Combined Effects of Homogenization and Singular Perturbations: Quantitative Estimates

Weisheng Niu* Zhongwei Shen†

Abstract

We investigate quantitative estimates in periodic homogenization of second-order elliptic systems of elasticity with singular fourth-order perturbations. The convergence rates, which depend on the scale κ that represents the strength of the singular perturbation and on the length scale ε of the heterogeneities, are established. We also obtain the large-scale Lipschitz estimate, down to the scale ε and independent of κ. This large-scale estimate, when combined with small-scale estimates, yields the classical Lipschitz estimate that is uniform in both ε and κ.

Keywords: Homogenization; Singular Perturbation; Convergence Rate; Uniform Lipschitz Estimate.
AMS Subject Classification 2020: 35B27; 35B25.

1 Introduction

In this paper we aim to quantify the combined effects of homogenization and singular perturbations for the elliptic system,

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ $(d \ge 2)$ is a bounded domain and

$$\mathcal{L}_\varepsilon = \kappa^2 \Delta^2 - \text{div}(A(x/\varepsilon)\nabla), \quad 0 < \varepsilon, \kappa < 1. \tag{1.2}$$

The coefficient matrix (tensor) $A(y) = (a_{ij}^{\alpha\beta}(y))$, with $1 \le \alpha, \beta, i, j \le d$, is assumed to be real, bounded measurable and to satisfy the elasticity condition,

$$\begin{aligned} &a_{ij}^{\alpha\beta}(y) = a_{ji}^{\beta\alpha}(y) = a_{\alpha j}^{i\beta}(y), \\ &\nu_1|\xi|^2 \le a_{ij}^{\alpha\beta}\xi_i^\alpha \xi_j^\beta \le \nu_2|\xi|^2 \end{aligned} \tag{1.3}$$

for a.e. $y \in \mathbb{R}^d$ and for any symmetric matrix $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times d}$, where ν_1, ν_2 are positive constants. We also assume that A is 1-periodic; i.e.,

$$A(y+z) = A(y) \ \text{ for any } z \in \mathbb{Z}^d \text{ and a.e. } y \in \mathbb{R}^d. \tag{1.4}$$

*Supported by the NSF of China (11971031, 11701002).
†Supported in part by NSF grant DMS-1856235.

The elliptic operator in (1.2) arises in the study of the formation of the so-called shear bands in elastic materials subject to severe loadings [8]. Variational functionals associated with the related nonlinear operators are also used to model the heterogeneous thin films of martensitic materials [20, 7]. Homogenization of the elliptic system (1.1) was first studied by Bensoussan, Lions, and Papanicolaou in [5], where qualitative results were obtained for the case $\kappa = \varepsilon$. Also see related work in G-convergence in [23, 24]. Later on, in [8] Francfort and Müller provided a systematic qualitative analysis in periodic homogenization of (1.1) and the related nonlinear functionals for the case $\kappa = \varepsilon^\gamma$, where $0 < \gamma < \infty$. See also [22] for the related work in the stochastic setting. Assume that A satisfies conditions (1.3) - (1.4) and $\kappa = \varepsilon^\gamma$. Let $u_\varepsilon \in H_0^2(\Omega; \mathbb{R}^d)$ be the weak solution of (1.1) with $F \in H^{-1}(\Omega; \mathbb{R}^d)$. Thanks to [5, 8], as $\varepsilon \to 0$, u_ε converges weakly in $H^1(\Omega; \mathbb{R}^d)$ to the weak solution u_0 in $H_0^1(\Omega)$ of the second-order elliptic system,

$$-\text{div}(\widehat{A}\nabla u_0) = F \quad \text{in } \Omega, \tag{1.5}$$

with constant coefficients. The effective coefficient matrix $\widehat{A}$ in (1.5) depends on κ, which represents the strength of the singular perturbation, in three cases: $0 < \gamma < 1$; $\gamma = 1$; and $\gamma > 1$. In the case $\gamma > 1$, the matrix $\widehat{A}$ agrees with the effective matrix for the second-order elliptic operator $-\text{div}(A(x/\varepsilon)\nabla)$, without singular perturbation. If $0 < \gamma < 1$, the matrix $\widehat{A}$ is simply given by the average of A over its periodic cell. In the most interesting case $\gamma = 1$, the expression for the matrix $\widehat{A}$ depends on a corrector, which solves a cell problem for a fourth-order elliptic system. The same is true for a general $\kappa = \kappa(\varepsilon)$ under the assumption that

$$\kappa \to 0 \text{ as } \varepsilon \to 0, \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\kappa}{\varepsilon} = \rho. \tag{1.6}$$

The effective matrix $\widehat{A}$ in (1.5) depends on ρ in three cases: $\rho = 0$; $0 < \rho < \infty$; and $\rho = \infty$. See Section 3 for the details.

Our primary interest in this paper is in the quantitative homogenization of the elliptic system (1.1). The qualitative results described above show that the singular perturbation and the homogenization have combined effects in determining the effective equation for (1.1). So a natural question is to understand the combined effects in a quantitative way. More precisely, we shall be interested in the sharp convergence rate of u_ε to u_0 in terms of ε and κ, as well as regularity estimates of u_ε, which are uniform in ε and κ. Although much work has been done on the quantitative homogenization for the second-order elliptic system $-\text{div}(A(x/\varepsilon)\nabla u_\varepsilon) = F$ in recent years, to the best of our knowledge, the question has not been previously addressed, with the exception of [15], where an $O(\varepsilon)$ rate in $L^2(\Omega)$ was obtained in the case $\kappa = \varepsilon$ for Dirichlet problems with homogeneous boundary conditions.

Our first main result provides a convergence rate in $L^2(\Omega)$ for a general κ satisfying (1.6).

Theorem 1.1. *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^d, d \geq 2$, and A satisfy (1.3)-(1.4). Suppose (1.6) holds and if $\rho = 0$, we also assume that A is Lipschitz continuous, i.e.,*

$$|A(x) - A(y)| \leq L|x - y|, \quad \text{for any } x, y \in \mathbb{R}^d. \tag{1.7}$$

For $F \in L^2(\Omega; \mathbb{R}^d)$ and $G \in H^2(\Omega; \mathbb{R}^d)$, let $u_\varepsilon \in H^2(\Omega; \mathbb{R}^d)$ be a weak solution of (1.1) with $u_\varepsilon - G \in H_0^2(\Omega; \mathbb{R}^d)$, and $u_0 \in H^1(\Omega; \mathbb{R}^d)$ the weak solution of its homogenized problem (1.5) with

$u_0 - G \in H_0^1(\Omega;\mathbb{R}^d)$. *Then*

$$
\begin{aligned}
&\|u_\varepsilon - u_0\|_{L^2(\Omega)} \\
&\le \big\{\|F\|_{L^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\}
\begin{cases}
C_1\left\{\kappa+\varepsilon+\left(\dfrac{\varepsilon}{\kappa}\right)^2\right\} & \text{if } \rho=\infty,\\
C_2\left\{\kappa+\varepsilon+\rho^{-2}\left|\left(\dfrac{\kappa}{\varepsilon}\right)^2-\rho^2\right|\right\} & \text{if } 0<\rho<\infty,\\
C_3\left\{\kappa+\varepsilon+\left(\dfrac{\kappa}{\varepsilon}\right)^2\right\} & \text{if } \rho=0,
\end{cases}
\end{aligned}
\tag{1.8}
$$

where C_1, C_2 depend only on d, ν_1, ν_2 and Ω, and C_3 depends only on d, ν_1, ν_2, Ω, and L.

The $O(\varepsilon)$ convergence rate in $L^2(\Omega)$ has been established for second-order elliptic systems with highly oscillating coefficients in various contexts. Following a general approach developed in [21, 19] (see [18] for references on the related work), one first establishes an $O(\varepsilon^{1/2})$ rate in $H^1(\Omega)$ for a two-scale expansion of u_ε, and then uses a duality argument to improve the rate to $O(\varepsilon)$ in $L^2(\Omega)$. To carry this out, we introduce an operator,

$$
\mathcal{L}_\varepsilon^\lambda = \lambda^2\varepsilon^2\Delta^2 - \operatorname{div}(A(x/\varepsilon)\nabla), \tag{1.9}
$$

where $0<\lambda<\infty$ is fixed. Let $\mathcal{L}_0^\lambda = -\operatorname{div}\big(\widehat{A^\lambda}\nabla\big)$ denote the effective operator for $\mathcal{L}_\varepsilon^\lambda$ in (1.9). In Section 4 we will show that if $\mathcal{L}_\varepsilon^\lambda(u_{\varepsilon,\lambda}) = F$ and $u_{\varepsilon,\lambda} - G \in H_0^2(\Omega;\mathbb{R}^d)$, then

$$
\|u_{\varepsilon,\lambda} - u_{0,\lambda}\|_{L^2(\Omega)} \le C(1+\lambda)\varepsilon\big\{\|F\|_{L^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\}, \tag{1.10}
$$

where $u_{0,\lambda}$ is the weak solution of $\mathcal{L}_0^\lambda(u_{0,\lambda}) = F$ in Ω with $u_{0,\lambda} - G \in H_0^1(\Omega;\mathbb{R}^d)$. To complete the proof of Theorem 1.1, we observe that

$$
\mathcal{L}_\varepsilon^\lambda = \mathcal{L}_\varepsilon \quad \text{and} \quad u_{\varepsilon,\lambda} = u_\varepsilon \quad \text{if } \lambda = \kappa\varepsilon^{-1}, \tag{1.11}
$$

and use energy estimates to bound $\|u_{0,\lambda} - u_0\|_{L^2(\Omega)}$.

We note that the convergence rate in (1.8) involves three terms. The first term κ is caused by the singular perturbation, the second term ε by homogenization, while the third term is generated by $|\widehat{A^\lambda} - \widehat{A}|$. One may find examples in the one-dimensional case, which show that both the perturbation error $O(\kappa)$ and the homogenization error $O(\varepsilon)$ are sharp. Our estimates of $|\widehat{A^\lambda} - \widehat{A}|$ in Section 3 should also be sharp as $\lambda \to 0$ or ∞. As a result, we believe the convergence rates obtained in Theorem 1.1 are sharp. On the other hand, in view of (1.10), it is interesting to point out that for any $\varepsilon>0$ and $\kappa>0$, the solution u_ε may be approximated with an $O(\kappa+\varepsilon)$ error in $L^2(\Omega)$ by the solution of a second-order elasticity system with constant coefficients satisfying (1.3). However, the coefficients depend on $\lambda = \kappa\varepsilon^{-1}$. We remark that the proof of Theorem 1.1 also gives a sharp estimate for a two-scale expansion,

$$
u_\varepsilon - u_0 - \varepsilon\chi(x/\varepsilon)S_\varepsilon(\nabla u_0), \tag{1.12}
$$

in $H^1(\Omega;\mathbb{R}^d)$, where $\chi(y)$ is the corrector for $\mathcal{L}_\varepsilon^\rho$ if $0<\rho<\infty$, $\chi(y)=0$ if $\rho=\infty$, and $\chi(y)$ is the corrector for the operator $-\operatorname{div}(A(x/\varepsilon)\nabla)$ if $\rho=0$. See Remark 4.5.

Our second main result gives the large-scale Lipschitz estimate down to the microscopic scale ε.

Theorem 1.2. *Assume that A satisfies (1.3) and (1.4). Let $u_\varepsilon \in H^2(B_R;\mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in B_R, where $B_R = B(x_0,R)$, $R > \varepsilon$, and $F \in L^p(B_R;\mathbb{R}^d)$ for some $p > d$. Then for $\varepsilon \le r < R$,*

$$\left(\fint_{B_r} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left\{\left(\fint_{B_R} |\nabla u_\varepsilon|^2\right)^{1/2} + R\left(\fint_{B_R} |F|^p\right)^{1/p}\right\}, \tag{1.13}$$

where C depends only on d, ν_1, ν_2, and p.

Under the additional smoothness condition that A is Hölder continuous:

$$|A(x) - A(y)| \le M|x-y|^\sigma \quad \text{for any } x, y \in \mathbb{R}^d, \tag{1.14}$$

we obtain the classical Lipschitz estimate, which is uniform in both ε and κ, for $\mathcal{L}_\varepsilon(u_\varepsilon) = F$.

Theorem 1.3. *Assume that A satisfies conditions (1.3), (1.4), and (1.14) for some $\sigma \in (0,1)$. Let $u_\varepsilon \in H^2(B_r;\mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B_r = B(x_0,r)$, where $F \in L^p(B_r;\mathbb{R}^d)$ for some $p > d$. Then*

$$|\nabla u_\varepsilon(x_0)| \le C\left\{\left(\fint_{B_r} |\nabla u_\varepsilon|^2\right)^{1/2} + r\left(\fint_{B_r} |F|^p\right)^{1/p}\right\}, \tag{1.15}$$

where C depends only on d, ν_1, ν_2, p, and (M,σ).

Under the conditions (1.3), (1.4) and (1.14), the interior Lipschitz estimate (1.15) as well as the boundary Lipschitz estimate with the Dirichlet condition was proved by Avellaneda and Lin in a seminal work [3], using a compactness method. The boundary Lipschitz estimate with Neumann conditions was established in [14]. Related work in the stochastic setting may be found in [11, 2, 1, 6, 13].

To prove Theorem 1.2, we use an approach found in [12, 6]. As in [3], the idea is to utilize correctors to establish a large-scale $C^{1,\alpha}$ estimate for $0 < \alpha < 1$, from which the large-scale Lipschitz estimate (1.13) follows. Unlike the compactness method used in [3, 14], the approach requires a (suboptimal) convergence rate in $H^1(\Omega)$ for a two-scale expansion of u_ε. In order to reach down to the microscopic scale ε, which is necessary for obtaining the classical Lipschitz estimate in Theorem 1.3, we introduce an intermediate equation,

$$\lambda^2\varepsilon^2\Delta^2 v_{\varepsilon,\lambda} - \operatorname{div}(\widehat{A^\lambda}\nabla v_{\varepsilon,\lambda}) = F, \tag{1.16}$$

with $\lambda > 0$ fixed, where $\widehat{A^\lambda}$ is the effective matrix for $\mathcal{L}_\varepsilon^\lambda$ in (1.9). The key observation is to use the solution of (1.16), instead of the homogenized equation (1.5), in the two-scale expansion of u_ε. The purpose is two-fold. Firstly, with the added higher-order term in the equation (1.16), one eliminates the error caused by the singular perturbation. As a result, we are able to establish a convergence rate in $H^1(\Omega)$, uniformly in λ. Secondly, since $\widehat{A^\lambda}$ is constant, one may prove the $C^{1,\alpha}$ estimate, uniformly in λ, for (1.16) by classical methods. We remark that as in [12, 6], the same approach may be used to establish the large-scale $C^{k,\alpha}$ estimates down to the scale ε for any $k \ge 2$.

A more interesting problem would be the extension of the results in this paper to the stochastic setting.

The paper is organized as follows. In Section 2 we collect some regularity estimates, which are uniform in λ, for the operator (2.1) without the periodicity assumption. The materials in this section are more or less known. In Section 3 we present the qualitative homogenization for the operator (1.2) under the assumption (1.6). The proof of Theorem 1.1 is given in Section 4. In Section 5 we establish an approximation result in $H^1(\Omega)$ for $u_{\varepsilon,\lambda}$ by solutions of (1.16), while the result is used in Section 6 to prove the large-scale $C^{1,\alpha}$ estimate. Finally, the proofs of Theorems 1.2 and 1.3 are given in Section 7.

The summation convention is used throughout. We also use $\fint_E u$ to denote the L^1 average of u over the set E.

2 Preliminaries

Consider the operator,

$$\mathcal{L}^\lambda = \mathcal{L}_1^\lambda = \lambda^2 \Delta^2 - \operatorname{div}(A(x)\nabla), \tag{2.1}$$

with $0 < \lambda < \infty$ fixed and $A = A(x)$ satisfying the elasticity condition (1.3). The periodicity condition (1.4) is not used in this section with the exception of Lemma 2.11 and Theorem 2.12. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. For $F \in H^{-1}(\Omega;\mathbb{R}^d)$ and $G \in H^2(\Omega;\mathbb{R}^d)$, there exists a unique $u \in H^2(\Omega;\mathbb{R}^d)$ such that $\mathcal{L}^\lambda(u) = F$ in Ω and $u - G \in H_0^2(\Omega;\mathbb{R}^d)$. Moreover, the solution u satisfies the energy estimate,

$$\lambda \|\nabla^2 u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \le C\big\{ \|F\|_{H^{-1}(\Omega)} + \|\nabla G\|_{L^2(\Omega)} + \lambda \|\nabla^2 G\|_{L^2(\Omega)} \big\}, \tag{2.2}$$

where C depends only on d, ν_1, ν_2, and Ω. To see this, one considers $v = u - G$ and applies the Lax-Milgram Theorem to the bilinear form,

$$a(\phi,\psi) = \lambda^2 \int_\Omega \nabla^2 \phi \cdot \nabla^2 \psi \, dx + \int_\Omega A(x) \nabla \phi \cdot \nabla \psi \, dx, \tag{2.3}$$

on the Hilbert space $H_0^2(\Omega;\mathbb{R}^d)$. The first Korn inequality is needed for proving (2.2).

2.1 Caccioppoli's inequalities

Theorem 2.1. *Let $u \in H^2(B_{2r};\mathbb{R}^d)$ be a weak solution of $\mathcal{L}^\lambda(u) = F + \operatorname{div}(f)$ in $B_{2r} = B(x_0,2r)$, where $F \in L^2(B_{2r};\mathbb{R}^d)$ and $f \in L^2(B_{2r};\mathbb{R}^{d\times d})$. Then*

$$\lambda^2 \int_{B_r} |\nabla^2 u|^2 \, dx \le \frac{C}{r^2} \left(\frac{\lambda^2}{r^2} + 1 \right) \int_{B_{2r}} |u|^2 \, dx + C \int_{B_{2r}} |F||u| \, dx + C \int_{B_{2r}} |f|^2 \, dx, \tag{2.4}$$

$$\int_{B_r} |\nabla u|^2 \, dx \le \frac{C}{r^2} \int_{B_{2r}} |u|^2 \, dx + C \int_{B_{2r}} |F||u| \, dx + C \int_{B_{2r}} |f|^2 \, dx, \tag{2.5}$$

where C depends only on d, ν_1 and ν_2.

Proof. By translation and dilation we may assume that $x_0 = 0$ and $r = 1$. For $1 < s < t < 2$, let φ be a cut-off function in $C_0^\infty(B(0,t))$ such that $0 \le \varphi \le 1$, $\varphi = 1$ on B_s and $|\nabla^k \varphi| \le C(t-s)^{-k}$ for $k = 1, \dots, 4$. By taking the test function $u\varphi^4$ in the weak formulation of the equation $\mathcal{L}^\lambda(u) = F + \operatorname{div}(f)$ and using the Cauchy inequality, we deduce that

$$\begin{aligned} &\lambda^2 \int_{B_s} |\nabla^2 u|^2 \, dx + \int_{B_s} |\nabla u|^2 \, dx \\ &\le C \int_{B_2} \big(|F||u| + |f|^2 \big) \, dx + C\lambda^2 (t-s)^{-2} \int_{B_t} |\nabla(u\varphi)|^2 \, dx \\ &\quad + C\big((t-s)^{-2} + \lambda^2 (t-s)^{-4} \big) \int_{B_2} |u|^2 \, dx. \end{aligned} \tag{2.6}$$

To eliminate the term involving $|\nabla(u\varphi)|$ in the right-hand side of (2.6), we use an iteration technique found in [4], where an improved Caccioppoli inequality for a general higher-order elliptic system was proved. We point out that Theorem 2.1 does not follow directly from [4], since we require the constant C to be independent of the parameter λ.

Using the identity,

$$u\varphi \cdot \Delta(u\varphi) = (u\Delta u)\varphi^2 + 2u\nabla(u\varphi)\nabla\varphi - 2|u|^2|\nabla\varphi|^2 + |u|^2 \varphi \Delta\varphi,$$

and integration by parts as well as the Cauchy inequality, we may show that

$$\begin{aligned} \int_{B_t} |\nabla(u\varphi)|^2 \, dx \le C \left(\int_{B_t} |u\varphi|^2 \, dx \right)^{1/2} \left(\int_{B_t} |\varphi \Delta u|^2 \, dx \right)^{1/2} \\ + C \int_{B_t} |u|^2 |\nabla\varphi|^2 \, dx + C \int_{B_t} |u|^2 |\varphi| |\Delta\varphi| \, dx, \end{aligned} \tag{2.7}$$

where C depends only on d. This, together with (2.6), gives

$$\begin{aligned} &\lambda^2 \int_{B_s} |\nabla^2 u|^2 \, dx + \int_{B_s} |\nabla u|^2 \, dx \\ &\le C \int_{B_2} \big(|F||u| + |f|^2 \big) \, dx + \frac{\lambda^2}{2} \int_{B_t} |\nabla^2 u|^2 \, dx \\ &\quad + C\big((t-s)^{-2} + \lambda^2 (t-s)^{-4} \big) \int_{B_2} |u|^2 \, dx. \end{aligned} \tag{2.8}$$

For $j \ge 1$, let $t_j = 2 - \tau^j$, where $\tau \in (0,1)$ is to be determined. It follows from (2.8) that

$$\begin{aligned} &\lambda^2 \int_{B_{t_j}} |\nabla^2 u|^2 \, dx + \int_{B_{t_j}} |\nabla u|^2 \, dx \\ &\le C \int_{B_2} \big(|F||u| + |f|^2 \big) \, dx + \frac{\lambda^2}{2} \int_{B_{t_{j+1}}} |\nabla^2 u|^2 \, dx \\ &\quad + C\big((\tau^j - \tau^{j+1})^{-2} + \lambda^2 (\tau^j - \tau^{j+1})^{-4} \big) \int_{B_2} |u|^2 \, dx. \end{aligned} \tag{2.9}$$

By iteration this leads to

$$\begin{aligned}
&\lambda^2 \int_{B_{t_1}} |\nabla^2 u|^2 \, dx + \int_{B_{t_1}} |\nabla u|^2 \, dx \\
&\le C \sum_{i=1}^{j} \frac{1}{2^{i-1}} \int_{B_2} \left(|F||u| + |f|^2 \right) dx + \frac{\lambda^2}{2^j} \int_{B_{t_{j+1}}} |\nabla^2 u|^2 \, dx \\
&\quad + C \sum_{i=1}^{j} \frac{1}{2^{i-1}} \left((\tau^i - \tau^{i+1})^{-2} + \lambda^2 (\tau^i - \tau^{i+1})^{-4} \right) \int_{B_2} |u|^2 \, dx
\end{aligned} \tag{2.10}$$

for $j \ge 1$. We now choose $\tau \in (0,1)$ so that $2\tau^4 > 1$. By letting $j \to \infty$ in (2.10) we obtain (2.4) with $r = 1$, and

$$\int_{B_1} |\nabla u|^2 \, dx \le C(\lambda^2 + 1) \int_{B_2} |u|^2 \, dx + C \int_{B_2} \left(|F||u| + |f|^2 \right) dx, \tag{2.11}$$

which gives (2.5) if $\lambda \le 1$. Finally, if $\lambda > 1$, we note that (2.10) yields

$$\lambda^2 \int_{B_{t_1}} |\nabla^2 u|^2 \, dx \le C \int_{B_2} \left(|F||u| + |f|^2 \right) dx + C(1 + \lambda^2) \int_{B_2} |u|^2 \, dx. \tag{2.12}$$

By (2.7) we have

$$\begin{aligned}
\int_{B_1} |\nabla u|^2 \, dx &\le C \int_{B_{t_1}} |u|^2 \, dx + C \int_{B_{t_1}} |\Delta u|^2 \, dx \\
&\le C \int_{B_2} \left(|F||u| + |f|^2 + |u|^2 \right) dx,
\end{aligned} \tag{2.13}$$

where we have used (2.12) for the last inequality. □

Remark 2.2. Let u be a solution of $\mathcal{L}^\lambda(u) = F + \operatorname{div}(f)$ in B_{2r}. Let $w = \lambda^2 \Delta u$. Since

$$\Delta w = F + \operatorname{div}(f) + \operatorname{div}(A \nabla u),$$

it follows from the Caccioppoli inequality for Δ that

$$\begin{aligned}
\int_{B_r} \lambda^4 |\nabla \Delta u|^2 \, dx &\le \frac{C\lambda^4}{r^2} \int_{B_{3r/2}} |\Delta u|^2 \, dx + Cr^2 \int_{B_{3r/2}} |F|^2 \, dx \\
&\quad + C \int_{B_{3r/2}} |f|^2 \, dx + C \int_{B_{3r/2}} |\nabla u|^2 \, dx \\
&\le \frac{C}{r^2} \left(\frac{\lambda}{r} + 1 \right)^4 \int_{B_{2r}} |u|^2 \, dx + C \left(\frac{\lambda}{r} + 1 \right)^2 \int_{B_{2r}} |f|^2 \, dx \\
&\quad + Cr^2 \int_{B_{2r}} |F|^2 \, dx,
\end{aligned} \tag{2.14}$$

where we have used (2.4) and (2.5) for the last inequality.

2.2 Reverse Hölder inequalities

Theorem 2.3. *Let $u \in H^2(B_{2r}; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}^\lambda(u) = F + \operatorname{div}(f)$ in $B_{2r} = B(x_0, 2r)$, where $F \in L^2(B_{2r}; \mathbb{R}^d)$ and $f \in L^2(B_{2r}; \mathbb{R}^{d\times d})$. Then there exists some $p > 2$, depending only on d, ν_1 and ν_2, such that*

$$\left(\fint_{B_r} |\nabla u|^p\right)^{1/p} \le C\left\{\left(\fint_{B_{2r}} |\nabla u|^2\right)^{1/2} + \left(\fint_{B_{2r}} |f|^p\right)^{1/p} + Cr\left(\fint_{B_{2r}} |F|^2\right)^{1/2}\right\}, \tag{2.15}$$

where C depends only on d, ν_1 and ν_2.

Proof. This follows from (2.5) by the self-improvement property of the (weak) reverse Hölder inequalities. Let $B' = B(z,t)$ be a ball such that $2B' \subset B(x_0, 2r)$. Choose $1 < q_1 < 2 < q_2 < \infty$ such that

$$\left(\fint_{2B'} |u - E|^{q_2}\right)^{1/q_2} \le Ct\left(\fint_{2B'} |\nabla u|^{q_1}\right)^{1/q_1},$$

where E is the L^1 average of u over $2B'$. Since $\mathcal{L}^\lambda(u - E) = \mathcal{L}^\lambda(u)$, it follows from (2.5) that

$$\left(\fint_{B'} |\nabla u|^2\right)^{1/2} \le C\left(\fint_{2B'} |\nabla u|^{q_1}\right)^{1/q_1} + Ct\left(\fint_{2B'} |F|^{q_2'}\right)^{1/q_2'} + C\left(\fint_{2B'} |f|^2\right)^{1/2}, \tag{2.16}$$

where C depends only on d, ν_1 and ν_2. The fact that (2.16) holds for any ball $2B' \subset B$ implies (2.15) [10]. □

Remark 2.4. Let Ω be a bounded Lipschitz domain. Fix $x_0 \in \partial\Omega$ and define

$$D_r = B(x_0, r) \cap \Omega \quad \text{and} \quad \Delta_r = B(x_0, r) \cap \partial\Omega,$$

where $0 < r < r_0 = c_0 \operatorname{diam}(\Omega)$. Let $u \in H^2(D_{2r}; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}^\lambda(u) = F + \operatorname{div}(f)$ in D_{2r} with $u = 0$ and $\nabla u = 0$ on Δ_{2r}. Then

$$\lambda^2 \int_{D_r} |\nabla^2 u|^2\, dx \le \frac{C}{r^2}\left(\frac{\lambda^2}{r^2} + 1\right)\int_{D_{2r}} |u|^2\, dx + C\int_{D_{2r}} |F||u|\, dx + C\int_{D_{2r}} |f|^2\, dx, \tag{2.17}$$

$$\int_{D_r} |\nabla u|^2\, dx \le \frac{C}{r^2}\int_{D_{2r}} |u|^2\, dx + C\int_{D_{2r}} |F||u|\, dx + C\int_{D_{2r}} |f|^2\, dx, \tag{2.18}$$

where C depends only on d, ν_1 and ν_2. Note that since $u = 0$ and $\nabla u = 0$ on Δ_{2r}, we have $u\varphi \in H_0^2(D_{2r}; \mathbb{R}^d)$ for any $\varphi \in C_0^2(B_{2r})$. The proof of (2.17) and (2.18) is exactly the same as that of Theorem 2.1. As a consequence, we also obtain the boundary reverse Hölder inequality,

$$\left(\fint_{D_r} |\nabla u|^p\right)^{1/p} \le C\left\{\left(\fint_{D_{2r}} |\nabla u|^2\right)^{1/2} + \left(\fint_{D_{2r}} |f|^p\right)^{1/p} + Cr\left(\fint_{D_{2r}} |F|^2\right)^{1/2}\right\}, \tag{2.19}$$

where $C > 0$ and $p > 2$ depend only on d, ν_1, ν_2 and the Lipschitz constant of $B(z, r_0) \cap \partial\Omega$.

Theorem 2.5. *Suppose A satisfies (1.3) and Ω is a bounded Lipschitz domain. Let $u \in H_0^2(\Omega; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}^\lambda(u) = \operatorname{div}(f)$ in Ω. Then there exists $p > 2$, depending only on d, ν_1, ν_2 and Ω, such that*

$$\|\nabla u\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}, \tag{2.20}$$

where C depends only on d, ν_1, ν_2, and Ω.

Proof. The Meyers estimate (2.20) was proved in [8] by an interpolation argument. It also follows readily from the reverse Hölder estimates (2.15) and (2.19). Indeed, by using (2.15), (2.19) and a simple covering argument, we see that for some $p > 2$,

$$\|\nabla u\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)} + C\|\nabla u\|_{L^2(\Omega)} \le C\|f\|_{L^p(\Omega)},$$

where we have used the energy estimate and Hölder's inequality for the last step. □

2.3 $C^{1,\alpha}$ estimates

Lemma 2.6. *Suppose A satisfies conditions (1.3) and (1.14). Let $u \in H^2(B_2; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}^\lambda(u) = 0$ in $B_2 = B(0, 2)$. Then*

$$\|u\|_{C^{1,\alpha}(B_1)} \le C_\alpha \left(\fint_{B_2} |u|^2 \right)^{1/2}, \tag{2.21}$$

where $0 < \alpha < \sigma$ and C_α depends only on d, ν_1, ν_2, α, and (M, σ).

Proof. We first observe that if A is a constant matrix satisfying the elasticity condition (1.3), then

$$\max_{B_1} |\nabla^k u| \le C_k \left(\fint_{B_{3/2}} |u|^2 \right)^{1/2}, \tag{2.22}$$

where C_k depends on d, ν_1, ν_2 and k. To see this, we note that since A is constant, $\nabla^k u$ is a solution. Thus, by (2.5) and an iteration argument,

$$\|u\|_{H^k(B_1)} \le C_k \|u\|_{L^2(B_{3/2})}$$

for any $k \ge 1$. By Sobolev imbedding, this gives (2.22). Next, we use a standard perturbation argument to show that if A is uniformly continuous and $\gamma > 0$,

$$\int_{B_\rho} |\nabla u|^2 \, dx \le C_\gamma \left(\frac{\rho}{R} \right)^{d-2\gamma} \int_{B_R} |\nabla u|^2 \, dx \tag{2.23}$$

for $0 < \rho < R < r$. To do this, we let $v \in H^2(B_R; \mathbb{R}^d)$ be the solution of

$$\lambda^2 \Delta^2 v - \operatorname{div}(\overline{A} \nabla v) = 0 \quad \text{in } B_R \quad \text{and } v - u \in H_0^2(B_R; \mathbb{R}^d), \tag{2.24}$$

where $\overline{A} = \fint_{B_R} A$. Since

$$\lambda^2 \Delta^2 (v - u) - \operatorname{div}(\overline{A} \nabla (v - u)) = \operatorname{div}((\overline{A} - A) \nabla u) \quad \text{in } B_R,$$

by energy estimates,

$$\int_{B_R} |\nabla u - \nabla v|^2 \, dx \le C \|\overline{A} - A\|_{L^\infty(B_R)}^2 \int_{B_R} |\nabla u|^2 \, dx.$$

By (2.22), for $0 < \rho < R < r$,

$$\fint_{B_\rho} |\nabla v|^2 \le C \fint_{B_R} |\nabla v|^2.$$

The rest of the argument for (2.23) is exactly the same as in the case of second-order elliptic systems [10, pp.84-88]. An argument similar to that in [10, pp.84-88] also shows that if A satisfies (1.14), then

$$\left(\fint_{B_r} |\nabla u - \fint_{B_r} \nabla u|^2 \right)^{1/2} \le C_\alpha r^\alpha \left(\fint_{B_2} |u|^2 \right)^{1/2}$$

for any $\alpha \in (0, \sigma)$ and $0 < r < 1$. This implies (2.21). □

The following theorem gives the $C^{1,\alpha}$ estimate, uniform in λ, for the operator $\mathcal{L}^\lambda$.

Theorem 2.7. *Suppose A satisfies conditions (1.3) and (1.14). Let $u \in H^2(B_2; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}^\lambda(u) = F$ in B_2, where $F \in L^p(B_2; \mathbb{R}^d)$ for some $p > d$. Then, if $0 < \alpha < \min(\sigma, 1 - \frac{d}{p})$,*

$$\|u\|_{C^{1,\alpha}(B_1)} \le C_\alpha \Big\{ \|u\|_{L^2(B_2)} + \|F\|_{L^p(B_2)} \Big\}, \tag{2.25}$$

where C_α depends on d, ν_1, ν_2, p, α, and (M, σ).

Proof. The case $F = 0$ was given by Lemma 2.6. The general case is proved by a perturbation argument as in the case of second-order elliptic systems. Let $0 < r < R < 1$. Let $v \in H^2(B_R; \mathbb{R}^d)$ be the weak solution of $\mathcal{L}^\lambda(v) = 0$ in B_R such that $v - u \in H_0^2(B_R; \mathbb{R}^d)$. Since $\mathcal{L}^\lambda(u - v) = F$ in B_R, by the energy estimate,

$$\int_{B_R} |\nabla u - \nabla v|^2 \, dx \le C R^2 \int_{B_R} |F|^2 \, dx \le C R^{d + 2(1 - \frac{d}{p})} \|F\|_{L^p(B_2)}^2, \tag{2.26}$$

where C depends only on d, ν_1, ν_2, and p. By Lemma 2.6,

$$\int_{B_r} |\nabla v - \fint_{B_r} \nabla v|^2 \, dx \le C \left(\frac{r}{R} \right)^{d + 2\alpha} \int_{B_R} |\nabla v - \fint_{B_R} \nabla v|^2 \, dx$$

for any $0 < \alpha < \sigma$. This, together with (2.26), leads to

$$\begin{aligned} \int_{B_r} |\nabla u - \fint_{B_r} \nabla u|^2 \, dx \le C \left(\frac{r}{R} \right)^{d + 2\alpha} & \int_{B_R} |\nabla u - \fint_{B_R} \nabla u|^2 \, dx \\ & + C R^{d + 2(1 - \frac{d}{p})} \|F\|_{L^p(B_2)}^2, \end{aligned}$$

from which the estimate (2.25) follows, as in [10, pp.88-89]. We omit the details. □

2.4 Singular perturbations

For $\Omega \subset \mathbb{R}^d$ and $0 < t < c_0 \text{diam}(\Omega)$, let

$$\Omega_t = \{x \in \Omega : \text{dist}(x, \partial\Omega) < t\}. \tag{2.27}$$

Lemma 2.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Then,*

$$\|u\|_{L^2(\Omega_t)} \le Ct \|\nabla u\|_{L^2(\Omega_{2t})} \qquad \text{for } u \in H_0^1(\Omega), \tag{2.28}$$

$$\|u\|_{L^2(\Omega_t)} \le Ct^{1/2} \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2} \qquad \text{for } u \in H^1(\Omega), \tag{2.29}$$

and for $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\|u\|_{L^2(\Omega_t)} \le Ct^{3/2} \|u\|_{H^1(\Omega)}^{1/2} \|u\|_{H^2(\Omega)}^{1/2}, \tag{2.30}$$

where C depends on d and Ω.

Proof. The inequalities (2.28) and (2.29) may be proved by a localization argument, while (2.30) follows readily from (2.28)-(2.29). □

Lemma 2.9. *Let $u_\lambda \in H^2(\Omega; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}^\lambda(u_\lambda) = F$ with $u_\lambda - G \in H_0^2(\Omega; \mathbb{R}^d)$, where $F \in L^2(\Omega; \mathbb{R}^d)$, $G \in H^2(\Omega; \mathbb{R}^d)$, and Ω is a bounded Lipschitz domain. Let $u_0 \in H^1(\Omega; \mathbb{R}^d)$ be the weak solution of $-\text{div}(A\nabla u_0) = F$ in Ω and $u_0 - G \in H_0^1(\Omega; \mathbb{R}^d)$. Suppose $u_0 \in H^2(\Omega; \mathbb{R}^d)$. Then for $0 < \lambda \le 1$,*

$$\|\nabla u_\lambda - \nabla u_0\|_{L^2(\Omega)} \le C\sqrt{\lambda} \big\{ \|u_0\|_{H^2(\Omega)} + \|G\|_{H^2(\Omega)} \big\}, \tag{2.31}$$

where C depends only on d, ν_1, ν_2, and Ω.

Proof. Let η_t be a cut-off function in $C_0^\infty(\Omega)$ such that $0 \le \eta_t \le 1$, $\eta_t(x) = 1$ if $x \in \Omega \backslash \Omega_{2t}$, $\eta_t(x) = 0$ if $x \in \Omega_t$, and $|\nabla^k \eta_t| \le Ct^{-k}$ for $k = 1, 2$, where $t > 0$ is to be determined. Let $\widetilde{u}_0 = u_0 - G$ and

$$w = u_\lambda - G - (u_0 - G)\eta_t = u_\lambda - u_0 + \widetilde{u}_0(1 - \eta_t). \tag{2.32}$$

Note that $w \in H_0^2(\Omega; \mathbb{R}^d)$ and

$$\begin{aligned} \mathcal{L}^\lambda(w) &= \mathcal{L}^\lambda(u_\lambda) - \mathcal{L}^\lambda(u_0) + \mathcal{L}^\lambda\big[\widetilde{u}_0(1 - \eta_t)\big] \\ &= -\lambda^2 \Delta^2 u_0 + \lambda^2 \Delta^2 (\widetilde{u}_0(1 - \eta_t)) - \text{div}\big[A\nabla(\widetilde{u}_0(1 - \eta_t))\big]. \end{aligned}$$

It follows that for any $\psi \in H_0^2(\Omega; \mathbb{R}^d)$,

$$\begin{aligned} |\langle \mathcal{L}^\lambda(w), \psi \rangle| \le \lambda^2 \int_\Omega |\Delta u_0||\Delta \psi| \, dx + \lambda^2 \int_\Omega |\Delta(\widetilde{u}_0(1 - \eta_t))||\Delta \psi| \, dx \\ + C \int_\Omega |\nabla(\widetilde{u}_0(1 - \eta_t))||\nabla \psi| \, dx. \end{aligned}$$

By using the Cauchy inequality and Lemma 2.8, we obtain

$$\begin{aligned} |\langle \mathcal{L}^\lambda(w), \psi \rangle| \le \lambda^2 \|u_0\|_{H^2(\Omega)} \|\Delta \psi\|_{L^2(\Omega)} + C\lambda^2 t^{-1/2} \|\widetilde{u}_0\|_{H^2(\Omega)} \|\Delta \psi\|_{L^2(\Omega_{2t})} \\ + Ct^{1/2} \|\widetilde{u}_0\|_{H^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega_{2t})}. \end{aligned} \tag{2.33}$$

By taking $\psi = w$ in (2.33), $t = c_0\lambda$, and using the Cauchy inequality, we see that

$$\lambda \|\Delta w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} \le C\lambda^{1/2} \big\{ \|u_0\|_{H^2(\Omega)} + \|G\|_{H^2(\Omega)} \big\}. \tag{2.34}$$

In view of (2.32) this gives (2.31). □

Theorem 2.10. *Let u_λ and u_0 be the same as in Lemma 2.9. Also assume that Ω is a bounded $C^{1,1}$ domain and $\|\nabla A\|_\infty \le L < \infty$. Then for $0 < \lambda \le 1$,*

$$\|u_\lambda - u_0\|_{L^2(\Omega)} \le C\lambda \big\{ \|F\|_{L^2(\Omega)} + \|G\|_{H^2(\Omega)} \big\}, \tag{2.35}$$

where C depends on d, ν_1, ν_2, L, and Ω.

Proof. Let w be given by (2.32) with $t = c_0\lambda$. For $\widetilde{F} \in L^2(\Omega; \mathbb{R}^d)$, let $\widetilde{w} = v_\lambda - v_0 \widetilde{\eta}_t$, where $v_\lambda \in H_0^2(\Omega; \mathbb{R}^d)$ is the weak solution of $\mathcal{L}^\lambda(v_\lambda) = \widetilde{F}$ in Ω and $v_0 \in H_0^1(\Omega; \mathbb{R}^d)$ the weak solution of $-\text{div}(A\nabla v_0) = \widetilde{F}$ in Ω. The function $\widetilde{\eta}_t \in C_0^\infty(\Omega)$ is chosen so that $0 \le \widetilde{\eta}_t \le 1$, $\widetilde{\eta}_t = 1$ in $\Omega \setminus \Omega_{3t}$, $\widetilde{\eta}_t = 0$ in Ω_{2t}, and $|\nabla^k \widetilde{\eta}_t| \le Ct^{-k}$ for $k = 1, 2$. Note that

$$\begin{aligned} \Big| \int_\Omega w \cdot \widetilde{F} \, dx \Big| &= |\langle \mathcal{L}^\lambda(w), v_\lambda \rangle| \\ &\le |\langle \mathcal{L}^\lambda(w), \widetilde{w} \rangle| + |\langle \mathcal{L}^\lambda(w), v_0 \widetilde{\eta}_t \rangle|. \end{aligned}$$

It follows from (2.34) that

$$\begin{aligned} |\langle \mathcal{L}^\lambda(w), \widetilde{w} \rangle| &\le C \big\{ \lambda \|\Delta w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} \big\} \big\{ \lambda \|\Delta \widetilde{w}\|_{L^2(\Omega)} + \|\nabla \widetilde{w}\|_{L^2(\Omega)} \big\} \\ &\le C\lambda \big\{ \|u_0\|_{H^2(\Omega)} + \|G\|_{H^2(\Omega)} \big\} \|v_0\|_{H^2(\Omega)}. \end{aligned}$$

Also, by (2.33) and the fact that $\widetilde{\eta}_t = 0$ in Ω_{2t},

$$\begin{aligned} |\langle \mathcal{L}^\lambda(w), v_0 \widetilde{\eta}_t \rangle| &\le \lambda^2 \|u_0\|_{H^2(\Omega)} \|v_0 \widetilde{\eta}_t\|_{H^2(\Omega)} \\ &\le C\lambda^2 t^{-1/2} \|u_0\|_{H^2(\Omega)} \|v_0\|_{H^2(\Omega)}, \end{aligned}$$

where we have used Lemma 2.8 for the last inequality. As a result, we have proved that

$$\begin{aligned} \Big| \int_\Omega w \cdot \widetilde{F} \, dx \Big| &\le C\lambda \big\{ \|u_0\|_{H^2(\Omega)} + \|G\|_{H^2(\Omega)} \big\} \|v_0\|_{H^2(\Omega)} \\ &\le C\lambda \big\{ \|u_0\|_{H^2(\Omega)} + \|G\|_{H^2(\Omega)} \big\} \|\widetilde{F}\|_{L^2(\Omega)}, \end{aligned}$$

where, for the last step, we have used the H^2 estimate $\|v_0\|_{H^2(\Omega)} \le C\|\widetilde{F}\|_{L^2(\Omega)}$, which holds under the assumption that A is Lipschitz continuous and Ω is $C^{1,1}$. The estimate (2.35) now follows readily by duality. □

A proof for Theorem 2.10 in the case $d = 2$ may be found in [16]. As pointed out by A. Friedman in [9], the one-dimensional example,

$$\begin{cases} \lambda^2 \dfrac{d^4 u}{dx^4} - \dfrac{d^2 u}{dx^2} = 1 \quad \text{in } (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

shows that the $O(\lambda)$ rate in (2.35) is sharp. However, in the case of periodic boundary conditions, the rates in Lemma 2.9 and Theorem 2.10 can be improved.

Let $C^\infty_{\text{per}}(\mathbb{R}^d;\mathbb{R}^d)$ denote the space of C^∞, 1-periodic $\mathbb{R}^d$-valued functions in $\mathbb{R}^d$. Let $H^k_{\text{per}}(Y;\mathbb{R}^d)$ be the closure of $C^\infty_{\text{per}}(\mathbb{R}^d;\mathbb{R}^d)$ in $H^k(Y;\mathbb{R}^d)$, where $k \ge 1$ and $Y = [0,1]^d$. Note that for any $F \in L^2(Y;\mathbb{R}^d)$ with $\int_Y F\,dx = 0$, there exists a unique $u_\lambda \in H^2_{\text{per}}(Y;\mathbb{R}^d)$ such that $\mathcal{L}^\lambda(u_\lambda) = F$ in Y and $\int_Y u_\lambda\,dx = 0$.

Lemma 2.11. *Suppose A satisfies conditions (1.3) and (1.4). Let $u_\lambda \in H^2_{per}(Y;\mathbb{R}^d)$ be a weak solution of $\mathcal{L}^\lambda(u_\lambda) = F$ in Y with $\int_Y u_\lambda\,dx = 0$, where $F \in L^2(Y;\mathbb{R}^d)$ and $\int_Y F\,dx = 0$. Let $u_0 \in H^1_{per}(Y;\mathbb{R}^d)$ be the weak solution of $-\text{div}(A\nabla u_0) = F$ in Y with $\int_Y u_0\,dx = 0$. Suppose $u_0 \in H^2_{per}(Y;\mathbb{R}^d)$. Then*

$$\|\nabla u_\lambda - \nabla u_0\|_{L^2(Y)} \le C\lambda \|u_0\|_{H^2(Y)}, \tag{2.36}$$

where C depends only on d, ν_1 and ν_2.

Proof. Let $w = u_\lambda - u_0$. Then

$$\mathcal{L}^\lambda(w) = -\lambda^2 \Delta^2 u_0.$$

It follows that for any $\psi \in H^2_{\text{per}}(Y;\mathbb{R}^d)$,

$$|\langle \mathcal{L}^\lambda(w), \psi\rangle| \le \lambda^2 \|\Delta u_0\|_{L^2(Y)} \|\Delta \psi\|_{L^2(Y)}. \tag{2.37}$$

By taking $\psi = w$ in (2.37) and using the Cauchy inequality, we obtain

$$\lambda \|\Delta w\|_{L^2(Y)} + \|\nabla w\|_{L^2(Y)} \le C\lambda \|u_0\|_{H^2(Y)}, \tag{2.38}$$

which yields (2.36). □

Theorem 2.12. *Suppose A satisfies (1.3) and (1.4). Also assume that $\|\nabla A\|_\infty \le L < \infty$. Let u_λ and u_0 be the same as in Lemma 2.11. Then*

$$\|u_\lambda - u_0\|_{L^2(Y)} \le C\lambda^2 \|F\|_{L^2(Y)}, \tag{2.39}$$

where C depends on d, ν_1, ν_2, and L.

Proof. The proof is similar to that of Theorem 2.10. For $\widetilde{F} \in L^2(Y;\mathbb{R}^d)$ with $\int_Y \widetilde{F}\,dx = 0$, let $\widetilde{w} = v_\lambda - v_0$, where $v_\lambda \in H^2_{\text{per}}(Y;\mathbb{R}^d)$ is the weak solution of $\mathcal{L}^\lambda(v_\lambda) = \widetilde{F}$ in Y with $\int_Y v_\lambda\,dx = 0$, and $v_0 \in H^1_{\text{per}}(Y;\mathbb{R}^d)$ the solution of $-\text{div}(A\nabla v_0) = \widetilde{F}$ in Y with $\int_Y v_0\,dx = 0$. Note that

$$\begin{aligned}\left| \int_Y w \cdot \widetilde{F}\,dx \right| &= |\langle \mathcal{L}^\lambda(w), v_\lambda\rangle| \\ &\le |\langle \mathcal{L}^\lambda(w), \widetilde{w}\rangle| + |\langle \mathcal{L}^\lambda(w), v_0\rangle|.\end{aligned}$$

It follows from (2.37) that

$$\begin{aligned}|\langle \mathcal{L}^\lambda(w), \widetilde{w}\rangle| &\le \lambda^2 \|\Delta w\|_{L^2(Y)} \|\Delta \widetilde{w}\|_{L^2(Y)} + C\|\nabla w\|_{L^2(Y)} \|\nabla \widetilde{w}\|_{L^2(Y)} \\ &\le C\lambda^2 \|u_0\|_{H^2(Y)} \|v_0\|_{H^2(Y)}.\end{aligned}$$

By (2.37) we obtain

$$|\langle \mathcal{L}^\lambda(w), v_0\rangle| \le \lambda^2 \|\Delta u_0\|_{L^2(Y)} \|\Delta v_0\|_{L^2(Y)}.$$

Since $\|\nabla A\|_\infty \le L < \infty$, the H^2 estimates, $\|u_0\|_{H^2(Y)} \le C\|F\|_{L^2(Y)}$ and $\|v_0\|_{H^2(Y)} \le C\|\widetilde{F}\|_{L^2(Y)}$ hold. As a result, we have proved that

$$\Big| \int_Y w \cdot \widetilde{F}\, dx \Big| \le C\lambda^2 \|F\|_{L^2(Y)} \|\widetilde{F}\|_{L^2(Y)},$$

which, by duality, gives (2.39). □

3 Qualitative homogenization

The qualitative homogenization for the elliptic system (1.1) was established in [5, 8] for $\kappa = \varepsilon^\gamma$, where $0 < \gamma < \infty$. Here we consider a general case $\kappa = \kappa(\varepsilon)$ under the condition (1.6). Denoting $\kappa\varepsilon^{-1}$ as $\lambda = \lambda(\varepsilon)$, the system (1.1) may be written as

$$\lambda^2 \varepsilon^2 \Delta^2 u_{\varepsilon,\lambda} - \operatorname{div}(A(x/\varepsilon)\nabla u_{\varepsilon,\lambda}) = F. \tag{3.1}$$

We first fix $0 < \lambda < \infty$ and investigate the homogenization of the system (3.1).

For $1 \le \beta, j \le d$, let $P_j^\beta = y_j(0, \cdots, 1, \ldots, 0)$ with 1 in the β^{th} position. Consider the cell problem,

$$\begin{cases} \lambda^2 \Delta^2 \chi_j^{\lambda,\beta} - \operatorname{div}\big[A(y)\nabla(P_j^\beta + \chi_j^{\lambda,\beta})\big] = 0 \quad \text{in } \mathbb{R}^d, \\ \chi_j^{\lambda,\beta}(y) \text{ is 1-periodic in } y, \\ \int_Y \chi_j^{\lambda,\beta}(y)\, dy = 0, \end{cases} \tag{3.2}$$

where $Y = [0,1]^d$. Under conditions (1.3) and (1.4), for each $\lambda > 0$, (3.2) admits a unique solution $\chi_j^{\lambda,\beta} = (\chi_j^{\lambda,1\beta}, ..., \chi_j^{\lambda,d\beta})$ in $H^3_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$. This may be proved by using the Lax-Milgram Theorem on $H^2_{\text{per}}(Y;\mathbb{R}^d)$. Moreover, let $\chi^\lambda = \big(\chi_j^{\lambda,\alpha\beta}\big)$, then

$$\begin{aligned} \|\chi^\lambda\|_{H^1(Y)} &\le C(1+\lambda)^{-2}, \\ \|\nabla^2 \chi^\lambda\|_{L^2(Y)} &\le C\lambda^{-1}(1+\lambda)^{-1}, \\ \|\nabla^3 \chi^\lambda\|_{L^2(Y)} &\le C\lambda^{-2}, \end{aligned} \tag{3.3}$$

for some constant C depending only on d, ν_1 and ν_2. Estimates in (3.3) follow from energy estimates. Indeed, by using the test functions χ^λ and $\Delta\chi^\lambda$ and a Korn inequality, one obtains

$$\lambda \|\nabla^2 \chi^\lambda\|_{L^2(Y)} + \|\nabla \chi^\lambda\|_{L^2(Y)} \le C,$$

and $\|\nabla^3 \chi^\lambda\|_{L^2(Y)} \le C\lambda^{-2}$. The remaining estimates in (3.3) follow readily by Poincaré's inequality. If $\lambda = 0$, it is well known that (3.2) has a unique solution in $H^1_{\text{loc}}(\mathbb{R}^d)$ and $\|\chi^0\|_{H^1(Y)} \le C$.

Thanks to [5], for each fixed $\lambda \ge 0$, the homogenized operator of $\mathcal{L}_\varepsilon^\lambda$ in (1.9) is given by

$$\mathcal{L}_0^\lambda = -\operatorname{div}(\widehat{A^\lambda}\nabla), \tag{3.4}$$

where

$$\widehat{A^\lambda} = \fint_Y [A(y) + A(y)\nabla \chi^\lambda(y)]\, dy. \tag{3.5}$$

In view of (3.3), we have $|\widehat{A^\lambda}| \le C$, where C depends only on d, ν_1 and ν_2.

Lemma 3.1. *The constant matrix $\widehat{A^\lambda}$ satisfies the elasticity condition (1.3) with the same ν_1 and ν_2.*

Proof. Let $\widehat{A^\lambda} = \big(\widehat{A^\lambda}_{ij}^{\alpha\beta}\big)$ with $1 \le \alpha, \beta, i, j \le d$. Note that

$$\begin{aligned}
\widehat{A^\lambda}_{ij}^{\alpha\beta} &= \fint_Y A_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k} \left[P_j^{\gamma\beta} + \chi_j^{\lambda,\gamma\beta} \right] dy \\
&= \fint_Y A_{\ell k}^{t\gamma} \frac{\partial}{\partial y_k} \left[P_j^{\gamma\beta} + \chi_j^{\lambda,\gamma\beta} \right] \cdot \frac{\partial}{\partial y_\ell} \left[P_i^{t\alpha} + \chi_i^{\lambda,t\alpha} \right] dy + \lambda^2 \fint_Y \Delta \chi_j^{\lambda,\beta} \cdot \Delta \chi_i^{\lambda,\alpha}\, dy,
\end{aligned}$$

where $P_j^{\gamma\beta} = y_j \delta^{\gamma\beta}$ and we have used (3.2) for the last step. It follows that $\widehat{A^\lambda}$ satisfies the symmetry conditions in (1.3). To prove the ellipticity condition in (1.3), we introduce the bilinear form,

$$a_{\text{per}}(\phi, \psi) = \fint_Y A \nabla \phi \cdot \nabla \psi\, dy + \lambda^2 \fint_Y \Delta \phi \cdot \Delta \psi\, dy,$$

which is symmetric and nonnegative. It is known that the elasticity condition (1.3) implies

$$\frac{\nu_1}{4} |\zeta + \zeta^T|^2 \le A\zeta \cdot \zeta^T \le \frac{\nu_2}{4} |\zeta + \zeta^T|^2 \tag{3.6}$$

for any matrix $\zeta \in \mathbb{R}^{d\times d}$, where ζ^T denotes the transpose of ζ. Let $\xi = (\xi_j^\beta) \in \mathbb{R}^{d\times d}$ be a symmetric matrix. Let $\phi = \xi_j^\beta P_j^\beta$ and $\psi = \xi_j^\beta \chi_j^{\lambda,\beta}$. Then

$$\begin{aligned}
\widehat{A^\lambda}_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta &= a_{\text{per}}(\phi + \psi, \phi + \psi) \\
&\ge \fint_Y A \nabla(\phi + \psi) \cdot \nabla(\phi + \psi)\, dy \\
&\ge \frac{\nu_1}{4} \fint_Y |\nabla \phi + \nabla \psi + (\nabla \phi)^T + (\nabla \psi)^T|^2\, dy \\
&= \frac{\nu_1}{4} \fint_Y |\nabla \phi + (\nabla \phi)^T|^2\, dy + \frac{\nu_1}{4} \fint_Y |\nabla \psi + (\nabla \psi)^T|^2\, dy \\
&\ge \nu_1 |\xi|^2,
\end{aligned}$$

where we have used (3.6) and the fact $\int_Y \nabla \chi^\lambda\, dy = 0$. Also, note that

$$\begin{aligned}
\widehat{A^\lambda}_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta &= a_{\text{per}}(\phi + \psi, \phi - \psi) \\
&= a_{\text{per}}(\phi, \phi) - a_{\text{per}}(\psi, \psi) \le a_{\text{per}}(\phi, \phi) \le \nu_2 |\xi|^2,
\end{aligned}$$

where we have used (3.6) for the last inequality. □

Define

$$\overline{A} = \fint_Y A(y)\, dy. \tag{3.7}$$

Lemma 3.2. *Assume A satisfies (1.3) and (1.4). Let $\widehat{A^\lambda}$ be defined by (3.5). Then*

$$\big|\widehat{A^\lambda} - \overline{A}\big| \le C\lambda^{-2} \quad \text{for } 1 \le \lambda < \infty, \tag{3.8}$$

$$\big|\widehat{A^{\lambda_1}} - \widehat{A^{\lambda_2}}\big| \le C|1 - (\lambda_1/\lambda_2)^2| \quad \text{for } 0 < \lambda_1, \lambda_2 < \infty, \tag{3.9}$$

$$\big|\widehat{A^\lambda} - \widehat{A^0}\big| \le \widetilde{C}\lambda^2 \quad \text{for } 0 < \lambda \le 1, \text{ if in addition } \|\nabla A\|_\infty \le L, \tag{3.10}$$

where C depends only on d, ν_1, ν_2, and $\widetilde{C}$ depends on d, ν_1, ν_2 and L.

Proof. By the definitions of $\widehat{A^\lambda}$ and $\overline{A}$,

$$\big|\widehat{A^\lambda} - \overline{A}\big| = \big|\fint_Y A(y)\nabla\chi^\lambda(y)dy\big| \le C\|\nabla\chi^\lambda\|_{L^2(Y)},$$

which, together with (3.3), gives (3.8). Similarly, by the definition of $\widehat{A^\lambda}$,

$$\big|\widehat{A^{\lambda_1}} - \widehat{A^{\lambda_2}}\big| \le C\|\nabla\chi^{\lambda_1} - \nabla\chi^{\lambda_2}\|_{L^2(Y)}. \tag{3.11}$$

Since

$$-\mathrm{div}\big(A(y)\nabla(\chi^{\lambda_1} - \chi^{\lambda_2})\big) + \lambda_1^2\Delta^2(\chi^{\lambda_1} - \chi^{\lambda_2}) = (\lambda_2^2 - \lambda_1^2)\Delta^2\chi^{\lambda_2},$$

by energy estimates and the H^3 estimate for χ^λ in (3.3),

$$\begin{aligned}\|\nabla\chi^{\lambda_1} - \nabla\chi^{\lambda_2}\|_{L^2(Y)} &\le C|\lambda_2^2 - \lambda_1^2|\|\nabla^3\chi^{\lambda_2}\|_{L^2(Y)}\\ &\le C|1 - (\lambda_1/\lambda_2)^2|,\end{aligned}$$

which, combined with (3.11), gives (3.9).

We now turn to (3.10). Note that

$$\big|\widehat{A^\lambda} - \widehat{A^0}\big| = \big|\fint_Y A(\nabla\chi^\lambda - \nabla\chi^0)dy\big| \le C\|\nabla A\|_\infty\|\chi^\lambda - \chi^0\|_{L^2(Y)}, \tag{3.12}$$

where we have used the integration by parts for the last inequality. It follows by Theorem 2.12 that

$$\|\chi^\lambda - \chi^0\|_{L^2(Y)} \le C\lambda^2, \tag{3.13}$$

where C depends only on d, ν_1, ν_2 and L. This, combined with (3.12), gives (3.10). □

Define $\mathcal{L}_0 = -\mathrm{div}(\widehat{A}\nabla)$, where

$$\widehat{A} = \begin{cases} \overline{A} = \fint_Y A(y)\, dy & \text{if } \rho = \infty,\\ \widehat{A^\rho} & \text{if } 0 \le \rho < \infty, \end{cases} \tag{3.14}$$

where $\widehat{A^\rho}$ is given by (3.5).

Lemma 3.3. *Suppose that $\lambda \to \rho$. Then $\widehat{A^\lambda} \to \widehat{A}$.*

Proof. In view of Lemma 3.2, this is obvious if $0 < \rho \le \infty$. In the case $\rho = 0$, where $\widehat{A} = \widehat{A^0}$, the estimate (3.10) requires that A is Lipschitz continuous. The condition may be removed by an approximation argument. Indeed, let B be a smooth matrix satisfying (1.3)-(1.4). Then

$$\big|\widehat{A^\lambda} - \widehat{A^0}\big| \le \big|\widehat{A^\lambda} - \widehat{B^\lambda}\big| + \big|\widehat{B^\lambda} - \widehat{B^0}\big| + \big|\widehat{B^0} - \widehat{A^0}\big|. \tag{3.15}$$

Let τ^λ be the weak solution of the cell problem (3.2) with A being replaced by B. Then

$$\lambda^2 \Delta^2(\chi^\lambda - \tau^\lambda) - \operatorname{div}(A(y)\nabla(\chi^\lambda - \tau^\lambda)) = \operatorname{div}\big((A-B)\nabla(y+\tau^\lambda)\big).$$

By the reverse Hölder estimate (2.15), there exist some $p > 2$ and $C > 0$, depending only on d, ν_1 and ν_2, such that $\|\nabla \tau^\lambda\|_{L^p(Y)} \le C$. By energy estimates,

$$\begin{aligned}
\|\nabla(\chi^\lambda - \tau^\lambda)\|_{L^2(Y)} &\le C\|A-B\|_{L^2(Y)} + C\Big(\fint_Y |A-B|^2 |\nabla \tau^\lambda|^2 dy\Big)^{1/2} \\
&\le C\|A-B\|_{L^2(Y)} + C\|A-B\|_{L^q(Y)} \|\nabla \tau^\lambda\|_{L^p(Y)} \\
&\le C\|A-B\|_{L^q(Y)},
\end{aligned}$$

where $q = 2p/(p-2)$. By the definitions of $\widehat{A^\lambda}$ and $\widehat{B^\lambda}$, we obtain that

$$\big|\widehat{A^\lambda} - \widehat{B^\lambda}\big| \le \|A-B\|_{L^2(Y)} + \|\nabla(\chi^\lambda - \tau^\lambda)\|_{L^2(Y)} \le C\|A-B\|_{L^q(Y)}. \tag{3.16}$$

Similarly, one can prove that

$$\big|\widehat{A^0} - \widehat{B^0}\big| \le \|A-B\|_{L^2(Y)} + \|\nabla(\chi^0 - \tau^0)\|_{L^2(Y)} \le C\|A-B\|_{L^q(Y)},$$

which, combined with (3.15), (3.16) and (3.10) for B, gives

$$\big|\widehat{A^\lambda} - \widehat{A^0}\big| \le C\|A-B\|_{L^q(Y)} + C_B \lambda^2,$$

where C_B depends on $\|\nabla B\|_\infty$. By approximating A in $L^q(Y)$ with a sequence of smooth matrix satisfying (1.3) and (1.4), we obtain $\widehat{A^\lambda} \to \widehat{A^0}$ as $\lambda \to 0$. □

The following theorem shows that the effective equation for (1.1) is given by $\mathcal{L}_0(u_0) = F$.

Theorem 3.4. *Suppose that A satisfies (1.3)-(1.4) and κ satisfies (1.6). Let $F \in H^{-1}(\Omega;\mathbb{R}^d)$ and $G \in H^2(\Omega;\mathbb{R}^d)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^d$. Let $u_\varepsilon \in H^2(\Omega;\mathbb{R}^d)$ be the weak solution of (1.1) such that $u_\varepsilon - G \in H_0^2(\Omega;\mathbb{R}^d)$. Let $u_0 \in H^1(\Omega;\mathbb{R}^d)$ be the weak solution of $-\operatorname{div}(\widehat{A}\nabla u_0) = F$ in Ω with $u_0 - G \in H_0^1(\Omega;\mathbb{R}^d)$, where $\widehat{A}$ is given by (3.14). Then as $\varepsilon \to 0$, $u_\varepsilon \to u_0$ weakly in $H^1(\Omega;\mathbb{R}^d)$, and $A(x/\varepsilon)\nabla u_\varepsilon \to \widehat{A}\nabla u_0$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$.*

Proof. This is proved by using Tartar's method of test functions. Note that since $\kappa < 1$, by the energy estimate (2.2),

$$\kappa\|\nabla^2 u_\varepsilon\|_{L^2(\Omega)} + \|u_\varepsilon\|_{H^1(\Omega)} \le C\big\{\|F\|_{H^{-1}(\Omega)} + \|G\|_{H^2(\Omega)}\big\}, \tag{3.17}$$

where C depends on d, ν_1, ν_2 and Ω. Let $\{u_{\varepsilon'}\}$ be a sequence such that $u_{\varepsilon'} \to u$ weakly in $H^1(\Omega;\mathbb{R}^d)$ and $A(x/\varepsilon')\nabla u_{\varepsilon'} \to H$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$. We will show that $H = \widehat{A}\nabla u$ in Ω. Since $-\text{div}(H) = F$ in Ω, we see that $-\text{div}(\widehat{A}\nabla u) = F$ in Ω. By the uniqueness of weak solutions in $H^1(\Omega;\mathbb{R}^d)$ for $\mathcal{L}_0$, we deduce that $u = u_0$. As a result, we obtain that $u_\varepsilon \to u_0$ weakly in $H^1(\Omega;\mathbb{R}^d)$ and $A(x/\varepsilon)\nabla u_\varepsilon \to \widehat{A}\nabla u_0$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$, as $\varepsilon \to 0$.

To show $H = \widehat{A}\nabla u$, for notational simplicity, we let $\varepsilon = \varepsilon'$ and $\lambda = \kappa/\varepsilon$. Note that

$$\mathcal{L}_\varepsilon\Big\{P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon)\Big\} = 0 \quad \text{in } \mathbb{R}^d. \tag{3.18}$$

It follows that

$$\begin{aligned}\lambda^2\varepsilon^2 \int_\Omega \Delta\big\{P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon)\big\}\cdot \Delta(u_\varepsilon\psi)\,dx \\ + \int_\Omega A(x/\varepsilon)\nabla\big(P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon)\big)\cdot\nabla(u_\varepsilon\psi)\,dx = 0,\end{aligned} \tag{3.19}$$

for any $\psi \in C_0^\infty(\Omega)$. Also note that

$$\begin{aligned}&\lambda^2\varepsilon^2\int_\Omega \Delta u_\varepsilon\cdot\Delta\big\{(P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon))\psi\big\}\,dx \\ &+\int_\Omega A(x/\varepsilon)\nabla u_\varepsilon\cdot\nabla\big\{(P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon))\psi\big\}\,dx = \langle F, (P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon))\psi\rangle.\end{aligned} \tag{3.20}$$

By subtracting (3.19) from (3.20), we obtain

$$\begin{aligned}&2\lambda^2\varepsilon^2\int_\Omega \Delta u_\varepsilon\cdot\nabla(P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon))\nabla\psi\,dx \\ &\quad - 2\lambda^2\varepsilon^2\int_\Omega \Delta(P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon))\cdot\nabla u_\varepsilon\cdot\nabla\psi\,dx \\ &\quad + \lambda^2\varepsilon^2\int_\Omega \Delta u_\varepsilon\cdot(P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon))\Delta\psi\,dx \\ &\quad - \lambda^2\varepsilon^2\int_\Omega \Delta(P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon))u_\varepsilon\cdot\Delta\psi\,dx \\ &\quad + \int_\Omega A(x/\varepsilon)\nabla u_\varepsilon\cdot(P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon))\nabla\psi\,dx \\ &\quad - \int_\Omega A(x/\varepsilon)\nabla(P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon))u_\varepsilon\cdot\nabla\psi\,dx \\ &= \langle F, (P_j^\beta + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon))\psi\rangle.\end{aligned} \tag{3.21}$$

We now let $\varepsilon \to 0$ in (3.21). Using (3.17) and (3.3), it is not hard to see that the first four terms in the left-hand side of (3.21) converge to zero, while the right-hand side converges to $\langle F, P_j^\beta\psi\rangle$. Also, the fifth term in the left-hand side converges to

$$\int_\Omega H_i^\alpha\cdot P_j^{\alpha\beta}\frac{\partial\psi}{\partial x_i}\,dx = \langle F, P_j^\beta\psi\rangle - \int_\Omega H_j^\beta\psi\,dx.$$

Finally, we observe that by Lemma 3.3, $\widehat{A^\lambda} \to \widehat{A}$ as $\varepsilon \to 0$, and that $u_\varepsilon \to u$ strongly in $L^2(\Omega;\mathbb{R}^d)$. This implies that the last term in the left-hand side of (3.21) converges to

$$-\int_\Omega \widehat{A}_{ij}^{\alpha\beta} u^\alpha \frac{\partial \psi}{\partial x_i}\, dx = \int_\Omega \widehat{A}_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \psi \, dx,$$

where we have used integration by parts. Since $\psi \in C_0^\infty(\Omega)$ is arbitrary, we see that

$$H_j^\beta = \widehat{A}_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} = \widehat{A}_{ji}^{\beta\alpha} \frac{\partial u^\alpha}{\partial x_i},$$

where we have used the symmetry conditions of $\widehat{A}$. Hence, $H = \widehat{A}\nabla u$. □

4 Convergence rates

In this section we give the proof of Theorem 1.1. To this end, we fix $0 < \lambda < \infty$ and consider the Dirichlet problem,

$$\mathcal{L}_\varepsilon^\lambda(u_{\varepsilon,\lambda}) = F \quad \text{in } \Omega \quad \text{and} \quad u_{\varepsilon,\lambda} - G \in H_0^2(\Omega;\mathbb{R}^d), \tag{4.1}$$

where $\mathcal{L}_\varepsilon^\lambda$ is given by (1.9), $F \in L^2(\Omega;\mathbb{R}^d)$ and $G \in H^2(\Omega;\mathbb{R}^d)$. Let $u_{0,\lambda} \in H^1(\Omega;\mathbb{R}^d)$ be the solution of the homogenized problem,

$$-\mathrm{div}\big(\widehat{A^\lambda}\nabla u_{0,\lambda}\big) = F \quad \text{in } \Omega \quad \text{and} \quad u_{0,\lambda} - G \in H_0^1(\Omega;\mathbb{R}^d), \tag{4.2}$$

where $\widehat{A^\lambda}$ is given by (3.5). We shall study the convergence rate of $u_{\varepsilon,\lambda}$ to $u_{0,\lambda}$ as $\varepsilon \to 0$.

Let $\eta_t \in C_0^\infty(\Omega)$ be a cut-off function such that

$$\begin{aligned} &0 \le \eta_t \le 1, |\nabla^k \eta_t| \le C t^{-k} \text{ for } k = 1, 2, \\ &\eta_t = 1 \ \text{ if } x \in \Omega \backslash \Omega_{4t} \quad \text{and} \quad \eta_t(x) = 0 \ \text{ if } x \in \Omega_{3t}, \end{aligned} \tag{4.3}$$

where $\varepsilon \le t < 1$ and Ω_t is defined in (2.27). Let

$$w_{\varepsilon,\lambda} = u_{\varepsilon,\lambda} - u_{0,\lambda} + (u_{0,\lambda} - G)(1 - \eta_t) - \varepsilon \chi^\lambda(x/\varepsilon)\eta_t S_\varepsilon(\nabla u_{0,\lambda}), \tag{4.4}$$

where $t = (1+\lambda)\varepsilon$ and χ^λ is the corrector given by (3.2). The ε-smoothing operator S_ε in (4.4) is defined by

$$S_\varepsilon(f)(x) = \int_{\mathbb{R}^d} f(x - \varsigma)\varphi_\varepsilon(\varsigma) d\varsigma,$$

where $\varphi_\varepsilon(\varsigma) = \varepsilon^{-d}\varphi(\varsigma/\varepsilon)$ and φ is a fixed function in $C_0^\infty(B(0,1/2))$ such that $\varphi \ge 0$ and $\int_{\mathbb{R}^d} \varphi dx = 1$.

Lemma 4.1. *Let $f \in W^{1,p}(\mathbb{R}^d)$ for some $1 \le p \le \infty$. Then*

$$\|S_\varepsilon(f) - f\|_{L^p(\mathbb{R}^d)} \le \varepsilon \|\nabla f\|_{L^p(\mathbb{R}^d)}. \tag{4.5}$$

Suppose that $f, g \in L^p_{loc}(\mathbb{R}^d)$ for some $1 \le p < \infty$ and g is 1-periodic. Then

$$\|g^\varepsilon \nabla^k S_\varepsilon(f)\|_{L^p(\mathcal{O})} \le C_k \varepsilon^{-k} \|g\|_{L^p(Y)} \|f\|_{L^p(\mathcal{O}^\varepsilon)} \tag{4.6}$$

for $k \ge 0$, where $g^\varepsilon(x) = g(x/\varepsilon)$, $\mathcal{O}^\varepsilon = \{x \in \mathbb{R}^d : \text{dist}(x, \mathcal{O}) < \varepsilon\}$, and C_k depends only on d, k and p.

Proof. See e.g., [17]. □

Lemma 4.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $u_{\varepsilon,\lambda}$, $u_{0,\lambda}$ and $w_{\varepsilon,\lambda}$ be given by (4.1), (4.2) and (4.4), respectively. Suppose $u_{0,\lambda} \in H^2(\Omega; \mathbb{R}^d)$. Then for any $\psi \in H^2_0(\Omega; \mathbb{R}^d)$ and $0 < \varepsilon < (1+\lambda)^{-1}$,*

$$\begin{aligned} |\langle \mathcal{L}^\lambda_\varepsilon(w_{\varepsilon,\lambda}), \psi \rangle| &\le C \|u_{0,\lambda}\|_{H^2(\Omega)} \Big\{ \varepsilon \|\nabla \psi\|_{L^2(\Omega)} + \varepsilon^2 \lambda^2 \|\Delta \psi\|_{L^2(\Omega)} \Big\} \\ &\quad + C t^{1/2} \Big\{ \|u_{0,\lambda}\|_{H^2(\Omega)} + \|G\|_{H^2(\Omega)} \Big\} \|\nabla \psi\|_{L^2(\Omega_{5t})} \\ &\quad + C \varepsilon^2 \lambda^2 t^{-1/2} \Big\{ \|u_{0,\lambda}\|_{H^2(\Omega)} + \|G\|_{H^2(\Omega)} \Big\} \|\Delta \psi\|_{L^2(\Omega_{5t})}, \end{aligned} \tag{4.7}$$

where $t = (1+\lambda)\varepsilon$ and C depends only on d, ν_1, ν_2, and Ω.

Proof. Note that $w_{\varepsilon,\lambda} \in H^2_0(\Omega; \mathbb{R}^d)$ and

$$\begin{aligned} \mathcal{L}^\lambda_\varepsilon(w_{\varepsilon,\lambda}) &= -\text{div}\big\{ (\widehat{A^\lambda} - A(x/\varepsilon)) \nabla u_{0,\lambda} \big\} - \lambda^2 \varepsilon^2 \Delta^2(u_{0,\lambda}) + \mathcal{L}^\lambda_\varepsilon\big\{ (u_{0,\lambda} - G)(1-\eta_t) \big\} \\ &\quad - \mathcal{L}^\lambda_\varepsilon\big\{ \varepsilon \chi^\lambda(x/\varepsilon) \eta_t S_\varepsilon(\nabla u_{0,\lambda}) \big\} \\ &= -\text{div}\big\{ (\widehat{A^\lambda} - A(x/\varepsilon)) (\nabla u_{0,\lambda} - \eta_t S_\varepsilon(\nabla u_{0,\lambda})) \big\} \\ &\quad - \lambda^2 \varepsilon^2 \Delta^2(u_{0,\lambda}) + \mathcal{L}^\lambda_\varepsilon\big\{ (u_{0,\lambda} - G)(1-\eta_t) \big\} \\ &\quad - \text{div}\big\{ B^\lambda(x/\varepsilon) \eta_t S_\varepsilon(\nabla u_{0,\lambda}) \big\} - \lambda^2 \varepsilon \,\text{div}\big\{ \Delta \chi^\lambda(x/\varepsilon) \nabla [\eta_t S_\varepsilon(\nabla u_{0,\lambda})] \big\} \\ &\quad - 2\lambda^2 \varepsilon^2 \Delta\big\{ \nabla \chi^\lambda(x/\varepsilon) \nabla [\eta_t S_\varepsilon(\nabla u_{0,\lambda})] \big\} - \lambda^2 \varepsilon^3 \Delta\big\{ \chi^\lambda(x/\varepsilon) \Delta [\eta_t S_\varepsilon(\nabla u_{0,\lambda})] \big\} \\ &\quad + \varepsilon \,\text{div}\big\{ \chi^\lambda(x/\varepsilon) A(x/\varepsilon) \nabla [\eta_t S_\varepsilon(\nabla u_{0,\lambda})] \big\}, \end{aligned}$$

where

$$B^\lambda(y) = \lambda^2 \nabla \Delta \chi^\lambda(y) - A \nabla \chi^\lambda(y) - A(y) + \widehat{A^\lambda}. \tag{4.8}$$

It follows that for any $\psi \in H_0^2(\Omega;\mathbb{R}^d)$,

$$
\begin{aligned}
|\langle \mathcal{L}_\varepsilon^\lambda(w_{\varepsilon,\lambda}),\psi\rangle| &\le C\int_\Omega |[\nabla u_{0,\lambda} - \eta_t S_\varepsilon(\nabla u_{0,\lambda})]\nabla\psi|\,dx + \varepsilon^2\lambda^2\int_\Omega |\Delta u_{0,\lambda}||\Delta\psi|\,dx\\
&\quad + \lambda^2\varepsilon^2\int_\Omega |\Delta[(u_{0,\lambda}-G)(1-\eta_t)]||\Delta\psi|\,dx\\
&\quad + C\int_\Omega |\nabla[(u_{0,\lambda}-G)(1-\eta_t)]||\nabla\psi|\,dx\\
&\quad + C\Big|\int_\Omega B^\lambda(x/\varepsilon)\eta_t S_\varepsilon(\nabla u_{0,\lambda})\nabla\psi\,dx\Big|\\
&\quad + C\varepsilon\int_\Omega |\chi^\lambda(x/\varepsilon)\nabla\big[\eta_t S_\varepsilon(\nabla u_{0,\lambda})\big]\nabla\psi|\,dx\\
&\quad + C\varepsilon\lambda^2\int_\Omega |\nabla^2\chi^\lambda(x/\varepsilon)\nabla\big[\eta_t S_\varepsilon(\nabla u_{0,\lambda})\big]\nabla\psi|\,dx\\
&\quad + C\varepsilon^2\lambda^2\int_\Omega |\nabla\chi^\lambda(x/\varepsilon)\nabla^2\big[\eta_t S_\varepsilon(\nabla u_{0,\lambda})\big]\nabla\psi|\,dx\\
&\quad + C\varepsilon^3\lambda^2\int_\Omega |\chi^\lambda(x/\varepsilon)\nabla^3\big[\eta_t S_\varepsilon(\nabla u_{0,\lambda})\big]\nabla\psi|\,dx\\
&= I_1 + I_2 + \cdots + I_9.
\end{aligned}
$$

Using Lemma 4.1 and the Cauchy inequality, it is not hard to see that

$$
I_1 \le C\Big\{\|\nabla u_{0,\lambda}\|_{L^2(\Omega_{5t})}\|\nabla\psi\|_{L^2(\Omega_{5t})} + \varepsilon\|\nabla^2 u_{0,\lambda}\|_{L^2(\Omega\setminus\Omega_{2t})}\|\nabla\psi\|_{L^2(\Omega)}\Big\}. \tag{4.9}
$$

Next, we observe that

$$
\begin{aligned}
I_2 + I_3 + I_4 &\le \varepsilon^2\lambda^2\|\Delta u_{0,\lambda}\|_{L^2(\Omega)}\|\Delta\psi\|_{L^2(\Omega)} + C\lambda^2\varepsilon^2 t^{-1/2}\|u_{0,\lambda}-G\|_{H^2(\Omega)}\|\Delta\psi\|_{L^2(\Omega_{4t})}\\
&\quad + Ct^{1/2}\|u_{0,\lambda}-G\|_{H^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega_{4t})}.
\end{aligned} \tag{4.10}
$$

To bound I_5, we note that by (3.3), we have $\|B^\lambda\|_{L^2(Y)} \le C$, where C depends only on d, ν_1 and ν_2. Moreover, by the definition of $B^\lambda = (B_{ij}^\lambda), 1 \le i,j \le d$,

$$
\partial_{y_i} B_{ij}^\lambda = 0 \quad \text{and} \qquad \int_Y B_{ij}^\lambda\,dy = 0. \tag{4.11}
$$

This allows us to construct a matrix of 1-periodic flux correctors $\mathfrak{B}_{kij}^\lambda(y)$ such that

$$
\mathfrak{B}_{kij}^\lambda = -\mathfrak{B}_{ikj}^\lambda, \quad \partial_{y_k}\mathfrak{B}_{kij}^\lambda(y) = B_{ij}^\lambda(y), \quad \|\mathfrak{B}_{kij}^\lambda\|_{H^1(Y)} \le C,
$$

with C depending only on d, ν_1 and ν_2. It follows that

$$
\begin{aligned}
I_5 &\le C\varepsilon\|\mathfrak{B}^\lambda(x/\varepsilon)\nabla(\eta_t S_\varepsilon(\nabla u_{0,\lambda}))\nabla\psi\|_{L^1(\Omega)}\\
&\le C\|\nabla u_{0,\lambda}\|_{L^2(\Omega_{5t})}\|\nabla\psi\|_{L^2(\Omega_{5t})} + C\varepsilon\|\nabla^2 u_{0,\lambda}\|_{L^2(\Omega\setminus\Omega_{2t})}\|\nabla\psi\|_{L^2(\Omega)},
\end{aligned} \tag{4.12}
$$

where we have used the fact $\varepsilon t^{-1} \le 1$. Using (3.3) and (4.6), we also obtain

$$
I_6 + I_7 + I_8 + I_9 \le C\|\nabla u_{0,\lambda}\|_{L^2(\Omega_{5t})}\|\nabla\psi\|_{L^2(\Omega_{5t})} + C\varepsilon\|\nabla^2 u_{0,\lambda}\|_{L^2(\Omega\setminus\Omega_{2t})}\|\nabla\psi\|_{L^2(\Omega)}. \tag{4.13}
$$

By collecting estimates for $I_1, I_2, \ldots, I_9$, we obtain the desired estimate (4.7). □

Lemma 4.3. *Let $u_{\varepsilon,\lambda}$, $u_{0,\lambda}$ and $w_{\varepsilon,\lambda}$ be the same as in Lemma 4.2. Assume that $u_{0,\lambda} \in H^2(\Omega;\mathbb{R}^d)$. Then*

$$\lambda\varepsilon\|\Delta w_{\varepsilon,\lambda}\|_{L^2(\Omega)} + \|\nabla w_{\varepsilon,\lambda}\|_{L^2(\Omega)} \le C\big((1+\lambda)\varepsilon\big)^{1/2}\big\{\|u_0\|_{H^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\}, \tag{4.14}$$

where C depends only on d, ν_1, ν_2, and Ω.

Proof. Note that $w_{\varepsilon,\lambda} \in H_0^2(\Omega;\mathbb{R}^d)$. The estimate (4.14) follows readily by letting $\psi = w_{\varepsilon,\lambda}$ in (4.7) and using the Cauchy inequality as well as the first Korn inequality. □

The next theorem gives the sharp convergence rate in $L^2(\Omega)$ for $\mathcal{L}_\varepsilon^\lambda$ with λ fixed.

Theorem 4.4. *Suppose A satisfies conditions (1.3) and (1.4). Let Ω be a bounded $C^{1,1}$ domain, $F \in L^2(\Omega;\mathbb{R}^d)$ and $G \in H^2(\Omega;\mathbb{R}^d)$. Let $u_{\varepsilon,\lambda}$ be the weak solution of (4.1) and $u_{0,\lambda}$ the solution of the homogenized problem (4.2), where $0 < \lambda < \infty$. Then for any $0 < \varepsilon < (1+\lambda)^{-1}$,*

$$\|u_{\varepsilon,\lambda} - u_{0,\lambda}\|_{L^2(\Omega)} \le C(1+\lambda)\varepsilon\big\{\|F\|_{L^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\}, \tag{4.15}$$

where C depends only on d, ν_1, ν_2, and Ω.

Proof. For $\widetilde{F} \in C_0^\infty(\Omega;\mathbb{R}^d)$, let $v_{\varepsilon,\lambda} \in H_0^2(\Omega;\mathbb{R}^d)$ be the weak solution of $\mathcal{L}_\varepsilon^\lambda(v_{\varepsilon,\lambda}) = \widetilde{F}$ in Ω and $v_{0,\lambda}$ the solution in $H_0^1(\Omega;\mathbb{R}^d)$ of the homogenized problem $-\mathrm{div}(\widehat{A^\lambda}\nabla v_{0,\lambda}) = \widetilde{F}$ in Ω. Note that since Ω is $C^{1,1}$, we have $\|v_{0,\lambda}\|_{H^2(\Omega)} \le C\|\widetilde{F}\|_{L^2(\Omega)}$ and

$$\|u_{0,\lambda}\|_{H^2(\Omega)} \le C\big\{\|F\|_{L^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\},$$

where C depends only on d, ν_1, ν_2, and Ω. Let

$$\widetilde{w}_{\varepsilon,\lambda} = v_{\varepsilon,\lambda} - v_{0,\lambda}\widetilde{\eta}_t - \varepsilon\chi^\lambda(x/\varepsilon)\widetilde{\eta}_t S_\varepsilon(\nabla v_{0,\lambda}), \tag{4.16}$$

where $t = (1+\lambda)\varepsilon$ and $\widetilde{\eta}_t$ is a function in $C_0^\infty(\Omega)$ such that $0 \le \widetilde{\eta}_t \le 1$, $|\nabla^k \widetilde{\eta}_t| \le Ct^{-k}$ for $k = 1, 2$, $\widetilde{\eta}_t(x) = 1$ if $x \in \Omega \setminus \Omega_{8t}$, and $\widetilde{\eta}_t(x) = 0$ if $x \in \Omega_{7t}$.

Let $w_{\varepsilon,\lambda}$ be given by (4.4). Note that

$$\begin{aligned}
\Big|\int_\Omega w_{\varepsilon,\lambda}\cdot\widetilde{F}\,dx\Big| &= |\langle\mathcal{L}_\varepsilon^\lambda(w_{\varepsilon,\lambda}), v_{\varepsilon,\lambda}\rangle| \\
&\le |\langle\mathcal{L}_\varepsilon^\lambda(w_{\varepsilon,\lambda}), \widetilde{w}_{\varepsilon,\lambda}\rangle| + |\langle\mathcal{L}_\varepsilon^\lambda(w_{\varepsilon,\lambda}), v_{0,\lambda}\widetilde{\eta}_t\rangle| + |\langle\mathcal{L}_\varepsilon^\lambda(w_{\varepsilon,\lambda}), \zeta_{\varepsilon,\lambda}\rangle| \\
&= J_1 + J_2 + J_3,
\end{aligned} \tag{4.17}$$

where

$$\zeta_{\varepsilon,\lambda} = \varepsilon\chi^\lambda(x/\varepsilon)\widetilde{\eta}_t S_\varepsilon(\nabla v_{0,\lambda}). \tag{4.18}$$

Observe that

$$\begin{aligned}
J_1 &\le \varepsilon^2\lambda^2\|\Delta w_{\varepsilon,\lambda}\|_{L^2(\Omega)}\|\Delta\widetilde{w}_{\varepsilon,\lambda}\|_{L^2(\Omega)} + C\|\nabla w_{\varepsilon,\lambda}\|_{L^2(\Omega)}\|\nabla\widetilde{w}_{\varepsilon,\lambda}\|_{L^2(\Omega)} \\
&\le C(1+\lambda)\varepsilon\big\{\|u_{0,\lambda}\|_{H^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\}\|v_{0,\lambda}\|_{H^2(\Omega)},
\end{aligned} \tag{4.19}$$

where we have used (4.14) for the last inequality. To bound J_2, we use (4.7) to obtain

$$J_2 \le C(1+\lambda)\varepsilon\big\{\|u_{0,\lambda}\|_{H^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\}\|v_{0,\lambda}\|_{H^2(\Omega)}. \tag{4.20}$$

To handle J_3, we note that by (3.3) and (4.6),

$$\|\nabla \zeta_{\varepsilon,\lambda}\|_{L^2(\Omega)} \le C(1+\lambda)^{-2}\|v_{0,\lambda}\|_{H^2(\Omega)}, \tag{4.21}$$

$$\|\Delta \zeta_{\varepsilon,\lambda}\|_{L^2(\Omega)} \le C\varepsilon^{-1}(1+\lambda)^{-2}\|v_{0,\lambda}\|_{H^2(\Omega)}. \tag{4.22}$$

Since $\zeta_{\varepsilon,\lambda}=0$ in Ω_{5t}, it follows from (4.7) that

$$\begin{aligned} J_3 &\le C\varepsilon \|u_{0,\lambda}\|_{H^2(\Omega)}\|\nabla \zeta_{\varepsilon,\lambda}\|_{L^2(\Omega)} + C\varepsilon^2\lambda^2\|u_{0,\lambda}\|_{H^2(\Omega)}\|\Delta \zeta_{\varepsilon,\lambda}\|_{L^2(\Omega)} \\ &\le C\varepsilon \|u_{0,\lambda}\|_{H^2(\Omega)}\|v_{0,\lambda}\|_{H^2(\Omega)}. \end{aligned} \tag{4.23}$$

In view of (4.17), (4.19), (4.20) and (4.23), we have proved that

$$\begin{aligned} \Big| \int_\Omega w_{\varepsilon,\lambda}\cdot \widetilde{F}\, dx\Big| &\le C(1+\lambda)\varepsilon\big\{\|u_{0,\lambda}\|_{H^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\}\|v_{0,\lambda}\|_{H^2(\Omega)} \\ &\le C(1+\lambda)\varepsilon\big\{\|F\|_{L^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\}\|\widetilde{F}\|_{L^2(\Omega)}. \end{aligned}$$

By duality this implies that

$$\|w_{\varepsilon,\lambda}\|_{L^2(\Omega)} \le C(1+\lambda)\varepsilon\big\{\|F\|_{L^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\}.$$

Hence,

$$\begin{aligned} \|u_{\varepsilon,\lambda} - u_{0,\lambda}\|_{L^2(\Omega)} &\le \|w_{\varepsilon,\lambda}\|_{L^2(\Omega)} + \|(u_{0,\lambda}-G)(1-\eta_t)\|_{L^2(\Omega)} \\ &\qquad + \|\varepsilon \chi^\lambda(x/\varepsilon)\eta_t S_\varepsilon(\nabla u_{0,\lambda})\|_{L^2(\Omega)} \\ &\le C(1+\lambda)\varepsilon\big\{\|F\|_{L^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\}, \end{aligned}$$

which completes the proof. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $u_\varepsilon \in H^2(\Omega;\mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon)=F$ in Ω with $u_\varepsilon - G \in H_0^2(\Omega)$, and $u_0 \in H^1(\Omega;\mathbb{R}^d)$ the solution of the homogenized equation $-\mathrm{div}(\widehat{A}\nabla u_0)=F$ in Ω with $u_0 - G \in H_0^1(\Omega;\mathbb{R}^d)$. Let $\lambda=\kappa/\varepsilon$. Then $\mathcal{L}_\varepsilon^\lambda(u_\varepsilon)=\mathcal{L}_\varepsilon(u_\varepsilon)=F$ in Ω. Let $u_{0,\lambda}\in H^1(\Omega;\mathbb{R}^d)$ be the solution of $-\mathrm{div}(\widehat{A^\lambda}\nabla u_{0,\lambda})=F$ in Ω with $u_{0,\lambda}-G\in H_0^1(\Omega;\mathbb{R}^d)$. Note that

$$\begin{aligned} \|u_\varepsilon - u_0\|_{L^2(\Omega)} &\le \|u_\varepsilon - u_{0,\lambda}\|_{L^2(\Omega)} + \|u_{0,\lambda}-u_0\|_{L^2(\Omega)} \\ &\le C(\kappa+\varepsilon)\big\{\|F\|_{L^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\} + \|u_{0,\lambda}-u_0\|_{L^2(\Omega)}, \end{aligned} \tag{4.24}$$

where we have used Theorem 4.4 for the last inequality. To estimate $u_{0,\lambda}-u_0$, we observe that $u_{0,\lambda}-u_0\in H_0^1(\Omega;\mathbb{R}^d)$ and

$$-\mathrm{div}(\widehat{A}\nabla(u_0-u_{0,\lambda})) = \mathrm{div}((\widehat{A}-\widehat{A^\lambda})\nabla u_{0,\lambda})$$

in Ω. By energy estimates,

$$\begin{aligned} \|u_0 - u_{0,\lambda}\|_{H^1(\Omega)} &\le C|\widehat{A}-\widehat{A^\lambda}|\|\nabla u_{0,\lambda}\|_{L^2(\Omega)} \\ &\le C|\widehat{A}-\widehat{A^\lambda}|\big\{\|F\|_{L^2(\Omega)} + \|G\|_{H^1(\Omega)}\big\}, \end{aligned} \tag{4.25}$$

where C depends only on d, ν_1, ν_2, and Ω. This, together with Lemma 3.2 and (4.24), gives (1.8). □

Remark 4.5. Let $\chi(y)$ be the corrector for $\mathcal{L}_\varepsilon^\rho$ if $0 < \rho < \infty$, $\chi(y) = 0$ if $\rho = \infty$, and $\chi(y)$ the corrector for the operator $-\text{div}(A(x/\varepsilon)\nabla)$ if $\rho = 0$. Under the same assumptions as in Theorem 1.1, one may show that

$$\begin{aligned}
&\|u_\varepsilon - u_0 - \varepsilon\chi(x/\varepsilon)S_\varepsilon(\nabla u_0)\|_{H^1(\Omega)} \\
&\le \big\{\|F\|_{L^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\} \begin{cases} C_1\left\{(\kappa+\varepsilon)^{1/2} + \left(\frac{\varepsilon}{\kappa}\right)^2\right\} & \text{if } \rho = \infty, \\ C_2\left\{(\kappa+\varepsilon)^{1/2} + \rho^{-2}\big|\big(\frac{\kappa}{\varepsilon}\big)^2 - \rho^2\big|\right\} & \text{if } 0 < \rho < \infty, \\ C_3\left\{(\kappa+\varepsilon)^{1/2} + \left(\frac{\kappa}{\varepsilon}\right)^2\right\} & \text{if } \rho = 0, \end{cases}
\end{aligned} \tag{4.26}$$

where C_1, C_2 depend only on d, ν_1, ν_2 and Ω, and C_3 depends only on d, ν_1, ν_2, Ω, and L. To see this, let $\omega_{\varepsilon,\lambda}$ be given by (4.4) with $\lambda = \kappa/\varepsilon$, and

$$\widetilde{\omega}_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi(x/\varepsilon)S_\varepsilon(\nabla u_0).$$

It follows by Lemma 4.3 that

$$\|w_{\varepsilon,\lambda}\|_{H^1(\Omega)} \le C(\varepsilon+\kappa)^{1/2}\big\{\|F\|_{L^2(\Omega)} + \|G\|_{H^2(\Omega)}\big\}.$$

Note that

$$\begin{aligned}
\|w_{\varepsilon,\lambda} - \widetilde{\omega}_\varepsilon\|_{H^1(\Omega)} &\le \|u_{0,\lambda} - u_0\|_{H^1(\Omega)} + \|(u_{0,\lambda} - G)(1-\eta_t)\|_{H^1(\Omega)} \\
&\quad + \varepsilon\|\chi^\lambda(x/\varepsilon)\eta_t S_\varepsilon(\nabla u_{0,\lambda}) - \chi(x/\varepsilon)\eta_t S_\varepsilon(\nabla u_0)\|_{H^1(\Omega)} \\
&\quad + \varepsilon\|\chi(x/\varepsilon)(1-\eta_t)S_\varepsilon(\nabla u_0)\|_{H^1(\Omega)}.
\end{aligned}$$

The desired estimate follows from (4.25) and Lemma 3.2 as well as the estimates of $\|\chi^\lambda - \chi\|_{H^1(Y)}$ in the proof of Lemma 3.2. We omit the details.

5 Approximation

Fix $0 < \lambda < \infty$. Let $\mathcal{L}_\varepsilon^\lambda$ be defined as in (3.1). The goal of this section is to establish the following.

Theorem 5.1. *Suppose A satisfies (1.3) and (1.4). Let $u_{\varepsilon,\lambda} \in H^2(B_{2r};\mathbb{R}^d)$ be a solution to $\mathcal{L}_\varepsilon^\lambda(u_{\varepsilon,\lambda}) = F$ in B_{2r}, where $F \in L^p(B_{2r};\mathbb{R}^d)$ and $B_{2r} = B(z,2r)$ for some $z \in \mathbb{R}^d$. Assume that $p > d$ and $\varepsilon \le r < \infty$. Then there exists $v_{\varepsilon,\lambda} \in H^2(B_r;\mathbb{R}^d)$ such that*

$$\varepsilon^2\lambda^2\Delta^2 v_{\varepsilon,\lambda} - \text{div}\big(\widehat{A^\lambda}\nabla v_{\varepsilon,\lambda}\big) = F \quad \text{in } B_r, \tag{5.1}$$

and

$$\begin{aligned}
&\left(\fint_{B_r} |\nabla v_{\varepsilon,\lambda}|^2\right)^{1/2} \le C\left(\fint_{B_{2r}} |\nabla u_{\varepsilon,\lambda}|^2\right)^{1/2}, \\
&\left(\fint_{B_r} |\nabla u_{\varepsilon,\lambda} - \nabla v_{\varepsilon,\lambda} - (\nabla\chi^\lambda)(x/\varepsilon)\nabla v_{\varepsilon,\lambda}|^2\right)^{1/2}
\end{aligned} \tag{5.2}$$

$$\le C\left(\frac{\varepsilon}{r}\right)^{\sigma}\left\{\left(\fint_{B_{2r}}|\nabla u_{\varepsilon,\lambda}|^2\right)^{1/2}+r\left(\fint_{B_{2r}}|F|^p\right)^{1/p}\right\}, \tag{5.3}$$

where $C>0$ and $0<\sigma<1$ depend only on d, ν_1, ν_2, and p.

To prove Theorem 5.1, we introduce an intermediate Dirichlet problem,

$$\lambda^2\varepsilon^2\Delta^2 v_{\varepsilon,\lambda}+\mathcal{L}_0^\lambda(v_{\varepsilon,\lambda})=F \quad \text{in } \Omega \quad \text{and} \quad v_{\varepsilon,\lambda}-G\in H_0^2(\Omega;\mathbb{R}^d), \tag{5.4}$$

where $\mathcal{L}_0^\lambda=-\text{div}(\widehat{A^\lambda}\nabla)$ and $\widehat{A^\lambda}$ is defined by (3.5). We will establish a (suboptimal) convergence rate in $H^1(\Omega)$ for $u_{\varepsilon,\lambda}-v_{\varepsilon,\lambda}$, where $u_{\varepsilon,\lambda}$ is the solution to the Dirichlet problem,

$$\mathcal{L}_\varepsilon^\lambda(u_{\varepsilon,\lambda})=F \quad \text{in } \Omega \quad \text{and} \quad u_{\varepsilon,\lambda}-G\in H_0^2(\Omega;\mathbb{R}^d), \tag{5.5}$$

with $F\in L^2(\Omega;\mathbb{R}^d)$ and $G\in H^2(\Omega;\mathbb{R}^d)$. Let

$$w_{\varepsilon,\lambda}=u_{\varepsilon,\lambda}-v_{\varepsilon,\lambda}-\varepsilon\chi^\lambda(x/\varepsilon)\eta_\varepsilon S_\varepsilon(\nabla v_{\varepsilon,\lambda}), \tag{5.6}$$

where η_ε, S_ε and χ^λ are the same as in (4.4).

Lemma 5.2. *Let Ω be a bounded Lipschitz domain. Let $u_{\varepsilon,\lambda}, v_{\varepsilon,\lambda}$ be the weak solutions of (5.5) and (5.4), respectively, and $w_{\varepsilon,\lambda}$ be given by (5.6). Then*

$$\lambda\varepsilon\|\Delta w_{\varepsilon,\lambda}\|_{L^2(\Omega)}+\|\nabla w_{\varepsilon,\lambda}\|_{L^2(\Omega)}\le C\|\nabla v_{\varepsilon,\lambda}\|_{L^2(\Omega_{5\varepsilon})}+C\varepsilon\|\nabla^2 v_{\varepsilon,\lambda}\|_{L^2(\Omega\setminus\Omega_{2\varepsilon})} \tag{5.7}$$

for $0<\varepsilon<1$, where C depends only on d, ν_1, ν_2, and Ω.

Proof. The proof is similar to that of (4.14). Let $(g)^\varepsilon=g(x/\varepsilon)$. By direct calculations, we deduce that

$$\begin{aligned}
\mathcal{L}_\varepsilon^\lambda(w_{\varepsilon,\lambda}) &= \mathcal{L}_0^\lambda(v_{\varepsilon,\lambda})+\lambda^2\varepsilon^2\Delta^2 v_{\varepsilon,\lambda}-\mathcal{L}_\varepsilon^\lambda(v_{\varepsilon,\lambda})-\mathcal{L}_\varepsilon^\lambda\big(\varepsilon(\chi^\lambda)^\varepsilon S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon\big)\\
&= -\text{div}\Big\{\widehat{A^\lambda}\nabla v_{\varepsilon,\lambda}-A^\varepsilon\nabla v_{\varepsilon,\lambda}+\lambda^2\varepsilon^3\Delta\nabla[(\chi^\lambda)^\varepsilon S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon]\\
&\qquad\qquad -\varepsilon A^\varepsilon\nabla[(\chi^\lambda)^\varepsilon S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon]\Big\}\\
&= -\text{div}\Big\{(A^\varepsilon-\widehat{A^\lambda})[S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon-\nabla v_{\varepsilon,\lambda}]+(B^\lambda)^\varepsilon S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon\\
&\quad+\lambda^2\varepsilon^3\Delta\nabla[S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon](\chi^\lambda)^\varepsilon+\lambda^2\varepsilon^2(\nabla\chi^\lambda)^\varepsilon\Delta[S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon]\\
&\quad+2\lambda^2\varepsilon^2\nabla^2[S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon](\nabla\chi^\lambda)^\varepsilon+\lambda^2\varepsilon(\Delta\chi^\lambda)^\varepsilon\nabla[S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon]\\
&\quad+2\lambda^2\varepsilon(\nabla^2\chi^\lambda)^\varepsilon\nabla[S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon]-\varepsilon A^\varepsilon(\chi^\lambda)^\varepsilon\nabla[S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon]\Big\},
\end{aligned} \tag{5.8}$$

where $(B^\lambda)^\varepsilon=B^\lambda(x/\varepsilon)$ and B^λ is given by (4.8). Thus for any $\psi\in H_0^2(\Omega;\mathbb{R}^d)$,

$$\begin{aligned}
|\langle\mathcal{L}_\varepsilon^\lambda(w_{\varepsilon,\lambda}),\ \psi\rangle| &\le C\int_\Omega\big|[\nabla v_{\varepsilon,\lambda}-S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon]\nabla\psi\big|\,dx+C\Big|\int_\Omega(B^\lambda)^\varepsilon\eta_\varepsilon S_\varepsilon(\nabla v_{\varepsilon,\lambda})\nabla\psi\,dx\Big|\\
&\quad+C\lambda^2\varepsilon^3\int_\Omega\big|(\chi^\lambda)^\varepsilon\nabla^3[S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon]\nabla\psi\big|\,dx
\end{aligned}$$

$$
\begin{aligned}
&+ C\lambda^2\varepsilon^2 \int_\Omega \big|(\nabla\chi^\lambda)^\varepsilon \nabla^2[S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon]\nabla\psi\big|\,dx \\
&+ C\lambda^2\varepsilon \int_\Omega \big|(\nabla^2\chi^\lambda)^\varepsilon \nabla[S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon]\nabla\psi\big|\,dx \\
&+ C\varepsilon \int_\Omega \big|(\chi^\lambda)^\varepsilon \nabla[S_\varepsilon(\nabla v_{\varepsilon,\lambda})\eta_\varepsilon]\nabla\psi\big|\,dx \\
&\doteq \mathcal{I}_1 + \cdots + \mathcal{I}_6. \qquad (5.9)
\end{aligned}
$$

It is not hard to see that

$$
\begin{aligned}
\mathcal{I}_1 &\le C\|\nabla v_{\varepsilon,\lambda} - S_\varepsilon(\nabla v_{\varepsilon,\lambda})\|_{L^2(\Omega\setminus\Omega_{3\varepsilon})}\|\nabla\psi\|_{L^2(\Omega)} + C\|\nabla v_{\varepsilon,\lambda}\|_{L^2(\Omega_{4\varepsilon})}\|\nabla\psi\|_{L^2(\Omega_{4\varepsilon})} \\
&\le C\big\{\|\nabla v_{\varepsilon,\lambda}\|_{L^2(\Omega_{4\varepsilon})} + C\varepsilon\|\nabla^2 v_{\varepsilon,\lambda}\|_{L^2(\Omega\setminus\Omega_{2\varepsilon})}\big\}\|\nabla\psi\|_{L^2(\Omega)}.
\end{aligned}
$$

To handle $\mathcal{I}_2$, we use the matrix of flux correctors, as in the proof of Lemma 4.2, to obtain

$$
\begin{aligned}
\mathcal{I}_2 &= C\Big|\int_\Omega \varepsilon\partial_{x_k}\big(\mathfrak{B}^\lambda_{kij}(x/\varepsilon)\partial_{x_i}\psi\big)S_\varepsilon(\partial_{x_j}v_{\varepsilon,\lambda})\eta_\varepsilon dx\Big| \\
&\le C\varepsilon\int_\Omega |\eta_\varepsilon\mathfrak{B}^\lambda(x/\varepsilon)S_\varepsilon(\nabla^2 v_{\varepsilon,\lambda})\nabla\psi|\,dx + C\varepsilon\int_\Omega |\mathfrak{B}^\lambda(x/\varepsilon)S_\varepsilon(\nabla v_{\varepsilon,\lambda})\nabla\eta_\varepsilon\nabla\psi|\,dx \\
&\le C\varepsilon\|\nabla\psi\|_{L^2(\Omega)}\|\nabla^2 v_{\varepsilon,\lambda}\|_{L^2(\Omega\setminus\Omega_{2\varepsilon})} + C\|\nabla\psi\|_{L^2(\Omega_{4\varepsilon})}\|\nabla v_{\varepsilon,\lambda}\|_{L^2(\Omega_{5\varepsilon})},
\end{aligned}
$$

where, for the last step, we have used (4.6).

To bound $\mathcal{I}_3$, we use the Cauchy inequality, (3.3) and (4.6) to deduce that

$$
\begin{aligned}
\mathcal{I}_3 &\le C\lambda^2\varepsilon^3\|(\chi^\lambda)^\varepsilon\nabla^3 S_\varepsilon(\nabla v_{\varepsilon,\lambda})\|_{L^2(\Omega\setminus\Omega_{3\varepsilon})}\|\nabla\psi\|_{L^2(\Omega)} \\
&\quad + C\lambda^2\Big\{\|S_\varepsilon(\nabla v_{\varepsilon,\lambda})(\chi^\lambda)^\varepsilon\|_{L^2(\Omega_{4\varepsilon})} + \varepsilon\|S_\varepsilon(\nabla^2 v_{\varepsilon,\lambda})(\chi^\lambda)^\varepsilon\|_{L^2(\Omega_{4\varepsilon})} \\
&\qquad\qquad + \varepsilon^2\|\nabla S_\varepsilon(\nabla^2 v_{\varepsilon,\lambda})(\chi^\lambda)^\varepsilon\|_{L^2(\Omega_{4\varepsilon})}\Big\}\|\nabla\psi\|_{L^2(\Omega_{4\varepsilon})} \\
&\le C\varepsilon\|\nabla^2 v_{\lambda,\varepsilon}\|_{L^2(\Omega\setminus\Omega_{2\varepsilon})}\|\nabla\psi\|_{L^2(\Omega)} + C\|\nabla v_{\varepsilon,\lambda}\|_{L^2(\Omega_{5\varepsilon})}\|\nabla\psi\|_{L^2(\Omega_{4\varepsilon})}.
\end{aligned}
$$

Likewise,

$$
\mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6 \le C\|\nabla v_{\varepsilon,\lambda}\|_{L^2(\Omega_{5\varepsilon})}\|\nabla\psi\|_{L^2(\Omega)} + C\varepsilon\|\nabla^2 v_{\varepsilon,\lambda}\|_{L^2(\Omega\setminus\Omega_{2\varepsilon})}\|\nabla\psi\|_{L^2(\Omega)}.
$$

By taking the estimates on $\mathcal{I}_1,\ldots,\mathcal{I}_6$ into (5.9), it yields

$$
|\langle\mathcal{L}^\lambda_\varepsilon(w_{\varepsilon,\lambda}),\psi\rangle| \le C\|\nabla v_{\varepsilon,\lambda}\|_{L^2(\Omega_{5\varepsilon})}\|\nabla\psi\|_{L^2(\Omega)} + C\varepsilon\|\nabla^2 v_{\varepsilon,\lambda}\|_{L^2(\Omega\setminus\Omega_{2\varepsilon})}\|\nabla\psi\|_{L^2(\Omega)},
$$

which gives (5.7) by choosing $\psi = w_{\varepsilon,\lambda} \in H^2_0(\Omega;\mathbb{R}^d)$ and using the Cauchy inequality. □

Now we are prepared to prove Theorem 5.1.

Proof of Theorem 5.1. By dilation and translation, it suffices to consider the case where $r = 1$ and $z = 0$. Let $u_{\varepsilon,\lambda}$ be a solution of $\mathcal{L}^\lambda_\varepsilon(u_{\varepsilon,\lambda}) = F$ in B_2, and $v_{\varepsilon,\lambda}$ the solution to the Dirichlet problem,

$$
\lambda^2\varepsilon^2\Delta^2 v_{\varepsilon,\lambda} + \mathcal{L}^\lambda_0(v_{\varepsilon,\lambda}) = F \quad \text{in } B_{3/2} \quad \text{and} \quad v_{\varepsilon,\lambda} - u_{\varepsilon,\lambda} \in H^2_0(B_{3/2};\mathbb{R}^d).
$$

Let $w_{\varepsilon,\lambda}$ be defined by (5.6). We apply Lemma 5.2 with $\Omega = B_{3/2}$ to obtain

$$\|\nabla w_{\varepsilon,\lambda}\|_{L^2(B_{3/2})} \le C\|\nabla v_{\varepsilon,\lambda}\|_{L^2(B_{3/2}\setminus B_{(3/2)-5\varepsilon})} + C\varepsilon\|\nabla^2 v_{\varepsilon,\lambda}\|_{L^2(B_{(3/2)-2\varepsilon})}. \tag{5.10}$$

Since $\widehat{A^\lambda}$ is constant, we may apply (2.5) to the function $\nabla v_{\varepsilon,\lambda}$. This gives

$$\int_B |\nabla^2 v_{\varepsilon,\lambda}|^2\,dx \le \frac{C}{r^2}\int_{2B} |\nabla v_{\varepsilon,\lambda}|^2\,dx + C\int_{2B}|F|^2\,dx,$$

for any $2B = B(x_0, 2r) \subset B_2$. It follows that

$$\begin{aligned}\int_{B_{(3/2)-2\varepsilon}} |\nabla^2 v_{\varepsilon,\lambda}|^2\,dx &\le C\int_{B_{(3/2)-\varepsilon}} \frac{|\nabla v_{\varepsilon,\lambda}(x)|^2}{[\delta(x)]^2}\,dx + C\int_{B_{(3/2)-\varepsilon}} |F|^2\,dx\\ &\le C_q\varepsilon^{-1-\frac{2}{q}}\left(\int_{B_{3/2}} |\nabla v_{\varepsilon,\lambda}|^q\,dx\right)^{2/q} + C\int_{B_2}|F|^2\,dx,\end{aligned}$$

where $\delta(x) = \text{dist}(x, \partial B_{3/2})$, $q > 2$ and we have used Hölder's inequality for the last step. In view of (5.10) we deduce that for any $q > 2$,

$$\|\nabla w_{\varepsilon,\lambda}\|_{L^2(B_{3/2})} \le C\varepsilon^{\frac{1}{2}-\frac{1}{q}}\|\nabla v_{\varepsilon,\lambda}\|_{L^q(B_{3/2})} + C\varepsilon\|F\|_{L^2(B_2)}. \tag{5.11}$$

Next, we observe that $u_{\varepsilon,\lambda} - v_{\varepsilon,\lambda} \in H_0^2(B_{3/2})$ and

$$\lambda^2\varepsilon^2\Delta^2(u_{\varepsilon,\lambda} - v_{\varepsilon,\lambda}) - \text{div}\big(\widehat{A^\lambda}\nabla(u_{\varepsilon,\lambda} - v_{\varepsilon,\lambda})\big) = \text{div}\big((A(x/\varepsilon) - \widehat{A^\lambda})\nabla u_{\varepsilon,\lambda}\big) \tag{5.12}$$

in $B_{3/2}$. By energy estimates this gives (5.2) with $r = 1$. It follows by Theorem 2.5 that there exist some $q > 2$ and $C > 0$, depending only on d, ν_1 and ν_2, such that

$$\int_{B_{3/2}} |\nabla(u_{\varepsilon,\lambda} - v_{\varepsilon,\lambda})|^q\,dx \le C\int_{B_{3/2}} |\nabla u_{\varepsilon,\lambda}|^q\,dx.$$

As a result, there exists some $q > 2$ such that

$$\|\nabla w_{\varepsilon,\lambda}\|_{L^2(B_{3/2})} \le C\varepsilon^{\frac{1}{2}-\frac{1}{q}}\|\nabla u_{\varepsilon,\lambda}\|_{L^q(B_2)} + C\varepsilon\|F\|_{L^2(B_2)}. \tag{5.13}$$

Note that for $x \in B_1$,

$$\nabla w_{\varepsilon,\lambda} = \nabla u_{\varepsilon,\lambda} - \nabla v_{\varepsilon,\lambda} - (\nabla\chi^\lambda)^\varepsilon S_\varepsilon(\nabla v_{\varepsilon,\lambda}) - \varepsilon(\chi^\lambda)^\varepsilon S_\varepsilon(\nabla^2 v_{\varepsilon,\lambda}).$$

It follows from (5.13) that

$$\begin{aligned}&\|\nabla u_{\varepsilon,\lambda} - \nabla v_{\varepsilon,\lambda} - (\nabla\chi^\lambda)^\varepsilon\nabla v_{\varepsilon,\lambda}\|_{L^2(B_1)}\\ &\le C\varepsilon^{\frac{1}{2}-\frac{1}{q}}\|\nabla u_{\varepsilon,\lambda}\|_{L^q(B_2)} + C\varepsilon\|F\|_{L^2(B_2)}\\ &\quad + \|(\nabla\chi^\lambda)^\varepsilon\big(\nabla v_{\varepsilon,\lambda} - S_\varepsilon(\nabla v_{\varepsilon,\lambda})\big)\|_{L^2(B_1)} + \varepsilon\|(\chi^\lambda)^\varepsilon S_\varepsilon(\nabla^2 v_{\varepsilon,\lambda})\|_{L^2(B_1)}.\end{aligned} \tag{5.14}$$

By (4.6), the last term in the right-hand side of (5.14) is bounded by

$$C\varepsilon\|\nabla^2 v_{\varepsilon,\lambda}\|_{L^2(B_{5/4})} \le C\varepsilon^{\frac{1}{2}-\frac{1}{q}}\|\nabla u_{\varepsilon,\lambda}\|_{L^q(B_2)} + C\varepsilon\|F\|_{L^2(B_2)}.$$

To handle the third term in the right-hand side of (5.14), we use the $C^{1,\sigma}$ estimate for the operator $\lambda^2\varepsilon^2\Delta^2 - \text{div}(\widehat{A^\lambda}\nabla)$ to obtain

$$\|\nabla v_{\varepsilon,\lambda}\|_{C^{0,\sigma}(B_{5/4})} \le C\|\nabla v_{\varepsilon,\lambda}\|_{L^2(B_{3/2})} + C\|F\|_{L^p(B_{3/2})}, \tag{5.15}$$

where $0<\sigma<1-\frac{d}{p}$. It follows that

$$\begin{aligned}
\|(\nabla\chi^\lambda)^\varepsilon\big(\nabla v_{\varepsilon,\lambda} - S_\varepsilon(\nabla v_{\varepsilon,\lambda})\big)\|_{L^2(B_1)} &\le C\|(\nabla\chi^\lambda)^\varepsilon\|_{L^2(B_1)}\|\nabla v_{\varepsilon,\lambda} - S_\varepsilon(\nabla v_{\varepsilon,\lambda})\|_{L^\infty(B_1)}\\
&\le C\varepsilon^\sigma\|\nabla v_{\varepsilon,\lambda}\|_{C^{0,\sigma}(B_{5/4})}\\
&\le C\varepsilon^\sigma\big\{\|\nabla v_{\varepsilon,\lambda}\|_{L^2(B_{3/2})} + \|F\|_{L^p(B_2)}\big\}\\
&\le C\varepsilon^\sigma\big\{\|\nabla u_{\varepsilon,\lambda}\|_{L^2(B_{3/2})} + \|F\|_{L^p(B_2)}\big\}.
\end{aligned} \tag{5.16}$$

In summary, we have proved that if $0<\sigma<\min(\frac{1}{2}-\frac{1}{q}, 1-\frac{d}{p})$, then

$$\|\nabla u_{\varepsilon,\lambda} - \nabla v_{\varepsilon,\lambda} - (\nabla\chi^\lambda)^\varepsilon\nabla v_{\varepsilon,\lambda}\|_{L^2(B_1)} \le C\varepsilon^\sigma\big\{\|\nabla u_{\varepsilon,\lambda}\|_{L^q(B_{3/2})} + \|F\|_{L^p(B_2)}\big\}, \tag{5.17}$$

where $2<q<\bar{q}$ and $\bar{q}$ depends only on d, ν_1 and ν_2.

Finally, we use the reverse Hölder estimate (2.15) to obtain

$$\|\nabla u_{\varepsilon,\lambda}\|_{L^q(B_{3/2})} \le C\big\{\|\nabla u_{\varepsilon,\lambda}\|_{L^2(B_2)} + \|F\|_{L^2(B_2)}\big\}, \tag{5.18}$$

where $q>2$ and C depends only on d, ν_1 and ν_2. This, together with (5.17), gives (5.3) with $r=1$. □

6 Large-scale $C^{1,\alpha}$ estimates

Recall that $P_j^\beta(x) = x_j(0,\dots,1,\dots,0)$ with 1 in the β^{th} position. Let

$$\begin{aligned}
\mathcal{H}_{1,\varepsilon}^\lambda = \Big\{ h(x):\ & h(x) = b + E_j^\beta(P_j^\beta(x) + \varepsilon\chi_j^{\lambda,\beta}(x/\varepsilon))\\
&\text{for some } b\in\mathbb{R}^d \text{ and } E=(E_j^\beta)\in\mathbb{R}^{d\times d}\Big\}.
\end{aligned} \tag{6.1}$$

Theorem 6.1. *Assume that A satisfies (1.3) and (1.4). Let $u_{\varepsilon,\lambda}\in H^1(B_R;\mathbb{R}^d)$ be a solution of $\mathcal{L}_\varepsilon^\lambda(u_{\varepsilon,\lambda}) = F$ in $B_R = B(x_0,R)$, where $R>\varepsilon$ and $F\in L^p(B_R;\mathbb{R}^d)$ for some $p>d$. Then for any $\varepsilon\le r<R$ and $0<\alpha<1-\frac{d}{p}$,*

$$\inf_{h\in\mathcal{H}_{1,\varepsilon}^\lambda}\left(\fint_{B_r}|\nabla u_{\varepsilon,\lambda}-\nabla h|^2\right)^{1/2} \le C\left(\frac{r}{R}\right)^\alpha\left\{\left(\fint_{B_R}|\nabla u_{\varepsilon,\lambda}|^2\right)^{1/2} + R\left(\fint_{B_R}|F|^p\right)^{1/p}\right\}, \tag{6.2}$$

where C depends only on d, ν_1, ν_2, p, and α.

Proof. By translation and dilation, we may assume that $x_0 = 0$ and $R = 2$. We also assume that $\varepsilon < r < (1/8)$, as the estimate (6.2) is trivial for $r \ge (1/8)$. Let $v_{\varepsilon,\lambda}$ be the weak solution of $\varepsilon^2\lambda^2\Delta^2 v_{\varepsilon,\lambda} + \mathcal{L}_0^\lambda(v_{\varepsilon,\lambda}) = F$ given by Theorem 5.1. Let $\varepsilon < tr < r < 1$, where $0 < t < (1/4)$ is to be determined, and

$$\overline{h} = \nabla v_{\varepsilon,\lambda}(0)(P(x) + \varepsilon\chi^\lambda(x/\varepsilon)),$$

where $P = (P_j^\beta(x))$. We obtain

$$\begin{aligned}
&\left(\fint_{B_{tr}} |\nabla u_{\varepsilon,\lambda} - \nabla\overline{h}|^2\right)^{1/2} + tr\left(\fint_{B_{tr}} |F|^p\right)^{1/p} \\
&\le \left(\fint_{B_{tr}} |\nabla u_{\varepsilon,\lambda} - \nabla v_{\varepsilon,\lambda} - (\nabla\chi^\lambda)^\varepsilon \nabla v_{\varepsilon,\lambda}|^2\right)^{1/2} \\
&\quad + \left(\fint_{B_{tr}} |\nabla v_{\varepsilon,\lambda} + (\nabla\chi^\lambda)^\varepsilon \nabla v_{\varepsilon,\lambda} - \nabla\overline{h}|^2\right)^{1/2} + Ct^{1-d/p} r\left(\fint_{B_{2r}} |F|^p\right)^{1/p}. \qquad (6.3)
\end{aligned}$$

Denote the first two terms in the right-hand side of (6.3) by $(6.3)_1, (6.3)_2$. Thanks to Theorem 5.1,

$$\begin{aligned}
(6.3)_1 &\le Ct^{-d/2}\left(\fint_{B_r} |\nabla u_{\varepsilon,\lambda} - \nabla v_{\varepsilon,\lambda} - (\nabla\chi^\lambda)^\varepsilon \nabla v_{\varepsilon,\lambda}|^2\right)^{1/2} \\
&\le Ct^{-d/2}\left(\frac{\varepsilon}{r}\right)^\sigma \left\{\left(\fint_{B_{2r}} |\nabla u_{\varepsilon,\lambda}|^2\right)^{1/2} + r\left(\fint_{B_{2r}} |F|^p\right)^{1/p}\right\}. \qquad (6.4)
\end{aligned}$$

On the other hand, by the $C^{1,\alpha}$ estimate of $v_{\varepsilon,\lambda}$,

$$\begin{aligned}
(6.3)_2 &\le \left(\fint_{B_{tr}} |\nabla v_{\varepsilon,\lambda} - \nabla v_{\varepsilon,\lambda}(0)|^2\right)^{1/2} + \left(\fint_{B_{tr}} |(\nabla\chi^\lambda)^\varepsilon [\nabla v_{\varepsilon,\lambda} - \nabla v_{\varepsilon,\lambda}(0)]|^2\right)^{1/2} \\
&\le C(tr)^\gamma \|\nabla v_{\varepsilon,\lambda}\|_{C^{0,\gamma}(B_{tr})} \\
&\le Ct^\gamma \left\{\left(\fint_{B_r} |\nabla v_{\varepsilon,\lambda}|^2\right)^{1/2} + r\left(\fint_{B_r} |F|^p\right)^{1/p}\right\} \\
&\le Ct^\gamma \left\{\left(\fint_{B_{2r}} |\nabla u_{\varepsilon,\lambda}|^2\right)^{1/2} + r\left(\fint_{B_{2r}} |F|^p\right)^{1/p}\right\}, \qquad (6.5)
\end{aligned}$$

where $0 < \gamma < 1 - \frac{d}{p}$ and we have used (5.2) for the last inequality.

Taking (6.4) and (6.5) into (6.3) and using the fact $\mathcal{L}_\varepsilon^\lambda(h) = 0$ for any $h \in \mathcal{H}_{1,\varepsilon}^\lambda$, we derive that

$$\begin{aligned}
&\inf_{h\in\mathcal{H}_{1,\varepsilon}^\lambda} \left\{\frac{1}{(rt)^\alpha}\left(\fint_{B_{tr}} |\nabla u_{\varepsilon,\lambda} - \nabla h|^2\right)^{1/2} + tr\left(\fint_{B_{tr}} |F|^p\right)^{1/p}\right\} \\
&\le C \inf_{h\in\mathcal{H}_{1,\varepsilon}^\lambda} \left\{t^{-d/2-\alpha}\left(\frac{\varepsilon}{r}\right)^\sigma + t^{\gamma-\alpha}\right\} \\
&\quad \times \frac{1}{(2r)^\alpha}\left\{\left(\fint_{B_{2r}} |\nabla u_{\varepsilon,\lambda} - \nabla h|^2\right)^{1/2} + r\left(\fint_{B_{2r}} |F|^2\right)^{1/2}\right\}.
\end{aligned}$$

For any $0 < \alpha < 1 - \frac{d}{p}$, we first choose $\gamma \in (\alpha, 1 - \frac{d}{p})$ and then $t > 0$ so small that $Ct^{\gamma - \alpha} \le 1/4$. As a result, if $r \ge N_0 \varepsilon$, where $N_0 > 1$ is so large that

$$Ct^{-d/2-\alpha} \left(\frac{\varepsilon}{r}\right)^{\sigma} \le 1/4,$$

then

$$\begin{aligned}
&\inf_{h \in \mathcal{H}^{\lambda}_{1,\varepsilon}} \left\{ \frac{1}{(tr)^{\alpha}} \left(\fint_{B_{tr}} |\nabla u_{\varepsilon,\lambda} - \nabla h|^2 \right)^{1/2} + tr \left(\fint_{B_{tr}} |F|^p \right)^{1/p} \right\} \\
&\le \frac{1}{2} \inf_{h \in \mathcal{H}^{\lambda}_{1,\varepsilon}} \left\{ \frac{1}{(2r)^{\alpha}} \left(\fint_{B_{2r}} |\nabla u_{\varepsilon,\lambda} - \nabla h|^2 \right)^{1/2} + r \left(\fint_{B_{2r}} |F|^p \right)^{1/p} \right\}.
\end{aligned}$$

By iteration, this implies that

$$\begin{aligned}
&\inf_{h \in \mathcal{H}^{\lambda}_{1,\varepsilon}} \left\{ \frac{1}{(tr)^{\alpha}} \left(\fint_{B_{tr}} |\nabla u_{\varepsilon,\lambda} - \nabla h|^2 \right)^{1/2} + tr \left(\fint_{B_{tr}} |F|^p \right)^{1/p} \right\} \\
&\le \inf_{h \in \mathcal{H}^{\lambda}_{1,\varepsilon}} \left\{ \left(\fint_{B_2} |\nabla u_{\varepsilon,\lambda} - \nabla h|^2 \right)^{1/2} + \left(\fint_{B_2} |F|^p \right)^{1/p} \right\}
\end{aligned} \tag{6.6}$$

for any $r \ge N_0 \varepsilon$. The case $\varepsilon \le r < N_0 \varepsilon$ follows easily from the case $r = N_0 \varepsilon$. □

As a corollary, we obtain a Liouville theorem for the operator $\mathcal{L}^{\lambda}$.

Theorem 6.2. *Suppose A satisfies conditions (1.3) and (1.4). Let $u \in H^2_{loc}(\mathbb{R}^d; \mathbb{R}^d)$ be a weak solution of*

$$\lambda^2 \Delta^2 u - \operatorname{div}(A \nabla u) = 0 \quad \text{in } \mathbb{R}^d.$$

Suppose that there exist $C > 0$ and $\sigma \in (0, 1)$ such that

$$\left(\fint_{B(0,R)} |u|^2 \right)^{1/2} \le C R^{1+\sigma}$$

for all $R > 1$. Then there exist $b \in \mathbb{R}^d$ and $E = (E_j^{\beta}) \in \mathbb{R}^{d \times d}$ such that

$$u(x) = b + E_j^{\beta} (P_j^{\beta} + \chi_j^{\lambda, \beta}(x)) \quad \text{in } \mathbb{R}^d.$$

Proof. This follows readily from Theorem 6.1 with $\varepsilon = 1$ and $F = 0$. □

7 Proof of Theorems 1.2 and 1.3

Theorem 7.1. *Assume that A satisfies (1.3) and (1.4). Let $u_{\varepsilon,\lambda} \in H^2(B_R; \mathbb{R}^d)$ be a solution of $\mathcal{L}^{\lambda}_{\varepsilon}(u_{\varepsilon,\lambda}) = F$ in B_R, where $F \in L^p(B_R; \mathbb{R}^d)$ for some $p > d$. Then for any $\varepsilon \le r < R$,*

$$\left(\fint_{B_r} |\nabla u_{\varepsilon,\lambda}|^2 \right)^{1/2} \le C \left\{ \left(\fint_{B_R} |\nabla u_{\varepsilon,\lambda}|^2 \right)^{1/2} + R \left(\fint_{B_R} |F|^p \right)^{1/p} \right\}, \tag{7.1}$$

where C depends only on d, ν_1, ν_2, and p.

Proof. This follows from Theorem 6.1, as in the case of second-order elliptic equations [12, 6]. We omit the details. □

Proof of Theorem 1.2. Since $\mathcal{L}_\varepsilon = \mathcal{L}_\varepsilon^\lambda$ with $\lambda = \kappa\varepsilon^{-1}$, Theorem 1.2 follows directly from Theorem 7.1. □

Proof of Theorem 1.3. By translation and dilation we may assume $r = 1$ and $x_0 = 0$. If $\varepsilon \ge (1/2)$, the Hölder norm of $A^\varepsilon = A(x/\varepsilon)$ is uniformly bounded. The Lipschitz estimate (1.15) follows directly from the $C^{1,\alpha}$ estimate in Theorem 2.7. Consider the case $0 < \varepsilon < (1/2)$. Let $u_\varepsilon \in H^2(B_1; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B_1 = B(0,1)$, where $F \in L^p(B_1; \mathbb{R}^d)$ for some $p > d$. Let $v(x) = \varepsilon u_\varepsilon(\varepsilon x)$. Then

$$(\kappa\varepsilon^{-1})^2 \Delta^2 v - \operatorname{div}(A\nabla v) = F_\varepsilon,$$

where $F_\varepsilon(x) = \varepsilon F(\varepsilon x)$. By Theorem 2.7,

$$\begin{aligned}
|\nabla u_\varepsilon(0)| = |\nabla v(0)| &\le C \left\{ \left(\fint_{B_1} |\nabla v|^2 \right)^{1/2} + \left(\fint_{B_1} |F_\varepsilon|^p \right)^{1/p} \right\} \\
&= C \left\{ \left(\fint_{B_\varepsilon} |\nabla u_\varepsilon|^2 \right)^{1/2} + \varepsilon \left(\fint_{B_\varepsilon} |F|^p \right)^{1/p} \right\} \\
&\le C \left\{ \left(\fint_{B_1} |\nabla u_\varepsilon|^2 \right)^{1/2} + \left(\fint_{B_1} |F|^p \right)^{1/p} \right\},
\end{aligned}$$

where we have used (1.13) with $R = 1$ for the last inequality. □

References

[1] S. N. Armstrong and J. C. Mourrat, *Lipschitz regularity for elliptic equations with random coefficients*, Arch. Ration. Mech. Anal. **219** (2016), no. 1, 255–348.

[2] S. N. Armstrong and C. K. Smart, *Quantitative stochastic homogenization of convex integral functionals*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 2, 423–481.

[3] M. Avellaneda and F. Lin, *Compactness methods in the theory of homogenization*, Comm. Pure Appl. Math. **40** (1987), no. 6, 803–847.

[4] A. Barton, *Gradient estimates and the fundamental solution for higher-order elliptic systems with rough coefficients*, Manuscripta Math. **151** (2016), no. 3-4, 375–418.

[5] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, AMS Chelsea Publishing, Providence, RI, 2011, Corrected reprint of the 1978 original.

[6] J. Fischer and F. Otto, *A higher-order large-scale regularity theory for random elliptic operators*, Comm. Partial Differential Equations **41** (2016), no. 7, 1108–1148.

[7] I. Fonseca, G. Francfort, and G. Leoni, *Thin elastic films: the impact of higher order perturbations*, Quart. Appl. Math. **65** (2007), no. 1, 69–98.

[8] G. A. Francfort and S. Müller, *Combined effects of homogenization and singular perturbations in elasticity*, J. Reine Angew. Math. **454** (1994), 1–35.

[9] A. Friedman, *Singular perturbations for partial differential equations*, Arch. Rational Mech. Anal. **29** (1968), 289–303.

[10] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Mathematics Studies, vol. 105, Princeton University Press, Princeton, NJ, 1983.

[11] A. Gloria, S. Neukamm, and F. Otto, *Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics*, Invent. Math. **199** (2015), no. 2, 455–515.

[12] ———, *A regularity theory for random elliptic operators*, Milan J. Math. **88** (2020), no. 1, 99–170.

[13] A. Gloria and F. Otto, *Quantitative results on the corrector equation in stochastic homogenization*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 11, 3489–3548.

[14] C. E. Kenig, F. Lin, and Z. Shen, *Homogenization of elliptic systems with Neumann boundary conditions*, J. Amer. Math. Soc. **26** (2013), no. 4, 901–937.

[15] W. Niu and Y. Yuan, *Convergence rate in homogenization of elliptic systems with singular perturbations*, J. Math. Phys. **60** (2019), no. 11, 111509, 7.

[16] Z. Schuss, *Singular perturbations and the transition from thin plate to membrane*, Proc. Amer. Math. Soc. **58** (1976), 139–147.

[17] Z. Shen, *Boundary estimates in elliptic homogenization*, Anal. PDE **10** (2017), no. 3, 653–694.

[18] ———, *Periodic homogenization of elliptic systems*, Operator Theory: Advances and Applications, vol. 269, Birkhäuser/Springer, Cham, 2018, Advances in Partial Differential Equations (Basel).

[19] Z. Shen and J. Zhuge, *Convergence rates in periodic homogenization of systems of elasticity*, Proc. Amer. Math. Soc. **145** (2017), no. 3, 1187–1202.

[20] Y. C. Shu, *Heterogeneous thin films of martensitic materials*, Arch. Ration. Mech. Anal. **153** (2000), no. 1, 39–90.

[21] T. A. Suslina, *Homogenization of the Dirichlet problem for elliptic systems: L^2-operator error estimates*, Mathematika **59** (2013), no. 2, 463–476.

[22] C. I. Zeppieri, *Stochastic homogenisation of singularly perturbed integral functionals*, Annali di Matematica Pura ed Applicata (1923-) **195** (2016), no. 6, 2183–2208.

[23] V. V. Zhikov, *G-convergence of elliptic operators*, Mat. Zametki **33** (1983), no. 3, 345–356.

[24] V. V. Zhikov and E. V. Krivenko, *Averaging of singularly perturbed elliptic operators*, Mat. Zametki **33** (1983), no. 4, 571–582.

Weisheng Niu
School of Mathematical Science, Anhui University, Hefei, 230601, CHINA.
E-mail:niuwsh@ahu.edu.cn

Zhongwei Shen
Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506, USA.
E-mail: zshen2@uky.edu

May 6, 2021