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AUSLANDER'S THEOREM FOR GROUP COACTIONS ON NOETHERIAN GRADED DOWN-UP ALGEBRAS

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Abstract. We prove a version of a theorem of Auslander for finite group coactions on noetherian graded down-up algebras.

0. Introduction

Maurice Auslander [3] proved that if G is a finite subgroup of $\mathrm{GL}_n(\Bbbk)$, containing no pseudo-reflections (e.g., subgroups of $\mathrm{SL}_n(\Bbbk)$), acting linearly on the commutative polynomial ring $A = \Bbbk[x_1, \ldots, x_n]$, with fixed subring A^G, then the natural map from the skew group algebra $A * G$ to $\mathrm{End}_{A^G}(A)$ is an isomorphism of graded algebras. This theorem is the main ingredient in the McKay correspondence, relating representations of G and A^G-modules. Noncommutative versions of this theorem of Auslander [4], [5] are an important ingredient in establishing a noncommutative McKay correspondence. One of the main open questions concerning a noncommutative version of Auslander's Theorem is the following conjecture that was stated in [4, Conj. 0.4] and [9, Conj. 0.2], where the condition that the homological determinant of the H-action is trivial generalizes the result for group actions by subgroups of $\mathrm{SL}_n(\Bbbk)$:

Let A be a connected graded noetherian Artin–Schelter regular algebra [1] *and H be a semisimple Hopf algebra acting on A inner-faithfully and homogeneously. If*

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the homological determinant of the H-action on A is trivial, then there is a natural graded algebra isomorphism

$$A\#H \cong \mathrm{End}_{A^H}(A).$$

By [9, Thm. 0.3] the above conjecture holds when A has global dimension two, which is one of the main results in [9]. It is natural to search for a proof of this conjecture for global dimension three (or higher). The paper [5] started this program by showing that the above conjecture holds for certain finite group actions on noetherian graded down-up algebras, which are Artin–Schelter regular algebras of global dimension three [5, Thm. 0.6]. Some interesting partial results concerning Auslander's Theorem have been proven in [4], [5], [12], [13], [21]. The goal of this paper is to verify the conjecture for finite group coactions on Artin–Schelter regular down-up algebras (Theorem 0.1). The idea of the proof is to use the pertinency introduced in [4] that has been one major tool for proving the noncommutative Auslander's Theorem.

Throughout the paper, let $\Bbbk$ be a base field of characteristic zero, and all objects are over $\Bbbk$.

Down-up algebras were introduced in 1998 by Benkart–Roby in [6], and, since then, these algebras have been studied extensively. Noetherian graded down-up algebras are Artin–Schelter regular algebras of global dimension three with two generators by a result of [20]. Let α and β be two scalars in $\Bbbk$. The graded down-up algebra, denoted by $\mathbb{D}(\alpha,\beta)$, is generated by two elements d and u and subject to two relations

$$d^2u = \alpha dud + \beta ud^2, \tag{E0.0.1}$$
$$du^2 = \alpha udu + \beta u^2d. \tag{E0.0.2}$$

This algebra is noetherian if and only if $\beta \neq 0$, and in this paper we always assume that $\beta \neq 0$. When $\alpha = 0$, we use $\mathbb{D}_\beta$ instead of $\mathbb{D}(0,\beta)$. The groups of graded algebra automorphisms of the down-up algebras were computed in [15]. Recently, the invariant theory of graded down-up algebras under finite group actions and coactions has been studied in [17], [11], [13].

In a general setting, let H be a semisimple Hopf algebra and let K be its $\Bbbk$-linear dual. Then K is also a semisimple Hopf algebra. It is well known that a left H-action on an algebra A is equivalent to a right K-coaction on A.

Suppose H is a semisimple Hopf algebra with integral $\textstyle\int$, and A is an algebra with $\mathrm{GKdim}\, A < \infty$. Here $\mathrm{GKdim}\, A$ denotes the Gelfand–Kirillov dimension of A. If H acts on A, by [4, Def. 0.1], the *pertinency* of the H-action on A is defined to be

$$\mathsf{p}(A,H) = \mathrm{GKdim}\, A - \mathrm{GKdim}((A\#H)/I) \tag{E0.0.3}$$

where I is the 2-sided ideal of $A\#H$ generated by $1\#\textstyle\int$. Define the fixed subring of the H-action to be

$$A^H = \{a \in A \mid h\cdot(a) = \epsilon(h)a, \forall h \in H\}$$

where ϵ is the counit of H. For any algebra A with H-action, there is a natural algebra homomorphism $\phi : A\#H \to \mathrm{End}_{A^H}(A)$ which sends $a\#h$ to an A^H-endomorphism of A:

$$\phi(a\#h) : x \mapsto a(h \cdot (x)), \ \ \forall\, x \in A.$$

By [4, Thm. 0.3], if A is a noetherian, connected graded, Artin–Schelter regular and Cohen-Macaulay domain of GKdim ≥ 2, then $\mathsf{p}(A, H) \geq 2$ if and only if the canonical map

$$\phi : \quad A\#H \to \mathrm{End}_{A^H}(A) \tag{E0.0.4}$$

is an isomorphism. For simplicity, if ϕ is an isomorphism, we say that (A, H) has the *isom-property.*

In this paper we are interested in the case when H is $\Bbbk^G := \mathrm{Hom}_{\Bbbk}(\Bbbk G, \Bbbk)$, or equivalently, K is the group algebra $\Bbbk G$ for some finite group G, and when A is the noetherian graded down-up algebra $\mathbb{D}(\alpha, \beta)$. Our main result is

Theorem 0.1. *Let $H := \Bbbk^G$ act on $A := \mathbb{D}(\alpha, \beta)$ homogeneously and inner-faithfully, where $\beta \neq 0$. If the action has trivial homological determinant, then the pertinency $\mathsf{p}(A, H) \geq 2$. As a consequence, Auslander's Theorem holds, namely, there is a natural isomorphism of graded algebras*

$$\phi : \quad A\#H \cong \mathrm{End}_{A^H}(A).$$

Theorem 0.1 fails without the hypothesis of "trivial homological determinant", see Remark 1.6(2). Theorem 0.1 suggests there is a McKay correspondence for down-up algebras $\mathbb{D}(\alpha, \beta)$; it follows from [9, Thm. A] that when Auslander's Theorem holds, there are bijections between several categories of modules, e.g., simple left H-modules and indecomposable direct summands of A as a left A^H-modules. The paper [22] shows that whenever Auslander's Theorem holds one can view $A\#H$ as a generalized noncommutative crepant resolution (NCCR) of A^H, and when A^H is a central subalgebra of $A\#H$, $A\#H$ is an NCCR of A^H.

The paper is organized as follows: Section 1 contains some preliminary results, Section 2 contains the proof of Theorem 0.1, and Section 3 contains some examples.

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1. Preliminaries

In this section we recall some basic definitions and make some comments. We will omit the definition of Artin–Schelter Gorensteinness and Artin–Schelter regularity

[1] since these can be found in many other papers and we will not need these in the proof of the main result. As mentioned in the introduction, noetherian graded down-up algebras are Artin–Schelter regular of global dimension three.

We introduce a temporary concept. For a graded module C and an integer w, the wth shift of C, denoted by $C(w)$, is defined by $C(w)_m = C_{w+m}$ for all $m \in \mathbb{Z}$.

Definition 1.1. Let H be a semisimple Hopf algebra acting on a connected graded algebra A homogeneously and inner-faithfully. Decompose A into

$$\left(\bigoplus_{i=1}^{s} A^H(-w_i)\right) \oplus B \tag{E1.1.1}$$

as a right A^H-module for some integer $s \geq 1$, where B has no direct summand that is isomorphic to $A^H(w)$ for some integer w. If $B = 0$, H is called a *reflection Hopf algebra* with respect to A. If $s \geq 2$ (but $B \neq 0$), we say H is a *fractional-reflection Hopf algebra* with respect to A, since part (but not all) of A is a graded free A^H-module.

Lemma 1.2. *Suppose H acts on a connected graded algebra A as a reflection* (*or fractional-reflection*) *Hopf algebra. Then:*

(1) *(A, H) does not have the isom-property.*
(2) *If A is a noetherian Artin–Schelter Gorenstein algebra, then the H-action on A does not have trivial homological determinant.*

Proof. (1) Since H is a fractional-reflection Hopf algebra, $s \geq 2$ in (E1.1.1). We write $A = A^H \oplus A^H(-w_2) \oplus C$ where C is a right A^H-module. Note that w_2 is necessarily positive since A is connected graded. There is a homogeneous A^H-module map of degree $-w_2$:

$$A \xrightarrow{\mathrm{pr}_{A^H(-w_2)}} A^H(-w_2) \xrightarrow{\text{shift by degree } w_2} A^H \xrightarrow{\text{inclusion}} A.$$

Then $\mathrm{End}_{A^H}(A)$ has a nonzero element of negative degree. On the other hand, every nonzero homogeneous element in $A\#H$ has nonnegative degree. Therefore $A\#H \not\cong \mathrm{End}_{A^H}(A)$.

(2) We now assume that A is noetherian and Artin–Schelter Gorenstein. If the H-action on A has trivial homological determinant, then, by [16, Thm. 3.6] and the proof of [16, Lem. 3.5(d)], we have

(a) A^H is noetherian and Artin–Schelter Gorenstein,
(b) $\mathrm{injdim}\, A = \mathrm{injdim}\, A^H =: d$, and
(c) the AS indices of A and A^H are the same, denoted by ℓ.

Let $\mathfrak{m}$ be the graded maximal ideal of A^H. We consider the local cohomology $R^d\Gamma_{\mathfrak{m}}(A)^*$ as in [2], [16]. Since $A^H(-w_2)$ is a direct summand of A (as a right A^H-module), $R^d\Gamma_{\mathfrak{m}}(A^H(-w_2))^*$ is a direct summand of $R^d\Gamma_{\mathfrak{m}}(A)^*$. If both A and A^H are Artin–Schelter Gorenstein, by [23, Lem. 3.5],

$$R^d\Gamma_{\mathfrak{m}}(A)^* \cong A(-\ell) \quad \text{and} \quad R^d\Gamma_{\mathfrak{m}}(A^H(-w_2))^* \cong A^H(-\ell + w_2).$$

The lowest degree of nonzero element in $R^d\Gamma_{\mathfrak{m}}(A^H(-w_2))^*$ is $\ell - w_2$ and the lowest degree of nonzero element in $R^d\Gamma_{\mathfrak{m}}(A)^*$ is ℓ. Since w_2 is positive, this is impossible. Therefore the H-action on A does not have trivial homological determinant. □

Remark 1.3. Lemma 1.2(2) is a generalization of [8, Thm. 2.3].

The definition of maximal Cohen-Macaulay modules was extended to this context in [10, Def. 3.5].

Proposition 1.4. *Let A be connected graded and suppose that $(A, \Bbbk^G)$ has the isom-property. Write $A = \bigoplus_{g\in G} A_g$. If $g \neq h$, then A_g is not isomorphic to $A_h(w)$ for any $w \in \mathbb{Z}$. As a consequence, if A is noetherian Artin–Schelter Gorenstein, there are at least $|G|$ non-isomorphic graded maximal Cohen-Macaulay modules over $A^{\mathrm{co}\, G}$, up to degree shift.*

Proof. Let $B = A^{\mathrm{co}\, G}$. Suppose to the contrary that $A_g \cong A_h(w)$ for some $g \neq h$. If $w \neq 0$, then $\mathrm{End}_B(A)$ has an element of negative degree. So $A \# \Bbbk^G \ncong \mathrm{End}_B(A)$, a contradiction. If $w = 0$, then the degree zero part of $\mathrm{End}_B(A)$ contains a 2×2 matrix algebra which is not commutative. However the degree 0 part of $A \# \Bbbk^G$ is $\Bbbk^G$, which is commutative. Therefore $A \# \Bbbk^G \ncong \mathrm{End}_B(A)$, a contradiction. Therefore A_g is not isomorphic to $A_h(w)$ if $g \neq h$.

The consequence is clear. □

The homological (co)determinant is defined in [16]. We need some facts about the homological (co)determinant of group coactions on down-up algebras. Suppose that $\mathbb{D}_\beta$ is G-graded with $\deg_G d = g_1$ and $\deg_G u = g_2$ (or equivalently, G coacts on $\mathbb{D}_\beta$). Assume that the G-coaction on $\mathbb{D}_\beta$ is inner-faithful, which is equivalent to the condition that G is generated by g_1 and g_2, in this case.

Lemma 1.5. *Retain the above notation. The homological (co)determinant of the $\Bbbk^G$-action (or G-coaction) on $\mathbb{D}_\beta$ is $g_1^2 g_2^2$, and is trivial if and only if $g_1^2 g_2^2 = 1$, where 1 is the unit of G.*

Proof. Let $A = \mathbb{D}_\beta$. Since G coacts on A homogeneously, A is a $\mathbb{Z} \times G$-graded algebra. Recall that $\mathbb{D}_\beta$ is generated by d and u subject to relations

$$d^2 u = \beta u d^2, \quad du^2 = \beta u^2 d.$$

By using the generators and relations of A, one checks that the G-graded resolution of the trivial A-module $\Bbbk$ is

$$0 \to A(g_1^{-2} g_2^{-2}) \to A(g_1^{-1} g_2^{-2}) \oplus A(g_1^{-2} g_2^{-1}) \to A(g_1^{-1}) \oplus A(g_2^{-1}) \to A \to \Bbbk \to 0.$$

Using this resolution to compute the Ext-group, one sees that $\mathrm{Ext}^3_A(\Bbbk, \Bbbk) \cong \Bbbk(g_1^2 g_2^2)$ as a G-graded vector space. Hence the G-coaction maps a basis element $\mathfrak{e} \in \mathrm{Ext}^3_A(\Bbbk, \Bbbk)$ to $\mathfrak{e} \otimes g_1^2 g_2^2$. By definition, the homological codeterminant of the G-coaction is $g_1^2 g_2^2$. The assertion follows. □

Next we make some comments about [11, Example 2.1].

Remark 1.6. Consider the algebra $\mathbb{D} := \mathbb{D}_1$ as in [11, Example 2.1].

(1) By [5, Thm. 0.6], if $H = \Bbbk G$ for any finite group G acting on $\mathbb{D}$, then $\mathsf{p}(\mathbb{D}, G) \geq 2$ and $\mathbb{D} * G \cong \mathrm{End}_{\mathbb{D}^G}(\mathbb{D})$, so that Auslander's Theorem holds for group actions on $\mathbb{D}$; this result was expected because all finite groups acting on $\mathbb{D}$ are

"small", since they have no reflections, in a sense made precise in [17]. But, when $H = \Bbbk^G$ as in [11, Example 2.1 and Lemma 2.2], H is a fractional-reflection Hopf algebra with respect to $\mathbb{D}$, so by Lemma 1.2(1), $(\mathbb{D}, H)$ does not have the isom-property, namely, Auslander's Theorem fails. By [4, Thm. 0.3], $\mathsf{p}(\mathbb{D}, \Bbbk^G) \leq 1$, and by Lemma 1.2(2), the $\Bbbk^G$-action does not have trivial homological determinant. Hence group actions behave differently from group coactions.

(2) By [16, Cor. 4.11] or [17, Prop. 0.2(2)], if $H = \Bbbk G$ for some finite group G, then $\mathbb{D}^G$ is Gorenstein if and only if the G-action on $\mathbb{D}$ has trivial homological determinant. This result was expected because, again, these groups contain no reflections of $\mathbb{D}$. However, [11, Example 2.1] shows that when $H = \Bbbk^G$, this statement fails, namely, $\mathbb{D}^{\mathrm{co}\ G}$ is Gorenstein, but the $\Bbbk^G$-action does not have trivial homological determinant. This result is surprising, and there might be a relationship between the facts in parts (1) and (2).

(3) Theorem 0.1 implies that if the $\Bbbk^G$-action on $\mathbb{D}$ has trivial homological determinant, then $\mathsf{p}(\mathbb{D}, \Bbbk^G) \geq 2$ and $\mathbb{D} \# \Bbbk^G \cong \mathrm{End}_{\mathbb{D}^{\mathrm{co}\ G}}(\mathbb{D})$.

(4) In the commutative case, when a semisimple Hopf algebra acts on a polynomial ring $A := \Bbbk[x_1, \ldots, x_n]$, Auslander's Theorem fails if and only if there is a nontrivial Hopf subalgebra $H_0 \subseteq H$ (in this case, H and H_0 are group algebras $\Bbbk G$ and $\Bbbk G_0$ respectively for some $G_0 \subseteq G$) such that A^{H_0} is Artin–Schelter regular; this happens if and only if G contains a reflection, or equivalently, G is not small. Recall that a finite subgroup G of $\mathrm{GL}_n(\Bbbk)$ is *small* if it does not contain any reflections. Hence one might conjecture that Auslander's Theorem holds for a semisimple Hopf algebra if and only if there is no such Hopf subalgebra, and that this definition is the generalization for Hopf algebras of the notion of a "small group". However, [11, Example 2.1], where $H = \Bbbk^G$, shows that this definition of an analogue of a "small subgroup" does not work, since in this example Auslander's Theorem fails, but as one can easily check, or use [17, Prop. 0.2(2)], that there is NO nontrivial Hopf subalgebra $H_0 \subseteq H$ such that $\mathbb{D}^{H_0}$ is Artin–Schelter regular. So it is not clear how to generalize Auslander's Theorem beyond our noncommutative analogue of subgroups of $\mathrm{SL}_n(\Bbbk)$ (namely H-actions with trivial homological determinant) to a noncommutative analogue for Hopf algebras of the notion of "small" groups (groups containing no reflections).

Question 1.7. For actions (and coactions) by semisimple Hopf algebras H on Artin–Schelter regular algebras A, is there an analogue of the action on $\Bbbk[x_1, \ldots, x_n]$ by a finite "small" subgroup of $\mathrm{GL}_n(\Bbbk)$ (a condition for Hopf actions with nontrivial homological determinant for which Auslander's Theorem holds)?

To prove Theorem 0.1, we only need to show that $\mathsf{p}(\mathbb{D}(\alpha, \beta), \Bbbk^G) \geq 2$. The pertinency $\mathsf{p}(A, H)$ is defined in (E0.0.3).

Let $\int$ be the integral of a semisimple Hopf algebra H, and I be the two-sided ideal of $A \# H$ generated by $1 \# \int$. Recall that a $\Bbbk^G$-action on an algebra A is equivalent to a G-grading on A.

We recall the following result from [4] that will be used in the pertinency computation.

Lemma 1.8. *Let $H := (kG)^\circ$ act on A inner-faithfully, and write $A = \oplus_{g \in G} A g$.*

(1) [4, Lem. 5.1(3)] *If $f \in \cap_{g \in G} A A_g$ then $f \# 1 \in I$.*

(2) [4, Lem. 5.1 (6)]

$$\mathsf{p}(R,H) \geq d - \operatorname{GKdim} A/(\cap_{g\in G} AA_g) \geq d - \max\{\operatorname{GKdim} A/AA_g | g \in G\}.$$

The following is a modification of [4, Lem. 5.1(4)].

Lemma 1.9. *Let G be a finite group and $\Bbbk^G$ act on A inner-faithfully and homogeneously. Let $z \in A$.*

(1) *Suppose that, for each $g \in G$, there is an $x \in A$ of G-degree g and $y \in A$ such that $z = yx$. Then $z\#1$ is in the ideal of $A\#\Bbbk^G$ generated by $e := 1\#\int$.*
(2) *Suppose $z = f_n \cdots f_1$ is such that the collection (with possible repetitions)*

$$\{1, \deg_G(f_1), \deg_G(f_2 f_1), \ldots, \deg_G(f_{n-1} \cdots f_2 f_1), \deg_G(z)\}$$

includes all elements in G. Then $z\#1$ is in the ideal of $A\#\Bbbk^G$ generated by $e := 1\#\int$.

Proof. (1) Since $z = yx \in AA_g$ for each g, we have $z \in \bigcap_{g\in G} AA_g$. The assertion follows from Lemma 1.8.

(2) This is a special case of part (1). $\square$

In the next lemma we use some arguments from Bergman's Diamond Lemma [7]. Recall that $\mathbb{D}(\alpha,\beta)$ is generated by d and u. We use the ordering $d < u$ in this paper. Two relations of $\mathbb{D}(\alpha,\beta)$, namely, (E0.0.1)–(E0.0.2) can be written as

$$\begin{aligned} ud^2 &= \text{lower terms},\\ u^2 d &= \text{lower terms} \end{aligned}$$

where "lower terms" stands for a linear combination of monomials that have lower degree (in the lexicographic order) than the terms explicitly appearing in the same equation.

Lemma 1.10. *Retain the above notation.*

(1) *Let W be an ideal of $\mathbb{D}(\alpha,\beta)$ such that, in the factor ring $\mathbb{D}(\alpha,\beta)/W$, there are relations*

$$\begin{aligned} d^s(ud)^i &= \text{lower terms},\\ u^t &= \text{lower terms} \end{aligned}$$

for some $i, s, t \geq 0$. Then $\operatorname{GKdim} \mathbb{D}(\alpha,\beta)/W \leq 1$.
(2) *Let W be an ideal of $\mathbb{D}_\beta$ such that, in the factor ring $\mathbb{D}_\beta/W$, there are relations*

$$\begin{aligned} d^{2s}(du)^i &= \text{lower terms},\\ (ud)^j u^{2t} &= \text{lower terms} \end{aligned}$$

for some $i, j, s, t \geq 0$. Then $\operatorname{GKdim} \mathbb{D}_\beta/W \leq 1$.

Proof. (1) Together with (E0.0.1)-(E0.0.2), we have at least four relations

$$\begin{aligned} ud^2 &= \text{lower terms}, \\ u^2 d &= \text{lower terms}, \\ d^s (ud)^i &= \text{lower terms}, \\ u^t &= \text{lower terms} \end{aligned}$$

in the factor ring $\mathbb{D}(\alpha, \beta)/W$. By the Diamond Lemma [7] and using the first two relations, $\mathbb{D}(\alpha, \beta)/W$ has a $\Bbbk$-linear basis consisting of monomials of the form

$$d^a (ud)^b u^c, \quad a, b, c \geq 0$$

with some constraints. (A similar statement is [11, Lem. 1.1(3)] where we use the order $u < d$.) Two of the constraints are (i) either $a < s$ or $b < i$ and (ii) $c < t$, which follows from the last two relations of $\mathbb{D}(\alpha, \beta)/W$. Therefore, for each $\mathbb{N}$-degree d, the $\Bbbk$-dimension of $(\mathbb{D}(\alpha, \beta)/W)_d$ is uniformly bounded. As a consequence of a Gelfand–Kirillov dimension computation [5, (E1.1.6)], $\operatorname{GKdim} \mathbb{D}(\alpha, \beta)/W \leq 1$.

(2) The proof is similar to the one of part (1) and uses the fact that d^2 and u^2 are normal elements of $\mathbb{D}_\beta$.

Without loss of generality, we can assume that $s = t = i = j =: a > 0$ and re-use the letters i and j. Let

$$\begin{aligned} d^{2a} (du)^a &= \text{lower terms}, \\ (ud)^a u^{2a} &= \text{lower terms} \end{aligned}$$

in $\mathbb{D}_\beta/W$. Then

$$d^{4a} u^{4a} = \lambda (d^{2a} (du)^a)((ud)^a u^{2a}) = \text{lower terms}$$

in $\mathbb{D}_\beta/W$, for some $\lambda \in \Bbbk$. Then $d^{4a} u^{4a} = d^{4a+1} f$ in $\mathbb{D}_\beta/W$ for some f. Since u^2 skew-commutes with d and u, we obtain that

$$d^{4a} (ud)^j u^{4a} = d^{4a+1} f'$$

or

$$d^{4a} (ud)^j u^{4a} = \text{lower terms}.$$

Therefore

$$d^i (ud)^j u^k = \text{lower terms}$$

in $\mathbb{D}_\beta/W$ when at least two of indices i, j, k are larger than $4a$. By the Diamond Lemma argument as in the proof of part (1), for each $\mathbb{N}$-degree d, the $\Bbbk$-dimension of $(\mathbb{D}_\beta/W)_d$ is uniformly bounded. By [5, (E1.1.6)], $\operatorname{GKdim} \mathbb{D}_\beta/W \leq 1$. $\square$

2. Proof of Theorem 0.1

In this section we prove the main result, the theorem of Auslander for group coactions on down-up algebras. First we recall a result from [11].

Let $\mathbb{F}$ be the algebra generated by x and y, subject to two relations

$$x^3 = yxy \quad \text{and} \quad y^3 = xyx. \tag{E2.0.1}$$

As a graded algebra, $\mathbb{F}$ is isomorphic to $\mathbb{D}_{-1}$ [11, Lem. 1.5(1)]. Let $\mathbb{H}$ be the algebra generated by x and y, subject to two relations

$$x^2y + yx^2 - 2y^3 = 0 \quad \text{and} \quad -2x^3 + xy^2 + y^2x = 0. \tag{E2.0.2}$$

Then, as a graded algebra, $\mathbb{H}$ is isomorphic to $\mathbb{D}(-2,-1)$ [11, Lem. 1.9(1)].

Lemma 2.1 ([11, Prop. 1.12])**.** *Suppose G is a finite non-cyclic group coacting on $A := \mathbb{D}(\alpha, \beta)$ homogeneously and inner-faithfully. Then one of the following occurs.*

1. (1) $\alpha = 0$ *and u and d are G-homogeneous after a change of variables.*
2. (2) *A is isomorphic to $\mathbb{F}$ and using the generators of $\mathbb{F}$, both x and y are G-homogeneous.*
3. (3) *A is isomorphic to $\mathbb{H}$ and using the generators of $\mathbb{H}$, both x and y are G-homogeneous.*
4. (4) *G is abelian and there are linearly independent elements x and y of $\mathbb{D}(\alpha, -1)$ of degree one such that*
$$\begin{aligned} &\alpha x^2y + (-2-\alpha)xyx + \alpha yx^2 + (2-\alpha)y^3 = 0, \\ &(2-\alpha)x^3 + \alpha xy^2 + (-2-\alpha)yxy + \alpha y^2x = 0 \end{aligned}$$
and x and y are G-homogeneous.
5. (5) *G is abelian and u and d are G-homogeneous after a change of variables.*

The above lemma shows that there are plenty of interesting examples of finite group coactions on noetherian down-up algebras.

Note that the hypothesis of "G being non-cyclic" is needed in the above lemma which was proved in [11]. In the present paper we will also consider cyclic cases. In particular, our main theorem does not need the hypothesis of "G being non-cyclic".

We separate the proof of Theorem 0.1 into subcases according to the above lemma. In Cases 1 and 2 we assume that G is not cyclic; the cyclic cases will be included in Case 3.

2.1. Case 1: $\alpha = 0$, u and d are G-homogeneous

In this subsection, as $\alpha = 0$, A is the down-up algebra

$$\mathbb{D}_\beta := \Bbbk\langle d, u\rangle/(d^2u - \beta ud^2, du^2 - \beta u^2 d), \quad \beta \in \Bbbk^\times.$$

Suppose that $\mathbb{D}_\beta$ is G-graded with $\deg_G d = g_1$ and $\deg_G u = g_2$. Since $\mathbb{D}_\beta$ is generated by d and u, G is generated by g_1 and g_2. Let

$$X_1 := \{(g_2g_1)^i \mid i \geq 0\} \cup \{g_1(g_2g_1)^i \mid i \geq 0\} \subseteq G \tag{E2.1.1}$$

and

$$X_2 := \{(g_1g_2)^i \mid i \geq 0\} \cup \{g_2(g_1g_2)^i \mid i \geq 0\} \subseteq G. \tag{E2.1.2}$$

As in [4, Lem. 5.1] let

$$J := \text{the ideal of } A \text{ generated by } \bigcap_{g\in G} AA_g \tag{E2.1.3}$$

when a group G coacts on A.

Lemma 2.2. *Suppose that $\langle g_1\rangle X_1 = G = \langle g_2\rangle X_2$. Then $\mathsf{p}(\mathbb{D}_\beta, \Bbbk^G) \geq 2$.*

Proof. Let $A = \mathbb{D}_\beta$. By Lemma 1.8 (2) it suffices to show that

$$\operatorname{GKdim} A/J \leq 1$$

where J is defined as in (E2.1.3). By Lemma 1.10(2), it suffices to show that $v := d^{2a}(du)^{a+1}$ and $w := u^{2a}(ud)^a$ are in the ideal J, where $a = |G|$. By symmetry, we show only that v is in J.

Since $v = dd^{2a}(ud)^a u$, it suffices to show that $f := d^{2a}(ud)^a$ is in J. By hypothesis $\langle g_1\rangle X_1 = G$, every element g in G is of the form $g_1^i(g_2g_1)^j$ for some $a \geq i, j \geq 0$. Since d^2 is normal, we can write f as $c(ud)^{a-j}d^{2a-i} \cdot d^i(ud)^j$, for some $c \in \Bbbk^\times$ with $\deg_G d^i(du)^j = g_1^i(g_2g_1)^j = g$. Then $f \in AA_g$ for all g, which implies that $f \in J$ as required. $\square$

Lemma 2.3. *Suppose G is generated by g_1, g_2 and $\langle g_1^2\rangle = \langle g_2^2\rangle$. Then $\langle g_1\rangle X_1 = G = \langle g_2\rangle X_2$.*

Proof. Let N be the normal subgroup of G generated by g_1^2 and g_2^2. Then G/N is a dihedral group D_{2n}. In this case the image of X_1 in G/N consists of all elements in G/N. Then $G = NX_1$. Under the hypothesis, we have $N = \langle g_1^2\rangle$. Hence $\langle g_1\rangle X_1 = G$. By symmetry, $G = \langle g_2\rangle X_2$. $\square$

Now we are ready to prove a part of Theorem 0.1.

Proposition 2.4. *Retain the notation as in Theorem 0.1. Suppose further that $\alpha = 0$ and u and d are G-homogeneous. Then $\mathsf{p}(A, H) \geq 2$.*

Proof. By Lemma 1.5, when the $\Bbbk^G$-action on $\mathbb{D}_\beta$ has trivial homological determinant, $g_1^2 = g_2^{-2}$. Hence $\langle g_1^2\rangle = \langle g_2^2\rangle$. By Lemma 2.3, $\langle g_1\rangle X_1 = G = \langle g_2\rangle X_2$. Now the main assertion follows from Lemma 2.2. $\square$

2.2. Case 2: $A = \mathbb{H}$, x and y are G-homogeneous

In this subsection we have that $A = \mathbb{H}$ and that x and y in $\mathbb{H}$ are G-homogeneous. Let $g_1 = \deg_G x$ and $g_2 = \deg_G y$. By the relations of $\mathbb{H}$, one sees that $g_1^2 = g_2^2$. Two relations of $\mathbb{H}$ can be written as

$$x(x^2 - y^2) = -(x^2 - y^2)x \quad \text{and} \quad y(x^2 - y^2) = -(x^2 - y^2)y.$$

Define a filtration $\mathcal{F}$ on $\mathbb{H}$ by

$$F_i\mathbb{H} = (\Bbbk x + \Bbbk y + \Bbbk z)^i, i \geq 0$$

where $z = x^2 - y^2$. It is easy to see that the G-coaction preserves this filtration. Let B be $\operatorname{gr}_{\mathcal{F}}\mathbb{H}$. Then $B \cong (\Bbbk\langle x, y\rangle/(x^2 - y^2))[z, \sigma]$ where σ maps $x \to -x$ and $y \to -y$. Then G coacts on B by $\deg_G x = g_1, \deg_G y = g_2$ and $\deg_G z = g_1^2$. The following lemma follows from [4, Lem. 3.6].

Lemma 2.5. *Retain the above notation. Then* $\mathsf{p}(\mathbb{H}, \Bbbk^G) \geq \mathsf{p}(B, \Bbbk^G)$.

By the above lemma, it suffices to show that $\mathsf{p}(B, \Bbbk^G) \geq 2$. For the rest of the proof we follow the proof in Case 1.

Lemma 2.6. *Let J be an ideal of B containing both $x^{2s}(yx)^i$ and $(xy)^j z^t$ for some $i, j, s, t \geq 0$. Then* $\operatorname{GKdim} B/J \leq 1$.

Proof. Without loss of generality, we can assume that $s = t = i = j =: a > 0$ and re-use letters i and j. Let $f_1 = x^{2a}(yx)^a$ and $f_2 = (xy)^a z^a$ in J. Then $x^{6a}z^a \in \Bbbk f_1 f_2 \subseteq J$. Note that B has a $\Bbbk$-linear basis

$$\{x^i(yx)^j z^k \mid i, j, k \geq 0\} \cup \{x^i(yx)^j z^k y \mid i, j, k \geq 0\}.$$

Since $x^2(= y^2)$ and z are skew-commuting with x, y, z, every element is of the form $x^i(yx)^j z^k$ or $x^i(yx)^j z^k y$ is 0 in B/J when at least two of indices i, j, k are larger than $6a$. An elementary counting argument shows that $\operatorname{GKdim} B/J \leq 1$. $\square$

Use the notation introduced in (E2.1.1) and (E2.1.2):

$$X_1 := \{(g_2 g_1)^i \mid i \geq 0\} \cup \{g_1(g_2 g_1)^i \mid i \geq 0\} \subseteq G$$

and

$$X_2 := \{(g_1 g_2)^i \mid i \geq 0\} \cup \{g_2(g_1 g_2)^i \mid i \geq 0\} \subseteq G.$$

Lemma 2.7. *Retain the above notation.*

(1) $\langle g_1^2 \rangle X_1 = G = \langle g_2^2 \rangle X_2$.
(2) $\mathsf{p}(B, \Bbbk^G) \geq 2$.
(3) $\mathsf{p}(\mathbb{H}, \Bbbk^G) \geq 2$.

Proof. (1) Let N be the normal subgroup of G generated by g_1^2 (or by g_2^2). Then G/N is a dihedral group D_{2n}. In this case the image of X_1 in G/N consists of all elements in G/N. Then $G = NX_1 = \langle g_1^2 \rangle X_1$. Similarly, $G = \langle g_2^2 \rangle X_2$.

(2) By Lemma 1.8(2) it suffices to show that

$$\operatorname{GKdim} B/J \leq 1$$

where J is the ideal of B generated by $\bigcap_{g \in G} BB_g$. By Lemma 2.6, it suffices to show that $f_1 := x^{2a}(yx)^a$ and $f_2 := (xy)^a z^a$ are in the ideal J, where $a = |G|$. By part (1), $\langle g_1 \rangle X_1 = G$, every element in G is of the form $g_1^i (g_2 g_1)^j$ for some $0 \leq i, j \leq a$. By the fact that x^2 commutes with y, we obtain that $f_1 = f_1'(x^i(yx)^j)$ for some $f_1' \in B$. Then $f_1 \in BB_g$ for all g, which implies that $f_1 \in J$. Since z skew-commutes with x and y, a similar argument shows that $f_2 \in J$. Now the assertion follows by Lemma 2.6.

(3) This follows from part (2) and Lemma 2.5. $\square$

Part (3) of the above lemma says that Auslander's Theorem holds in this case, even without the hypothesis that the homological determinant of the H-action is trivial in this special case. For the sake of completeness we calculate the homological (co)determinants of the G-coactions easily in the next lemma.

Lemma 2.8. *Suppose a finite group G coacts on A.*

(1) *If $A = \mathbb{H}$ and x and y are G-homogeneous with G-degree g_1 and g_2 respectively, then the homological codeterminant of the G-coaction is g_1^4, which is also g_2^4.*
(2) *If $A = \mathbb{F}$ and x and y are G-homogeneous with G-degree g_1 and g_2 respectively, then the homological codeterminant of the G-coaction is g_1^4, which is also g_2^4.*
(3) *Let $A = \mathbb{D}(\alpha, -1)$, for $\alpha \neq 2$, and $x = \frac{1}{2}(d + u)$ and $y = \frac{1}{2}(d - u)$. By* [11, Prop. 1.12(4)], *A is generated by x and y and subject to relations*

$$\alpha x^2 y + (-2 - \alpha)xyx + \alpha yx^2 + (2 - \alpha)y^3 = 0, \tag{E2.8.1}$$
$$(2 - \alpha)x^3 + \alpha xy^2 + (-2 - \alpha)yxy + \alpha y^2 x = 0. \tag{E2.8.2}$$

Suppose that G is abelian and that x and y are G-homogeneous with G-degree g_1 and g_2 respectively. Then the homological codeterminant of the G-coaction is g_1^4, which is also g_2^4.

Proof. Since the proofs are similar to the proof of Lemma 1.5, the details are omitted. $\square$

2.3. Case 3: G is abelian

Let G be a finite abelian group and let $\widehat{G}$ be the character group $\mathrm{Hom}_{groups}(G, \Bbbk^\times)$. Since $\Bbbk$ is algebraically closed of characteristic zero, $\widehat{G}$ is isomorphic to G as an abstract group. As a consequence, $(\Bbbk G)^*$ is isomorphic to $\Bbbk G$ as a Hopf algebra.

Let A be a down-up algebra $\mathbb{D}(\alpha, \beta)$ generated by d and u. Every graded algebra automorphism g of A can be written as a 2×2-matrix with respect to the basis $\{d, u\}$. We say g is *diagonal* (respectively, *non-diagonal*) if its matrix presentation with respect to $\{d, u\}$ is diagonal (respectively, non-diagonal). When the basis $\{d, u\}$ is replaced by $\{d', u'\} = \{c_1 d, c_2 u\}$ for some $c_1, c_2 \in \Bbbk^\times$, the matrix presentation of g could change accordingly, but the diagonal property of g will not change. We call this kind of change of basis a *scalar base change* which we use in the proof of Lemma 2.9.

Let G be a finite abelian group that coacts on A inner-faithfully and homogeneously. This G-coaction on A is equivalent to a $\widehat{G}$-action on A preserving the $\mathbb{N}$-grading. Therefore we can consider the $\widehat{G}$-action instead of the G-coaction. The theorem of Auslander was proved for finite group actions on graded noetherian down-up algebras in [5, Thm. 0.6] except for the case $A = \mathbb{D}(\alpha, -1)$ for $\alpha \neq 2$. In fact their proof [5, Proof of Thm. 0.6] works for any diagonal automorphisms of $\mathbb{D}(\alpha, -1)$, too, and [13, Prop. 4.6] handles another special class of groups acting on $A = \mathbb{D}(\alpha, -1)$ for $\alpha \neq 2$. In this subsection we prove Auslander's Theorem only for a finite abelian group $\widehat{G}$ of graded automorphisms of $\mathbb{D}(\alpha, -1)$ with $\alpha \neq 2$ that is not all diagonal. Combining with the results in [5], we take care of all abelian groups (including cyclic ones).

Throughout the rest of this subsection let A be $\mathbb{D}(\alpha, -1)$ for some $\alpha \neq 2$. The next lemma classifies all possible finite abelian groups that are not diagonal having trivial homological determinant.

Lemma 2.9. *Consider the following subgroup of* $\mathrm{GL}_2(\Bbbk)$

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} : a, b \in \{\pm 1, \pm i\} \right\}.$$

The following hold.

(1) *T is an abelian group acting naturally on A, with respect to the basis $\{d, u\}$, inner-faithfully and homogeneously with trivial homological determinant.*

(2) *Let $\widehat{G}$ be a finite abelian group acting on A inner-faithfully and homogeneously with trivial homological determinant. If $\widehat{G}$ contains a non-diagonal matrix, then $\widehat{G}$ is a subgroup of T after a scalar base change.*

(3) *Let $\widehat{G}$ be as in part* (2)*. Then, up to a scalar base change, $\widehat{G}$ is one of the following:*

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} : a, b \in \{\pm 1\} \right\},$$

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} : a \in \{\pm 1\}, b \in \{\pm i\} \right\}, \quad \text{or} \quad T.$$

Proof. (1) This follows by a direct computation.

(2) Suppose that $f := \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and $g := \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ are in $\widehat{G}$. The commutativity of G forces $a = d$. By [15, Thm. 1.5], the homological determinant of $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ is a^4. Thus $a \in \{\pm 1, \pm i\}$ as $\widehat{G}$ has trivial homological determinant. In other words, $f \in T$. After a scalar base change, we may assume that $b = c$ in the matrix g. By [15, Thm. 1.5], the homological determinant of g (with $b = c$) is b^4. Then $b \in \{\pm 1, \pm i\}$ and $g \in T$. Now assume that $\widehat{G}$ contains another non-diagonal automorphism $h := \begin{pmatrix} 0 & c' \\ c & 0 \end{pmatrix}$. Then the equation $gh = hg$ implies that $c' = c$. So $h \in T$ and $\widehat{G}$ is a subgroup of T.

(3) This follows by a direct computation. $\square$

Using the classification in Lemma 2.9, we can work out the corresponding coactions. Let $x = \frac{1}{2}(d + u)$ and $y = \frac{1}{2}(d - u)$, or equivalently, $d = x + y$ and $u = x - y$. By the proof of [11, Prop. 1.12(4)], we have the following.

Lemma 2.10. *Suppose G is a finite abelian group coacting on $A := \mathbb{D}(\alpha, -1)$, for $\alpha \neq 2$, such that*

(a) *the G-coaction has trivial homological codeterminant, and*

(b) *the corresponding $\widehat{G}$-action contains a non-diagonal matrix with respect to the basis $\{d, u\}$.*

Then the following hold.

(1) *There are linearly independent elements x and y of $\mathbb{D}(\alpha, -1)$ of degree one such that*

$$\alpha x^2 y + (-2 - \alpha) xyx + \alpha y x^2 + (2 - \alpha) y^3 = 0,$$
$$(2 - \alpha) x^3 + \alpha x y^2 + (-2 - \alpha) yxy + \alpha y^2 x = 0$$

and x and y are G-homogeneous.

(2) *Let $\deg_G x = g_1$, $\deg_G y = g_2$. Then $g_1^2 = g_2^2$ and $g_1^4 = g_2^4 = 1$ in G.*

Proof. (1) A part of the proof appeared in the proof of [11, Prop. 1.12], so we give only a sketch of the argument here.

First, the $\widehat{G}$-action on A has the special forms as listed in Lemma 2.9(3). Using the forms given there, let $x = \frac{1}{2}(d+u)$ and $y = \frac{1}{2}(d-u)$, or $d = x+y$ and $u = x-y$. Then both x and y are $\widehat{G}$-eigenvectors. This means that both x and y are G-homogeneous in the corresponding G-coaction. The two relations are obtained in the proof of [11, Prop. 1.12] by direct computation, which we will not repeat here.

(2) By the relations and the hypothesis that $\alpha \neq 2$, one sees that $g_1^2 = g_2^2$. The second assertion is Lemma 2.8(3). □

Ueyama [24] introduced the notion of a *graded isolated singularity*, and we recall his definition here. For a graded algebra A, let $\operatorname{grmod} A$ denote the category of finitely generated graded left A-modules. For a graded finitely generated A-module an element $x \in M$ is called *torsion* if there exists a positive integer n such that $A_{\geq n} x = 0$. The module M is called a *torsion module* if every element of M is torsion. Let $\operatorname{tors} A$ denote the full subcategory of $\operatorname{grmod} A$ consisting of torsion modules. We can then define the quotient category $\operatorname{tails} A = \operatorname{grmod} A / \operatorname{tors} A$. Following [24], we say that A^G has a *graded isolated singularity* if $\operatorname{gldim}(\operatorname{tails} A^G) < \infty$. Mori and Ueyama prove that if the Auslander map is an isomorphism, then A^G has a graded isolated singularity if and only if $A\#G/I$ is finite-dimensional [21, Thm. 3.10]. Examples of graded isolated singularities are of particular interest, since when A^G has a graded isolated singularity, the category of graded CM A^G-modules has several nice properties (see [25]).

Next we compute the pertinency for G-coactions.

Lemma 2.11. *Retain the hypothesis of Lemma* 2.10.

(1) *If $g_2 = 1$ and $g_1 \neq 1$ then $\mathsf{p}(A, \Bbbk^G) = 3$. As a consequence, $A^{\mathrm{co}\, G}$ has a graded isolated singularity.*
(2) *If $g_2 \neq 1$, $g_1 \neq 1$, $g_1 \neq g_2$, and $g_1^2 = g_2^2 = 1$, then $\mathsf{p}(A, \Bbbk^G) \geq 2$.*
(3) *If $g_1 \neq 1$, $g_1^2 = 1$ and $g_2 = g_1$, then $\mathsf{p}(A, \Bbbk^G) = 3$. As a consequence, $A^{\mathrm{co}\, G}$ has a graded isolated singularity.*
(4) *If $g_1^2 \neq 1$, $g_2 = g_1$, then $\mathsf{p}(A, \Bbbk^G) = 3$. As a consequence, $A^{\mathrm{co}\, G}$ has a graded isolated singularity.*
(5) *If $g_1^2 \neq 1$, and $g_2 = g_1^{-1}$, then $\mathsf{p}(A, \Bbbk^G) \geq 2$.*
(6) *If $G = T$, then $\mathsf{p}(A, \Bbbk^G) \geq 2$.*

Proof. Let J be the ideal generated by $\bigcap_{g\in G} AA_g$ as defined in (E2.1.3).

(1) In this case $\deg_G x = g_1 \neq 1$ and $\deg_G y = 1$. Then

$$\deg_G x^2 = \deg_G xyx = \deg_G xy^2x = 1.$$

It is easy to see that $x^2, xyx, xy^2x \in J$. By the first relation of A, $y^3 \in J$. Thus A/J is finite-dimensional, or $\operatorname{GKdim} A/J = 0$. This means that $\mathsf{p}(A, \Bbbk^G) = 3$, and by [4, Cor. 3.8], $A^{\mathrm{co}\, G}$ has a graded isolated singularity.

(2) It is easy to check that $xyx, yxy \in AA_g$ for all $g \in G$. So $xyx, yxy \in J$. Using relations of A, we have, in A/J,

$$y^3 = ax^2y + byx^2,$$
$$y^2x = xy^2 + cx^3$$

for some $a, b, c \in \Bbbk$. By using Bergman's Diamond Lemma [7] with degree lexicographic monomial order with $y > x$, A/J has a monomial basis and each of the monomials does not contain subwords y^3, y^2x, xyx, yxy. This implies that A/J is spanned by

$$\{x^i : i \geq 0\} \cup \{yx^i, x^iy, x^iy^2 : i \geq 0\} \cup \{yx^iy, yx^iy^2 : i \geq 0\}.$$

Thus $\operatorname{GKdim} A/J \leq 1$, and hence $\mathsf{p}(A, \Bbbk^G) \geq 2$.

(3,4) In these cases, every monomial of degree 4 is in J. So $\operatorname{GKdim} A/J = 0$ as required.

(5) In this case, one can show that $x^3 \in J$ as $\deg_G x$ generates the group G. Similarly, we have $y^3 \in J$. Using the relations in A, one sees that, in A/J,

$$\begin{aligned} x^3 &= 0, \\ y^3 &= 0, \\ yx^2 &= -x^2y + axyx, \\ y^2x &= -xy^2 + ayxy \end{aligned}$$

for some $a \in \Bbbk$. By Bergman's Diamond Lemma [7], A/J is spanned by

$$\{x^i(yx)^jy^k : 0 \leq i, k \leq 2, j \geq 0\}.$$

Therefore $\operatorname{GKdim} A/J \leq 1$ and $\mathsf{p}(A, \Bbbk^G) \geq 2$.

(6) Let $\{d_7, \ldots, d_1\}$ be an ordered set of elements (possibly with repetitions) in G such that the set $\{\prod_{s=1}^7 d_s, \prod_{s=1}^6 d_s, \cdots, d_2d_1, d_1\}$ is equal to $G \setminus \{1\}$. Suppose $f_s \in A$ are homogeneous of degree d_s for all $s = 1, \ldots, 7$. By Lemma 1.9(2), the product $f_7f_6 \cdots f_1$ is in J. Using this observation one sees that the following elements are in J:

$$y^2xy^3x,\ xyx^3yx,\ yx^3yx^2,\ x^3yx^3,\ x^2yx^3y,\ yxy^3xy,\ xy^3xy^2,\ y^3xy^3.$$

(The reason for verifying a product of 7 letters is that any subword of these monomials does not have G-degree 1. This list is all degree 7 monomials in J.) Using the fact that $x = \frac{1}{2}(d+u)$ and $y = \frac{1}{2}(d-u)$, we obtain the following relation in A/J:

$$\begin{aligned} 0 &= 2^7(x^3yx^3 - y^3xy^3) \\ &= (-2)u^7 + \text{lower terms}, \end{aligned}$$

or equivalently,

$$u^7 = \text{lower terms}.$$

In other words, we can write u^7 in terms of terms in lower degree in the lexicographic order. Similarly, by using $x = \frac{1}{2}(d+u)$ and $y = \frac{1}{2}(d-u)$, we calculate the following in A/J:

$$\begin{aligned}
0 &= 2^7(x^3yx^3+y^3xy^3)\\
&= (-2a^5-2a^4+6a^3+8a^2-4a-6)udu^5\\
&\quad+(-2a^6+2a^5+6a^4-6a^3-6a^2+4a+2)udududu\\
&\quad+\text{lower terms};\\
0 &= 2^7(y^2xy^3x+x^2yx^3y)\\
&= (-2a^5+2a^4+10a^3-4a^2-12a+2)udu^5\\
&\quad+(-2a^6-2a^5+6a^4+6a^3-6a^2-4a+2)udududu\\
&\quad+\text{lower terms};\\
0 &= 2^7(xyx^3yx+yxy^3xy)\\
&= (2a^5-2a^4-6a^3+8a^2+4a-6)udu^5\\
&\quad+(-2a^6-2a^5+6a^4+6a^3-6a^2-4a+2)udududu\\
&\quad+\text{lower terms};\\
0 &= 2^7(yx^3yx^2+xy^3xy^2)\\
&= (2a^5+2a^4-10a^3-4a^2+12a+2)udu^5\\
&\quad+(-2a^6+2a^5+6a^4-6a^3-6a^2+4a+2)udududu\\
&\quad+\text{lower terms};
\end{aligned}$$

where "lower term" means a linear combination of monomials of degree 7 that have lower degrees than terms appearing in the expression (in this case, $udududu$) with respect to lexicographic order. Recall that a is the scalar that appeared in one of the relations of A,

$$y^3 = ax^2y + byx^2,$$

see the proof of part (2). If $a^2 \neq 1$ and $a^2 \neq 2$, then by a linear algebra computation, both udu^5 and $udududu$ can be expressed as "lower terms":

$$\begin{aligned}
udu^5 &= \text{lower terms},\\
udududu &= \text{lower terms}.
\end{aligned}$$

Since u^7 and $udududu$ (and then $(ud)^4$) are equal to lower terms in A/J, Lemma 1.10(1) implies that GKdim $A/J \leq 1$, as required.

If $a = 1$, then we have

$$\begin{aligned}
0 &= 2^7(x^3yx^3+y^3xy^3)\\
&= -6du^6+8d^3u^4-8d^4udu+8d^5u^2+2d^7;\\
0 &= 2^7(y^2xy^3x+x^2yx^3y)\\
&= -4udu^5+2du^6-4d^2udu^3+4d^3u^4-4d^3udud+4d^4udu-8d^5u^2+2d^7;\\
0 &= 2^7(xyx^3yx+yxy^3xy)\\
&= -2du^6-4d^3u^4+2d^7;\\
0 &= 2^7(yx^3yx^2+xy^3xy^2)\\
&= 4udu^5-2du^6+4d^2udu^3-4d^3udud+4d^4udu-8d^5u^2+2d^7.
\end{aligned}$$

By a linear algebra computation, we have

$$d^3(ud)^2 = \text{lower terms}.$$

Since u^7 and $d^3(ud)^2$ are equal to lower terms in A/J, Lemma 1.10(1) implies that $\operatorname{GKdim} A/J \le 1$, as required.

If $a = -1$, then we have

$$\begin{aligned}
0 &= 2^7(x^3yx^3 + y^3xy^3) \\
&= 2du^6 - 4d^3u^4 + 2d^7; \\
0 &= 2^7(y^2xy^3x + x^2yx^3y) \\
&= 4udu^5 + 2du^6 - 4d^2udu^3 - 4d^3udud + 4d^4udu + 8d^5u^2 + 2d^7; \\
0 &= 2^7(xyx^3yx + yxy^3xy) \\
&= 6du^6 + 8d^3u^4 - 8d^4udu - 8d^5u^2 + 2d^7; \\
0 &= 2^7(yx^3yx^2 + xy^3xy^2) \\
&= -4udu^5 - 2du^6 + 4d^2udu^3 + 4d^3u^4 - 4d^3udud + 4d^4udu + 8d^5u^2 + 2d^7.
\end{aligned}$$

By a linear algebra computation, we have

$$d^3(ud)^2 = \text{lower terms}.$$

Since u^7 and $d^3(ud)^2$ are equal to lower terms in A/J, Lemma 1.10(1) implies that $\operatorname{GKdim} A/J \le 1$, as required.

If $a^2 = 2$, we need to use a different set of elements in J. By a similar argument as before and by Lemma 1.9(2), the following elements are in J:

$$\begin{gathered}
x^3y^3x^3, y^3x^3y^3, y^2xy^3x^3, x^2yx^3y^3, y^3x^3yx^2, x^3y^3xy^2, yxy^3xyx^2, xyx^3yxy^2, \\
y^2xy^3xy^2, x^2yx^3yx^2, yxy^3xy^3, xyx^3yx^3, yx^4yx^3, xy^4xy^3, xy^4xyx^2, yx^4yxy^2.
\end{gathered}$$

By a Sage computation, we have, in A/J,

$$\begin{aligned}
0 &= 2^9(x^3y^3x^3 - y^3x^3y^3) \\
&= (-2)u^9 + (8+4a)ududu^5 + (18+12a)udududu + \text{lower terms}, \\
0 &= 2^9(y^2xy^3x^3 - x^2yx^3y^3) \\
&= (-2)u^9 + (-4-4a)ududu^5 + (-2-4a)udududu + \text{lower terms}, \\
0 &= 2^9(y^3x^3yx^2 - x^3y^3xy^2) \\
&= 2u^9 + (-4)ududu^5 + 2udududu + \text{lower terms}, \\
0 &= 2^9(y^2xy^3xy^2 - x^2yx^3yx^2) \\
&= (-2)u^9 + 2udududu + \text{lower terms}.
\end{aligned}$$

An easy linear algebra computation shows that

$$\begin{aligned}
u^9 &= \text{lower terms}, \\
(ud)^2u^5 &= \text{lower terms}, \\
(ud)^4u &= \text{lower terms}.
\end{aligned}$$

Since u^9 and $(ud)^4u$ (and then $(ud)^5$) are equal to lower terms in A/J, Lemma 1.10(1) implies that $\operatorname{GKdim} A/J \le 1$, as required. This finishes the proof. □

2.4. Case 4: $A = \mathbb{F}$, x and y are G-homogeneous

The final case is when $A = \mathbb{F}$ and the proof is also quite tricky. We start with a result of [11].

Lemma 2.12 ([11, Lem. 1.6]). *Let $\mathbb{F}$ be generated by x and y and subject to two relations* (E2.0.1).

(1) *Define an order on monomials by extending $x < y$ lexicographically. Then we have a complete set of five relations that is the reduction system in the sense of* [7, p.180].

$$\begin{aligned} y^3 &= xyx, \\ yxy &= x^3, \\ y^2x^3 &= xyx^2y, \\ yx^2yx &= x^3y^2, \\ yx^4 &= x^4y. \end{aligned}$$

(2) *We also have the other relations:*

$$\begin{aligned} y^4 &= x^4, \\ yxyx &= x^4, \\ xyxy &= y^4. \end{aligned}$$

(3) *There is a $\Bbbk$-linear basis consisting of the monomials of the form*

$$x^i(yx^3)^j(yx^2)^\epsilon(y^2x^2)^k y^a x^b$$

where $i, j, k \geq 0$, ϵ is either 0 or 1, and

$$(a, b) = (0, 0), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 1), (2, 2)$$

if $j + \epsilon + k = 0$,

$$(a, b) = (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 1), (2, 2)$$

if $j > 0$ and $\epsilon + k = 0$, and

$$(a, b) = (1, 0), (2, 0), (2, 1), (2, 2)$$

if $\epsilon + k > 0$.

Let G be a finite group coacting on $A := \mathbb{F}$ such that x and y are G-homogeneous. Let J be defined as in (E2.1.3).

Lemma 2.13. *Suppose there are $\alpha, \beta, \alpha', \beta' \geq 0$ such that*

$$(yx^3)^\alpha(y^2x^2)^\beta \text{ and } (yx^3)^{\alpha'}(yx^2)(y^2x^2)^{\beta'} \in J,$$

then $\operatorname{GKdim} A/J \leq 1$.

Proof. We again use the Diamond Lemma [7]. By the fact that $x^4 = y^4$ is central in A (see Lemma 2.12(2)), there is a monomial f in x and y such that $f(yx^3)^i(y^2x^2)^k = x^{4w}$ for some w. So we have the following equations in A/J:

$$\begin{aligned} x^{4w} &= 0, \\ (yx^3)^{\alpha}(y^2x^2)^{\beta} &= 0, \\ (yx^3)^{\alpha'}(yx^2)(y^2x^2)^{\beta'} &= 0. \end{aligned}$$

Therefore, if there are i, j, k with $x^i(yx^3)^j(yx^2)^{\epsilon}(y^2x^2)^k y^a x^b \neq 0$ in A/J for (a, b) as in Lemma 2.12(b), then there is a uniform bound on at least two of $\{i, j, k\}$. Then $\operatorname{GKdim} A/J \leq 1$ by [5, (E1.1.6)] and Lemma 2.12(3). $\square$

The goal of the rest of this subsection is to find some monomials $(yx^3)^i(y^2x^2)^k$ and $(yx^3)^{i'}(yx^2)(y^2x^2)^{k'}$ in J. We introduce the following notation. Let

$$\begin{aligned} X_0 &= x, \\ X_1 &= y. \end{aligned}$$

We will use X_i for $i \in \mathbb{Z}/(2)$. Let

$$\begin{aligned} V_0 &= yx^3, \\ V_1 &= xyx^2, \\ V_2 &= x^2yx, \\ V_3 &= x^3y, \end{aligned}$$

and

$$\begin{aligned} W_0 &= y^2x^2, \\ W_1 &= xy^2x, \\ W_2 &= x^2y^2, \\ W_3 &= yx^2y. \end{aligned}$$

We will use V_i and W_i for $i \in \mathbb{Z}/(4)$. The following lemma follows from a direct computation and the relations given in Lemma 2.12.

Lemma 2.14. *Retain the above notation.*

(1) $X_iV_j = V_{j+1}X_i$ *and* $X_iW_j = W_{j+1}X_i$ *for all* $i \in \mathbb{Z}/(2)$ *and* $j \in \mathbb{Z}/(4)$.
(2) *Elements* $\{V_0, \ldots, V_3, W_0, \ldots, W_3\}$ *are pairwise commutative.*
(3) *The following relations hold*
  (a) $V_0V_2 = x^8 = V_1V_3$.
  (b) $W_0W_2 = x^8 = W_1W_3$.
  (c) $V_0V_1 = x^4W_0$.

Now let G be a finite group coacting on $\mathbb{F}$ such that x and y are G-homogeneous. We also assume that the G-coaction has trivial homological codeterminant, namely, $\deg_G(x^4) = 1$. Let

$$\begin{aligned} x_i &= \deg_G X_i, \\ v_j &= \deg_G V_j, \\ w_k &= \deg_G W_k \end{aligned}$$

for $i \in \mathbb{Z}/(2)$ and $j, k \in \mathbb{Z}/(4)$. Let N be the subgroup generated by $\{v_i\}_{i=0}^3 \cup \{w_k\}_{k=0}^3$. By the lemma above, we have

Lemma 2.15. *Retain the above notation.*

1. *G is generated by $x_0(= g_1)$ and $x_1(= g_2)$.*
2. *N is an abelian subgroup of G.*
3. *N is a normal subgroup of G.*
4. *N is generated by v_0 and v_1 and G/N is generated by the image $\overline{x_0}$ of x_0; and $\overline{x_0} = \overline{x_1}$ in G/N.*
5. *N is also generated by $\{v_i, w_j\}$ for any pair (i, j).*
6. *$G = N \cup Nx_{i_1} \cup Nx_{j_1}x_{j_2} \cup Nx_{k_1}x_{k_2}x_{k_3}$ for any fixed $i_s, j_s, k_s \in \mathbb{Z}/(2)$.*
7. *Let n be the order of v_0. For any fixed $i_s \in \mathbb{Z}/(2)$ and $j_s, k_s \in \mathbb{Z}/(4)$ for $s = 1, 2, 3, 4$, any element in G is a right subword of*

$$v_{j_4}^n w_{k_4}^n x_{i_3} v_{j_3}^n w_{k_3}^n x_{i_2} v_{j_2}^n w_{k_2}^n x_{i_1} v_{j_1}^n w_{k_1}^n.$$

Proof. (1) Since G-coaction is inner-faithful, G is generated by $\deg_G x$ and $\deg_G y$.

(2) This follows from Lemma 2.14(2).

(3) This follows from Lemma 2.14(1) and part (1).

(4) Since $\deg_G x^4 = 1$, $x_0^4 = x_1^4 = 1$. By Lemma 2.14(3), $v_2 = v_0^{-1}$, $v_3 = v_1^{-1}$, and $w_0 = v_0 v_1$. It is easy to check from Lemma 2.14(1) that $w_1 = v_0^{-1} v_1$, $w_2 = v_0^{-1} v_1^{-1}$ and $w_3 = v_0 v_1^{-1}$. Therefore N is generated by v_0 and v_1. It is clear that in G/N, $\overline{x_0} = \overline{x_1}$. So G/N is generated by $\overline{x_0}$.

(5) By the proof of part (4), we have

$$\{v_0, v_1, v_2, v_3\} = \{v_0, v_1, v_0^{-1}, v_1^{-1}\},$$
$$\{w_0, w_1, w_2, w_3\} = \{v_0 v_1, v_0^{-1} v_1, v_0^{-1} v_1^{-1}, v_0 v_1^{-1}\}.$$

Therefore N is generated by $\{v_i, w_j\}$ for any pair (i, j).

(6) This follows from the fact that $\overline{x_0} = \overline{x_1}$ in the quotient group G/N and that $G/N \cong \mathbb{Z}/(4)$.

(7) This follows from parts (5,6). □

We have a version of Lemma 2.15 for monomials in $\mathbb{F}$.

Lemma 2.16. *Retain the above notation.*

1. *For any fixed $i_s \in \mathbb{Z}/(2)$ and $j_s, k_s \in \mathbb{Z}/(4)$ for $s = 1, 2, 3, 4$, any element in G is the degree of some right subword of*

$$\Phi := V_{j_4}^n W_{k_4}^n X_{i_3} V_{j_3}^n W_{k_3}^n X_{i_2} V_{j_2}^n W_{k_2}^n X_{i_1} V_{j_1}^n W_{k_1}^n.$$

   As a consequence, $\Phi \in J$.
2. *$(yx^3)^{4n}(yx^2)(y^2x^2)^{4n}$ and $(yx^3)^{4n}(y^2x^2)^{4n+1}$ are elements in J.*

Proof. (1) This is a slightly stronger version of Lemma 2.15(7). For example, if an element g is of the form

$$v_{j_2}^a w_{k_2}^b x_0 v_{j_1}^n w_{k_1}^n$$

then we take a right subword of the form $f := V_{j_2}^a W_{k_2}^b X_0 V_{j_1}^n W_{k_1}^n$. Since V_s and W_t all commute, f is a subword of Φ. Clearly, $\deg_G f = v_{j_2}^a w_{k_2}^b x_0 v_{j_1}^n w_{k_1}^n$. The consequence follows from Lemma 1.9(1).

(2) By using Lemma 2.14(1), Φ equals

$$V_{j_4}^n V_{j_3+1}^n V_{j_2+2}^n V_{j_1+3}^n X_{i_3} X_{i_2} X_{i_1} W_{k_4-3} W_{k_3-2} W_{k_2-1} W_{k_1}.$$

By taking $j_4 = j_3+1 = j_2+2 = j_1+3 = 0$ and $k_4-3 = k_3-2 = k_2-1 = k_1 = 0$ and $i_3 = 1$, $i_2 = i_1 = 0$, we have that

$$(yx^3)^{4n}(yx^2)(y^2x^2)^{4n} \in J.$$

By taking $j_4 = j_3+1 = j_2+2 = j_1+3 = 0$ and $k_4-3 = k_3-2 = k_2-1 = k_1 = 1$ and $i_3 = i_2 = 1$ and $i_1 = 0$, we have that

$$(yx^3)^{4n}(y^2x)(xy^2x)^{4n} \in J.$$

Then $(yx^3)^{4n}(y^2x^2)^{4n+1} \in J$. $\square$

Now we can prove the result of this subsection.

Proposition 2.17. *Retain the notation as in Theorem* 0.1. *Suppose that $A = \mathbb{F}$ and x and y are G-homogeneous. Then $\mathsf{p}(A, \Bbbk^G) \geq 2$.*

Proof. Combining Lemma 2.16(2) with Lemma 2.13, $\operatorname{GKdim} A/J \leq 1$. This is equivalent to $\mathsf{p}(A, \Bbbk^G) \geq 2$. $\square$

Putting all these pieces together we have a proof of Theorem 0.1.

Proof of Theorem 0.1. First we assume that G is abelian (that could be cyclic). If A is not $\mathbb{D}(\alpha, -1)$ for some $\alpha \neq 2$, the assertion follows from [5, Thm. 0.6]. Now we assume that $A = \mathbb{D}(\alpha, -1)$ for some $\alpha \neq 2$. Using notations introduced in subsection 2.3, the character group of G is denoted by $\widehat{G}$. If $\widehat{G}$ acts on A diagonally with respect to the basis $\{d, u\}$, then the assertion follows from [5, Proof of Thm. 0.6]. Otherwise, $\widehat{G}$ contains non-diagonal matrices with respect to the basis $\{d, u\}$. In this case the assertion follows from Lemmas 2.9, 2.10 and 2.11. This takes care of the case when G is abelian.

Next we assume that G is not abelian. As a consequence, G is not cyclic, so we can apply Lemma 2.1. Using the classification given in Lemma 2.1, we only need to show the assertion for the first three cases as in the last two cases G is abelian.

In case Lemma 2.1(1), this is Proposition 2.4.

In case Lemma 2.1(2), this is Proposition 2.17.

In case Lemma 2.1(3), this is Lemma 2.7(3).

This finishes the proof. $\square$

3. Examples

We conclude this paper with several examples that indicate the variety of covariant subrings that can be obtained from coactions on down-up algebras, and give some concluding comments.

Example 3.1. A group $G = \langle a, b\rangle$ coacts on $\mathbb{D}_\beta$, where $\deg_G d = a$ and $\deg_G u = b$, with trivial homological codeterminant if $a^2b^2 = 1$ [Lemma 1.5]. One such family of groups is the dihedral groups, for $n \geq 2$,

$$D_{2n} = \langle a, b \mid a^2 = b^2 = (ba)^n = 1\rangle.$$

Each element of $\mathbb{D}_\beta$ can be written as a linear combination of monomials of the form $d^i(ud)^j u^k$, and such a monomial is in the identity component of $\mathbb{D}_\beta$ exactly when

$$a^i(ba)^j b^k = 1.$$

Clearly d^2 and u^2 are covariants, so it suffices to consider the four cases $(i, k) = (0, 0), (1, 0), (0, 1), (1, 1)$, and it is easy to check that the subring of covariants is generated as a $\Bbbk$-algebra by $d^2, u^2, (du)^n, (ud)^n$, i.e.

$$\mathbb{D}_\beta^{\mathrm{co}\ D_{2n}} = \Bbbk\langle d^2, u^2, (du)^n, (ud)^n\rangle.$$

When $\beta = \pm 1$, $\mathbb{D}_{\pm 1}^{\mathrm{co}\ D_{2n}}$ is isomorphic to the commutative algebra

$$\mathbb{D}_{\pm 1}^{\mathrm{co}\ D_{2n}} \cong \frac{\Bbbk[X, Y, Z, W]}{(X^nY^n - ZW)},$$

a hypersurface in $\mathbb{A}^4$. When $\beta \neq \pm 1$, the ring of covariants is a hypersurface in a noncommutative skew polynomial ring in the sense of [18, Def. 1.3(c)]. For any $\beta \neq 0$, $\mathbb{D}_\beta \# \Bbbk^{D_{2n}}$ is a noncommutative quasi-resolution or NQR (a generalization of NCCR) of $\mathbb{D}_\beta^{\mathrm{co}\ D_{2n}}$ in the sense of [22]. When $n = 4$, this example should be compared to [11, Example 2.1], where a different coaction of D_8 (without trivial homological codeterminant) on $\mathbb{D}_1$ is given; in that example the ring of covariants is a commutative hypersurface in $\mathbb{A}^4$, but Auslander's Theorem fails (Remark 1.6(1)).

Next we consider a second coaction on $\mathbb{D}_\beta$, where the ring of covariants is quite different.

Example 3.2. The quaternion group $G = Q_8$ of order 8 coacts on $\mathbb{D}_\beta$ by $\deg_G d = i$ and $\deg_G u = k$ with trivial homological determinant (Lemma 1.5). A monomial $d^{e_1}(ud)^{e_2}u^{e_3}$ has group grade the identity of Q_8 exactly when

$$i^{e_1} j^{e_2} k^{e_3} = 1$$

holds in Q_8. It is not hard to check that the covariants are generated by the following 9 monomials:

$$d^4,\ u^4,\ d^2u^2,\ d^2(ud)^2,\ (ud)^2u^2,$$

$$d(ud)u^3 = (du)^2u^2,\ d^3(ud)u = d^2(du)^2,\ d(ud)^3u = (du)^4,\ (ud)^4.$$

When $\beta \neq 0$, $\mathbb{D}_\beta \# \Bbbk^{Q_8}$ is a noncommutative quasi-resolution (NQR) of the covariant subring $\mathbb{D}_\beta^{\mathrm{co}\ Q_8}$ in the sense of [22].

Finally we consider the down-up algebra $\mathbb{H}$.

Example 3.3. The dihedral groups D_{2n} coact on $\mathbb{H}$ homogeneously with trivial homological codeterminant, although our proof of Auslander's Theorem holds for any group coaction in this case (see Lemma 2.7 and the comments after that). Suppose that $G := D_{2n} = \langle a, b \mid a^2 = b^2 = 1 = (ab)^n\rangle$ and that $\deg_G x = a$ and $\deg_G y = b$. The relations in $\mathbb{H}$ (E2.0.2) can be written as

$$(x^2 - y^2)y = -y(x^2 - y^2),$$
$$x(x^2 - y^2) = -(x^2 - y^2)x,$$

and hence $x^2 - y^2$ is a normal element of $\mathbb{H}$, and, moreover, xy and yx commute with $y^2 - x^2$. It is clear that x^2 and y^2 are covariants under this action, and that $\mathbb{H}/(x^2 - y^2) \cong \Bbbk\langle x, y\rangle/(x^2 - y^2)$. Since $x^2 - y^2$ is also a normal element of $\mathbb{H}^{\text{co } D_{2n}}$, we obtain that

$$\frac{\mathbb{H}^{\text{co } D_{2n}}}{(x^2 - y^2)} \cong \left(\frac{\Bbbk\langle x, y\rangle}{(x^2 - y^2)}\right)^{\text{co } D_{2n}}.$$

It follows that the generators of $\mathbb{H}^{\text{co } D_{2n}}$ are the 4 elements

$$x^2,\ y^2,\ (yx)^n,\ x(yx)^{n-1}y = (xy)^n.$$

Next we show that $\mathbb{H}^{\text{co } D_{2n}}$ is a hypersurface [18, Def. 1.3(c)] in an iterated Ore extension that is an AS-regular algebra of dimension 4.

Multiplying the relation $x^2y + yx^2 - 2y^3 = 0$ by y on each side and subtracting gives the relation $x^2y^2 - y^2x^2 = 0$. We next give a number of relations in $\mathbb{H}$; note that the defining relations of $\mathbb{H}$ are symmetric in x and y, so the relations with x and y interchanged also hold. It is easy to check the following relation

$$x^2y - yx^2 = 2y(y^2 - x^2);$$

multiplying by x on left gives:

$$[x^2, (xy)] = 2xy(y^2 - x^2) = (y^2 - x^2)(2xy).$$

Similarly

$$[y^2, (xy)] = 2xy(y^2 - x^2).$$

Inductively one can show that

$$[y^2, (xy)^n] = 2n(xy)^n(y^2 - x^2).$$

Further

$$(yx)(xy) = y^2(y^2 + (y^2 - x^2)),$$

and inductively we get

$$\begin{aligned}&(yx)^n(xy)^n\\ &= y^2(y^2+(y^2-x^2))(y^2+2(y^2-x^2))(y^2+3(y^2-x^2))\cdots(y^2+(2n-1)(y^2-x^2)).\end{aligned}$$

Let the right side of the above equation be denoted by $w(x^2, y^2)$. We claim that the subalgebra generated by x^2, y^2, $(yx)^n$, $(xy)^n$ is a generalized Weyl algebra (or ambiskew polynomial ring, as in [14]), i.e., it is an iterated Ore algebra modulo one relation.

To simplify notation, let $X = x^2$, $Y = y^2$, $Z^+ = (yx)^n$, $Z^- = (xy)^n$. From the relations above we have the following relations:

$$\begin{aligned} XY &= YX, \\ Z^+X &= (X + 2n(Y - X))Z^+, \\ Z^+Y &= (Y + 2n(Y - X))Z^+. \end{aligned}$$

In this notation $Z^+Z^- = w(X, Y)$ and $Z^-Z^+ = w(Y, X)$. Let B be the algebra generated by X, Y, Z^+ defined by the first three relations above. Then B is the Ore extension $\Bbbk[X, Y][Z^+; \sigma]$, where σ is the automorphism of $\Bbbk[X, Y]$ given by $\sigma(X) = X + 2n(Y - X)$ and $\sigma(Y) = Y + 2n(Y - X)$. Adjoining Z^- to B adds the following three relations:

$$\begin{aligned} Z^-X &= (X - 2n(Y - X))Z^-, \\ Z^-Y &= (Y - 2n(Y - X))Z^-, \\ Z^-Z^+ &= Z^+Z^- - f(X, Y). \end{aligned}$$

where $f(X, Y) = w(X, Y) - w(Y, X) \in \Bbbk[X, Y]$. One checks that $\sigma^{-1}(X) = X - 2n(Y - X)$, $\sigma^{-1}(Y) = Y - 2n(Y - X)$ and $\sigma^{-1}(Z^+) = Z^+$ defines an algebra automorphism of B. Define the map δ on B, by $\delta(X) = \delta(Y) = 0$ and $\delta(Z^+) = -f(X, Y)$. One checks that δ extends to a σ^{-1}-skew derivation of B, preserving the three relations defining B. Hence we have

$$\mathbb{H}^{\text{co } D_{2n}} \cong \frac{\Bbbk[X, Y][Z^+; \sigma][Z^-; \sigma^{-1}, \delta]}{(Z^+Z^- - w(X, Y))},$$

and $\mathbb{H}^{\text{co } D_{2n}}$ is a hypersurface [18, Def. 1.3(c)] in an AS-regular algebra of dimension 4, with Hilbert series

$$\frac{1 - t^{4n}}{(1 - t^2)^2(1 - t^{2n})^2}.$$

Down-up algebras have no reflections [19, Prop. 6.4], so we would expect Auslander's Theorem to hold for all finite group actions; this has been proved for almost all finite group actions, except for some finite groups acting on $\mathbb{D}(\alpha, -1)$ for $\alpha \neq 2$ (abelian groups with trivial homological determinant are covered by Theorem 0.1). Theorem 0.1 also covers finite group coactions with trivial homological determinant, and we have shown Theorem 0.1 does not hold for all group coactions (Remark 1.6(1)). It remains to examine actions by other Hopf algebras on down-up algebras.

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