mathcal{T}'_{m,n}\\ \gamma,\eta\neq 0}}\langle f\rangle^{\nu dV}_{\hat{K}'_{\gamma,\eta}}\langle\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\langle g|w_2|^{p-1}\rangle^{|w_2|^{2-p}\mu du}_{\hat{K}'_{\gamma,\eta}}\langle|w_2|^{2-p}\mu\rangle^{du}_{\hat{K}'_{\gamma,\eta}}u(\hat{K}_{\gamma,\eta})\\
&=\sum_{\substack{(\gamma,\eta)\in\mathcal{T}'_{m,n}\\ \gamma,\eta\neq 0}}\left(\langle\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\right)^{p-1}\langle|w_2|^{2-p}\mu\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\langle f\rangle^{\nu dV}_{\hat{K}'_{\gamma,\eta}}\langle g|w_2|^{p-1}\rangle^{|w_2|^{-p}\mu dV}_{\hat{K}'_{\gamma,\eta}}\\
&\quad\times u(\hat{K}'_{\gamma,\eta})\left(\langle\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\right)^{2-p}\\
&\leq[\mu,\nu]_p^{1,1}\sum_{\substack{(\gamma,\eta)\in\mathcal{T}'_{m,n}\\ \gamma,\eta\neq 0}}\langle f\rangle^{\nu dV}_{\hat{K}'_{\gamma,\eta}}\langle g|w_2|^{p-1}\rangle^{|w_2|^{-p}\mu dV}_{\hat{K}'_{\gamma,\eta}}\left(u(\hat{K}'_{\gamma,\eta})\right)^{p-1}\left(\nu(\hat{K}'_{\gamma,\eta})\right)^{2-p}.
\end{aligned}
\tag{3.9}
$$

Recall from Lemma 2.1 that $|\hat{K}_\alpha|\approx|K_\alpha|$ for the tree structure $\mathcal{T}$ with Lebesgue measure σ on the unit disc. Hence for the induced tree structure $\mathcal{T}'_{m,n}$ with the induced weighted measure u on the Hartogs triangle, we also have $u(\hat{K}'_{\gamma,\eta})\approx u(K'_{\gamma,\eta})$. The facts that $p\geq 2$ and $K'_{\gamma,\eta}\subseteq\hat{K}'_{\gamma,\eta}$ gives the inequality $\left(\nu(\hat{K}'_{\gamma,\eta})\right)^{2-p}\leq\left(\nu(K'_{\gamma,\eta})\right)^{2-p}$. Combining these facts, we have

$$\left(u(\hat{K}'_{\gamma,\eta})\right)^{p-1}\left(\nu(\hat{K}'_{\gamma,\eta})\right)^{2-p} \lesssim \left(u(K'_{\gamma,\eta})\right)^{p-1}\left(\nu(K'_{\gamma,\eta})\right)^{2-p}. \tag{3.10}$$

By Hölder's inequality,

$$u(K'_{\gamma,\eta}) \leq \left(\nu(K'_{\gamma,\eta})\right)^{\frac{1}{p'}}\left(\int\limits_{K'_{\gamma,\eta}} |w_2|^{-p}\mu dV\right)^{\frac{1}{p}}.$$

Therefore,

$$\left(u(K'_{\gamma,\eta})\right)^{p-1}\left(\nu(K'_{\gamma,\eta})\right)^{2-p} \leq \left(\nu(K'_{\gamma,\eta})\right)^{\frac{1}{p}}\left(\int\limits_{K'_{\gamma,\eta}} |w_2|^{-p}\mu dV\right)^{\frac{1}{p'}}. \tag{3.11}$$

Applying these inequalities to the last line of (3.9), we have

$$\begin{aligned}
&[\mu,\nu]_p^{1,1}\sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}}\langle f\rangle^{\nu dV}_{\hat{K}'_{\gamma,\eta}}\langle g|w_2|^{p-1}\rangle^{|w_2|^{-p}\mu dV}_{\hat{K}'_{\gamma,\eta}}\left(u(\hat{K}'_{\gamma,\eta})\right)^{p-1}\left(\nu(\hat{K}'_{\gamma,\eta})\right)^{2-p}\\
\lesssim&[\mu,\nu]_p^{1,1}\sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}}\langle f\rangle^{\nu dV}_{\hat{K}'_{\gamma,\eta}}\langle g|w_2|^{p-1}\rangle^{|w_2|^{-p}\mu dV}_{\hat{K}'_{\gamma,\eta}}\left(\nu(K'_{\gamma,\eta})\right)^{\frac{1}{p}}\left(\int\limits_{K'_{\gamma,\eta}} |w_2|^{-p}\mu dV\right)^{\frac{1}{p'}}.
\end{aligned} \tag{3.12}$$

Applying Hölder's inequality again to the sum above yields:

$$\begin{aligned}
&\sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}}\langle f\rangle^{\nu dV}_{\hat{K}'_{\gamma,\eta}}\langle g|w_2|^{p-1}\rangle^{|w_2|^{-p}\mu dV}_{\hat{K}'_{\gamma,\eta}}\left(\nu(K'_{\gamma,\eta})\right)^{\frac{1}{p}}\left(\int\limits_{K'_{\gamma,\eta}} |w_2|^{-p}\mu dV\right)^{\frac{1}{p'}}\\
&\leq\left(\sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}}\left(\langle f\rangle^{\nu dV}_{\hat{K}'_{\gamma,\eta}}\right)^p\nu(K'_{\gamma,\eta})\right)^{\frac{1}{p}}\\
&\quad\times\left(\sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}}\left(\langle g|w_2|^{p-1}\rangle^{|w_2|^{-p}\mu dV}_{K'_{\gamma,\eta}}\right)^{p'}\int\limits_{K'_{\gamma,\eta}} |w_2|^{-p}\mu dV\right)^{\frac{1}{p'}}.
\end{aligned} \tag{3.13}$$

By the disjointness of $K'_{\gamma,\eta}$ and Lemma 2.6, we have

$$\sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}}\left(\langle f\rangle^{\nu dV}_{\hat{K}'_{\gamma,\eta}}\right)^p\nu(K'_{\gamma,\eta}) \leq \int\limits_{\mathbb{H}}(\mathcal{M}_{\mathcal{T}'_{m,n},\nu}f)^p\nu dV \leq (p')^{2p}\|f\|^{2p}_{L^p(\mathbb{H},\nu dV)}. \tag{3.14}$$

Note that $\|g|w_2|^{p-1}\|_{L^{p'}(\mathbb{H},|w_2|^{-p}\mu dV)} = \|g\|_{L^{p'}(\mathbb{H},\mu dV)}$. A similar argument using the maximal function $\mathcal{M}_{\mathcal{T}'_{m,n},|w_2|^{-p}\mu}$ will also give the inequality

$$\sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}} \left(\langle g|w_2|^{p-1}\rangle_{\hat{K}'_{\gamma,\eta}}^{|w_2|^{-p}\mu dV}\right)^p \int_{K'_{\gamma,\eta}} |w_2|^{-p}\mu dV \leq (p)^{2p'}\|g\|^{p'}_{L^{p'}(\mathbb{H},\mu dV)}. \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13) and (3.9) finally yields

$$\left\langle Q^{1,1}_{m,n,\nu}f, g\mu\right\rangle \lesssim [\mu,\nu]_p^{1,1}(pp')^2\|f\|_{L^p(\mathbb{H},\nu dV)}\|g\|_{L^{p'}(\mathbb{H},\mu dV)}. \tag{3.16}$$

Therefore $\|Q^{1,1}_{m,n,\nu}\|_{L^p(\mathbb{H},\nu dV)\to L^p(\mathbb{H},\mu dV)} \lesssim (pp')^2[\mu,\nu]_p^{1,1}$.

For the case $1 < p < 2$ and we claim that

$$\left\langle Q^{1,1}_{m,n,\nu}f, g\mu\right\rangle \lesssim ([\mu,\nu]_p^{1,1})^{\frac{1}{p-1}}\|f\|_{L^p(\mathbb{H},\nu dV)}\|g\|_{L^{p'}(\mathbb{H},\mu dV)}, \tag{3.17}$$

for all $f \in L^p(\mathbb{H},\nu dV)$ and $g \in L^{p'}(\mathbb{H},\mu dV)$. By the definition of $Q^{1,1}_{m,n,\nu}$,

$$\begin{aligned}
\left\langle Q^{1,1}_{m,n,\nu}f, g\mu\right\rangle &= \left\langle \sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}} 1_{\hat{K}'_{\gamma,\eta}}(w_1,w_2)|w_2|^{-1}\langle f\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,\eta}}, g\mu\right\rangle \\
&= \sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}} \left\langle 1_{\hat{K}'_{\gamma,\eta}}(w_1,w_2)\langle f\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,\eta}}, g|w_2|^{-1}\mu\right\rangle \\
&= \sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}} \langle f\nu|w_2|^2\rangle^{du}_{K'_{\gamma,\eta}}\langle g|w_2|\mu\rangle^{du}_{\hat{K}'_{\gamma,\eta}} u(\hat{K}'_{\gamma,\eta}) \\
&= \sum_{(\gamma,\eta)\in\mathcal{T}'_{m,n}} \left\langle 1_{\hat{K}'_{\gamma,\eta}}(w_1,w_2)|w_2|^{-1}\langle g|w_2|^{p-1}|w_2|^{-p}\mu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,\eta}}|w_2|, f\nu\right\rangle \\
&= \left\langle M_{|z_2|}Q^{1,1}_{m,n,|w_2|^{-p}\mu}(g|w_2|^{p-1}), f\nu\right\rangle.
\end{aligned} \tag{3.18}$$

Set $h = g|w_2|^{p-1}$ and $\psi = |w_2|^{-p}\mu$. Then $\|h\|_{L^{p'}(\mathbb{H},\psi dV)} = \|g\|_{L^{p'}(\mathbb{H},\mu dV)}$. Setting the weight ω to satisfies $|w_2|^{-p}\omega^{\frac{-p}{-p'}} = \psi$, we have $\omega = \mu^{\frac{p'}{p}} = \nu|z_2|^{p'}$. Replacing p by p', μ by ω, and ν by ψ and going through same argument for the case $p \geq 2$ yields that

$$\begin{aligned}
&\|M_{|z_2|}Q^{1,1}_{m,n,|w_2|^{-p}\mu}\|_{L^{p'}(\mathbb{H},\nu dV)} \\
&= \|Q^{1,1}_{m,n,|w_2|^{-p}\mu}\|_{L^{p'}(\mathbb{H},|w_2|^{p'}\nu dV)} \\
&\lesssim (pp')^2 \sup_{(\gamma,\eta)\in\mathcal{T}'_{m,n}} \left(\langle\mu|w_2|^{2-p}\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\right)^{p'-1}\langle|w_2|^{2-p'}\nu|z_2|^{p'}\rangle^{du}_{\hat{K}'_{\gamma,\eta}} \\
&= (pp')^2\left(\sup_{(\gamma,\eta)\in\mathcal{T}'_{m,n}} \langle\mu|w_2|^{2-p}\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\left(\langle|w_2|^2\nu\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\right)^{p-1}\right)^{\frac{1}{p-1}}
\end{aligned}$$

$$= (pp')^2([\mu,\nu]_p^{1,1})^{\frac{1}{p-1}}. \tag{3.19}$$

Thus we have

$$\left\langle Q_{m,n,\nu}^{1,1} f, g\mu \right\rangle \lesssim (pp')^2([\mu,\nu]_p^{1,1})^{\frac{1}{p-1}} \|g\|_{L^{p'}(\mathbb{H},\mu dV)} \|f\|_{L^p(\mathbb{H},\nu dV)},$$

and

$$\|Q_{m,n,\nu}^{1,1}\|_{L^p(\mathbb{H},\mu dV)} \lesssim (pp')^2([\mu,\nu]_p^{1,1})^{\frac{1}{p-1}}.$$

Combining the results for $1 < p < 2$ and $p \geq 2$ gives:

$$\|Q_{m,n,\nu}^{1,1}\|_{L^p(\mathbb{H},\nu dV)\to L^p(\mathbb{H},\mu dV)} \lesssim (pp')^2([\mu,\nu]_p^{1,1})^{\max\{1,\frac{1}{p-1}\}}. \tag{3.20}$$

To estimate $\|Q_{m,n,\nu}^{1,0}\|_{L^p(\mathbb{H},\nu dV)\to L^p(\mathbb{H},\mu dV)}$, we combine the above arguments for $Q_{m,n,\nu}^{0,0}$ and $Q_{m,n,\nu}^{1,1}$. For arbitrary $g \in L^{p'}(\mathbb{H},\mu)$,

$$\begin{aligned}
&\left\langle Q_{m,n,\nu}^{1,0} f(z_1,z_2), g(z_1,z_2)\mu \right\rangle \\
&= \int_{\mathbb{H}} Q_{m,n,\nu}^{1,0} f(z_1,z_2) g(z_1,z_2) \mu dV(z_1,z_2) \\
&= \int_{\mathbb{H}} \sum_{(\gamma,0)\in\mathcal{T}'_{m,n}} 1_{\hat{K}'_{\gamma,0}}(z_1,z_2)|z_2|^{-1} \langle f\nu|w_2|^2\rangle^{du}_{\hat{K}'_{\gamma,0}} g(z_1,z_2)\mu dV(z_1,z_2) \\
&= \sum_{(\gamma,0)\in\mathcal{T}'_{m,n}} (u(\hat{K}'_{\gamma,0}))^{-1} \int_{\hat{K}'_{\gamma,0}} |f(z_1,z_2)|\nu dV(z_1,z_2) \int_{\hat{K}'_{\gamma,0}} |z_2|^{-1}|g(z_1,z_2)|\mu dV(z_1,z_2) \\
&\approx \sum_{(\gamma,0)\in\mathcal{T}'_{m,n}} |\hat{K}_\gamma|^{-1} \int_{\hat{K}'_{\gamma,0}} |f(z_1,z_2)|\nu dV(z_1,z_2) \int_{\hat{K}'_{\gamma,0}} |g(z_1,z_2)||z_2|^{-1}\mu dV(z_1,z_2).
\end{aligned} \tag{3.21}$$

Recall that $\mu(z_1,z_2) = \mu_1(z_1/z_2)\mu_2(z_2)$. There holds $\nu(z_1,z_2) = \nu_1(z_1/z_2)\nu_2(z_2)$ by the definition of ν. Hence

$$\begin{aligned}
&\int_{\hat{K}'_{\gamma,0}} |f(z_1,z_2)|\nu dV(z_1,z_2) \\
&= \int_{\hat{K}_\gamma\times\mathbb{D}} |f(z_2 t, z_2)|\nu_1(t)\nu_2(z_2)|z_2|^2 dV(t,z_2) \\
&\leq \left(\int_{\mathbb{D}} \left| \int_{\hat{K}_\gamma} |f(z_2 t, z_2)|\nu_1(t) dV(t) \right|^p \nu_2(z_2)|z_2|^2 dV(z_2) \right)^{\frac{1}{p}} \left(\int_{\mathbb{D}} \nu_2(z_2)|z_2|^2 dV(z_2) \right)^{\frac{1}{p'}}
\end{aligned}$$

$$\approx \nu_1(\hat{K}_\gamma)\left(\int\limits_{\mathbb{D}} \frac{|\int_{\hat{K}_\gamma}|f(z_2t,z_2)|\nu_1(t)dV(t)|^p}{(\nu_1(\hat{K}_\gamma))^p}\nu_2(z_2)|z_2|^2dV(z_2)\right)^{\frac{1}{p}}$$
$$\times\left(\int\limits_{\mathbb{D}}\nu_2(z_2)|z_2|^2dV(z_2)\right)^{\frac{1}{p'}}. \tag{3.22}$$

Similarly,

$$\int\limits_{\hat{K}'_{\gamma,0}} |g(z_1,z_2)||z_2|^{-1}\mu dV(z_1,z_2)$$
$$\lesssim \mu_1(\hat{K}_\gamma)\left(\int\limits_{\mathbb{D}} \frac{|\int_{\hat{K}_\gamma}|g(z_2t,z_2)|\mu_1(t)dV(t)|^{p'}}{(\mu_1(\hat{K}_\gamma))^{p'}}\mu_2(z_2)|z_2|^2dV(z_2)\right)^{\frac{1}{p'}}$$
$$\times\left(\int\limits_{\mathbb{D}}\mu_2(z_2)|z_2|^{2-p}dV(z_2)\right)^{\frac{1}{p}}. \tag{3.23}$$

Set $f^*(t,z_2)=f(z_2t,z_2)$ and $g^*(t,z_2)=g(z_2t,z_2)$. Recall the boundedness of $\mathcal{M}_{\mathcal{T}_m,\mu_1}$ from the proof of Lemma 2.6 and the fact that $(\nu_1(\hat{K}_\gamma))^{2-p}\le(\nu_1(K_\gamma))^{2-p}$ for $p\ge 2$. Applying these facts, substituting (3.22) and (3.23) into (3.21), and following the computation for the operator $Q^{1,1}_{m,n,\nu}$ then yields that for $p\ge 2$,

$$\begin{aligned}
&\left\langle Q^{1,0}_{m,n,\nu}f(z_1,z_2),g(z_1,z_2)\mu\right\rangle\\
&\lesssim \left(\langle|z_2|^{2-p}\mu_2\rangle^{dV}_{\mathbb{D}}(\langle\nu_2\rangle^{dV}_{\mathbb{D}})^{p-1}\right)^{\frac{1}{p}}\sup_{0\neq\gamma\in\mathcal{T}_m}\langle\mu_1\rangle^{dV}_{\hat{K}_\gamma}\left(\langle\nu_1\rangle^{dV}_{\hat{K}_\gamma}\right)^{p-1}\times\\
&\quad\|\mathcal{M}_{\mathcal{T}_m,\nu_1}(|f^*(\cdot,z_2)|)\|_{L^p(\mathbb{D}^2,|z_2|^2\nu_1\nu_2)}\|\mathcal{M}_{\mathcal{T}_m,\mu_1}(|g^*(\cdot,z_2)|)\|_{L^{p'}(\mathbb{D}^2,|z_2|^2\mu_1\mu_2)}\\
&\lesssim \left(\langle|z_2|^{2-p}\mu_2\rangle^{dV}_{\mathbb{D}}(\langle\nu_2\rangle^{dV}_{\mathbb{D}})^{p-1}\right)^{\frac{1}{p}}\sup_{0\neq\gamma\in\mathcal{T}_m}\langle\mu_1\rangle^{dV}_{\hat{K}_\gamma}\left(\langle\nu_1\rangle^{dV}_{\hat{K}_\gamma}\right)^{p-1}\\
&\quad\times pp'\|f^*\|_{L^p(\mathbb{D}^2,|z_2|^2\nu_1\nu_2)}\|g^*\|_{L^{p'}(\mathbb{D}^2,|z_2|^2\mu_1\mu_2)}\\
&\lesssim \left(\langle|z_2|^{2-p}\mu_2\rangle^{dV}_{\mathbb{D}}(\langle\nu_2\rangle^{dV}_{\mathbb{D}})^{p-1}\right)^{\frac{1}{p}}\sup_{0\neq\gamma\in\mathcal{T}_m}\langle\mu_1\rangle^{dV}_{\hat{K}_\gamma}\left(\langle\nu_1\rangle^{dV}_{\hat{K}_\gamma}\right)^{p-1}pp'\|f\|_{L^p(\mathbb{H},\nu)}\|g\|_{L^{p'}(\mathbb{H},\mu)}.
\end{aligned} \tag{3.24}$$

The same duality argument as in (3.18) implies that for $1<p<2$,

$$\left\langle Q^{1,0}_{m,n,\nu}f(z_1,z_2),g(z_1,z_2)\mu\right\rangle$$
$$\lesssim \left(\langle|z_2|^{2-p}\mu_2\rangle^{dV}_{\mathbb{D}}(\langle\nu_2\rangle^{dV}_{\mathbb{D}})^{p-1}\right)^{\frac{1}{p}}\left(\sup_{0\neq\gamma\in\mathcal{T}_m}\langle\mu_1\rangle^{dV}_{\hat{K}_\gamma}\left(\langle\nu_1\rangle^{dV}_{\hat{K}_\gamma}\right)^{p-1}\right)^{\frac{1}{p-1}}$$

$$\times pp'\|f\|_{L^p(\mathbb{H},\nu)}\|g\|_{L^{p'}(\mathbb{H},\mu)}.$$

Combining these inequalities, we obtain

$$\begin{aligned}
&\|Q^{1,0}_{m,n,\nu}\|_{L^p(\mathbb{H},\nu dV)\to L^p(\mathbb{H},\mu dV)}\\
\lesssim& pp'\left(\langle|z_2|^{2-p}\mu_2\rangle^{dV}_{\mathbb{D}}(\langle\nu_2\rangle^{dV}_{\mathbb{D}})^{p-1}\right)^{\frac{1}{p}}\left(\sup_{0\neq\gamma\in\mathcal{T}_m}\langle\mu_1\rangle^{dV}_{\hat{K}_\gamma}\left(\langle\nu_1\rangle^{dV}_{\hat{K}_\gamma}\right)^{p-1}\right)^{\max\{1,\frac{1}{p-1}\}}\\
=&pp'[\mu,\nu]^{1,0}_p
\end{aligned}\tag{3.25}$$

By a similar argument, one can obtain the estimate for $\|Q^{0,1}_{m,n,\nu}\|$:

$$\begin{aligned}
&\|Q^{0,1}_{m,n,\nu}\|_{L^p(\mathbb{H},\nu dV)\to L^p(\mathbb{H},\mu dV)}\\
\lesssim& pp'\left(\sup_{0\neq\eta\in\mathcal{T}_n}\langle|z_2|^{2-p}\mu_2\rangle^{dV}_{\hat{K}_\eta}(\langle\nu_2\rangle^{dV}_{\hat{K}_\eta})^{p-1}\right)^{\max\{1,\frac{1}{p-1}\}}\left(\langle\mu_1\rangle^{dV}_{\mathbb{D}}\left(\langle\nu_1\rangle^{dV}_{\mathbb{D}}\right)^{p-1}\right)^{\frac{1}{p}}\\
=&pp'[\mu,\nu]^{0,1}_p.
\end{aligned}\tag{3.26}$$

Combining (3.8), (3.20), (3.25) and (3.26), we obtain the upper bound in Theorem 1.2:

$$\|P^+\|_{L^p(\mathbb{H},\mu)}\lesssim([\mu,\nu]^{0,0}_p)^{1/p}+pp'([\mu,\nu]^{1,0}_p+[\mu,\nu]^{0,1}_p)+(pp')^2([\mu,\nu]^{1,1}_p)^{\max\{1,\frac{1}{p-1}\}}$$

3.2. Proof for the lower bound

Now we turn to show the lower bound

$$[\mu,\nu]_p^{\frac{1}{2p}}\lesssim\|P\|_{L^p(\mathbb{H},\mu dV)}$$

in Theorem 1.2. By the proof of Lemma 2.7,

$$\|P\|_{L^p(\mathbb{H},\mu dV)}=\|M_{z_2}QM_\nu:L^p(\mathbb{H},\nu dV)\to L^p(\mathbb{H},\mu dV)\|.\tag{3.27}$$

It suffices to show that $[\mu,\nu]_p\leq\|M_{z_2}QM_\nu:L^p(\mathbb{H},\nu dV)\to L^p(\mathbb{H},|z_2|^{-p}\mu dV)\|^{2p}$. For simplicity, we set $\mathcal{A}:=\|M_{z_2}QM_\nu:L^p(\mathbb{H},\nu dV)\to L^p(\mathbb{H},|z_2|^{-p}\mu dV)\|$. Set $|z_2|^{-p}\mu=\mu_p$. If $\mathcal{A}<\infty$, then we have a weak-type (p,p) estimate:

$$\mu_p\{(w_1,w_2)\in\mathbb{H}:|M_{z_2}QM_\nu f(w_1,w_2)|>\lambda\}\lesssim\frac{\mathcal{A}^p}{\lambda^p}\|f\|^p_{L^p(\mathbb{H},\nu dV)}.\tag{3.28}$$

We choose $f(w_1,w_2)=1_{\hat{K}'_{\gamma,\eta}}(w_1,w_2)$ with γ and η to be determined. Then

$$\begin{aligned}
&|M_{z_2} Q M_\nu 1_{\hat{K}'_{\gamma,\eta}}(z_1, z_2)| \\
&= \left| \int\limits_{\hat{K}'_{\gamma,\eta}} \frac{1}{\pi^2 (1 - \frac{z_1 \bar{w}_1}{z_2 \bar{w}_2})^2 (1 - z_2 \bar{w}_2)^2} \nu(w_1, w_2) dV(w_1, w_2) \right| \\
&= \left| \int\limits_{\hat{K}_{\gamma,\eta}} \frac{1}{\pi^2 (1 - \frac{z_1}{z_2} \bar{t}_1)^2 (1 - z_2 \bar{w}_2)^2} \nu(t_1 w_2, w_2) |w_2|^2 dV(t_1, w_2) \right| \\
&= \left| P_{\mathbb{D}^2}(|w_2|^2 \nu(t_1 w_2, w_2) 1_{\hat{K}_\gamma \times \hat{K}_\eta}(t_1, w_2))(z_1/z_2, z_2) \right|. \qquad (3.29)
\end{aligned}$$

Here $P_{\mathbb{D}^2}$ is the Bergman projection on the polydisc $\mathbb{D}^2$.

Recall that for a point $z \in \mathbb{D}$ and a tree structure $\mathcal{T}$, the generation $\text{gen}(z)$ equals N if $z \in K_j^N$ for some j. By [4, Lemma 5], there exists an integer N so that for $\gamma \in \mathcal{T}_m$ with $\text{gen}(\gamma) > N$, there is a $\gamma' \in \mathcal{T}_{m'}$ with $\text{gen}(\gamma) = \text{gen}(\gamma')$ such that for any fixed $z \in \hat{K}_{\gamma'}$ and all $w \in \hat{K}_\gamma$ there holds,

$$(1 - z\bar{w})^{-2} = (1 - z\bar{\gamma})^{-2} + ((1 - z\bar{w})^{-2} - (1 - z\bar{\gamma})^{-2}),$$

where $|(1 - z\bar{w})^{-2} - (1 - z\bar{\gamma})^{-2}| \leq 2^{-1}|1 - z\bar{\gamma}|^{-2}$ and $|1 - z\bar{\gamma}|^2 \approx |\hat{K}_\gamma|$. Moreover, an elementary geometric argument yields that $\arg((1 - z\bar{w})^{-2}, (1 - z\bar{\gamma})^{-2}) \leq \pi/6$ for all $w \in \hat{K}_\gamma$. Thus for $(\gamma, \eta) \in \mathcal{T}_m \times \mathcal{T}_n$ with $\text{gen}(\gamma), \text{gen}(\eta) > N$, there is a $(\gamma', \eta') \in \mathcal{T}_{m'} \times \mathcal{T}_{n'}$ with $\text{gen}(\gamma) = \text{gen}(\gamma')$ and $\text{gen}(\eta) = \text{gen}(\eta')$ such that for any fixed $(z_1/z_2, z_2) \in \hat{K}_{\gamma'} \times \hat{K}_{\eta'}$ there holds:

$$\arg\left(\left(1 - \frac{z_1}{z_2}\bar{t}_1\right)^{-2} (1 - z_2\bar{w}_2)^{-2}, \left(1 - \frac{z_1}{z_2}\bar{\gamma}\right)^{-2} (1 - z_2\bar{\eta})^{-2}\right) \leq \pi/3,$$

for all $(t_1, w_2) \in \hat{K}_\gamma \times \hat{K}_\eta$. Hence

$$\begin{aligned}
&|P_{\mathbb{D}^2}(|w_2|^2 \nu(t_1 w_2, w_2) 1_{\hat{K}_\gamma \times \hat{K}_\eta}(t_1, w_2))(z_1/z_2, z_2)| \\
&= \left| \int\limits_{\hat{K}_{\gamma,\eta}} \frac{1}{\pi^2 (1 - \frac{z_1}{z_2}\bar{t}_1)^2 (1 - z_2\bar{w}_2)^2} \nu(t_1 w_2, w_2) |w_2|^2 dV(t_1, w_2) \right| \\
&\geq 16^{-1} \int\limits_{\hat{K}_{\gamma,\eta}} \frac{1}{\pi^2 |1 - \frac{z_1}{z_2}\bar{\gamma}|^2 |1 - z_2\bar{\eta}|^2} \nu(t_1 w_2, w_2) |w_2|^2 dV(t_1, w_2) \\
&> c_1 \langle |w_2|^2 \nu(t_1 w_2, w_2) \rangle^{dV}_{\hat{K}_\gamma \times \hat{K}_\eta},
\end{aligned}$$

for some constant c_1. Thus via the biholomorphism between $\mathbb{D} \times \mathbb{D}^*$ and $\mathbb{H}$, the following containment holds:

$$\hat{K}'_{\gamma',\eta'} \subseteq \{(w_1,w_2) \in \mathbb{H} : |M_{z_2} Q M_\nu f(w_1,w_2)| > c_1 \langle |w_2|^2 \nu(t_1 w_2, w_2)\rangle^{dV}_{\hat{K}_\gamma \times \hat{K}_\eta}\}. \tag{3.30}$$

By [4, Lemma 4], there holds that $\nu(\mathbb{H}) < \infty$. Hence

$$\langle |w_2|^2 \nu(t_1 w_2, w_2)\rangle^{dV}_{\hat{K}_\gamma \times \hat{K}_\eta} = \langle |w_2|^2 \nu\rangle^{du}_{\hat{K}'_{\gamma,\eta}} < \infty.$$

Inequality (3.28) then implies

$$\mu_p(\hat{K}'_{\gamma',\eta'}) \leq \mathcal{A}^p \left(\langle |w_2|^2 \nu\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\right)^{-p} \nu(\hat{K}'_{\gamma,\eta}), \tag{3.31}$$

which is equivalent to $\langle |w_2|^{2-p}\mu\rangle^{du}_{\hat{K}'_{\gamma',\eta'}} \left(\langle |w_2|^2 \nu\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\right)^{p-1} \lesssim \mathcal{A}^p$. Since one can interchange the roles of γ,η and γ',η' in the proof of [4, Lemma 5], there holds

$$\langle |w_2|^{2-p}\mu\rangle^{du}_{\hat{K}'_{\gamma,\eta}} \left(\langle |w_2|^2 \nu\rangle^{du}_{\hat{K}'_{\gamma',\eta'}}\right)^{p-1} \lesssim \mathcal{A}^p.$$

Combining these two inequalities, we have

$$\left(\langle |w_2|^{2-p}\mu\rangle^{du}_{\hat{K}'_{\gamma,\eta}} \left(\langle |w_2|^2 \nu\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\right)^{p-1}\right) \left(\langle |w_2|^{2-p}\mu\rangle^{du}_{\hat{K}'_{\gamma',\eta'}} \left(\langle |w_2|^2 \nu\rangle^{du}_{\hat{K}'_{\gamma',\eta'}}\right)^{p-1}\right) \lesssim \mathcal{A}^{2p}. \tag{3.32}$$

By Hölder's inequality,

$$u(\hat{K}'_{\gamma,\eta})^p \leq \int_{\hat{K}'_{\gamma,\eta}} |w_2|^{2-p}\mu du \left(\int_{\hat{K}'_{\gamma,\eta}} |w_2|^2 \nu du\right)^{p-1} \tag{3.33}$$

for any $(\gamma,\eta) \in \mathcal{T}_{m,n}$. Therefore $\langle |w_2|^{2-p}\mu\rangle^{du}_{\hat{K}'_{\gamma,\eta}} \left(\langle |w_2|^2 \nu\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\right)^{p-1} \gtrsim 1$ for all $\gamma,\eta \in \mathcal{T}_{m,n}$. Applying this to (3.32) and taking the supremum of the left side of (3.32) for $\text{gen}(\gamma) > N$ and $\text{gen}(\eta) > N$, there holds

$$\sup_{\substack{(\gamma,\eta)\in\mathcal{T}_{m,n},\\ \text{gen}(\gamma),\text{gen}(\eta)>N}} \langle |w_2|^{2-p}\mu\rangle^{du}_{\hat{K}'_{\gamma,\eta}} \left(\langle |w_2|^2 \nu\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\right)^{p-1} \lesssim \mathcal{A}^{2p}. \tag{3.34}$$

We turn to show that (3.34) also holds when the supremum is taken over tents where either $\text{gen}(\gamma) \leq N$ or $\text{gen}(\eta) \leq N$.

Suppose that both $\text{gen}(\gamma) \leq N$ and $\text{gen}(\eta) \leq N$. Then $\hat{K}_\gamma$ and $\hat{K}_\eta$ are big tents on the unit disk $\mathbb{D}$ and $|\hat{K}_\gamma| = |\hat{K}_\eta| \approx 1$. Set $B_{1/4} = \{z \in \mathbb{C} : |z| < 1/4\}$. Then for any given $z \in \mathbb{D}$, $|z\bar{w}| < 1/4$ for $w \in B_{1/4}$. Therefore $\text{Arg}((1 - z\bar{w})^2) \subseteq [-\frac{\pi}{6}, \frac{\pi}{6}]$. Applying this fact, we obtain

$$
\begin{aligned}
&\left|P_{\mathbb{D}^2}(|w_2|^2\nu(t_1w_2,w_2)1_{B_{1/4}\times B_{1/4}}(t_1,w_2))\left(\frac{z_1}{z_2},z_2\right)\right| \\
=&\left|\int\limits_{B_{1/4}\times B_{1/4}}\frac{|w_2|^2}{\pi^2(1-\frac{z_1}{z_2}\bar{t}_1)^2(1-z_2\bar{w}_2)^2}\nu(t_1w_2,w_2)dV(t_1,w_2)\right| \\
\geq&16^{-1}\left|\int\limits_{B_{1/4}\times B_{1/4}}\pi^{-2}|w_2|^2\nu(t_1w_2,w_2)dV(t_1,w_2)\right|\geq c_2\langle|w_2|^2\nu(t_1w_2,w_2)\rangle^{dV}_{B_{1/4}\times B_{1/4}}
\end{aligned} \tag{3.35}
$$

for some constant c_2. Therefore,

$$
\begin{aligned}
\mathbb{D}^2=\big\{(z_1,z_2)\in\mathbb{D}^2:&|P_{\mathbb{D}}(|w_2|^2\nu(t_1w_2,w_2)1_{B_{1/4}\times B_{1/4}})(z_1,z_2)| \\
&>c_2\langle|z_2|^2\nu(t_1z_2,z_2)\rangle^{dV}_{B_{1/4}\times B_{1/4}}\big\}.
\end{aligned}
$$

Let $B'_{1/4,1/4}$ denote the set $\{(w_1,w_2)\in\mathbb{H}:(\frac{w_1}{w_2},w_2)\in B_{1/4}\times B_{1/4}\}$. Via the bihomomorphism between $\mathbb{D}\times\mathbb{D}^*$ and $\mathbb{H}$, we obtain

$$
\begin{aligned}
\mu_p(\mathbb{H})&=\mu_p\left\{(z_1,z_2)\in\mathbb{H}:|P_{\mathbb{H}}(\nu 1_{B'_{1/4}})(z_1,z_2)|>c_2\langle|z_2|^2\nu\rangle^{du}_{B'_{1/4,1/4}}\right\} \\
&\leq\frac{\mathcal{A}^p\|1_{B'_{1/4}}\|^p_{L^p(\mathbb{H},\nu dV)}}{c_2^p\left(\langle|z_2|^2\nu\rangle^{du}_{B'_{1/4,1/4}}\right)^p}.
\end{aligned}
$$

Thus

$$
\langle|w_2|^{2-p}\mu\rangle^{du}_{\mathbb{H}}\left(\langle|w_2|^2\nu\rangle^{du}_{B'_{1/4,1/4}}\right)^{p-1}\lesssim\mathcal{A}^p.
$$

Interchanging the role of variables z and w, we also have

$$
\begin{aligned}
\mu_p(B'_{1/4,1/4})&=\mu_p\left\{(w_1,w_2)\in B'_{1/4,1/4}:|P_{\mathbb{H}}(\nu 1_{\mathbb{H}})(w_1,w_2)|>c_2\langle|w_2|^2\nu\rangle^{du}_{\mathbb{H}}\right\} \\
&\leq\frac{\mathcal{A}^p\|1\|^p_{L^p(\mathbb{H},\nu dV)}}{c_2^p\left(\langle|w_2|^2\nu\rangle^{du}_{\mathbb{H}}\right)^p}.
\end{aligned}
$$

Thus

$$
\langle|w_2|^{2-p}\mu\rangle^{du}_{B'_{1/4,1/4}}(\langle|w_2|^2\nu\rangle^{du}_{\mathbb{H}})^{p-1}\lesssim\mathcal{A}^p.
$$

By Hölder's inequality

$$
\langle|w_2|^{2-p}\mu\rangle^{du}_{B'_{1/4,1/4}}\left(\langle|w_2|^2\nu\rangle^{du}_{B'_{1/4,1/4}}\right)^{p-1}\geq 1.
$$

Combining these inequalities yields that

$$\begin{aligned}&\langle |w_2|^{2-p}\mu\rangle_{\mathbb{H}}^{du}(\langle |w_2|^2\nu\rangle_{\mathbb{H}}^{du})^{p-1}\\ \lesssim&\langle |w_2|^{2-p}\mu\rangle_{B'_{1/4,1/4}}^{du}(\langle |w_2|^2\nu\rangle_{\mathbb{H}}^{du})^{p-1}\langle |w_2|^{2-p}\mu\rangle_{\mathbb{H}}^{du}\left(\langle |w_2|^2\nu\rangle_{B'_{1/4,1/4}}^{du}\right)^{p-1}\lesssim \mathcal{A}^{2p} \end{aligned}\tag{3.36}$$

Therefore, $|\hat{K}'_{\gamma,\eta}|\approx 1$ implies

$$\langle |w_2|^{2-p}\mu\rangle_{\hat{K}'_{\gamma,\eta}}^{du}\left(\langle |w_2|^2\nu\rangle_{\hat{K}'_{\gamma,\eta}}^{du}\right)^{p-1}\lesssim \langle |w_2|^{2-p}\mu\rangle_{\mathbb{H}}^{du}(\langle |w_2|^2\nu\rangle_{\mathbb{H}}^{du})^{p-1}\lesssim \mathcal{A}^{2p}. \tag{3.37}$$

For the case $\text{gen}(\gamma)\le N$ and $\text{gen}(\eta)>N$, we combine the arguments for both the big tents and the small tents. There exists an η' with $\text{gen}(\eta)=\text{gen}(\eta')$ such that for all $\frac{z_1}{z_2}\in\mathbb{D}$ and $z_2\in\hat{K}_{\eta'}$, there holds:

$$\begin{aligned}&|P_{\mathbb{D}^2}(|w_2|^2\nu(t_1w_2,w_2)1_{B_{1/4}\times\hat{K}_\eta}(t_1,w_2))(z_1/z_2,z_2)|\\ =&\left|\int\limits_{B_{1/4}\times\hat{K}_\eta}\frac{|w_2|^2\nu(t_1w_2,w_2)}{\pi^2(1-\frac{z_1}{z_2}\bar{t}_1)^2(1-z_2\bar{w}_2)^2}dV(t_1,w_2)\right|\\ \ge&16^{-1}\int\limits_{B_{1/4}\times\hat{K}_\eta}\frac{|w_2|^2\nu(t_1w_2,w_2)}{\pi^2|1-z_2\bar{\eta}|^2}dV(t_1,w_2)>c_3\langle |w_2|^2\nu(t_1w_2,w_2)\rangle_{\hat{K}_0\times\hat{K}_\eta}^{dV},\end{aligned}$$

for some constant c_3. Set $B'_{1/4,\eta}=\{(w_1,w_2)\in\mathbb{H}:(\frac{w_1}{w_2},w_2)\in B_{1/4}\times\hat{K}_\eta\}$. Via the biholomorphism between $\mathbb{D}\times\mathbb{D}^*$ and $\mathbb{H}$ again, the following containment holds:

$$\hat{K}'_{0,\eta'}\subseteq\left\{(w_1,w_2)\in\mathbb{H}:|M_{z_2}QM_\nu 1_{B'_{1/4,\eta}}(w_1,w_2)|>\frac{c_3}{32}\langle |w_2|^2\nu(t_1w_2,w_2)\rangle_{\hat{K}_0\times\hat{K}_\eta}^{dV}\right\}. \tag{3.38}$$

Applying the proof for inequalities (3.34) and (3.37) to (3.38) gives

$$\langle |w_2|^{2-p}\mu\rangle_{\hat{K}'_{\gamma,\eta}}^{du}\left(\langle |w_2|^2\nu\rangle_{\hat{K}'_{\gamma,\eta}}^{du}\right)^{p-1}\lesssim \langle |w_2|^{2-p}\mu\rangle_{\hat{K}'_{0,\eta}}^{du}\left(\langle |w_2|^2\nu\rangle_{\hat{K}'_{0,\eta}}^{du}\right)^{p-1}\lesssim \mathcal{A}^{2p}. \tag{3.39}$$

The last case $\text{gen}(\eta)\le N_n$ and $\text{gen}(\gamma)>N_m$ follows from a similar argument with the role of γ and η interchanged. Combining all these estimates, we obtain the desired lower bound:

$$[\mu,\nu]_p=\sup_{(\gamma,\eta)\in\mathcal{T}_{m,n}}\langle |w_2|^{2-p}\mu\rangle_{\hat{K}'_{\gamma,\eta}}^{du}\left(\langle |w_2|^2\nu\rangle_{\hat{K}'_{\gamma,\eta}}^{du}\right)^{p-1}\lesssim \mathcal{A}^{2p}, \tag{3.40}$$

which completes the proof of Theorem 1.2.

4. Examples

We begin by providing a sharp example for the upper bound estimate in Theorem 1.2.

4.1. A sharp example for the upper bound

We give an example for the case $1 < p \leq 2$ here. The case $p > 2$ follows from a duality argument. The idea is based on the construction of the sharp examples in [28] and [29]. Recall $B_{1/4} = \{z \in \mathbb{C} : |z| < 1/4\}$. Given a number $1 > s > 0$, we set

$$\mu(w_1, w_2) = |w_2|^{p-2} \frac{|(1 - w_1/w_2)(1 - w_2)|^{2(p-1)(1-s)}}{|w_1/w_2|^{2-s}|w_2|^{2-s}}, \tag{4.1}$$

$$f(w_1, w_2) = \bar{w}_2 \mu^{\frac{1}{1-p}}(w_1, w_2) 1_{T_{\frac{1}{2}}}(w_1/w_2) 1_{T_{\frac{1}{2}}}(w_2). \tag{4.2}$$

Then for $(t_1, t_2) \in \mathbb{D}^2$ with T'_{t_1,t_2} not intersecting the tent $T'_{|t_1|,|t_2|}$ and away from the point $(0, 0)$, there holds $\mu(w_1, w_2) \approx 1$ and hence $\langle \mu |w_2|^{2-p} \rangle^{du}_{T'_{t_1,t_2}} \left(\langle |w_2|^2 \nu \rangle^{du}_{T'_{t_1,t_2}} \right)^{p-1} \approx 1$.

When T'_{t_1,t_2} intersects the tent $T'_{|t_1|,|t_2|}$ with $|t_1|, |t_2| \geq 1/2$, there exists a positive constant $c > 0$ such that $T'_{t_1,t_2} \subseteq T'_{c|t_1|,c|t_2|}$ and $u(T'_{t_1,t_2}) \approx u(T'_{c|t_1|,c|t_2|})$. Thus

$$\begin{aligned}
&\int_{T'_{t_1,t_2}} |w_2|^{2-p} \mu(w_1, w_2) du(w_1, w_2) \\
&\lesssim \int_{T'_{c|t_1|,c|t_2|}} |w_2|^{2-p} \mu(w_1, w_2) du(w_1, w_2) \\
&= \int_{T_{c|t_1|} \times T_{c|t_2|}} |(1 - w_1)(1 - w_2)|^{(p-1)(2-2s)} dV(w_1, w_2) \\
&= \prod_{j=1}^{2} \int_{\{w_j \in \mathbb{D} : |1-w_j| < 1-c|t_j|\}} |1 - w_j|^{(p-1)(2-2s)} dV(w_j).
\end{aligned} \tag{4.3}$$

Using the changes of variables $z_j = i\frac{1-w_j}{1+w_j}$, we have

$$\begin{aligned}
&\int_{\{w_j \in \mathbb{D} : |1-w_j| < 1-c|t_j|\}} \left| \frac{1 - w_j}{1 + w_j} \right|^{(p-1)(2-2s)} dV(w_j) \\
&\approx \int_{\{z_j \in \mathbb{C} : |z_j| < 1-c|t_j|, \operatorname{Im} z_j > 0\}} |z_j|^{(p-1)(2-2s)} dV(z_j) \approx \frac{(1 - c|t_j|)^{(p-1)(2-2s)+2}}{(p-1)(2-2s) + 2}.
\end{aligned} \tag{4.4}$$

Thus

$$\int_{T'_{t_1,t_2}} |w_2|^{2-p} \mu(w_1, w_2) du(w_1, w_2) \approx \prod_{j=1}^{2} \frac{(1 - c|t_j|)^{(p-1)(2-2s)+2}}{(p-1)(2-2s) + 2}. \tag{4.5}$$

Similarly, for $\nu = |w_2|^{-p'} \mu^{\frac{-p'}{p}}$,

$$\int_{T'_{t_1,t_2}} |w_2|^2 \nu(w_1, w_2) du(w_1, w_2) \approx \prod_{j=1}^{2} \frac{(1 - c|t_j|)^{2s}}{2s}. \tag{4.6}$$

Since $1 < p \leq 2$ and $0 < s < 1$, we have

$$[\mu, \nu]_p^{1,1} = \sup_{\substack{t_1,t_2 \in \mathbb{D} \\ |t_1|,|t_2| \geq 1/2}} \langle |w_2|^{2-p} \mu \rangle^{du}_{T'_{t_1,t_2}} \left(\langle |w_2|^2 \nu \rangle^{du}_{T'_{t_1,t_2}} \right)^{p-1} \lesssim s^{-2(p-1)}.$$

Moreover,

$$[\mu, \nu]_p^{0,0} = \langle |w_2|^{2-p} \mu \rangle^{du}_{\mathbb{H}} \left(\langle \nu |w_2|^2 \rangle^{du}_{\mathbb{H}} \right)^{p-1} \approx s^{-2p};$$

$$[\mu, \nu]_p^{1,0} = \left(\langle |z_2|^{2-p} \mu_2 \rangle^{dV}_{\mathbb{D}} (\langle \nu_2 \rangle^{dV}_{\mathbb{D}})^{p-1} \right)^{\frac{1}{p}} \left(\sup_{\substack{z \in \mathbb{D}, \\ |z| > 1/2}} \langle \mu_1 \rangle^{dV}_{T_z} \left(\langle \nu_1 \rangle^{dV}_{T_z} \right)^{p-1} \right)^{\max\{1, \frac{1}{p-1}\}} \approx s^{-2};$$

$$[\mu, \nu]_p^{0,1} = \left(\sup_{\substack{z \in \mathbb{D}, \\ |z| > 1/2}} \langle |z_2|^{2-p} \mu_2 \rangle^{dV}_{T_z} (\langle \nu_2 \rangle^{dV}_{T_z})^{p-1} \right)^{\max\{1, \frac{1}{p-1}\}} \left(\langle \mu_1 \rangle^{dV}_{\mathbb{D}} \left(\langle \nu_1 \rangle^{dV}_{\mathbb{D}} \right)^{p-1} \right)^{\frac{1}{p}} \approx s^{-2}.$$

Thus the upper bound

$$([\mu, \nu]_p^{0,0})^{\frac{1}{p}} + pp'([\mu, \nu]_p^{1,0} + [\mu, \nu]_p^{0,1}) + (pp')^2([\mu, \nu]_p^{1,1})^{\max\{1, \frac{1}{p-1}\}} \lesssim s^{-2}.$$

When $w_1/w_2, w_2 \in T_{1/2}$, there hold $|w_1|/|w_2|, |w_2| \approx 1$. Therefore

$$\|f\|^p_{L^p(\mathbb{H},\mu)} \approx \int_{T'_{1/2,1/2}} \mu^{\frac{p}{1-p}}(w_1, w_2) \mu(w_1, w_2) dV(w_1, w_2) \approx \langle |w_2|^2 \nu \rangle^{du}_{T'_{1/2,1/2}} \approx s^{-2}. \tag{4.7}$$

For $z_1/z_2, z_2 \in B_{1/4}$, we claim that

$$|P(f)(z_1, z_2)| \gtrsim |z_2|^{-1} \langle f \rangle^{dV}_{T'_{1/2,1/2}}. \tag{4.8}$$

Note that for $w \in T_{\frac{1}{2}}$ and $z \in B_{1/4}$, there holds that $|1 - z\bar{w}| \approx 1$ and

$$\arg\{(1 - z\bar{w}), 1\} \in (-\arcsin(1/4), \arcsin(1/4)).$$

Using these facts and the formula (2.2) for the Bergman projection, we have

$$\begin{aligned}
|P(f)(z_1,z_2)| &= \left| \int\limits_{T'_{1/2,1/2}} \frac{f(w_1,w_2)}{\pi^2 z_2\bar{w}_2(1-\frac{z_1\bar{w}_1}{z_2\bar{w}_2})^2(1-z_2\bar{w}_2)^2} dV(w_1,w_2) \right| \\
&= |z_2|^{-1} \left| \int\limits_{T_{1/2,1/2}} \frac{f(w_1w_2,w_2)w_2}{\pi^2(1-\frac{z_1}{z_2}\bar{w}_1)^2(1-z_2\bar{w}_2)^2} dV(w_1,w_2) \right| \\
&\gtrsim |z_2|^{-1} \int\limits_{T_{1/2,1/2}} |1-w_1/w_2|^{2s-2}|1-w_2|^{2s-2} dV(w_1,w_2) \\
&\approx |z_2|^{-1} \langle f \rangle^{dV}_{T'_{1/2,1/2}}. \qquad (4.9)
\end{aligned}$$

Since $|w_1|/|w_2|, |w_2| \approx 1$ for $(w_1,w_2) \in T'_{1/2,1/2}$, we have

$$\langle f \rangle_{T'_{1/2,1/2}} \approx \int\limits_{T'_{1/2,1/2}} |(1-w_1/w_2)(1-w_2)|^{2(s-1)} \, dV(w_1,w_2) \approx \langle |w_2|^2 \nu \rangle^{du}_{T'_{1/2,1/2}} \approx s^{-2}. \qquad (4.10)$$

Thus $|P(f)(z_1,z_2)| \gtrsim |z_2|^{-1}s^{-2}$. Moreover,

$$\begin{aligned}
\|Pf\|^p_{L^p(\mathbb{H},\mu dV)} &= \int\limits_{\mathbb{H}} |Pf(z_1,z_2)|^p \mu(z_1,z_2) dV(z_1,z_2) \\
&\geq s^{-2p} \int\limits_{\{(z_1/z_2,z_2)\in B_{1/4}\times B_{1/4}\}} |z_2|^{-p}\mu(z_1,z_2)dV(z_1,z_2) \\
&\approx s^{-2p} \int\limits_{\{(t,z_2)\in B_{1/4}\times B_{1/4}\}} |t|^{s-2}|z_2|^{s-2} dV(t,z_2') \approx s^{-2p-2}. \qquad (4.11)
\end{aligned}$$

Thus the desired estimates hold:

$$\begin{aligned}
\frac{\|Pf\|^p_{L^p(\mathbb{H},\mu dV)}}{\|f\|^p_{L^p(\mathbb{H},\mu dV)}} &\gtrsim s^{-2p} \\
&\gtrsim \left(([\mu,\nu]^{0,0}_p)^{\frac{1}{p}} + pp'([\mu,\nu]^{1,0}_p + [\mu,\nu]^{0,1}_p) + (pp')^2([\mu,\nu]^{1,1}_p)^{\max\{1,\frac{1}{p-1}\}} \right)^p.
\end{aligned}$$

4.2. L^p regularity of the Bergman projection on the Hartogs triangle

If weight μ is identically 1, then μdV is the Lebesgue measure on the Hartogs triangle, and $\|P^+\|_{L^p(\mathbb{H},\mu)}$ is the unweighted L^p norm of the Bergman projection. Chakrabarti and Zeytuncu showed in [6] that the Bergman projection on the Hartogs triangle is L^p regular if and only if $\frac{4}{3} < p < 4$. Using Theorem 1.2, we give an alternative proof of this L^p regularity result.

Set $\mu \equiv 1$. Then $\nu = |w_2|^{-p'}$ and

$$[\mu,\nu]_p = \sup_{\substack{(\gamma,\eta)\in\mathcal{T}_{m,n}\\ 1\le m,n\le M}} \langle |w_2|^{2-p}\rangle^{du}_{\hat{K}'_{\gamma,\eta}} \left(\langle |w_2|^{2-p'}\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\right)^{p-1}.$$

When $p \geq 4$ or $p \leq 4/3$, we have $\int_{\mathbb{H}} |w_2|^{2-p} du \int_{\mathbb{H}} |w_2|^{2-p'} du = \infty$. Thus $[\mu,\nu]_p = \infty$ for $p \notin (\frac{4}{3}, 4)$. By Theorem 1.2, the Bergman projection P is not bounded on $L^p(\mathbb{H})$.

When $p \in (\frac{4}{3}, 4)$, we have for $(\gamma,\eta) \in \mathcal{T}_{m,n}$ where $1 \leq m,n \leq M$ that

$$\langle |w_2|^{2-p}\rangle^{du}_{\hat{K}'_{\gamma,\eta}} \left(\langle |w_2|^{2-p'}\rangle^{du}_{\hat{K}'_{\gamma,\eta}}\right)^{p-1} = \langle |w_2|^{2-p}\rangle^{dV}_{\hat{K}_\eta} \left(\langle |w_2|^{2-p'}\rangle^{dV}_{\hat{K}_\eta}\right)^{p-1}.$$

Therefore for all $w_2 \in \hat{K}_\eta$ with $|\eta| \geq 1/2$, we have $|w_2| \gtrsim 1$ and

$$\langle |w_2|^{2-p}\rangle^{dV}_{\hat{K}_\eta} \left(\langle |w_2|^{2-p'}\rangle^{dV}_{\hat{K}_\eta}\right)^{p-1} \approx 1. \tag{4.12}$$

When $\eta = 0$, $\hat{K}_\eta = \mathbb{D}$ and

$$\langle |w_2|^{2-p}\rangle^{dV}_{\mathbb{D}} \left(\langle |w_2|^{2-p'}\rangle^{dV}_{\mathbb{D}}\right)^{p-1} = [\mu,\nu]^{0,0}_p = \frac{2}{4-p}\left(\frac{2(p-1)}{3p-4}\right)^{p-1}. \tag{4.13}$$

Similarly, we have the estimates for the other $[\mu,\nu]^{i,j}_p$:

$$[\mu,\nu]^{1,0}_p \approx \left(\langle |w_2|^{2-p}\rangle^{dV}_{\mathbb{D}} (\langle |w_2|^{2-p'}\rangle^{dV}_{\mathbb{D}})^{p-1}\right)^{\frac{1}{p}} = \left(\frac{2}{4-p}\left(\frac{2(p-1)}{3p-4}\right)^{p-1}\right)^{1/p};$$

$$[\mu,\nu]^{0,1}_p \approx [\mu,\nu]^{1,1}_p \approx 1.$$

Hence for $p \in (3/4, 4)$,

$$\begin{aligned}&([\mu,\nu]^{0,0}_p)^{\frac{1}{p}} + pp'([\mu,\nu]^{1,0}_p + [\mu,\nu]^{0,1}_p) + (pp')^2([\mu,\nu]^{1,1}_p)^{\max\{1,\frac{1}{p-1}\}}\\ &\approx \left(\frac{2}{4-p}\left(\frac{2(p-1)}{3p-4}\right)^{p-1}\right)^{1/p} \approx (4-p)^{-1/p}(3p-4)^{-1/p'} < \infty.\end{aligned} \tag{4.14}$$

Theorem 1.2 gives the norm estimate of the Bergman projection P for $\frac{4}{3} < p < 4$,

$$\|P\|_{L^p(\mathbb{H})} \lesssim (4-p)^{-1/p}(3p-4)^{-1/p'}, \tag{4.15}$$

which implies the blowing up of $\|P\|_{L^p(\mathbb{H})}$ as $p \to \frac{4}{3}^+$ or $p \to 4^-$. This fact can also be checked by computing the quotient $\|P(|z_2|^{-p'}\bar{z}_2)\|^p_{L^p(\mathbb{H})}/\||z_2|^{-p'}\bar{z}_2\|^p_{L^p(\mathbb{H})}$ for the cases $p \to 4^-$ and $p \to \frac{4}{3}^+$. Moreover, the estimate (4.15) is sharp in the sense that

$$\frac{\|P(|z_2|^{-p'}\bar{z}_2)\|_{L^p(\mathbb{H})}}{\||z_2|^{-p'}\bar{z}_2\|_{L^p(\mathbb{H})}} \approx \frac{1}{(4-p)^{1/p}(3p-4)^{1/p'}}.$$

4.3. The case $\mu(w_1, w_2) = |w_1|^a|w_2|^b$

When the weight $\mu(w_1, w_2) = |w_1|^a|w_2|^b$, the weight $\nu = |w_1|^{-ap'/p}|w_2|^{-p'(1+b/p)}$. The singularity of the weight only occurs at places where w_1 or w_2 vanishes. Hence the blowing up of weights at both $z_1 = 0$ and $z_2 = 0$ can only be captured by computing the average of the weights over the entire Hartogs triangle. Thus $[\mu, \nu]_p^{0,0}$ will be the largest term among $[\mu, \nu]_p^{i,j}$. By a change of variables we obtain

$$\begin{aligned}[\mu,\nu]_p^{0,0} &= \langle |w_1|^a|w_2|^{2-p+b}\rangle_{\mathbb{H}}^{du}\left(\langle |w_1|^{-ap'/p}|w_2|^{2-p'(1+b/p)}\rangle_{\mathbb{H}}^{du}\right)^{p-1} \\ &= \langle |w_1|^a\rangle_{\mathbb{D}}^{dV}\left(\langle |w_1|^{-\frac{ap'}{p}}\rangle_{\mathbb{D}}^{dV}\right)^{p-1}\langle |w_2|^{2+a-p+b}\rangle_{\mathbb{D}}^{dV}\left(\langle |w_2|^{2-p'(1+\frac{a+b}{p})}\rangle_{\mathbb{D}}^{dV}\right)^{p-1}.\end{aligned} \tag{4.16}$$

A computation using polar coordinates implies the following estimate for $[\mu, \nu]_p^{0,0}$:

- $[\mu, \nu]_p^{0,0} \approx (a+2)^{-1}(2-\frac{ap'}{p})^{1-p}(4-p+a+b)^{-1}(4-p'(1+\frac{a+b}{p}))^{1-p}$, for $-2 < a < 2(p-1)$ and $p-4 < a+b < 3p-4$;
- $[\mu, \nu]_p^{0,0} = \infty$ otherwise.

Rearranging these inequalities, we conclude that the Bergman projection P is L^p regular for $p \geq 2$ if and only if $\max\{1, \frac{a+1}{2}, \frac{a+b+4}{3}\} < p < a+b+4$. The p range we obtain here is not of form $(\alpha, \frac{\alpha}{\alpha-1})$. This is because, for the case a or b is not zero, the Bergman projection operator is not self-adjoint on $L^p(\mathbb{H}, \mu dV)$, and is not necessarily L^2 bounded.

4.4. L^p *regularity of the Bergman projection on the generalized Hartogs triangle*

In [14], Edholm and McNeal studied the L^p boundedness of the Bergman projection on the generalized Hartogs triangle

$$\mathbb{H}_{m/n} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^m < |z_2|^n < 1\},$$

where $m, n \in \mathbb{Z}^+$ with $\gcd(m, n) = 1$. A crucial step in their paper (see [14, Proposition 3.4]) is to analyze the L^p regularity of the integral operator $\mathcal{K}_A$ defined by

$$\mathcal{K}_A(f)(z_1, z_2) := \int\limits_{\mathbb{H}_{m/n}} \frac{|z_2\bar{w}_2|^A}{|1-z_2\bar{w}_2|^2|z_2^n\bar{w}_2^n - z_1^m\bar{w}_1^m|^2} f(w_1, w_2)dV(w_1, w_2).$$

Using the proper map $h : (w_1, w_2) \mapsto (w_1^m w_2^{n-1}, w_2)$ from $\mathbb{H}_{m/n}$ to $\mathbb{H}$, we can relate the L^p norm of $\mathcal{K}_A$ on $\mathbb{H}_{m/n}$ to the weighted L^p norm of the absolute Bergman projection on $\mathbb{H}$:

$$\begin{aligned}\|\mathcal{K}_A\|_{L^p(\mathbb{H}_{m/n})} &= m^{-1}\|M_h P^+ M_h : L^p(\mathbb{H}, \omega_1 dV) \to L^p(\mathbb{H}, \omega_2 dV)\| \\ &= m^{-1}\|P^+ M_h : L^p(\mathbb{H}, \omega_1 dV) \to L^p(\mathbb{H}, \omega_2 h^p dV)\|,\end{aligned}$$

where the weights $\omega_1(w_1, w_2) = |w_1|^{\frac{2m-2}{m}(p-1)}|w_2|^{\frac{2}{m}(n-1)(1-p)}$, $\omega_2(w_1, w_2) = |w_1|^{-2+\frac{2}{m}} \times |w_2|^{\frac{2}{m}(n-1)}$, and $h(w_1, w_2) = |w_2|^{A-2n+1}$. Setting $\mu := \omega_2 h^p$ and $\nu := \omega_1^{-\frac{p'}{p}}|w_2|^{(A-2n)p'}$, we obtain

$$\|P^+ M_h : L^p(\mathbb{H}, \omega_1 dV) \to L^p(\mathbb{H}, \omega_2 h^p dV)\| = \|Q^+ M_\nu : L^p(\mathbb{H}, \nu dV) \to L^p(\mathbb{H}, \mu dV)\|. \tag{4.17}$$

Here ν is no longer equal to $|w_2|^{-p'}\mu^{-\frac{p'}{p}}$ which is the dual weight of $|w_2|^{-p}\mu$ with respect to the measure u. Still, μ and ν are blowing or vanishing at points where z_1 or z_2 vanishes. Thus $[\mu, \nu]_p^{1,1} \approx 1$. Moreover, the constants $[\mu, \nu]_p^{1,0}$ and $[\mu, \nu]_p^{0,1}$ only measure the singularity of the weight in either z_1 or z_2 variable. Therefore

$$([\mu, \nu]_p^{0,0})^{1/p} = \left(\langle \mu|w_2|^{2-p}\rangle_{\mathbb{H}}^{du} \left(\langle |w_2|^2\nu\rangle_{\mathbb{H}}^{du}\right)^{p-1}\right)^{1/p}$$

is the main term in the upper bound of the weighted norm of $Q^+ M_\nu$. By a simple integration, we obtain for $p \in \left(\frac{2n+2m}{Am+2n+2m-2nm}, \frac{2n+2m}{2nm-Am}\right)$,

$$\begin{aligned}[\mu, \nu,]_p^{0,0} \approx &(2n-A)^{-p}\left(\frac{2m+2n}{2mn-Am} - p\right)^{-1}\left(\frac{2m+2n}{2mn-Am} - p'\right)^{1-p} \\ =&(2n-A)^{-p}\left(\frac{2m+2n+Am-2mn}{(p-1)(2mn-Am)}\right)^{1-p} \times \\ &\left(\frac{2m+2n}{2mn-Am} - p\right)^{-1}\left(p - \frac{2m+2n}{Am+2n+2m-2mn}\right)^{1-p} < \infty. \end{aligned} \tag{4.18}$$

Hence, we recover [14, Proposition 3.4]:

$$\mathcal{K}_A \text{ is bounded on } L^p(\mathbb{H}_{m/n}) \text{ if } p \in \left(\frac{2n+2m}{Am+2n+2m-2nm}, \frac{2n+2m}{2nm-Am}\right)$$
$$\text{whenever } Am + 2n + 2m - 2nm > 2nm - Am > 0,$$

and obtain an L^p norm estimate for such a bounded $\mathcal{K}_A$:

$$\|\mathcal{K}_A\|_{L^p(\mathbb{H}_{m/n})} \lesssim m^{-1}([\mu, \nu]_p^{0,0})^{\frac{1}{p}}. \tag{4.19}$$

By [14, Theorem 3.4], the Bergman projection $|P_{\mathbb{H}_{m/n}}(f)(z)| \lesssim m^2\mathcal{K}_A(|f|)(z)$ with $A = 2n-1+\frac{1-n}{m}$. Applying (4.19) and (4.18) to this inequality of $P_{\mathbb{H}_{m/n}}$, we recover the L^p regularity result of the Bergman projection on $\mathbb{H}_{m/n}$, obtained in [14, Corollary 4.7], and obtain an estimate for the L^p norm of $P_{\mathbb{H}_{m/n}}$:

Theorem 4.1. *For $p \in \left(\frac{2m+2n}{m+n+1}, \frac{2m+2n}{m+n-1}\right)$,*

$$\|P_{\mathbb{H}_{m/n}}\|_{L^p(\mathbb{H}_{m/n})} \lesssim m^2(p-1)^{\frac{1}{p'}}\left((2m+2n)-(m+n-1)p\right)^{-\frac{1}{p}} \times \left((n+m+1)p-(2m+2n)\right)^{-\frac{1}{p'}}.$$

It's not clear for us if the norm estimate above is sharp or not especially when the constant m is large. Each operator $\mathcal{K}_A$ above corresponds to a sub-Bergman projection induced by an orthogonal decomposition of the Bergman space in [14]. Since the ranges of these sub-Bergman projections are orthogonal to each other, the norm bound obtained using the inequality $|P_{\mathbb{H}_{m/n}}(f)(z)| \lesssim m^2\mathcal{K}_A(|f|)(z)$ might not be optimal for large m.

5. Remarks and generalizations

1. The assumption $\mu(z_1,z_2)=\mu_1(z_1/z_2)\mu_2(z_2)$ in Theorem 1.2 is used only in the proof of Lemma 2.6. Because of this fact, our lower bound in Theorem 1.2 holds without this assumption:

Corollary 5.1. *Let μ be a weight on $\mathbb{H}$ and set $\nu = |w_2|^{-p'}\mu^{\frac{-p'}{p}}$. If the Bergman projection P is bounded on the corresponding weighted space $L^p(\mathbb{H},\mu dV)$, then $[\mu,\nu]_p<\infty$. Moreover, there holds*

$$\|P\|_{L^p(\mathbb{H},\mu dV)} \gtrsim \left([\mu,\nu]_p\right)^{\frac{1}{2p}}.$$

Using inequality (1.11) and a similar argument for $Q^{1,1}_{m,n,\nu}$ in the proof of Theorem 1.2, we can also generalize our upper bound estimate for P and P^+ as follows:

Corollary 5.2. *Let μ be a weight on $\mathbb{H}$ and set $\nu = |w_2|^{-p'}\mu^{\frac{-p'}{p}}$. Suppose the quantities $\|\mathcal{M}_{\mathcal{T}'_{m,n},\nu}\|_{L^p(\mathbb{H},\nu dV)}$, $\|\mathcal{M}_{\mathcal{T}'_{m,n},|w_2|^{-p}\mu}\|_{L^{p'}(\mathbb{H},|z_2|^{-p}\mu dV)}$, and $[\mu,\nu]_p$ are all finite. Then the operators P and P^+ are bounded on $L^p(\mathbb{H},\mu dV)$. Moreover,*

$$\|P\|_{L^p(\mathbb{H},\mu dV)} \leq \|P^+\|_{L^p(\mathbb{H},\mu dV)} \lesssim \|\mathcal{M}_{\mathcal{T}'_{m,n},\nu}\|_{L^p(\mathbb{H},\nu dV)}\|\mathcal{M}_{\mathcal{T}'_{m,n},|w_2|^{-p}\mu}\|_{L^{p'}(\mathbb{H},|w_2|^{-p}\mu dV)} \times \left([\mu,\nu]_p\right)^{\max\{1,\frac{1}{p-1}\}}.$$

In [17], Fefferman gave a sufficient condition for the boundedness of the maximal operator $M^{(n)}_\mu$ on $\mathbb{R}^n$ defined by

$$\mathcal{M}_{\mu}^{(n)}(f)(x) = \sup_{x \in R} \frac{\int_R |f(t)|\mu(t)dV(t)}{\int_R \mu(t)dV(t)},$$

where R is any rectangle in $\mathbb{R}^n$ with sides parallel to the coordinate axes. He showed that, if the weight μ on $\mathbb{R}^n$ is uniformly in the class A_∞ in each variable separately, then $\mathcal{M}_{\mu}^{(n)}$ is L^p bounded on $L^p(\mathbb{R}^n, \mu)$ for all $1 < p < \infty$. In [1], Aleman, Pott, and Reguera studied the B_∞ weights on the unit disc which is the analogue of the A_∞ weights in the Bergman setting. Using their results and Fefferman's proof, it is possible to give a sufficient condition for the boundedness of $\|\mathcal{M}_{\mathcal{T}'_{m,n},\nu}\|_{L^p(\mathbb{H},\nu dV)}$ and $\|\mathcal{M}_{\mathcal{T}'_{m,n},|w_2|^{-p}\mu}\|_{L^p(\mathbb{H},|z_2|^{-p}\mu dV)}$ in the corollary above. To obtain an upper bound estimate, one also needs to understand the dependence of the quantities $\|\mathcal{M}_{\mathcal{T}'_{m,n},\nu}\|_{L^p(\mathbb{H},\nu dV)}$ and $\|\mathcal{M}_{\mathcal{T}'_{m,n},|w_2|^{-p}\mu}\|_{L^p(\mathbb{H},|z_2|^{-p}\mu dV)}$ on the sufficient condition for the weight ν and $|z_2|^{-p}\mu$.

2. The example in Section 4.1 showed the upper bound estimate in Theorem 1.2 is sharp. It is not clear if the lower bound estimates given in Theorem 1.2, or in [28] and [29] are sharp. It would be interesting to see what a sharp lower bound is in terms of the Bekollé-Bonami constant.

3. We focus on the weighted estimates for the Bergman projection on the Hartogs triangle for the simplicity of the computation. In [29], Rahm, Tchoundja, and Wick obtained the weighted estimates for operators $S_{a,b}$ and $S^+_{a,b}$ defined by

$$S_{a,b}f(z) := (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{f(w)(1-|w|^2)^b}{(1-z\bar{w})^{n+1+a+b}} dV(w);$$

$$S^+_{a,b}f(z) := (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{f(w)(1-|w|^2)^b}{|1-z\bar{w}|^{n+1+a+b}} dV(w),$$

on the weighted space $L^p(\mathbb{B}_n, (1-|w|^2)^b \mu dV)$. Using the methods in this paper, it is possible to obtain weighted estimates for analogues of $S_{a,b}$ and $S^+_{a,b}$ in the Hartogs triangle setting. When the domain Ω is covered by the polydisc through a rational proper holomorphic map as in [11], an induced dyadic structure on Ω can be obtained via the proper map. One direction for generalization is to obtain the Bekollé-Bonami type estimates for the Bergman projection, and analogues of $S_{a,b}$ and $S^+_{a,b}$ on such a domain Ω.

4. In the proof of Theorem 1.2, the positive dyadic operator $Q^+_{m,n,\nu}$ is used to relate the Bergman projection to the maximal operator. The constant $\frac{p^4}{(p-1)^2}$ appeared in Theorem 1.2 dominates the L^p and $L^{p'}$ norms of the maximal operator on the $\mathbb{D}^2$. In [12], Čučković showed that the L^p-norm of the Bergman projection on a smooth bounded strongly pseudoconvex domain is dominated by $\frac{p^2}{p-1}$. This fact suggests the possibility to relate the Bergman projection to the maximal function via a dyadic harmonic analysis argument. It would be interesting to see what is the appropriate dyadic structure and

the dyadic operator for the Bergman projection on the strongly pseudoconvex domain, and establish Bekollé-Bonami estimates for weighted L^p norm of the projection.

5. Although the main result is expressed on the Hartogs triangle, there is a theorem on the bidisc and other product domains in disguise. Note that on the unit ball $\mathbb{B}_n$, the following estimate [29] holds

$$\|P_{\mathbb{B}_n}\|_{L^p(\mathbb{B}_n,\mu)} \lesssim \|P^+_{\mathbb{B}_n}\|_{L^p(\mathbb{B}_n,\mu)} \lesssim B_p(\mu)^{\max\{1,\frac{1}{p-1}\}}.$$

Fubini's theorem then yields that for a product domain $\Omega = \mathbb{B}_{n_1} \times \mathbb{B}_{n_2} \times \cdots \times \times \mathbb{B}_{n_d}$ and a weight μ on Ω of the form $\mu(\mathbf{z}_1, \dots, \mathbf{z}_d) = \mu_1(\mathbf{z}_1) \cdots \mu_d(\mathbf{z}_d)$ with $\mathbf{z}_j \in \mathbb{C}^{n_j}$ for each j,

$$\|P_\Omega\|_{L^p(\Omega,\mu)} \lesssim \|P^+_\Omega\|_{L^p(\Omega,\mu)} \lesssim \prod_{j=1}^{n} B_p(\mu_j)^{\max\{1,\frac{1}{p-1}\}}.$$

For more general weights there are challenges in determining the sharp behavior of the weight condition in the multiparameter setting and there are additional issues that appear in these questions. We plan to undertake a deeper study of these questions in a forthcoming project.

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