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Bekollé-Bonami estimates on some pseudoconvex domains ☆

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We establish a weighted L^p norm estimate for the Bergman projection for a class of pseudoconvex domains. We obtain an upper bound for the weighted L^p norm when the domain is, for example, a bounded smooth strictly pseudoconvex domain, a pseudoconvex domain of finite type in $\mathbb{C}^2$, a convex domain of finite type in $\mathbb{C}^n$, or a decoupled domain of finite type in $\mathbb{C}^n$. The upper bound is related to the Bekollé-Bonami constant and is sharp. When the domain is smooth, bounded, and strictly pseudoconvex, we also obtain a lower bound for the weighted norm. As an additional application of the method of proof, we obtain the result that the Bergman projection is weak-type $(1, 1)$ on these domains.

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1. Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. Let dV denote the Lebesgue measure on $\mathbb{C}^n$. The Bergman projection P is the orthogonal projection from $L^2(\Omega)$ onto the Bergman space $A^2(\Omega)$, the space of all square-integrable holomorphic functions. Associated with P, there is a unique function K_Ω on $\Omega \times \Omega$ such that for any $f \in L^2(\Omega)$:

$$P(f)(z) = \int_\Omega K_\Omega(z; \bar{w}) f(w) dV(w). \tag{1.1}$$

Let P^+ denote the positive Bergman operator defined by:

$$P^+(f)(z) := \int_\Omega |K_\Omega(z; \bar{w})| f(w) dV(w). \tag{1.2}$$

A question of importance in analytic function theory and harmonic analysis is to understand the boundedness of P or P^+ on the space $L^p(\Omega, \sigma dV)$, where σ is some non-negative locally integrable function on Ω. In [1,3], Bekollé and Bonami established the following for P and P^+ on the unit ball $\mathbb{B}_n \subseteq \mathbb{C}^n$:

Theorem 1.1 *(Bekollé-Bonami). Let T_z denote the Carleson tent over z in $\mathbb{B}_n \in \mathbb{C}^n$ defined as below:*

- $T_z := \left\{ w \in \mathbb{B}_n : \left| 1 - \bar{w} \frac{z}{|z|} \right| < 1 - |z| \right\}$ *for* $z \neq 0$, *and*
- $T_z := \mathbb{B}_n$ *for* $z = 0$.

Let the weight σ be a positive, locally integrable function on $\mathbb{B}_n$. Let $1 < p < \infty$. Then the following conditions are equivalent:

(1) $P : L^p(\mathbb{B}_n, \sigma) \mapsto L^p(\mathbb{B}_n, \sigma)$ *is bounded;*
(2) $P^+ : L^p(\mathbb{B}_n, \sigma) \mapsto L^p(\mathbb{B}_n, \sigma)$ *is bounded;*
(3) *The Bekollé-Bonami constant $\mathcal{B}_p(\sigma)$ is finite where:*

$$\mathcal{B}_p(\sigma) := \sup_{z \in \mathbb{B}_n} \frac{\int_{T_z} \sigma(w) dV(w)}{\int_{T_z} dV(w)} \left(\frac{\int_{T_z} \sigma^{-\frac{1}{p-1}}(w) dV(w)}{\int_{T_z} dV(w)} \right)^{p-1}.$$

In [19], we generalized Bekollé and Bonami's result to a wide class of pseudoconvex domains of finite type. To do so, we combined the methods of Bekollé [3] with McNeal [30]. This method of proof is qualitative, showing that the Bekollé-Bonami class is sufficient for the weighted inequality of the projection to hold on those domains, and also necessary if the domain is strictly pseudoconvex. However, the method of good-lambda inequalities in [3] seems unlikely to give optimal estimates for the norm of the Bergman projection.

In this paper, we address the quantitative side of this question using sparse domination techniques.

Motivated by recent developments on the A_2-Conjecture by Hytönen [20] for singular integrals in the setting of Muckenhoupt weighted L^p spaces, people have made progress on the dependence of the operator norm $\|P\|_{L^p(\mathbb{B}_n,\sigma)}$ on $\mathcal{B}_p(\sigma)$. In [39], Pott and Reguera gave a weighted L^p estimate for the Bergman projection on the upper half plane. Their estimates are in terms of the Bekollé-Bonami constant and the upper bound is sharp. Later, Rahm, Tchoundja, and Wick [40] generalized the results of Pott and Reguera to the unit ball case, and also obtained estimates for the Berezin transform. Weighted norm estimates of the Bergman projection have also been obtained [18] on the Hartogs triangle.

We use the known estimates of the Bergman kernel in [13,6,36,26,29,27] to establish the Bekollé-Bonami type estimates for the Bergman projection on some classes of finite type domains. By finite type we mean that the D'Angelo 1-type [10] is finite. The domains of finite type we focus on are:

(1) domains of finite type in $\mathbb{C}^2$;
(2) strictly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$;
(3) convex domains of finite type in $\mathbb{C}^n$;
(4) decoupled domain of finite type in $\mathbb{C}^n$.

Given functions of several variables f and g, we use $f \lesssim g$ to denote that $f \leq Cg$ for a constant C. If $f \lesssim g$ and $g \lesssim f$, then we say f is comparable to g and write $f \approx g$.

The main result obtained in this paper is:

Theorem 1.2. *Let $1 < p < \infty$, and p' denote the Hölder conjugate to p. Let $\sigma(z)$ be a positive, locally integrable function on Ω. Set $\nu = \sigma^{-p'/p}(z)$. Then the Bergman projection P satisfies the following norm estimate on the weighted space $L^p(\Omega, \sigma)$:*

$$\|P\|_{L^p(\Omega,\sigma)} \leq \|P^+\|_{L^p(\Omega,\sigma)} \lesssim [\sigma]_p, \tag{1.3}$$

where

$$[\sigma]_p = \left(\langle\sigma\rangle_\Omega^{dV}\left(\langle\nu\rangle_\Omega^{dV}\right)^{p-1}\right)^{1/p} + pp'\left(\sup_{\epsilon_0>\delta>0, z\in \mathbf{b}\Omega}\langle\sigma\rangle_{B^\#(z,\delta)}^{dV}\left(\langle\nu\rangle_{B^\#(z,\delta)}^{dV}\right)^{p-1}\right)^{\max\{1,\frac{1}{p-1}\}}.$$

The tent $B^\#(z, \delta)$ above is slightly different from the tent we use in [19] in order to fit in the machinery of dyadic harmonic analysis. These two tents are essentially equivalent. The construction of $B^\#(z, \delta)$ uses the existence of the projection map onto $\mathbf{b}\Omega$ which is defined in a small tubular neighborhood of $\mathbf{b}\Omega$. Hence the restriction $\delta < \epsilon_0$ is needed to make sure that $B^\#(z, \delta)$ is inside the tubular neighborhood. See Lemma 3.2 and Definition 3.3 in Section 3. For the detailed definition of the constant $[\sigma]_p$ and its connection with the Bekollé-Bonami constant $\mathcal{B}_p(\sigma)$, see Definition 3.4 and Remark 3.5. We provide a sharp example for the upper bound above. See Section 6.

Using the asymptotic expansion of the Bergman kernel on a strictly pseudoconvex domain [13,6], we showed in [19] that when Ω is smooth, bounded, and strictly pseudoconvex, the boundedness of the Bergman projection P on the weighted space $L^p(\Omega, \sigma)$ implies that the weight σ is in the $\mathcal{B}_p$ class. Here we also provide the corresponding quantitative result, giving a lower bound of the weighted norm of P:

Theorem 1.3. *Let Ω be a smooth, bounded, strictly pseudoconvex domain. Let $1 < p < \infty$, and p' denote the Hölder conjugate to p. Let $\sigma(z)$ be a positive, locally integrable function on Ω. Set $\nu = \sigma^{-p'/p}(z)$. Suppose the projection P is bounded on $L^p(\Omega, \sigma)$. Then we have*

$$\left(\sup_{\epsilon_0 > \delta > 0, z \in \mathbf{b}\Omega} \langle \sigma \rangle^{dV}_{B^\#(z,\delta)} \left(\langle \nu \rangle^{dV}_{B^\#(z,\delta)} \right)^{p-1} \right)^{\frac{1}{2p}} \lesssim \|P\|_{L^p(\Omega,\sigma)}. \tag{1.4}$$

If Ω is also Reinhardt, then

$$(\mathcal{B}_p(\sigma))^{\frac{1}{2p}} \lesssim \|P\|_{L^p(\Omega,\sigma)}, \tag{1.5}$$

where $\mathcal{B}_p(\sigma) = \max \left\{ \langle \sigma \rangle^{dV}_{\Omega} \left(\langle \nu \rangle^{dV}_{\Omega} \right)^{p-1}, \ \sup_{\epsilon_0 > \delta > 0, z \in \mathbf{b}\Omega} \langle \sigma \rangle^{dV}_{B^\#(z,\delta)} \left(\langle \nu \rangle^{dV}_{B^\#(z,\delta)} \right)^{p-1} \right\}$.

When Ω is the unit ball in $\mathbb{C}^n$, the estimate (1.5) was obtained in [40]. When Ω is smooth, bounded, and strictly pseudoconvex, it was proven in [19] that if P is bounded on $L^p(\Omega, \sigma)$, then the constant $\mathcal{B}_p(\sigma)$ is finite. It remains unclear to us that, for a general strictly pseudoconvex domain Ω, how $\|P\|_{L^p(\Omega,\sigma)}$ dominates the constant $\langle \sigma \rangle^{dV}_{\Omega} \left(\langle \nu \rangle^{dV}_{\Omega} \right)^{p-1}$.

The approach we employ in this paper is similar to the ones in [39] and [40]. To prove (1.3), we show that P and P^+ are controlled by a positive dyadic operator. Then an analysis of the weighted L^p norm of the dyadic operator yields the desired estimate. The construction of the dyadic operator uses a doubling quasi-metric on the boundary $\mathbf{b}\Omega$ of the domain Ω and a result of Hytönen and Kairema [17]. For the domains we consider, estimates of the Bergman kernel function in terms of those quasi-metrics are known so that a domination of the Bergman projection by the dyadic operator is possible. There are other domains where estimates for the Bergman kernel function are known. We just focus on the above four cases and do not attempt to obtain the most general result.

The paper is organized as follows: In Section 2, we recall the definitions and known results concerning the non-isotropic metrics and balls on the boundary of the domain. In Section 3, we give the definition of tents and the dyadic tents structure based on the non-isotropic balls in Section 2. In Section 4, we recall known estimates for the Bergman kernel function, and prove a pointwise domination of the (positive) Bergman kernel function by a positive dyadic kernel. In Section 5, we prove Theorem 1.2. In Section 6, we provide a sharp example for the upper bound in Theorem 1.2. In Section 7, we prove Theorem 1.3. In Section 8, we provide an additional (unweighted) application of the pointwise dyadic domination to show the Bergman projection is weak-type $(1, 1)$. We point out some directions for generalization in Section 9.

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2. Non-isotropic balls on the boundary

In this section, we recall various definitions of quasi-metrics and their associated balls on the boundary of Ω. When Ω is of finite type in $\mathbb{C}^2$ or strictly pseudoconvex in $\mathbb{C}^n$, distances on the boundary can be well described using sub-Riemannian geometry. Properties and equivalence of these distances can be found in [37,38,33,2]. For discussions about the sub-Riemannian geometry, see for example [4,16,32].

For convex or decoupled domains of finite type in $\mathbb{C}^n$, the boundary geometry could be more complicated. We use quasi-metrics in [29,28,27]. In fact, all four classes of domains we consider in this paper can be referred to as the so-called "simple domains" in [30]. There it has been shown that estimates for the kernel function on these domains fall into a unified framework. When Ω is of finite type in $\mathbb{C}^2$ or strictly pseudoconvex in $\mathbb{C}^n$, the boundary geometry of Ω is relatively straightforward. The quasi-metric induced by special coordinates systems in [26] and [30] is essentially the same as the sub-Riemannian metric.

It is worth mentioning that estimates expressed using some quasi-metrics for the Bergman kernel function are known in other settings. See for example [8,22,24].

2.1. Balls on the boundaries of domains of finite type in $\mathbb{C}^2$ or strictly pseudoconvex domains in $\mathbb{C}^n$

Let Ω be a bounded domain in $\mathbb{C}^n$ with C^∞-smooth boundary. A defining function ρ of Ω is a real-valued C^∞ function on $\mathbb{C}^n$ with the following properties:

(1) $\rho(z) < 0$ for all $z \in \Omega$ and $\rho(z) > 0$ for all $z \notin \Omega$.
(2) $\partial\rho(z) \neq 0$ when $\rho(z) = 0$.

Such a ρ can be constructed, for instance, using the Euclidean distance between the point z and $\mathbf{b}\Omega$, the boundary of Ω. One can also normalize ρ so that $|\partial\rho| = 1$. Let $T(\mathbf{b}\Omega)$ denote the tangent bundle of $\mathbf{b}\Omega$ and $\mathbb{C}T(\mathbf{b}\Omega) = T(\mathbf{b}\Omega) \otimes \mathbb{C}$ its complexification. Let $T^{1,0}(\mathbf{b}\Omega)$ denote the subbundle of $\mathbb{C}T(\mathbf{b}\Omega)$ whose sections are linear combinations of $\partial/\partial z_j$, and $T^{0,1}(\mathbf{b}\Omega)$ be its complex conjugate bundle. Their sum $H(\mathbf{b}\Omega) := T^{1,0}(\mathbf{b}\Omega) + T^{0,1}(\mathbf{b}\Omega)$ is a bundle of real codimension one in the complex tangent bundle $\mathbb{C}T(\mathbf{b}\Omega)$. Let $\langle \cdot, \cdot \rangle$ denote the contraction of the one forms and vector fields, and let $[\cdot, \cdot]$ denote the Lie bracket of two vector fields. Let λ denote the Levi form, the Hermitian form on $T^{1,0}(\mathbf{b}\Omega)$ defined by

$$\lambda(L, \bar{K}) = \langle \frac{1}{2}(\partial - \bar{\partial})\rho, [L, \bar{K}] \rangle \quad \text{for} \quad L, K \in T^{1,0}(\mathbf{b}\Omega).$$

By the Cartan formula for the exterior derivative of a one form, one obtains

$$\lambda(L, \bar{K}) = \left\langle -d\left(\frac{1}{2}(\partial - \bar{\partial})\rho\right), L \wedge \bar{K} \right\rangle = \langle \partial\bar{\partial}\rho, L \wedge \bar{K} \rangle.$$

Hence, the Levi form is the complex Hessian $(\rho_{i\bar{j}})$ of ρ, restricted to $T^{1,0}(\mathbf{b}\Omega)$.

The domain Ω is called pseudoconvex (resp. strictly pseudoconvex) if λ is positive semidefinite (resp. definite), i.e., the complex Hessian $(\rho_{i\bar{j}})$ is positive semidefinite (resp. definite) when restricted to $T^{1,0}(\mathbf{b}\Omega)$.

Given $L \in T^{1,0}(\mathbf{b}\Omega)$, we say the type of L at a point $p \in \mathbf{b}\Omega$ is k and write $\text{Type}_p L = k$ if k is the smallest integer such that there is a iterated commutator

$$[\dots[[L_1, L_2], L_3], \dots, L_k] = \Psi_k,$$

where each L_j is either L or $\bar{L}$ such that $\langle \Psi_k, (\partial - \bar{\partial})\rho \rangle \neq 0$.

When $\Omega \subseteq \mathbb{C}^2$, the subbundle $T^{1,0}(\mathbf{b}\Omega)$ has dimension one at each boundary point p and $\text{Type}_p L$ defines the type of the point p: A point $q \in \mathbf{b}\Omega$ is of finite type m in the sense of Kohn [23] if $\text{Type}_p L = m$. A domain is of Kohn finite type m if every point $q \in \mathbf{b}\Omega$ is of Kohn finite type at most m. In the $\mathbb{C}^2$ case, Kohn's type and D'Angelo's 1-type are equivalent. See [11] for the proof.

When Ω is strictly pseudoconvex, the Levi form λ is positive definite. Thus for every $L \in T^{1,0}(\mathbf{b}\Omega)$ and $p \in \mathbf{b}\Omega$, one has that $\text{Type}_p L = 2$.

Using the defining function ρ, a local basis of $H(\mathbf{b}\Omega)$ can be chosen as follows. Let $p \in \mathbf{b}\Omega$ be a boundary point. We may assume that, after a unitary rotation, $\partial\rho(p) = dz_n$. Then there is a neighborhood U of p such that $\frac{\partial\rho}{\partial z_n} \neq 0$ on U. We define $n-1$ local tangent vector fields on $\mathbf{b}\Omega \cap U$:

$$L_j = \rho_{z_n} \frac{\partial}{\partial z_j} - \rho_{z_j} \frac{\partial}{\partial z_n} \qquad j = 1, 2, 3 \dots, n-1;$$

and their conjugates:

$$\bar{L}_j = \rho_{\bar{z}_n}\frac{\partial}{\partial \bar{z}_j} - \rho_{\bar{z}_j}\frac{\partial}{\partial \bar{z}_n} \qquad j = 1, 2, 3 \ldots, n-1.$$

Then the L_j's span $T^{1,0}(\mathbf{b}\Omega)$ and the $\bar{L}_j$'s span $T^{0,1}(\mathbf{b}\Omega)$. We set

$$S = \sum_{j=1}^{n} \rho_{\bar{z}_j}\frac{\partial}{\partial z_j} \quad \text{and} \quad T = S - \bar{S}.$$

Then the L_j's, $\bar{L}_j$'s and T span $\mathbb{C}T(\mathbf{b}\Omega)$. Let X_j, X_{n-1+j} be real vector fields such that

$$L_j = X_j - iX_{n-1+j}$$

for $j = 1, \ldots, n-1$. Then X_j's and T span the real tangent space of $\mathbf{b}\Omega$ near the point p.

For every k-tuple of integers $l^{(k)} = (l_1, \ldots, l_k)$ with $k \geq 2$ and $l_j \in \{1, \ldots, 2n-2\}$, we define $\lambda_{l^{(k)}}$ to be the smooth function such that

$$[X_{l_k}, [X_{l_{k-1}}, [\ldots [X_{l_2}, X_{l_1}] \ldots]]] = \lambda_{l^{(k)}} T \mod X_1, \ldots, X_{2n-2},$$

and define Λ_k to be the smooth function

$$\Lambda_k(q) = \left(\sum_{\text{all } l^{(k)}} |\lambda_{l^{(k)}}(q)|^2 \right)^{1/2}.$$

For $q \in U$ and $\delta > 0$, we set

$$\Lambda(q, \delta) = \sum_{j=2}^{m} \Lambda_j(q)\delta^j. \tag{2.1}$$

In the $\mathbb{C}^2$ case, a point $q \in \mathbf{b}\Omega$ is of finite type m if and only if $\Lambda_2(q) = \cdots = \Lambda_{m-1}(q) = 0$ but $\Lambda_m(q) \neq 0$. When Ω is strictly pseudoconvex in $\mathbb{C}^n$, the function $\Lambda_2 \neq 0$ on $\mathbf{b}\Omega$.

Though the function Λ is locally defined, one can construct a global Λ that is defined on $\mathbf{b}\Omega$ and is comparable to its every local piece. In the finite type in $\mathbb{C}^2$ case and the strictly pseudoconvex case, a global construction can be realized without using partitions of unity. We explain this now. When Ω is strictly pseudoconvex, Λ_2 does not vanish on the boundary of Ω, therefore we can simply set $\Lambda(q, \delta) = \delta^2$. When Ω is of finite type in $\mathbb{C}^2$, global tangent vector fields L_1 and S can be chosen on $\mathbf{b}\Omega$:

$$L_1 = \rho_{z_2}\frac{\partial}{\partial z_1} - \rho_{z_1}\frac{\partial}{\partial z_2},$$

$$S = \rho_{\bar{z}_1}\frac{\partial}{\partial z_1} + \rho_{\bar{z}_2}\frac{\partial}{\partial z_2}.$$

Then the function Λ induced by the above L_1 and S is a smooth function defined on $\mathbf{b}\Omega$. From now on, we choose Λ to be the smooth function induced by L_1 and S on $\mathbf{b}\Omega$ when Ω is of finite type in $\mathbb{C}^2$, and choose $\Lambda(q,\delta)=\delta^2$ when Ω is strictly pseudoconvex in $\mathbb{C}^n$.

We recall several non-isotropic metrics on $\mathbf{b}\Omega$ that are locally equivalent:

Definition 2.1. For $p,q\in\mathbf{b}\Omega$, the metric $d_1(\cdot,\cdot)$ is defined by:

$$d_1(p,q)=\inf\Big\{\int_0^1|\alpha'(t)|dt:\alpha \text{ is any piecewise smooth map from } [0,1] \text{ to } \mathbf{b}\Omega \\ \text{with } \alpha(0)=p,\alpha(1)=q, \text{ and } \alpha'(t)\in H_{\alpha(t)}(\mathbf{b}\Omega)\Big\}. \tag{2.2}$$

Equipped with the metric d_1, we define the ball B_1 centered at $p\in\mathbf{b}\Omega$ of radius r to be

$$B_1(p,r)=\{q\in\mathbf{b}\Omega:d_1(p,q)<r\}. \tag{2.3}$$

Definition 2.2. For $p,q\in\mathbf{b}\Omega$, the metric $d_2(\cdot,\cdot)$ is defined by:

$$d_2(p,q)=\inf\Big\{\delta: \text{ There is a piecewise smooth map } \alpha \text{ from } [0,1] \text{ to } \mathbf{b}\Omega \\ \text{with } \alpha(0)=p,\alpha(1)=q,\alpha'(t)=\sum_{j=1}^{2n-2}a_j(t)X_j(\alpha(t)), \text{ and } |a_j(t)|<\delta\Big\}. \tag{2.4}$$

Equipped with the metric d_2, we define the ball B_2 centered at $p\in\mathbf{b}\Omega$ of radius r to be

$$B_2(p,r)=\{q\in\mathbf{b}\Omega:d_2(p,q)<r\}. \tag{2.5}$$

Definition 2.3. For $p,q\in\mathbf{b}\Omega$, the metric $d_3(\cdot,\cdot)$ is defined by:

$$d_3(p,q)=\inf\Big\{\delta: \text{ There is a piecewise smooth map } \alpha \text{ from } [0,1] \text{ to } \mathbf{b}\Omega \text{ with} \\ \alpha(0)=p,\alpha(1)=q, \text{ and } \alpha'(t)=\sum_{j=1}^{2n-2}a_j(t)X_j(\alpha(t))+b(t)T(\alpha(t)), \\ \text{where } |a_j(t)|<\delta,|b(t)|<\Lambda(p,\delta)\Big\}. \tag{2.6}$$

Equipped with the metric d_3, we define the ball B_3 centered at $p\in\mathbf{b}\Omega$ of radius r to be

$$B_3(p,r)=\{q\in\mathbf{b}\Omega:d_3(p,q)<r\}. \tag{2.7}$$

It is known that when the domain is strictly pseudoconvex in $\mathbb{C}^n$, or of finite type in $\mathbb{C}^2$, the quasi-metrics d_1, d_2, and d_3 are locally equivalent (cf. [37,38,33]), i.e. there are positive constants C_1, C_2 and δ so that when $d_i(p,q) < \delta$ with $i \in \{1,2,3\}$,

$$C_1 d_j(p,q) < d_i(p,q) < C_2 d_j(p,q) \quad \text{for } j \in \{1,2,3\}.$$

As a consequence, the balls B_j are also equivalent in the sense that for small δ, there are positive constants C_1, C_2 such that

$$B_i(p, C_1\delta) \subseteq B_j(p,\delta) \subseteq B_i(p, C_2\delta) \quad \text{for } i,j \in \{1,2,3\}.$$

It is worth noting that the definition $d_1(\cdot,\cdot)$ does not depend on how we choose the local vector fields. Moreover, if $d_1(p,q) < \delta$, then for some positive constants C_1, C_2,

$$C_1\Lambda(p,\delta) \le \Lambda(q,\delta) \le C_2\Lambda(p,\delta). \tag{2.8}$$

To introduce the Ball-Box Theorem below, we also need to define balls using the exponential map.

Definition 2.4. For $q \in \mathbf{b}\Omega$ and $\delta > 0$, set

$$B_4(q,\delta) = \left\{ p \in \mathbf{b}\Omega : p = \exp\left(\sum_{j=1}^{2n-2} a_j X_j(q) + bT(q)\right), \right.$$
$$\left. \text{where } |a_j| < \delta, \text{ and } |b| < \Lambda(p,\delta) \right\}.$$

Theorem 2.5 *(Ball-Box Theorem). There exist positive constants C_1, C_2 such that for any $q \in \mathbf{b}\Omega$ and any sufficiently small $\delta > 0$,*

$$B_j(q, C_1\delta) \subseteq B_4(q,\delta) \subseteq B_j(q, C_2\delta) \quad \textit{for } j \in \{1,2,3\}.$$

The proof of this theorem can be found in for example [38,4,16,32]. Variants of the Ball-Box Theorem also exist in the literature. The following version of the Ball-Box Theorem is a consequence of Theorem 2.5 and can be found in [2].

Corollary 2.6 *(Ball-Box Theorem). Let Ω be a smooth, bounded, strictly pseudoconvex domain. There exist positive constants C_1, C_2 such that for any $q \in \mathbf{b}\Omega$ and any sufficiently small $\delta > 0$,*

$$Box(q, C_1\delta) \subseteq B_j(q,\delta) \subseteq Box(q, C_2\delta) \quad \textit{for } j \in \{1,2,3,4\}.$$

Here $Box(q,\delta) = \{q + Z_H + Z_N \in \mathbf{b}\Omega : |Z_H| < \delta, |Z_N| < \delta^2\}$ where $Z_H \in H_q(\mathbf{b}\Omega)$ and Z_N is orthogonal to $H_q(\mathbf{b}\Omega)$.

We will only use this corollary for the strictly pseudoconvex case. See for example [2].

The next theorem provides estimates for the surface volume of B_4, and hence also for B_j with $j = \{1, 2, 3\}$. See for example [38].

Lemma 2.7. *Let μ denote the Lebesgue surface measure on $\mathbf{b}\Omega$. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1\delta^{2n-2}\Lambda(p,\delta) \leq \mu(B_4(p,\delta)) \leq C_2\delta^{2n-2}\Lambda(p,\delta). \tag{2.9}$$

As a consequence of the definitions of d_1 and Λ and Lemma 2.7, we have the "doubling measure property" for the non-isotropic ball: There exists a positive constant C such that for each $p \in \mathbf{b}\Omega$ and $\delta > 0$,

$$\mu(B_j(p,\delta)) \leq C\mu(B_j(p,\delta/2)) \quad \text{for } j \in \{1,2,3,4\}. \tag{2.10}$$

2.2. Balls on the boundary of a convex/decoupled domain of finite type

When Ω is a convex/decoupled domain of finite type in $\mathbb{C}^n$, non-isotropic sets can be constructed using a special coordinate system of McNeal [29,27,30] near the boundary of Ω. Let $p \in \mathbf{b}\Omega$ be a point of finite type m. For a small neighborhood U of the point p, there exists a holomorphic coordinate system $z = (z_1, \dots, z_n)$ centered at a point $q \in U$ and defined on U and quantities $\tau_1(q,\delta), \tau_2(q,\delta), \dots, \tau_n(q,\delta)$ such that

$$\tau_1(q,\delta) = \delta \quad \text{and} \quad \delta^{1/2} \lesssim \tau_j(q,\delta) \lesssim \delta^{1/m} \quad \text{for} \quad j = 2,3,\dots,n. \tag{2.11}$$

Moreover, the polydisc $D(q,\delta)$ defined by:

$$D(q,\delta) = \{z \in \mathbb{C}^n : |z_j| < \tau_j(q,\delta), j = 1,\dots,n\} \tag{2.12}$$

is the largest one centered at q on which the defining function ρ changes by no more than δ from its value at q, i.e. if $z \in D(q,\delta)$, then $|\rho(z) - \rho(q)| \lesssim \delta$.

The polydisc $D(q,\delta)$ is known to satisfy several "covering properties" [28]:

(1) There exists a constant $C > 0$, such that for points $q_1, q_2 \in U \cap \Omega$ with $D(q_1,\delta) \cap D(q_2,\delta) \neq \emptyset$, we have

$$D(q_2,\delta) \subseteq CD(q_1,\delta) \text{ and } D(q_1,\delta) \subseteq CD(q_2,\delta). \tag{2.13}$$

(2) There exists a constant $c > 0$ such that for $q \in U \cap \Omega$ and $\delta > 0$, we have

$$D(q,2\delta) \subseteq cD(q,\delta). \tag{2.14}$$

It was also shown in [28] that $D(p,\delta)$ induces a global quasi-metric on Ω. Here we will use it to define a quasi-metric on $\mathbf{b}\Omega$.

For $q \in \mathbf{b}\Omega$ and $\delta > 0$, we define the non-isotropic ball of radius δ to be the set

$$B_5(q,\delta) = D(q,\delta) \cap \mathbf{b}\Omega.$$

Set containments (2.13), (2.14), and the compactness and smoothness of $\mathbf{b}\Omega$ imply the following properties for B_5:

(1) There exists a constant C such that for $q_1, q_2 \in U \cap \mathbf{b}\Omega$ with $B_5(q_1,\delta) \cap B_5(q_2,\delta) \neq \emptyset$,

$$B_5(q_2,\delta) \subseteq CB_5(q_1,\delta) \text{ and } B_5(q_1,\delta) \subseteq CB_5(q_2,\delta). \tag{2.15}$$

(2) There exists a constant $c > 0$ such that for $q \in U \cap \Omega$ and $\delta > 0$, we have

$$B_5(q,\delta) \subseteq cB_5(q,\delta/2) \qquad \text{and} \qquad \mu(B_5(q,\delta)) \approx \delta \prod_{j=2}^{n} \tau_j^2(q,\delta). \tag{2.16}$$

The balls B_5 induce a quasi-metric on $\mathbf{b}\Omega \cap U$. For $q, p \in \mathbf{b}\Omega \cap U$, we set $\tilde{d}_5(q,p) = \inf\{\delta > 0 : p \in B_5(q,\delta)\}$. To extend this quasi-metric $\tilde{d}_5(\cdot,\cdot)$ to a global quasi-metric $d_5(\cdot,\cdot)$ defined on $\mathbf{b}\Omega \times \mathbf{b}\Omega$, one can just patch the local metrics together in an appropriate way. The resulting quasi-metric is not continuous, but satisfies all the relevant properties. We refer the reader to [28] for more details on this matter. Since $d_5(\cdot,\cdot)$ and $\tilde{d}_5(\cdot,\cdot)$ are equivalent, we may abuse the notation B_5 for the ball on the boundary induced by d_5. Then (2.15) and (2.16) still hold true for B_5.

3. Tents and dyadic structures on Ω

From now on, the domain Ω will belong to one of the following cases:

- a bounded, smooth, pseudoconvex domain of finite type in $\mathbb{C}^2$,
- a bounded, smooth, strictly pseudoconvex domain in $\mathbb{C}^n$,
- a bounded, smooth, convex domain of finite type in $\mathbb{C}^n$, or
- a bounded, smooth, decoupled domain of finite type in $\mathbb{C}^n$.

Notations $d(\cdot,\cdot)$ and $B(p,\delta)$ will be used for

- the metric $d_1(\cdot,\cdot)$ and the ball $B_1(p,\delta)$ if Ω is pseudoconvex of finite type in $\mathbb{C}^2$ or strictly pseudoconvex in $\mathbb{C}^n$;
- the metric $d_5(\cdot,\cdot)$ and the ball $B_5(p,\delta)$ if Ω is a convex/decoupled domain of finite type.

Remark 3.1. It is worth noting that even though we use the same notation $B(p,\delta)$ for balls on the boundary of Ω, the constant δ has different geometric meanings in different settings. When Ω is a bounded, smooth, pseudoconvex domain of finite type in $\mathbb{C}^2$, or a bounded, smooth, strictly pseudoconvex domain in $\mathbb{C}^n$, δ represents the radius of the sub-Riemannian ball $B_1(p,\delta)$. When Ω is a bounded, smooth, convex (or decoupled) domain of finite type in $\mathbb{C}^n$, 2δ is the height in the z_1 coordinate of the polydisc $D(q,\delta)$ that defines $B_5(q,\delta)$. If Ω is the unit ball $\mathbb{B}_n$ which is strictly pseudoconvex, convex, and decoupled, the ball $B_1(q,\delta)$ will be of similar size as the ball $B_5(q,\sqrt{\delta})$.

3.1. Dyadic tents on Ω and the $\mathcal{B}_p(\sigma)$ constant

The non-isotropic ball $B(p,\delta)$ on the boundary $\mathbf{b}\Omega$ induces "tents" in the domain Ω. To define what "tents" are we need the orthogonal projection map near the boundary. Let $\operatorname{dist}(\cdot,\cdot)$ denote the Euclidean distance in $\mathbb{C}^n$. For small $\epsilon > 0$, set

$$N_\epsilon(\mathbf{b}\Omega) = \{w \in \mathbb{C}^n : \operatorname{dist}(w, \mathbf{b}\Omega) < \epsilon\}.$$

Lemma 3.2. *For sufficiently small $\epsilon_0 > 0$, there exists a map $\pi : N_{\epsilon_0}(\mathbf{b}\Omega) \to \mathbf{b}\Omega$ such that*

(1) *For each point $z \in N_{\epsilon_0}(\mathbf{b}\Omega)$ there exists a unique point $\pi(z) \in \mathbf{b}\Omega$ such that*

$$|z - \pi(z)| = \operatorname{dist}(z, \mathbf{b}\Omega).$$

(2) *For $p \in \mathbf{b}\Omega$, the fiber $\pi^{-1}(p) = \{p - \epsilon n(p) : -\epsilon_0 \le \epsilon < \epsilon_0\}$ where $n(p)$ is the outer unit normal vector of $\mathbf{b}\Omega$ at point p.*
(3) *The map π is smooth on $N_{\epsilon_0}(\mathbf{b}\Omega)$.*
(4) *If the defining function ρ is the signed distance function to the boundary, the gradient $\nabla\rho$ satisfies*

$$\nabla\rho(z) = n(\pi(z)) \quad \textit{for} \quad z \in N_{\epsilon_0}(\mathbf{b}\Omega).$$

A proof of Lemma 3.2 can be found in [2].

Definition 3.3. Let ϵ_0 and π be as in Lemma (3.2). For $z \in \mathbf{b}\Omega$ and sufficiently small $\delta > 0$, the "tent" $B^{\#}(z,\delta)$ over the ball $B(z,\delta)$ is defined to be the subset of $N_{\epsilon_0}(\mathbf{b}\Omega)$ as follows: When Ω is a pseudoconvex domain of finite type in $\mathbb{C}^2$ or a strictly pseudoconvex domain,

$$B^{\#}(z,\delta) = B_1^{\#}(z,\delta) = \{w \in \Omega : \pi(w) \in B_1(z,\delta), |\pi(w) - w| \le \Lambda(\pi(w),\delta)\}.$$

When Ω is a convex (or decoupled) domain of finite type in $\mathbb{C}^n$,

$$B^{\#}(z,\delta) = B_5^{\#}(z,\delta) = \{w \in \Omega : \pi(w) \in B_5(z,\delta), |\pi(w) - w| \le \delta\}.$$

For $\delta \gtrsim 1$ and any $z \in \mathbf{b}\Omega$, we set $B^{\#}(z,\delta) = \Omega$.

For the "tent" $B^{\#}(z,\delta)$ to be within $N_{\epsilon_0}(\mathbf{b}\Omega)$, the constant δ in Definition 3.3 needs to satisfy $\Lambda(z',\delta) < \epsilon_0$ for $z' \in B_1(z,\delta)$ when Ω is of finite type in $\mathbb{C}^2$ or strictly pseudoconvex; and satisfy $\delta < \epsilon_0$ when Ω is a convex (or decoupled) domain in $\mathbb{C}^n$.

Given a subset $U \in \mathbb{C}^n$, let $V(U)$ denote the Lebesgue measure of U. By (2.8) and the definitions of the tents $B_1^{\#}(z,\delta)$ and $B_5^{\#}(z,\delta)$, we have:

$$V(B_1^{\#}(z,\delta)) \approx \delta^{2n-2}\Lambda^2(z,\delta), \tag{3.1}$$

$$V(B_5^{\#}(z,\delta)) \approx \delta^2 \prod_{j=2}^{n} \tau_j^2(z,\delta), \tag{3.2}$$

and hence also the "doubling property":

$$V(B^{\#}(z,\delta)) \approx V(B^{\#}(z,\delta/2)). \tag{3.3}$$

We give the definition of the Bekollé-Bonami constant on Ω. For a weight σ and a subset $U \subseteq \Omega$, we set $\sigma(U) := \int_U \sigma dV$ and let $\langle f\rangle_U^{\sigma dV}$ denote the average of the function $|f|$ with respect to the measure σdV on the set U:

$$\langle f\rangle_U^{\sigma dV} = \frac{\int_U |f(w)|\sigma dV}{\sigma(U)}.$$

Definition 3.4. Given weights $\sigma(z)$ and $\nu = \sigma^{-p'/p}(z)$ on Ω, the characteristic $[\sigma]_p$ of the weight σ is defined by

$$[\sigma]_p := \left(\langle\sigma\rangle_\Omega^{dV}\left(\langle\nu\rangle_\Omega^{dV}\right)^{p-1}\right)^{1/p} + pp'\left(\sup_{\epsilon_0>\delta>0, z\in\mathbf{b}\Omega}\langle\sigma\rangle_{B^{\#}(z,\delta)}^{dV}\left(\langle\nu\rangle_{B^{\#}(z,\delta)}^{dV}\right)^{p-1}\right)^{\max\{1,\frac{1}{p-1}\}}. \tag{3.4}$$

Remark 3.5. A natural generalization of the $\mathcal{B}_p$ constant in the above setting will be

$$\mathcal{B}_p(\sigma) = \max\left\{\langle\sigma\rangle_\Omega^{dV}\left(\langle\nu\rangle_\Omega^{dV}\right)^{p-1}, \sup_{\epsilon_0>\delta>0, z\in\mathbf{b}\Omega}\langle\sigma\rangle_{B^{\#}(z,\delta)}^{dV}\left(\langle\nu\rangle_{B^{\#}(z,\delta)}^{dV}\right)^{p-1}\right\}.$$

It is not hard to see that $\mathcal{B}_p(\sigma)$ and $[\sigma]_p$ are qualitatively equivalent, i.e., $\mathcal{B}_p(\sigma)$ is finite if and only if $[\sigma]_p$ is finite. But they are not quantitatively equivalent. As one will see in the proof of Theorem 1.2, the products of averages of σ and $\sigma^{1/(1-p)}$ over the whole domain and over the small tents will have different impacts on the estimate for the weighted norm of the projection P. The $\mathcal{B}_p(\sigma)$ above fails to reflect such a difference, and hence is unable to give the sharp upper bound. For the same reason, the claimed sharpness of the Bekollé-Bonami bound in [40] is not quite correct. See Remark 6.1. This issue did not occur in the upper half plane case [39] since the average over the whole upper half plane is not included in the $\mathcal{B}_p$ constant there.

Now we are in the position of constructing dyadic systems on $\mathbf{b}\Omega$ and Ω. Note that the ball $B(\cdot,\delta)$ on $\mathbf{b}\Omega$ satisfies the "doubling property" as in (2.10) and (2.16). By (2.8) and (2.13), the surface area $\mu(B(q_1,\delta)) \approx \mu(B(q_2,\delta))$ for any $q_1, q_2 \in \mathbf{b}\Omega$ satisfying $d(q_1,q_2) \leq \delta$. Combining these facts yields that the metric $d(\cdot,\cdot)$ is a doubling metric, i.e. for every $q \in \mathbf{b}\Omega$ and $\delta > 0$, the ball $B(q,\delta)$ can be covered by at most M balls $B(x_i,\delta/2)$. Results of Hytönen and Kairema in [17] then give the following lemmas:

Lemma 3.6. *Let δ be a positive constant that is sufficiently small and let $s > 1$ be a parameter. There exist reference points $\{p_j^{(k)}\}$ on the boundary $\mathbf{b}\Omega$ and an associated collection of subsets $\mathcal{Q} = \{Q_j^k\}$ of $\mathbf{b}\Omega$ with $p_j^{(k)} \in Q_j^k$ such that the following properties hold:*

(1) *For each fixed k, $\{p_j^{(k)}\}$ is a largest set of points on $\mathbf{b}\Omega$ satisfying $d_1(p_j^{(k)}, p_i^{(k)}) > s^{-k}\delta$ for all i, j. In other words, if $p \in \mathbf{b}\Omega$ is a point that is not in $\{p_j^{(k)}\}$, then there exists an index j_o such that $d_1(p, p_{j_o}^{(k)}) \leq s^{-k}\delta$.*
(2) *For each fixed k, $\bigcup_j Q_j^k = \mathbf{b}\Omega$ and $Q_j^k \bigcap Q_i^k = \emptyset$ when $i \neq j$.*
(3) *For $k < l$ and any i, j, either $Q_j^k \supseteq Q_i^l$ or $Q_j^k \bigcap Q_i^l = \emptyset$.*
(4) *There exist positive constants c and C such that for all j and k,*

$$B(p_j^{(k)}, cs^{-k}\delta) \subseteq Q_j^k \subseteq B(p_j^{(k)}, Cs^{-k}\delta).$$

(5) *Each Q_j^k contains of at most N numbers of Q_i^{k+1}. Here N does not depend on k, j.*

Lemma 3.7. *Let δ and $\{p_j^{(k)}\}$ be as in Lemma 3.6. There are finitely many collections $\{\mathcal{Q}_l\}_{l=1}^N$ such that the following hold:*

(1) *Each collection $\mathcal{Q}_l$ is associated to some dyadic points $\{z_j^{(k)}\}$ and they satisfy all the properties in Lemma 3.6.*
(2) *For any $z \in \mathbf{b}\Omega$ and small $r > 0$, there exist $Q_{j_1}^{k_1} \in \mathcal{Q}_{l_1}$ and $Q_{j_2}^{k_2} \in \mathcal{Q}_{l_2}$ such that*

$$Q_{j_1}^{k_1} \subseteq B(z,r) \subseteq Q_{j_2}^{k_2} \quad \textit{and} \quad \mu(B(z,r)) \approx \mu(Q_{j_1}^{k_1}) \approx \mu(Q_{j_2}^{k_2}).$$

Setting the sets Q_j^k in Lemma 3.6 as the bases, we construct dyadic tents in Ω as follows:

Definition 3.8. Let δ, $\{p_j^{(k)}\}$ and $\mathcal{Q} = \{Q_j^k\}$ be as in Lemma 3.6. We define the collection $\mathcal{T} = \{\hat{K}_j^k\}$ of dyadic tents in the domain Ω as follows:

- When Ω is pseudoconvex of finite type in $\mathbb{C}^2$, or strictly pseudoconvex in $\mathbb{C}^n$, we define

$$\hat{K}_j^k := \{z \in \Omega : \pi(z) \in Q_j^k \text{ and } |\pi(z) - z| < \Lambda(\pi(z), s^{-k}\delta)\}.$$

- When Ω is a convex or decoupled domain of finite type in $\mathbb{C}^n$, we define

$$\hat{K}_j^k := \{z \in \Omega : \pi(z) \in Q_j^k \text{ and } |\pi(z) - z| < s^{-k}\delta\}.$$

Lemma 3.9. *Let $\mathcal{T} = \{\hat{K}_j^k\}$ be a collection of dyadic tents in Definition 3.8 and let $\{\mathcal{Q}_l\}_{l=1}^N$ be a collection of subsets in Lemma 3.7. The following statements hold true:*

(1) *For any $\hat{K}_j^k$, $\hat{K}_i^{k+1}$ in $\mathcal{T}$, either $\hat{K}_j^k \supseteq \hat{K}_i^{k+1}$ or $\hat{K}_j^k \bigcap \hat{K}_i^{k+1} = \emptyset$.*
(2) *For any $z \in \mathbf{b}\Omega$ and small $r > 0$, there exist $Q_{j_1}^{k_1} \in \mathcal{Q}_{l_1}$ and $Q_{j_2}^{k_2} \in \mathcal{Q}_{l_2}$ such that*

$$\hat{K}_{j_1}^{k_1} \subseteq B^{\#}(z, r) \subseteq \hat{K}_{j_2}^{k_2} \quad \text{and} \quad V(B^{\#}(z, r)) \approx V(\hat{K}_{j_1}^{k_1}) \approx V(\hat{K}_{j_2}^{k_2}).$$

Proof. Statement (1) is a consequence of the definition of $\hat{K}_j^k$ and Lemma 3.6(3). Statement (2) is a consequence of the definitions of $B^{\#}(z, r)$, $\hat{K}_j^k$, and Lemma 3.7(2). □

By Lemma 3.9(2), we can replace $B^{\#}(z, \delta)$ by $\hat{K}_j^k$ in the definition of $[\sigma]_p$ to obtain a quantity of comparable size:

$$[\sigma]_p \approx \left(\langle \sigma \rangle_\Omega^{dV} \left(\langle \nu \rangle_\Omega^{dV}\right)^{p-1}\right)^{1/p} + pp' \left(\sup_{1 \le l \le N} \sup_{\hat{K}_j^k \in \mathcal{T}_l} \langle \sigma \rangle_{\hat{K}_j^k}^{dV} \left(\langle \nu \rangle_{\hat{K}_j^k}^{dV}\right)^{p-1} \right)^{\max\{1, 1/(p-1)\}}. \tag{3.5}$$

From now on, we will abuse the notation $[\sigma]_p$ to represent both the supremum in $B_q^{\#}$ and the supremum in $\hat{K}_j^k$.

3.2. Dyadic kubes on Ω

By choosing the parameter s in Lemmas 3.6 and 3.7 to be sufficiently large, we can also assume that for any $p \in Q_i^{k+1} \subset Q_j^k$, one has

$$\Lambda(p, s^{-k-1}\delta) < \frac{1}{4}\Lambda(p, s^{-k}\delta). \tag{3.6}$$

Definition 3.10. For a collection $\mathcal{T}$ of dyadic tents, we define the center $\alpha_j^{(k)}$ of each tent $\hat{K}_j^k$ to be the point satisfying

- $\pi(\alpha_j^{(k)}) = p_j^{(k)}$; and
- $|p_j^{(k)} - \alpha_j^{(k)}| = \frac{1}{2} \sup_{\pi(p) = p_j^{(k)}} \operatorname{dist}(p, \mathbf{b}\Omega)$.

We set $K^k_{-1} = \Omega\backslash\left(\bigcup_j \hat{K}^0_j\right)$, and for each point $\alpha^{(k)}_j$ or its corresponding tent $\hat{K}^k_j$, we define the dyadic "kube" $K^k_j := \hat{K}^k_j\backslash\left(\bigcup_l \hat{K}^{k+1}_l\right)$, where l is any index with $p^{(k+1)}_l \in \hat{K}^k_j$.

The following lemma for dyadic kubes holds true:

Lemma 3.11. *Let $\mathcal{T} = \{\hat{K}^k_j\}$ be the system of tents induced by $\mathcal{Q}$ in Definition 3.8. Let K^k_j be the kubes of $\hat{K}^k_j$. Then*

(1) *K^k_j's are pairwise disjoint and $\bigcup_{j,k} K^k_j = \Omega$.*
(2) *When Ω is a finite type domain in $\mathbb{C}^2$ or a strictly pseudoconvex domain in $\mathbb{C}^n$,*

$$V(K^k_j) \approx V(\hat{K}^k_j) \approx s^{-k(2n-2)}\delta^{2n-2}\Lambda(p^{(k)}_j, s^{-k}\delta). \tag{3.7}$$

When Ω is a convex or decoupled domain of finite type in $\mathbb{C}^n$,

$$V(K^k_j) \approx V(\hat{K}^k_j) \approx s^{-2k}\delta^{-2}\prod_{j=2}^{n}\tau^2_j(p^{(k)}_j, s^{-k}\delta). \tag{3.8}$$

Proof. Statement (1) holds true by the definition of K^k_j. The estimates for $V(\hat{K}^k_j)$ in (3.7) and (3.8) follow from (3.1), (3.2) and Lemma 3.9(2). When the domain is convex or decoupled of finite type in $\mathbb{C}^n$, the height of $\hat{K}^k_j$ is s times the height of the tent $\hat{K}^k_j\backslash K^k_j$. Thus $V(\hat{K}^k_j) \approx V(\hat{K}^k_j\backslash K^k_j)$ which also implies $V(\hat{K}^k_j) \approx V(K^k_j)$. For the finite type in $\mathbb{C}^2$ case and strictly pseudoconvex case, it follows by (3.6) that $\Lambda(p, s^{-k-1}\delta) < \frac{1}{4}\Lambda(p, s^{-k}\delta)$ for any $p \in Q^{k+1}_i \subset Q^k_j$. Hence the height of $\hat{K}^k_j$ will be at least 4 times the height of $\hat{K}^k_j\backslash K^k_j$. Thus $V(\hat{K}^k_j) \approx V(K^k_j)$. □

3.3. Weighted maximal operator based on dyadic tents

Definition 3.12. Let σ be a positive integrable function on Ω. Let $\mathcal{T}_l$ be a collection of dyadic tents as in Definition 3.8. The weighted maximal operator $\mathcal{M}_{\mathcal{T}_l,\sigma}$ is defined by

$$\mathcal{M}_{\mathcal{T}_l,\sigma}f(w) := \sup_{\hat{K}^k_j\in\mathcal{T}_l}\frac{1_{\hat{K}^k_j}(w)}{\sigma(\hat{K}^k_j)}\int_{\hat{K}^k_j}|f(z)|\sigma(z)dV(z). \tag{3.9}$$

Lemma 3.13. *$\mathcal{M}_{\mathcal{T}_l,\sigma}$ is bounded on $L^p(\Omega,\sigma)$ for $1 < p \leq \infty$. Moreover*

$$\|M_{\mathcal{T}_l,\sigma}\|_{L^p(\sigma)} \lesssim p/(p-1). \tag{3.10}$$

Proof. It's obvious that $\mathcal{M}_{\mathcal{T}_l,\sigma}$ is bounded on $L^\infty(\Omega,\sigma)$. We claim $\mathcal{M}_{\mathcal{T}_l,\sigma}$ is of weak-type $(1,1)$, i.e. for $f \in L^1(\Omega,\sigma)$, the following inequality holds for all $\lambda > 0$:

$$\sigma(\{z \in \Omega : M_{\mathcal{T}_l,\sigma}(f)(z) > \lambda\}) \lesssim \frac{\|f\|_{L^1(\Omega,\sigma)}}{\lambda}. \tag{3.11}$$

Then the Marcinkiewicz Interpolation Theorem implies the boundedness of $\mathcal{M}_{\mathcal{T}_l,\sigma}$ on $L^p(\Omega, \sigma)$ for $1 < p \leq \infty$, and inequality (3.10) follows from a standard argument for the Hardy-Littlewood maximal operator.

For a point $w \in \{z \in \Omega : \mathcal{M}_{\mathcal{T}_l,\sigma} f(z) > \lambda\}$, there exists a unique maximal tent $\hat{K}_j^k \in \mathcal{T}$ that contains w and satisfies:

$$\frac{1_{\hat{K}_j^k}(w)}{\sigma(\hat{K}_j^k)} \int_{\hat{K}_j^k} |f(z)|\sigma(z)dV(z) > \frac{\lambda}{2}. \tag{3.12}$$

Let $\mathcal{I}_\lambda$ be the set of all such maximal tents $\hat{K}_j^k$. The union of these maximal tents covers the set $\{z \in \Omega : \mathcal{M}_{\mathcal{T}_l,\sigma} f(z) > \lambda\}$. Since the tents $\hat{K}_j^k$ are maximal, they are also pairwise disjoint. Hence

$$\sigma(\{z \in \Omega : \mathcal{M}_{\mathcal{T}_l,\sigma} f(z) > \lambda\}) \leq \sum_{\hat{K}_j^k \in \mathcal{I}_\lambda} \sigma(\hat{K}_j^k) \leq \sum_{\hat{K}_j^k \in \mathcal{I}_\lambda} \frac{2}{\lambda} \int_{\hat{K}_j^k} f(z)\sigma(z)dV(z) \leq \frac{2\|f\|_{L^1(\Omega,\sigma)}}{\lambda}.$$

Thus inequality (3.11) holds and $\mathcal{M}_{\mathcal{T}_l,\sigma}$ is weak-type (1,1). □

4. Estimates for the Bergman kernel function

We recall known estimates for the Bergman kernel function, and their relation with the volume of the tents in the previous section.

4.1. Finite type in $\mathbb{C}^2$ case

In [36], the estimate of the Bergman kernel has been expressed in terms of d_1 and $\Lambda(p, \delta)$. Similar results were also obtained in [26].

Theorem 4.1 *([36,26]). Let ϵ_0 be the same as in Lemma 3.2. Then for points $p, q \in N_{\epsilon_0}(\mathbf{b}\Omega)$, one has*

$$|K_\Omega(p; \bar{q})| \lesssim d_1(p, q)^{-2}\Lambda(\pi(p), d_1(p, q))^{-2}. \tag{4.1}$$

As a consequence, there is a constant c such that $p, q \in B_1^\#(\pi(p), cd_1(p, q))$ and

$$|K_\Omega(p; \bar{q})| \lesssim (V(B_1^\#(\pi(p), cd_1(p, q))))^{-1}. \tag{4.2}$$

4.2. Strictly pseudoconvex case

When Ω is bounded, strictly pseudoconvex with smooth boundary, the behavior of the Bergman kernel function is well understood. In [13,6], asymptotic expansions of the kernel function were obtained on and off the diagonal. To obtain the L^p mapping property of the Bergman projection, a weaker estimate as in [9] would suffice. The proof of the following theorem can be found in [30].

Theorem 4.2 *([30]). Let Ω be a smooth, bounded, strictly pseudoconvex domain in $\mathbb{C}^n$ with a defining function ρ. For each $p \in \mathbf{b}\Omega$, there exists a neighborhood U of p, holomorphic coordinates $(\zeta_1, \dots, \zeta_n)$ and a constant $C > 0$, such that for $p, q \in U \bigcap \Omega$,*

$$|K_\Omega(p; \bar{q})| \leq C \left(|\rho(p)| + |\rho(q)| + |p_n - q_n| + \sum_{j=1}^{n-1} |p_k - q_k|^2 \right)^{-n-1}. \tag{4.3}$$

Here $p = (p_1, \dots, p_n)$ is in ζ-coordinates.

Up to a unitary rotation and a translation, we may assume that, under the original z-coordinates, $\partial\rho(p) = dz_n$ and $p = 0$, then the holomorphic coordinates $(\zeta_1, \dots, \zeta_n)$ in Theorem 4.2 can be expressed as the biholomorphic mapping $\Phi(z) = \zeta$ with

$$\begin{aligned} &\zeta_1 = z_1 \\ &\vdots \\ &\zeta_{n-1} = z_{n-1} \\ &\zeta_n = z_n + \frac{1}{2} \sum_{k,l=1}^{n} \frac{\partial^2 \rho}{\partial z_l \partial z_k}(p) z_k z_l. \end{aligned}$$

The next theorem relates the estimate in Theorem 4.2 to the measure of the tents:

Theorem 4.3. *Let p, q, and $(p_1, \dots, p_n)$ be the same as in Theorem 4.2. There exists a constant $r > 0$ such that the tent $B_1^\#(p, r)$ contains points p and q, and*

$$r^2 \approx \max \left\{ |\rho(p)| + |\rho(q)|, |p_n - q_n| + \sum_{j=1}^{n-1} |p_k - q_k|^2 \right\}. \tag{4.4}$$

Moreover, $|K_\Omega(p; \bar{q})| \lesssim \left(V(B_1^\#(p, r)) \right)^{-1}$.

Proof. Note that Φ is biholomorphic and can be approximated by the identity map near p. For any points w, η in the neighborhood U of the point p in Theorem 4.2, the distance $d_1(w, \eta)$ is about the same when computed in coordinates $(\zeta_1, \dots, \zeta_n)$. Φ is also measure preserving since the complex Jacobian determinant $J_{\mathbb{C}}\Phi = 1$. Therefore we may assume that those results about d_1 and volumes of the tents in Sections 2 and 3 hold true in ζ-coordinates. Then by (3.1) and the strict pseudoconvexity ($\Lambda(\pi(p), \epsilon) = \epsilon^2$), the estimate

$$|K_\Omega(p; \bar{q})| \lesssim (V(B_1^{\#}(p, r)))^{-1}$$

holds true for $r > 0$ that satisfies (4.4). Therefore it is enough to show the existence of such a constant r. Set $r_1 = \sqrt{|p_n - q_n| + \sum_{j=1}^{n-1} |p_k - q_k|^2}$ and $r_2 = \sqrt{|\rho(p)| + |\rho(q)|}$. Note that $\partial/\partial\zeta_1, \dots, \partial/\partial\zeta_{n-1}$ are in $H_p(\mathbf{b}\Omega)$ and ∂/ζ_n is orthogonal to $H_p(\mathbf{b}\Omega)$. It follows from the fact $\Lambda(\pi(p), \epsilon) = \epsilon^2$ and Corollary 2.6 that there exists a constant C_1 such that the boundary point $\pi(q) \in \mathrm{Box}(\pi(p), C_1 r_1)$. On the other hand, $|\rho(p)| + |\rho(q)| \approx \mathrm{dist}(p, \mathbf{b}\Omega) + \mathrm{dist}(q, \mathbf{b}\Omega)$. Therefore there exists a constant C_2 such that $\Lambda(\pi(p), C_2 r_2) > |\rho(p)| + |\rho(q)|$. Set $r = \max\{C_1 r_1, C_2 r_2\}$. Then $B_1^{\#}(\pi(p), r)$ contains both points p, q and inequality (4.4) holds. □

4.3. Convex/decoupled finite type case

When Ω is a smooth, bounded, convex (or decoupled) domain of finite type in $\mathbb{C}^n$, estimates of the Bergman kernel function on Ω were obtained in [29,27,30]. See also [35] for a correction of a minor issue in [29].

Theorem 4.4. *Let Ω be a smooth, bounded, convex (or decoupled) domain of finite type in $\mathbb{C}^n$. Let p be a boundary point of Ω. There exists a neighborhood U of p so that for all $q_1, q_2 \in U \cap \Omega$,*

$$|K_\Omega(q_1; \bar{q}_2)| \lesssim \delta^{-2} \prod_{j=2}^{n} \tau_j(q_1, \delta)^{-2}, \tag{4.5}$$

where $\delta = |\rho(q_1)| + |\rho(q_2)| + \inf\{\epsilon > 0 : q_2 \in D(q_1, \epsilon)\}$.

We can reformulate Theorem 4.4 as below.

Theorem 4.5. *Let Ω be a smooth, bounded, convex (or decoupled) domain of finite type in $\mathbb{C}^n$. Let p be a boundary point of Ω. There exists a neighborhood U of p so that for all $q_1, q_2 \in U \cap \Omega$,*

$$|K_\Omega(q_1; \bar{q}_2)| \lesssim \Big(V(B_5^{\#}(\pi(q_1), \delta))\Big)^{-1}, \tag{4.6}$$

where $\delta = |\rho(q_1)| + |\rho(q_2)| + \inf\{\epsilon > 0 : q_2 \in D(q_1, \epsilon)\}$. Moreover, there exists a constant c such that $q_1, q_2 \in B_5^{\#}(\pi(q_1), c\delta)$.

Here the estimate (4.6) follows from (3.2). Recall that the polydisc $D(q, \delta)$ induces a global quasi-metric [28] on Ω. Then a triangle inequality argument using this quasi-metric yields the containment $q_2 \in B_5^{\#}(\pi(q_1), c\delta)$.

4.4. Dyadic operator domination

Theorem 4.6. *Let $\hat{K}_j^k$, K_j^k be the tents and kubes with respect to d and $B^{\#}$. Let $\{\mathcal{T}_l\}_{l=1}^N$ be the finite collections of tents induced by $\{\mathcal{Q}_l\}_{l=1}^N$ in Lemma 3.7. Then for $p, q \in \Omega$,*

$$|K_\Omega(p; \bar{q})| \lesssim (V(\Omega))^{-1} 1_{\Omega \times \Omega}(p, q) + \sum_{l=1}^{N} \sum_{\hat{K}_j^k \in \mathcal{T}_l} (V(\hat{K}_j^k))^{-1} 1_{\hat{K}_j^k \times \hat{K}_j^k}(p, q). \tag{4.7}$$

Proof. It suffices to show that for every p, q, there exists a $\hat{K}_j^k \in \mathcal{T}_l$ for some l such that

$$|K_\Omega(p; \bar{q})| \lesssim (V(\hat{K}_j^k))^{-1} 1_{\hat{K}_j^k \times \hat{K}_j^k}(p, q).$$

When $\operatorname{dist}(p, q) \approx 1$ or $\operatorname{dist}(p, \mathbf{b}\Omega) + \operatorname{dist}(q, \mathbf{b}\Omega) \approx 1$, the pair (p, q) is away from the boundary diagonal of $\Omega \times \Omega$. By Kerzman's Theorem [21,5], we have

$$|K_\Omega(p; \bar{q})| \lesssim 1 \approx (V(\Omega))^{-1} \approx (V(\Omega))^{-1} 1_{\Omega \times \Omega}(p, q).$$

We turn to the case when $\operatorname{dist}(p, q)$ and $\operatorname{dist}(p, \mathbf{b}\Omega) + \operatorname{dist}(q, \mathbf{b}\Omega)$ are both small and we may assume that both $p, q \in \Omega \cap N_{\epsilon_0}(\mathbf{b}\Omega)$. By Theorems 4.1, 4.3, and 4.5, there exists a small constant $r > 0$ such that $p, q \in B^{\#}(\pi(p), r)$ and

$$|K_\Omega(p; \bar{q})| \lesssim (V(B^{\#}(\pi(p), r)))^{-1}.$$

By Lemma 3.9, there exists a tent $\hat{K}_j^k \in \mathcal{T}_l$ for some l such that $B^{\#}(\pi(p), r) \subseteq \hat{K}_j^k$ and

$$V(\hat{K}_j^k) \approx V(B^{\#}(\pi(p), r)).$$

Thus $p, q \in \hat{K}_j^k$ and $|K_\Omega(p; \bar{q})| \lesssim (V(\hat{K}_j^k))^{-1} 1_{\hat{K}_j^k \times \hat{K}_j^k}(p, q)$. □

5. Proof of Theorem 1.2

Given a function h on Ω, we set M_h to be the multiplication operator by h:

$$M_h(f)(z) := h(z) f(z).$$

Let σ be a weight on Ω. Set $\nu(z) := \sigma^{-p'/p}(z)$ where p' is the Hölder conjugate index of p. Then it follows that the operator norms of P and P^+ on the weighted space $L^p(\Omega, \sigma)$ satisfy:

$$\|P : L^p(\Omega, \sigma) \to L^p(\Omega, \sigma)\| = \|PM_\nu : L^p(\Omega, \nu) \to L^p(\Omega, \sigma)\|; \tag{5.1}$$

$$\|P^+ : L^p(\Omega, \sigma) \to L^p(\Omega, \sigma)\| = \|P^+M_\nu : L^p(\Omega, \nu) \to L^p(\Omega, \sigma)\|. \tag{5.2}$$

It suffices to prove the inequality for $\|P^+M_\nu : L^p(\Omega, \nu) \to L^p(\Omega, \sigma)\|$.

Let $\{\mathcal{T}_l\}_{l=1}^N$ be the finite collections of tents in Theorem 4.6. Then inequality (4.7) holds: for $p, q \in \Omega$,

$$|K_\Omega(p, \bar{q})| \lesssim (V(\Omega))^{-1} 1_{\Omega\times\Omega}(p, q) + \sum_{l=1}^{N} \sum_{\hat{K}_j^k \in \mathcal{T}_l} (V(\hat{K}_j^k))^{-1} 1_{\hat{K}_j^k \times \hat{K}_j^k}(p, q). \tag{5.3}$$

Applying this inequality to the operator P^+M_ν yields

$$\begin{aligned}
\left|P^+M_\nu f(z)\right| &= \int_\Omega |K_\Omega(z; \bar{w})\nu(w)f(w)|dV(w) \\
&\lesssim \langle f\nu\rangle_\Omega^{dV} + \int_\Omega \sum_{l=1}^{N} \sum_{\hat{K}_j^k \in \mathcal{T}_l} \frac{1_{\hat{K}_j^k}(z) 1_{\hat{K}_j^k}(w) \, |\nu(w)f(w)|}{V(\hat{K}_j^k)} dV(w) \\
&= \langle f\nu\rangle_\Omega^{dV} + \sum_{l=1}^{N} \sum_{\hat{K}_j^k \in \mathcal{T}_l} 1_{\hat{K}_j^k}(z) \langle f\nu\rangle_{\hat{K}_j^k}^{dV}.
\end{aligned} \tag{5.4}$$

Set $Q_{0,\nu}^+(f)(z) := \langle f\nu\rangle_\Omega^{dV}$ and $Q_{l,\nu}^+ f(z) := \sum_{\hat{K}_j^k \in \mathcal{T}_l} 1_{\hat{K}_j^k}(z)\langle f\nu\rangle_{\hat{K}_j^k}^{dV}$. Then it suffices to estimate the norm for $Q_{l,\nu}^+$ with $l = 0, 1, \ldots, N$. The proof given below uses the argument for the upper bound of sparse operators in weighted theory of harmonic analysis, see for example [31] and [25]. An estimate for the norm of $Q_{0,\nu}^+$ is easy to obtain by Hölder's inequality:

$$\frac{\|Q_{0,\nu}^+(f)\|_{L^p(\Omega,\sigma)}^p}{\|f\|_{L^p(\Omega,\nu)}^p} \lesssim \frac{(\langle f\nu\rangle_\Omega^{dV})^p \langle\sigma\rangle_\Omega^{dV}}{\int_\Omega |f|^p \nu dV} \lesssim \langle\sigma\rangle_\Omega^{dV} (\langle\nu\rangle_\Omega^{dV})^{p-1}. \tag{5.5}$$

Now we turn to $Q_{l,\nu}^+$ for $l \neq 0$. Assume $p > 2$. For any $g \in L^{p'}(\Omega, \sigma)$,

$$\begin{aligned}
\left|\left\langle Q_{l,\nu}^+ f(z), g(z)\sigma(z)\right\rangle\right| &= \left|\int_\Omega Q_{l,\nu}^+ f(z) g(z)\sigma(z) dV(z)\right| \\
&= \left|\int_\Omega \sum_{\hat{K}_j^k \in \mathcal{T}_l} 1_{\hat{K}_j^k}(z) \langle f\nu\rangle_{\hat{K}_j^k}^{dV} g(z)\sigma(z) dV(z)\right| \\
&\leq \sum_{\hat{K}_j^k \in \mathcal{T}_l} \langle f\nu\rangle_{\hat{K}_j^k}^{dV} \int_{\hat{K}_j^k} |g(z)|\sigma(z) dV(z)
\end{aligned}$$

$$= \sum_{\hat{K}_j^k \in \mathcal{T}_l} \langle f\nu \rangle_{\hat{K}_j^k}^{dV} \langle g\sigma \rangle_{\hat{K}_j^k}^{dV} V(\hat{K}_j^k). \tag{5.6}$$

Since $\langle f\nu \rangle_{\hat{K}_j^k}^{dV} \langle g\sigma \rangle_{\hat{K}_j^k}^{dV} = \langle f \rangle_{\hat{K}_j^k}^{\nu dV} \langle \nu \rangle_{\hat{K}_j^k}^{dV} \langle g \rangle_{\hat{K}_j^k}^{\sigma dV} \langle \sigma \rangle_{\hat{K}_j^k}^{dV}$, it follows that

$$\begin{aligned}\sum_{\hat{K}_j^k \in \mathcal{T}_l} \langle f\nu \rangle_{\hat{K}_j^k}^{dV} \langle g\sigma \rangle_{\hat{K}_j^k}^{dV} V(\hat{K}_j^k) &= \sum_{\hat{K}_j^k \in \mathcal{T}_l} \langle f \rangle_{\hat{K}_j^k}^{\nu dV} \langle \nu \rangle_{\hat{K}_j^k}^{dV} \langle g \rangle_{\hat{K}_j^k}^{\sigma dV} \langle \sigma \rangle_{\hat{K}_j^k}^{dV} V(\hat{K}_j^k) \\ &= \sum_{\hat{K}_j^k \in \mathcal{T}_l} \left(\langle \nu \rangle_{\hat{K}_j^k}^{dV} \right)^{p-1} \langle \sigma \rangle_{\hat{K}_j^k}^{dV} \langle f \rangle_{\hat{K}_j^k}^{\nu dV} \langle g \rangle_{\hat{K}_j^k}^{\sigma dV} V(\hat{K}_j^k) \left(\langle \nu \rangle_{\hat{K}_j^k}^{dV} \right)^{2-p}. \end{aligned} \tag{5.7}$$

Then

$$\begin{aligned} &\sum_{\hat{K}_j^k \in \mathcal{T}_l} \left(\langle \nu \rangle_{\hat{K}_j^k}^{dV} \right)^{p-1} \langle \sigma \rangle_{\hat{K}_j^k}^{dV} \langle f \rangle_{\hat{K}_j^k}^{\nu dV} \langle g \rangle_{\hat{K}_j^k}^{\sigma dV} V(\hat{K}_j^k) \left(\langle \nu \rangle_{\hat{K}_j^k}^{dV} \right)^{2-p} \\ &\le \sup_{1\le l\le N} \sup_{\hat{K}_j^k \in \mathcal{T}_l} \langle \sigma \rangle_{\hat{K}_j^k}^{dV} \left(\langle \nu \rangle_{\hat{K}_j^k}^{dV} \right)^{p-1} \sum_{\hat{K}_j^k \in \mathcal{T}_l} \langle f \rangle_{\hat{K}_j^k}^{\nu dV} \langle g \rangle_{\hat{K}_j^k}^{\sigma dV} \left(V(\hat{K}_j^k) \right)^{p-1} \left(\nu(\hat{K}_j^k) \right)^{2-p}. \end{aligned} \tag{5.8}$$

By Lemma 3.11, one has $V(\hat{K}_j^k) \approx V(K_j^k)$. The fact that $p \ge 2$ and the containment $K_j^k \subseteq \hat{K}_j^k$ gives the inequality $\left(\nu(\hat{K}_j^k) \right)^{2-p} \le \left(\nu(K_j^k) \right)^{2-p}$. This inequality yields:

$$\left(V(\hat{K}_j^k) \right)^{p-1} \left(\nu(\hat{K}_j^k) \right)^{2-p} \lesssim \left(V(K_j^k) \right)^{p-1} \left(\nu(K_j^k) \right)^{2-p}. \tag{5.9}$$

By Hölder's inequality,

$$V(K_j^k) \le \left(\nu(K_j^k) \right)^{\frac{1}{p'}} \left(\sigma(K_j^k) \right)^{\frac{1}{p}}.$$

Therefore,

$$\left(V(K_j^k) \right)^{p-1} \left(\nu(K_j^k) \right)^{2-p} \le \left(\nu(K_j^k) \right)^{\frac{1}{p}} \left(\sigma(K_j^k) \right)^{\frac{1}{p'}}. \tag{5.10}$$

Substituting these inequalities into the last line of (5.8), we obtain

$$\begin{aligned} &\sum_{\hat{K}_j^k \in \mathcal{T}_l} \langle f \rangle_{\hat{K}_j^k}^{\nu dV} \langle g \rangle_{\hat{K}_j^k}^{\sigma dV} \left(V(\hat{K}_j^k) \right)^{p-1} \left(\nu(\hat{K}_j^k) \right)^{2-p} \\ &\lesssim \sum\nolimits_{\hat{K}_j^k \in \mathcal{T}_l} \langle f \rangle_{\hat{K}_j^k}^{\nu dV} \langle g \rangle_{\hat{K}_j^k}^{\sigma dV} \left(\nu(K_j^k) \right)^{\frac{1}{p}} \left(\sigma(K_j^k) \right)^{\frac{1}{p'}}. \end{aligned}$$

Applying Hölder's inequality again the sum above gives

$$\begin{aligned}&\sum_{\hat{K}_j^k\in\mathcal{T}_l}\langle f\rangle_{\hat{K}_j^k}^{\nu dV}\langle g\rangle_{\hat{K}_j^k}^{\sigma dV}\left(\nu(K_j^k)\right)^{\frac{1}{p}}\left(\sigma(K_j^k)\right)^{\frac{1}{p'}}\\ &\le\left(\sum_{\hat{K}_j^k\in\mathcal{T}_l}\left(\langle f\rangle_{\hat{K}_j^k}^{\nu dV}\right)^p\nu(K_j^k)\right)^{\frac{1}{p}}\left(\sum_{\hat{K}_j^k\in\mathcal{T}_l}\left(\langle g\rangle_{K_j^k}^{\sigma dV}\right)^{p'}\sigma(K_j^k)\right)^{\frac{1}{p'}}.\end{aligned}\tag{5.11}$$

By the disjointness of K_j^k and Lemma 3.13, we have

$$\sum_{\hat{K}_j^k\in\mathcal{T}_l}\left(\langle f\rangle_{\hat{K}_j^k}^{\nu dV}\right)^p\nu(K_j^k)\le\int_\Omega(\mathcal{M}_{\mathcal{T}_l,\nu}f)^p\nu dV\le(p')^p\|f\|_{L^p(\Omega,\nu)}^p.\tag{5.12}$$

Similarly, we also have

$$\sum_{\hat{K}_j^k\in\mathcal{T}_l}\left(\langle g\rangle_{\hat{K}_j^k}^{\sigma dV}\right)^{p'}\sigma(K_j^k)\le\int_\Omega(\mathcal{M}_{\mathcal{T}_l,\sigma}g)^{p'}\sigma dV\le(p)^{p'}\|g\|_{L^{p'}(\Omega,\sigma)}^{p'}.\tag{5.13}$$

Substituting (5.12) and (5.13) back into (5.11) and (5.6), we finally obtain

$$\left\langle Q_{l,\nu}^+f,g\sigma\right\rangle\lesssim pp'\sup_{1\le l\le N}\sup_{\hat{K}_j^k\in\mathcal{T}_l}\langle\sigma\rangle_{\hat{K}_j^k}^{dV}\left(\langle\nu\rangle_{\hat{K}_j^k}^{dV}\right)^{p-1}\|f\|_{L^p(\Omega,\nu)}\|g\|_{L^{p'}(\Omega,\sigma)}.$$

Therefore $\sum_{l=1}^N\|Q_{l,\nu}^+\|_{L^p(\Omega,\sigma)}\lesssim pp'\sup_{1\le l\le N}\sup_{\hat{K}_j^k\in\mathcal{T}_l}\langle\sigma\rangle_{\hat{K}_j^k}^{dV}\left(\langle\nu\rangle_{\hat{K}_j^k}^{dV}\right)^{p-1}$.

Now we turn to the case $1<p<2$ and show that for all $f\in L^p(\Omega,\nu)$ and $g\in L^{p'}(\Omega,\sigma)$

$$\left\langle Q_{l,\nu}^+f,g\sigma\right\rangle\lesssim\left(\sup_{1\le l\le N}\sup_{\hat{K}_j^k\in\mathcal{T}_l}\langle\sigma\rangle_{\hat{K}_j^k}^{dV}\left(\langle\nu\rangle_{\hat{K}_j^k}^{dV}\right)^{p-1}\right)^{\frac{1}{p-1}}\|f\|_{L^p(\Omega,\nu)}\|g\|_{L^{p'}(\Omega,\sigma)}.$$

By the definition of $Q_{l,\nu}^+$,

$$\begin{aligned}\left\langle Q_{l,\nu}^+f,g\sigma\right\rangle&=\left\langle\sum_{\hat{K}_j^k\in\mathcal{T}_l}1_{\hat{K}_j^k}(w)\langle f\nu\rangle_{\hat{K}_j^k}^{dV},g\sigma\right\rangle\\&=\sum_{\hat{K}_j^k\in\mathcal{T}_l}\langle f\nu\rangle_{\hat{K}_j^k}^{dV}\langle g\sigma\rangle_{\hat{K}_j^k}^{dV}V(\hat{K}_j^k)\\&=\sum_{\hat{K}_j^k\in\mathcal{T}_l}\left\langle 1_{\hat{K}_j^k}(w)\langle g\sigma\rangle_{\hat{K}_j^k}^{dV},f\nu\right\rangle\\&=\left\langle Q_{l,\sigma}^+(g),f\nu\right\rangle.\end{aligned}\tag{5.14}$$

Since $1 < p < 2$, $p' > 2$. Then, replacing p by p', interchanging σ and ν, and adopting the same argument for the case $p \geq 2$ yields that

$$\begin{aligned}\|Q^+_{l,\sigma}\|_{L^{p'}(\Omega,\nu)} &\lesssim p'p \sup_{1\leq l\leq N} \sup_{\hat{K}^k_j\in\mathcal{T}_l} \langle\nu\rangle^{dV}_{\hat{K}^k_j} \left(\langle\sigma\rangle^{dV}_{\hat{K}^k_j}\right)^{p'-1} \\ &= pp' \left(\sup_{1\leq l\leq N} \sup_{\hat{K}^k_j\in\mathcal{T}_l} \langle\sigma\rangle^{dV}_{\hat{K}^k_j} \left(\langle\nu\rangle^{dV}_{\hat{K}^k_j}\right)^{p-1}\right)^{\frac{1}{p-1}}.\end{aligned}$$

Thus we have

$$\left\langle Q^+_{l,\nu} f, g\sigma \right\rangle \lesssim pp' \left(\sup_{1\leq l\leq N} \sup_{\hat{K}^k_j\in\mathcal{T}_l} \langle\sigma\rangle^{dV}_{\hat{K}^k_j} \left(\langle\nu\rangle^{dV}_{\hat{K}^k_j}\right)^{p-1}\right)^{\frac{1}{p-1}} \|g\|_{L^{p'}(\Omega,\sigma)}\|f\|_{L^p(\Omega,\nu)},$$

and

$$\|Q^+_{l,\nu} : L^p(\Omega,\nu) \to L^p(\Omega,\sigma)\| \lesssim pp' \left(\sup_{1\leq l\leq N} \sup_{\hat{K}^k_j\in\mathcal{T}_l} \langle\sigma\rangle^{dV}_{\hat{K}^k_j} \left(\langle\nu\rangle^{dV}_{\hat{K}^k_j}\right)^{p-1}\right)^{\frac{1}{p-1}}.$$

Combining the results for $Q^+_{l,\nu}$ with $l \neq 0$ for $1 < p < 2$ and $p \geq 2$ and inequality (5.5) for $Q^+_{0,\nu}$ gives the estimate in Theorem 1.2:

$$\|P^+\|_{L^p(\Omega,\sigma)} \lesssim [\sigma]_p.$$

6. A sharp example

In this section, we provide an example to show that the estimate in Theorem 1.2 is sharp. Our example is for the case $1 < p \leq 2$. The case $p > 2$ follows by a duality argument. The idea is similar to the ones in [39,40]. Since Ω is a pseudoconvex domain of finite type, Kerzman's Theorem [21,5] implies that the kernel function K_Ω extends to a C^∞ function away from a neighborhood of the boundary diagonal of $\Omega\times\Omega$. Let $w_\circ \in \Omega$ be a point that is away from the set $N_{\epsilon_0}(\mathbf{b}\Omega)$ and satisfies $K_\Omega(w_\circ; \bar{w}_\circ) = 2C_1 > 0$ for some constant C_1. Then the maximum principle implies that $\{z \in \mathbf{b}\Omega : |K_\Omega(z; \bar{w}_\circ)| > C_1\}$ is a non-empty open subset of $\mathbf{b}\Omega$. We claim that there exists a strictly pseudoconvex point $z_\circ$ in the set $\{z \in \mathbf{b}\Omega : |K_\Omega(z; \bar{w}_\circ)| > C_1\}$. Let p be a point in $\{z \in \mathbf{b}\Omega : |K_\Omega(z; \bar{w}_\circ)| > C_1\}$. Since p is a point of finite 1-type in the sense of D'Angelo, the determinant of the Levi form does not vanish identically near point p. Thus the determinant of the Levi form is strictly positive at some point in every neighborhood of p, i.e., there is a sequence of strictly pseudoconvex points converging to p. Since $\{z \in \mathbf{b}\Omega : |K_\Omega(z; \bar{w}_\circ)| > C_1\}$ is a neighborhood of p, there exists a strictly pseudoconvex point $z_\circ$ such that

$$|K_\Omega(z_\circ;\bar{w}_\circ)| > C_1. \tag{6.1}$$

There are several possible proofs in the literature for the existence of the nonvanishing point of the determinant of the Levi form near a point of finite type. See for example [34] or the forthcoming thesis of Fassina [12]. Nevertheless, we choose the strictly pseudoconvex point $z_\circ$ above only for the simplicity of the construction of our example and it is not required. See Remark 6.2 below. By Kerzman's Theorem again, there exists a small constant δ_0 such that for any pair of points $(z,w) \in B^\#(z_\circ,\delta_0)\times\{w\in\Omega : \mathrm{dist}(w,w_\circ) < \delta_0\}$,

$$|K_\Omega(z,\bar{w}) - K_\Omega(z_\circ;\bar{w}_\circ)| \le C_1/10.$$

Thus for $(z,w) \in B^\#(z_\circ,\delta_0) \times \{w \in \Omega : \mathrm{dist}(w,w_\circ) < \delta_0\}$, one has

$$|K_\Omega(z,\bar{w})| \approx C_1.$$

Moreover, an elementary geometric reasoning yields that

$$\arg\{K_\Omega(z;\bar{w}), K_\Omega(z_\circ,w_\circ)\} \in [-\sin^{-1}(1/10), \sin^{-1}(1/10)]. \tag{6.2}$$

From now on, we let δ_0 be a fixed constant. For $z \in \Omega$, we set

$$h(z) = \inf\{\delta > 0 : z \in B^\#(z_\circ,\delta)\}, \text{ and } \quad l(z) = \mathrm{dist}(z,w_\circ). \tag{6.3}$$

Let $1 < p \le 2$. Let s be a positive constant that is sufficiently close to 0. We define the weight function σ on Ω to be

$$\sigma(w) = \frac{(h(w))^{(p-1)(2+2n-2s)}}{(l(w))^{2n-2s}}. \tag{6.4}$$

We claim that the constant $[\sigma]_p \approx s^{-1}$.

First, we consider the average of σ and $\sigma^{\frac{1}{1-p}}$ over the tent $B^\#(z,\delta)$. Note that

$$\sigma^{\frac{1}{1-p}}(w) = \frac{(h(w))^{(2s-2n-2)}}{(l(w))^{(2s-2n)/(p-1)}}.$$

If the tent $B^\#(z,\delta)$ does not intersect $B^\#(z_\circ,\delta)$, then for any $w \in B^\#(z,\delta)$ we have $h(w) \approx x+\delta$ and $l(w) \approx 1$ with $x \gtrsim \delta$. Thus we have

$$\langle\sigma\rangle^{dV}_{B^\#(z,\delta)} \left(\langle\sigma^{\frac{1}{1-p}}\rangle^{dV}_{B^\#(z,\delta)}\right)^{p-1} \approx (x+\delta)^{(p-1)(2+2n-2s)} \left((x+\delta)^{(2s-2n-2)}\right)^{p-1} = 1. \tag{6.5}$$

If $B^\#(z,\delta)$ intersects $B^\#(z_\circ,\delta)$, then there exists a constant C so that $B^\#(z_\circ,C\delta)$ contains $B^\#(z,\delta)$ with $|B^\#(z_\circ,C\delta)| \approx |B^\#(z,\delta)|$ by the doubling property of the ball $B^\#$. Hence

$$\langle\sigma\rangle^{dV}_{B^\#(z,\delta)}\left(\langle\sigma^{\frac{1}{1-p}}\rangle^{dV}_{B^\#(z,\delta)}\right)^{p-1}\lesssim\langle\sigma\rangle^{dV}_{B^\#(z_\circ,C\delta)}\left(\langle\sigma^{\frac{1}{1-p}}\rangle^{dV}_{B^\#(z_\circ,C\delta)}\right)^{p-1}.$$

Since $w_\circ$ is away from the set $N_{\epsilon_0}(\mathbf{b}\Omega)$ and hence is away from any tents, $l(w)\approx 1$ for any $w\in B^\#(z_\circ,C\delta)$. It follows that

$$\langle\sigma\rangle^{dV}_{B^\#(z_\circ,C\delta)}\approx\int\limits_{B^\#(z_\circ,C\delta)}h(w)^{(p-1)(2n+2-2s)}dV(w)(V(B^\#(z,\delta)))^{-1}.$$

Recall that $z_\circ$ is a strictly pseudoconvex point. There exist special holomorphic coordinates $(\zeta_1,\dots,\zeta_n)$ in a neighborhood of $z_\circ$ as in Theorem 4.2 so that $z_\circ=(0,\dots,0)$ and the tent

$$D(z_\circ,\delta):=\{w=(\zeta_1,\dots,\zeta_n)\in\Omega:|\zeta_n|<\delta^2,|\zeta_j|<\delta,j=1,\dots,n-1\},$$

is equivalent to $B^\#(z_\circ,C\delta)$ in the sense that there exist constants c_1 and c_2 so that

$$D(z_\circ,c_1\delta)\subseteq B^\#(z_\circ,C\delta)\subseteq D(z_\circ,c_2\delta).$$

Moreover, $h(w)\approx(|\zeta_1|^2+\cdots+|\zeta_{n-1}|^2+|\zeta_n|)^{\frac{1}{2}}$. Therefore

$$\langle\sigma\rangle^{dV}_{B^\#(z_\circ,C\delta)}\approx\int\limits_{B^\#(z_\circ,C\delta)}h(w)^{(p-1)(2n+2-2s)}dV(w)(V(B^\#(z,\delta)))^{-1}\approx\delta^{(p-1)(2n+2-2s)}. \tag{6.6}$$

Similarly,

$$\langle\sigma^{\frac{1}{1-p}}\rangle^{dV}_{B^\#(z_\circ,C\delta)}\approx\int\limits_{B^\#(z_\circ,C\delta)}h(w)^{2s-2n-2}dV(w)(V(B^\#(z,\delta)))^{-1}\approx s^{-1}\delta^{(2s-2n-2)}, \tag{6.7}$$

where s^{-1} comes from the power rule $\int_0^a t^{s-1}dt=a^s/s$. Combining these inequalities yields

$$\langle\sigma\rangle^{dV}_{B^\#(z,\delta)}\left(\langle\sigma^{\frac{1}{1-p}}\rangle^{dV}_{B^\#(z,\delta)}\right)^{p-1}\lesssim\langle\sigma\rangle^{dV}_{B^\#(z_\circ,C\delta)}(\langle\sigma^{\frac{1}{1-p}}\rangle^{dV}_{B^\#(z_\circ,C\delta)})^{p-1}\approx s^{-(p-1)}. \tag{6.8}$$

Now we turn to the average of σ and $\sigma^{\frac{1}{1-p}}$ over the entire domain Ω. Note that

$$\langle\sigma\rangle^{dV}_{\Omega}\approx\int\limits_{\Omega}l(w)^{2s-2n}dV(w).$$

A computation using polar coordinates yields that $\langle\sigma\rangle^{dV}_{\Omega}\approx s^{-1}$. Also,

$$\begin{aligned}\langle\sigma^{\frac{1}{1-p}}\rangle_{\Omega}^{dV} &\approx \int_{\Omega} h(w)^{2s-2n-2}dV(w)\\ &= \int_{B^{\#}(z_\circ,\delta_0)} h(w)^{2s-2n-2}dV(w) + \int_{\Omega\setminus B^{\#}(z_\circ,\delta_0)} h(w)^{2s-2n-2}dV(w)\\ &\approx \delta_0^{2s-2n-2}s^{-1} + \delta_0^{2s-2n-2} \approx s^{-1},\end{aligned}$$

where the third approximation sign follows by s being sufficiently small and δ_0 being a fixed constant. Thus

$$\langle\sigma\rangle_{\Omega}^{dV}\left(\langle\sigma^{\frac{1}{1-p}}\rangle_{\Omega}^{dV}\right)^{p-1} \approx s^{-1}(s^{-1})^{p-1} \approx s^{-p}. \tag{6.9}$$

This estimate together with inequalities (6.5) and (6.8) yields that $[\sigma]_p \approx s^{-1}$.

Now we consider the function

$$f(w) = \sigma^{\frac{1}{1-p}}(w)1_{B^{\#}(z_\circ,\delta_0)},$$

where δ_0 is the same fixed constant so that (6.2) holds. Since $z_\circ$ is a point away from $w_\circ$, $l(w) \approx 1$ for $w \in B^{\#}(z_\circ,\delta_0)$. Thus

$$\|f\|^p_{L^p(\Omega,\sigma)} = \langle\sigma^{\frac{1}{1-p}}\rangle^{dV}_{B^{\#}(z_\circ,\delta_0)}V(B^{\#}(z_\circ,\delta_0)) \approx s^{-1}.$$

When $z \in \{w \in \Omega : \mathrm{dist}(w, w_\circ) < \delta_0\}$, (6.1) and (6.2) imply that

$$\begin{aligned}|P(f)(z)| &= \left|\int_{B^{\#}(z_\circ,\delta_0)} K_\Omega(z;\bar{w})f(w)dV(w)\right|\\ &\approx \int_{B^{\#}(z_\circ,\delta_0)} |K_\Omega(z_0;\bar{w}_0)|f(w)dV(w)\\ &\approx \int_{B^{\#}(z_\circ,\delta_0)} f(w)dV(w) = \langle\sigma^{\frac{1}{1-p}}\rangle^{dV}_{B^{\#}(z_\circ,\delta_0)}V(B^{\#}(z_\circ,\delta_0)) \approx s^{-1}. \end{aligned}\tag{6.10}$$

By (6.10) and the fact that $\delta_0 \approx 1$, we obtain the desired estimate:

$$\begin{aligned}\|P(f)\|^p_{L^p(\Omega,\sigma)} &= \int_{\Omega} |P(f)(z)|^p\sigma(z)dV(z)\\ &\geq \int_{\{z\in\Omega:\mathrm{dist}(z,w_\circ)<\delta_0\}} |P(f)(z)|^p\sigma(z)dV(z)\\ &\gtrsim s^{-p}\int_{\{z\in\Omega:\mathrm{dist}(z,w_\circ)<\delta_0\}} (\mathrm{dist}(z,w_\circ))^{2s-2n}dV(z)\end{aligned}$$

$$\approx s^{-p}s^{-1} \approx ([\sigma]_p)^p \|f\|^p_{L^p(\Omega,\sigma)}. \tag{6.11}$$

Remark 6.1. In the particular case of the unit ball $\mathbb{B}_n$ we can make our example more explicit: the weight $\sigma(w) = |w - z_\circ|^{(p-1)(2+2n-2s)}/|w|^{2n-2s}$ with $z_\circ = (1, 0, \dots, 0)$ and the test function $f(w) = \sigma^{\frac{1}{1-p}}(w)1_{B^\#(z_\circ,1/2)}(w)$. One can compute explicitly in this case that σ is in the $\mathcal{B}_p$ class. We further remark that in [40], the authors produce an upper and lower bound in terms of a Bekollé-Bonami condition that doesn't utilize information about the large tents. The upper bound they produce is correct, however the claimed sharpness of the Bekollé-Bonami condition without testing the large tents is not quite correct. The example they construct does appropriately capture the behavior of small tents, but fails to do so in the case of large tents and this characteristic fails to capture the sharpness. It is for this reason that we have had to modify the definition of the Bekollé-Bonami characteristic in Definition 3.4 to reflect the behavior of both large and small tents.

Remark 6.2. In this example, we require $z_\circ$ to be a strictly pseudoconvex point only for the simplicity of the construction of the weight σ and the test function f, and the computation. For every $z \in \mathbf{b}\Omega$, the geometry of the tent $B^\#(z, \delta)$ is well understood. Thus σ and f can be modified accordingly so that the estimate (6.8) still holds true.

Remark 6.3. For a different example, we can also choose $z_\circ$ to be a point in Ω that is away from both $N_{\epsilon_0}(\mathbf{b}\Omega)$ and $w_\circ$, and change $h(w)$ in (6.3) to be $(\text{dist}(z_\circ, w))^{(p-1)(2n-2s)}$. The average of σ and $\sigma^{\frac{1}{1-p}}$ over tents is controlled by a constant since all tents are away from points $z_\circ$ and $w_\circ$. Moreover, $\langle\sigma\rangle^{dV}_\Omega(\langle\sigma^{\frac{1}{p-1}}\rangle^{dV}_\Omega)^{p-1} \approx s^{-p}$ by a computation using polar coordinates. Thus $[\sigma]_p \approx s^{-1}$ and a similar argument yields the sharpness of the bound in Theorem 1.2. We did not adopt this example since it does not reflect the connection between the weighted norm of the projection and the average of σ and $\sigma^{\frac{1}{1-p}}$ over small tents.

Remark 6.4. When the weight $\sigma \equiv 1$, the constant $[\sigma]_p \approx pp'$. Theorem 1.2 then gives an estimate for the L^p norm of the Bergman projection:

$$\|P\|_{L^p(\Omega)} \lesssim pp'.$$

For the strictly pseudoconvex case, such an estimate was obtained and proven to be sharp by Čučković [41]. Therefore, the constant pp' in $[\sigma]_p$ is necessary.

7. Proof of Theorem 1.3

We first show that the lower bound (1.4) in Theorem 1.3 with the assumption that Ω is bounded, smooth, and strictly pseudoconvex.

We begin by recalling the following two lemmas from [19].

Lemma 7.1. *Let Ω be a smooth, bounded, strictly pseudoconvex domain. If the Bergman projection P is bounded on the weighted space $L^p(\Omega, \sigma)$, then the weight σ and its dual weight $\nu = \sigma^{\frac{1}{1-p}}$ are integrable on Ω.*

Lemma 7.2. *Let Ω be a smooth, bounded, strictly pseudoconvex domain. Let δ be a small constant. For a boundary point z_1, let $B^\#(z_1, \delta)$ be a tent defined as in Definition 3.3. Then there exists a tent $B^\#(z_2, \delta)$ with $d(B(z_1, \delta), B(z_2, \delta)) \approx \delta$ so that if $f \geq 0$ is a function supported in $B^\#(z_i, \delta)$ and $z \in B^\#(z_j, \delta)$ with $i \neq j$ and $i, j \in \{1, 2\}$, then we have*

$$|P(f)(z)| \gtrsim \langle f \rangle^{dV}_{B^\#(z_i,\delta)}.$$

Recall that $\nu = \sigma^{1/(1-p)}$. By (5.1),

$$\|P\|_{L^p(\Omega,\sigma dV)} = \|PM_\nu : L^p(\Omega, \nu dV) \to L^p(\Omega, \sigma dV)\|.$$

It suffices to show that

$$\sup_{\epsilon_0 > \delta > 0, z \in \mathbf{b}\Omega} \langle \sigma \rangle^{dV}_{B^\#(z,\delta)} \left(\langle \nu \rangle^{dV}_{B^\#(z,\delta)} \right)^{p-1} \lesssim \|PM_\nu : L^p(\Omega, \nu dV) \to L^p(\Omega, \sigma dV)\|^{2p}.$$

For simplicity, we set $\mathcal{A} := \|PM_\nu : L^p(\Omega, \nu dV) \to L^p(\Omega, \sigma dV)\|$. If $\mathcal{A} < \infty$, then we have a weak-type (p, p) estimate:

$$\sigma\{w \in \Omega : |PM_\nu f(w)| > \lambda\} \lesssim \frac{\mathcal{A}^p}{\lambda^p} \|f\|^p_{L^p(\Omega,\nu dV)}. \tag{7.1}$$

Let δ_0 be a fixed constant so that Lemma 7.2 is true for all $\delta < \delta_0$. Set $f(w) = 1_{B^\#(z_1,\delta)}(w)$. Lemma 7.2 implies that for any $z \in B^\#(z_2, \delta)$,

$$|PM_\nu 1_{B^\#(z_1,\delta)}(z)| = \left| \int\limits_{B^\#(z_1,\delta)} K_\Omega(z; \bar{w}) \nu(w) dV(w) \right| > \langle \nu \rangle^{dV}_{B^\#(z_1,\delta)}. \tag{7.2}$$

It follows that

$$B^\#(z_2, \delta) \subseteq \{w \in \Omega : |PM_\nu f(w)| > \langle \nu \rangle^{dV}_{B^\#(z_1,\delta)}\}. \tag{7.3}$$

By Lemma 7.1, $\langle \nu \rangle^{dV}_{B^\#(z_1,\delta)}$ is finite. Then inequality (7.1) implies

$$\sigma(B^\#(z_2, \delta)) \leq \mathcal{A}^p \left(\langle \nu \rangle^{dV}_{B^\#(z_1,\delta)} \right)^{-p} \nu(B^\#(z_1, \delta)), \tag{7.4}$$

which is equivalent to $\langle \sigma \rangle^{dV}_{B^\#(z_2,\delta)} \left(\langle \nu \rangle^{dV}_{B^\#(z_1,\delta)} \right)^{p-1} \lesssim \mathcal{A}^p$. Since one can interchange the roles of z_1 and z_2 in Lemma 7.2, it follows that

$$\langle\sigma\rangle^{dV}_{B^\#(z_1,\delta)}\left(\langle\nu\rangle^{dV}_{B^\#(z_2,\delta)}\right)^{p-1}\lesssim\mathcal{A}^p.$$

Combining these two inequalities, we have

$$\langle\sigma\rangle^{dV}_{B^\#(z_1,\delta)}\left(\langle\nu\rangle^{dV}_{B^\#(z_2,\delta)}\right)^{p-1}\langle\sigma\rangle^{dV}_{B^\#(z_2,\delta)}\left(\langle\nu\rangle^{dV}_{B^\#(z_1,\delta)}\right)^{p-1}\lesssim\mathcal{A}^{2p}. \tag{7.5}$$

By Hölder's inequality,

$$V(B^\#(z_2,\delta))^p\le\int\limits_{B^\#(z_2,\delta)}\sigma dV\left(\int\limits_{B^\#(z_2,\delta)}\nu dV\right)^{p-1}. \tag{7.6}$$

Therefore $\langle\sigma\rangle^{dV}_{B^\#(z_2,\delta)}\left(\langle\nu\rangle^{dV}_{B^\#(z_2,\delta)}\right)^{p-1}\gtrsim 1$. Applying this to (7.5) and taking the supremum of the left side of (7.5) for all tents $B^\#(z_1,\delta)$ where $\delta<\delta_0$ yields

$$\sup_{\substack{\delta<\delta_0,\\ z_1\in\mathbf{b}\Omega}}\langle\sigma\rangle^{dV}_{B^\#(z_1,\delta)}\left(\langle\nu\rangle^{dV}_{B^\#(z_1,\delta)}\right)^{p-1}\lesssim\mathcal{A}^{2p}. \tag{7.7}$$

Since the constant ϵ_0 in Lemma 3.2 can be chosen to be δ_0, inequality (1.4) is proved.

Now we turn to prove (1.5) and assume in addition that Ω is Reinhardt. Since inequality (7.7) still holds true, it suffices to show

$$\langle\sigma\rangle^{dV}_{\Omega}\left(\langle\nu\rangle^{dV}_{\Omega}\right)^{p-1}\lesssim\mathcal{A}^{2p}. \tag{7.8}$$

Because Ω is Reinhardt, the monomials form a complete orthogonal system for the Bergman space $A^2(\Omega)$. Thus the kernel function K_Ω has the following series expression:

$$K_\Omega(z;\bar{w})=\sum_{\alpha\in\mathbb{N}^n}\frac{z^\alpha\bar{w}^\alpha}{\|z^\alpha\|^2_{L^2(\Omega)}}. \tag{7.9}$$

This implies that $K_\Omega(z;0)=\|1\|^{-2}_{L^2(\Omega)}$ for any $z\in\Omega$. By either the asymptotic expansion of K_Ω [13,6] or Kerzman's Theorem [21], we can find a precompact neighborhood U of the origin such that for any $z\in\Omega$ and $w\in U$,

$$|K_\Omega(z;\bar{w})|\approx 1\qquad\text{and}\qquad\arg\{K_\Omega(z;\bar{w}),K_\Omega(z;0)\}\in[-1/4,1/4]. \tag{7.10}$$

Let $f(w)=1_U(w)$. Then for any $z\in\Omega$,

$$|PM_\nu(f)(z)|=\left|\int\limits_U K_\Omega(z;\bar{w})\nu dV(w)\right|>c\|f\|_{L^1(\Omega,\nu dV)},$$

for some constant c. Therefore,

$$\Omega \subseteq \left\{z \in \Omega : |PM_\nu(f)(z)| > c\|f\|_{L^1(\Omega,\nu dV)}\right\}.$$

Applying this containment and the fact that $\|f\|_{L^1(\Omega,\nu dV)} = \|f\|^p_{L^p(\Omega,\nu dV)}$ to (7.1) yields

$$\sigma(\Omega) \leq \frac{\mathcal{A}^p}{c^p\|f\|^p_{L^1(\Omega,\nu dV)}}\|f\|^p_{L^p(\Omega,\nu dV)} \leq \frac{\mathcal{A}^p}{c^p\|f\|^{p-1}_{L^1(\Omega,\nu dV)}} < \infty. \tag{7.11}$$

Thus

$$\langle\sigma\rangle^{dV}_\Omega \left(\langle\nu\rangle^{dV}_U\right)^{p-1} \lesssim \mathcal{A}^p. \tag{7.12}$$

Interchanging the role of z and w in the argument above, we also have

$$U \subseteq \left\{w \in \Omega : |PM_\nu(1)(w)| > c\|1\|_{L^1(\Omega,\nu dV)}\right\},$$

and

$$\sigma(U) \leq \frac{\mathcal{A}^p}{c^p\|1\|^p_{L^1(\Omega,\nu dV)}}\|1\|^p_{L^p(\Omega,\nu dV)} \leq \frac{\mathcal{A}^p}{c^p\|1\|^{p-1}_{L^1(\Omega,\nu dV)}} < \infty. \tag{7.13}$$

Thus

$$\langle\sigma\rangle^{dV}_U \left(\langle\nu\rangle^{dV}_\Omega\right)^{p-1} \lesssim \mathcal{A}^p. \tag{7.14}$$

Combining (7.12), (7.14) and using the fact that

$$\langle\sigma\rangle^{dV}_U \left(\langle\nu\rangle^{dV}_U\right)^{p-1} \geq 1,$$

we obtain the desired estimate:

$$\langle\sigma\rangle^{dV}_\Omega \left(\langle\nu\rangle^{dV}_\Omega\right)^{p-1} \lesssim \langle\sigma\rangle^{dV}_\Omega \left(\langle\nu\rangle^{dV}_\Omega\right)^{p-1} \langle\sigma\rangle^{dV}_U \left(\langle\nu\rangle^{dV}_U\right)^{p-1} \lesssim \mathcal{A}^{2p}. \tag{7.15}$$

Estimates (7.15) and (7.7) then give (1.5). The proof is complete.

8. An application to the weak L^1 estimate

In [28], the weak-type $(1, 1)$ boundedness of the Bergman projection on simple domains was obtained using a Calderon-Zygmund type decomposition. In this section, we use Theorem 4.6 to provide an alternative approach to establish the weak-type bound for the Bergman projection. We follow the argument in [7] since we have a "sparse domination" for the Bergman projection.

Theorem 8.1. *There exists a constant $C > 0$ so that for all $f \in L^1(\Omega)$,*

$$\sup_{\lambda} \lambda V(\{z : |Pf(z)| > \lambda\}) < C\|f\|_{L^1(\Omega)}.$$

Proof. By a well-known equivalence of weak-type norms (see for example [15]), it suffices to show

$$\sup_{\substack{f_1 \\ \|f_1\|_{L^1(\Omega)}=1}} \sup_{G\subset\Omega} \inf_{\substack{G'\subset G \\ V(G)<2V(G')}} \sup_{\substack{f_2 \\ |f_2|\leq 1_{G'}}} |\langle Pf_1, f_2\rangle| < \infty. \tag{8.1}$$

In light of Theorem 4.6, we may replace P by $Q^+_{\ell_0,1}$ (using our previous notation) for some fixed ℓ_0 with $1 \leq \ell \leq N$. As in Definition 3.12, we consider the (now unweighted) dyadic maximal function $\mathcal{M}_{\mathcal{T}_{\ell_0},1}$. For convenience in what follows, we will simply write $\mathcal{M}_{\mathcal{T}_{\ell_0}}$. By Lemma 3.13, we know this operator is of weak-type $(1, 1)$. Fix f_1 with norm 1, $G \subset \Omega$ and constants C_1, C_2 to be chosen later. Define sets

$$H = \{z \in \Omega : \mathcal{M}_{\mathcal{T}_{\ell_0}} f_1(z) > C_1 V(G)^{-1}\}$$

and

$$\tilde{H} = \bigcup_{\hat{K}^k_j \in \mathcal{K}} \hat{K}^k_j$$

where

$$\mathcal{K} = \left\{\text{maximal tents } \hat{K}^k_j \text{ in } \mathcal{T}_{\ell_0} \text{ so } V(\hat{K}^k_j \cap H) > C_2 V(\hat{K}^k_j)\right\}.$$

Note if C_1 is chosen sufficiently large relative to C_2^{-1}, the weak-type estimate of $\mathcal{M}_{\mathcal{T}_{\ell_0}}$ implies

$$\begin{aligned} V(\tilde{H}) &= V\left(\bigcup_{\hat{K}^k_j \in \mathcal{K}} \hat{K}^k_j\right) \\ &\leq \sum_{\hat{K}^k_j \in \mathcal{K}} C_2^{-1} V(\hat{K}^k_j \cap H) \\ &\leq C_2^{-1} V(H) \\ &\leq C_2^{-1} C_1^{-1} V(G)\|f_1\|_{L^1(\Omega)} \\ &\leq \frac{1}{2} V(G). \end{aligned}$$

It is then clear if we let $G' = G \setminus \tilde{H}$, then $V(G) < 2V(G')$, so G' is a candidate set in the infimum in (8.1). If $z \in H^c$, then, by definition,

$$\mathcal{M}_{\mathcal{T}_{\ell_0}} f_1(z) \le C_1 V(G)^{-1}. \tag{8.2}$$

Using the distribution function,

$$\begin{aligned}
\|\mathcal{M}_{\mathcal{T}_{\ell_0}} f_1\|^2_{L^2(H^c)} &= 2 \int_0^{C_1 V(G)^{-1}} tV(\{z \in H^c : \mathcal{M}_{\mathcal{T}_{\ell_0}} f_1(z) > t\})dt \\
&\le 2 \int_0^{C_1 V(G)^{-1}} dt \|\mathcal{M}_{\mathcal{T}_{\ell_0}}\|_{L^{1,\infty}(H^c)} \|f_1\|_{L^1(\Omega)} \\
&\lesssim C_1 V(G)^{-1}.
\end{aligned} \tag{8.3}$$

Now let $|f_2| \le 1_G$ be fixed. We have

$$|\langle Q^+_{\ell_0,1} f_1, f_2\rangle| = \sum_{\hat{K}^k_j \in \mathcal{T}_{\ell_0}} V(\hat{K}^k_j)\langle f_1\rangle_{\hat{K}^k_j} \langle f_2\rangle_{\hat{K}^k_j}. \tag{8.4}$$

Note that for $\hat{K}^k_j \in \mathcal{T}_{\ell_0}$, if $V(\hat{K}^k_j \cap H) > C_2 V(\hat{K}^k_j)$ then $\hat{K}^k_j \subset \tilde{H}$. But f_2 is supported on $G' \subset \tilde{H}^c$, so for such a tent $\langle f_2\rangle_{\hat{K}^k_j} = 0$. Thus, examining (8.4), we may assume without loss of generality that if $\hat{K}^k_j \in \mathcal{T}_{\ell_0}$ then

$$V(\hat{K}^k_j \cap H) \le C_2 V(\hat{K}^k_j). \tag{8.5}$$

Then note that (8.5) implies the following holds true for the kubes K^k_j, provided C_2 is chosen sufficiently small:

$$\begin{aligned}
V(K^k_j \cap H^c) &= V(K^k_j) - V(K^k_j \cap H) \\
&\ge CV(\hat{K}^k_j) - V(\hat{K}^k_j \cap H) \\
&\gtrsim V(\hat{K}^k_j) \\
&\ge V(K^k_j)
\end{aligned}$$

where we let C be the implicit constant in Lemma 3.11. Thus we have

$$V(K^k_j) \lesssim V(K^k_j \cap H^c). \tag{8.6}$$

Therefore, continuing from (8.4) and using (8.3) and (8.6), we obtain

$$\begin{aligned}
|\langle Q^+_{\ell_0,1} f_1, f_2\rangle| &\lesssim \sum_{\hat{K}^k_j \in \mathcal{T}_{\ell_0}} V(K^k_j)\langle f_1\rangle_{\hat{K}^k_j} \langle f_2\rangle_{\hat{K}^k_j} \\
&\lesssim \sum_{\hat{K}^k_j \in \mathcal{T}_{\ell_0}} V(K^k_j \cap H^c)\langle f_1\rangle_{\hat{K}^k_j} \langle f_2\rangle_{\hat{K}^k_j}
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{H^c} (\mathcal{M}_{\mathcal{T}\ell_0} f_1)(\mathcal{M}_{\mathcal{T}\ell_0} f_2)\, dV \\
&\leq \|\mathcal{M}_{\mathcal{T}_{\ell_0}} f_1\|_{L^2(H^c)} \|\mathcal{M}_{\mathcal{T}_{\ell_0}} f_2\|_{L^2(\Omega)} \\
&\lesssim V(G)^{-\frac{1}{2}} \|f_2\|_{L^2(\Omega)} \\
&\leq V(G)^{-\frac{1}{2}} V(G)^{\frac{1}{2}} \\
&= 1,
\end{aligned}
$$

which establishes the result. □

9. Directions for generalization

1. The example in Section 6 showed the upper bound estimate in Theorem 1.2 is sharp. It is not clear if the lower bound estimates given in Theorem 1.2, or in [39] and [40] are sharp. It would be interesting to see what a sharp lower bound is in terms of the Bekollé-Bonami type constant.

2. Our lower bound estimate in Theorem 1.3 uses the asymptotic expansion of the Bergman kernel function and hence only works for bounded, smooth, strictly pseudoconvex domains. An interesting question would be whether similar lower bound estimates hold true for the Bergman projection when the domain is of finite type in $\mathbb{C}^2$, convex and of finite type in $\mathbb{C}^n$, or decoupled and of finite type in $\mathbb{C}^n$.

3. We focus on the weighted estimates for the Bergman projection for the simplicity of the computation. In [40], Rahm, Tchoundja, and Wick obtained the weighted estimates for operators $S_{a,b}$ and $S^+_{a,b}$ defined by

$$
S_{a,b} f(z) := (1 - |z|^2)^a \int_{\mathbb{B}_n} \frac{f(w)(1-|w|^2)^b}{(1 - z\bar{w})^{n+1+a+b}} dV(w);
$$

$$
S^+_{a,b} f(z) := (1 - |z|^2)^a \int_{\mathbb{B}_n} \frac{f(w)(1-|w|^2)^b}{|1 - z\bar{w}|^{n+1+a+b}} dV(w),
$$

on the weighted space $L^p(\mathbb{B}_n, (1-|w|^2)^b \mu dV)$. Using the methods in this paper, it is possible to obtain weighted estimates for analogues of $S_{a,b}$ and $S^+_{a,b}$ in the settings we considered in this paper.

Declaration of competing interest

There is no competing interest to disclose.

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