Canad. J. Math. Vol. 72 (3), 2020 pp. 708–731
http://dx.doi.org/10.4153/S0008414X1900004X

A Morita Cancellation Problem

D.-M. Lu, Q.-S. Wu, and J. J. Zhang

Abstract. We study a Morita-equivalent version of the Zariski cancellation problem.

1 Introduction

An algebra A is called *cancellative* if any algebra isomorphism $A[t] \cong B[t]$ of polynomial algebras for some algebra B implies that A is isomorphic to B. The famous Zariski Cancellation Problem (ZCP) asks

Is the commutative polynomial ring $\Bbbk[x_1, \dots, x_n]$ *over a field* $\Bbbk$ *cancellative for* $n \geq 1$?

See [Kr, BZ1, Gu3]. There is a long history of studying the cancellation property of affine commutative domains. For example, $\Bbbk[x_1]$ is cancellative by a result of Abhyankar, Eakin, and Heinzer in 1972 [AEH], while $\Bbbk[x_1, x_2]$ is cancellative by a result of Fujita in 1979 [Fu] and Miyanishi and Sugie in 1980 [MS] in characteristic zero, and by a result of Russell in 1981 [Ru] in positive characteristic. The ZCP for $n \geq 3$ has been open for many years. One remarkable achievement in this research area is a result of Gupta in 2014 [Gu1, Gu2], which settled the ZCP negatively in positive characteristic for $n \geq 3$. The ZCP in characteristic zero remains open for $n \geq 3$.

The ZCP (especially in dimension two) is closely related to the Automorphism Problem, the Characterization Problem, the Linearization Problem, the Embedding Problem, and the Jacobian Conjecture; see [Kr, EH, Gu3, BZ1] for history, partial results and references concerning the cancellation problem.

The ZCP for noncommutative algebras was introduced in [BZ1] and further investigated in [LWZ]. During the last few years, several researchers have been making significant contributions to the cancellation problem in the noncommutative setting and related topics; see, for example, [BZ1, BZ2, BY, CPWZ1, CPWZ2, CYZ1, CYZ2, Ga, GKM, GWY, LY, LWZ, LMZ, NTY, Ta1, Ta2, WZ].

The first goal of this paper is the introduction of a new cancellation property for noncommutative algebras. Let $\Bbbk$ be a base field; in the sequel, everything is over $\Bbbk$. For any algebra A, let $M(A)$ denote the category of right A-modules.

Received by the editors May 8, 2018; revised January 3, 2019.
Published online on Cambridge Core January 29, 2019.
D.-M. Lu was partially supported by the National Natural Science Foundation of China (Grant No. 11671351). Q.-S. Wu was partially supported by the National Natural Science Foundation of China (Grant No. 11771085 and Key Project No. 11331006). J.J. Zhang was partially supported by the US National Science Foundation (Nos. DMS-1402863 and DMS-1700825).
AMS subject classification: 16P99, 16W99.
Keywords: Zariski cancellation problem, Morita cancellation problem.

Definition 1.1 An algebra A is called *Morita cancellative* if for any algebra B,

$$M(A[t]) \text{ is equivalent to } M(B[t])$$

implies that

$$M(A) \text{ is equivalent to } M(B).$$

h is Morita version of the cancellation property is one of the natural generalizations of the original Zariski cancellation property when we study noncommutative algebras. Another generalization involves the derived category of modules. Let $D(A)$ denote the derived category of right A-modules for an algebra A.

Definition 1.2 An algebra A is called *derived cancellative* if for any algebra B,

$$D(A[t]) \text{ is triangulated equivalent to } D(B[t])$$

implies that

$$D(A) \text{ is triangulated equivalent to } D(B).$$

We will show [h eorem Ë8] that if Z is a commutative domain, then

$$Z \text{ is Morita cancellative if and only if } Z \text{ is cancellative}$$

and

$$Z \text{ is derived cancellative if and only if } Z \text{ is cancellative.}$$

In general, when A is noncommutative, the relationships between these three diòerent versions of cancellation property are not clear. Lemma ... (together with Example ..þ) provides noncommutative algebras that are neither cancellative, nor Morita cancellative, nor derived cancellative. We will introduce some general methods to handle the Morita cancellation problems for noncommutative algebras.

h e second aim of the paper is to show several classes of algebras are Morita (or derived) cancellative. First, we generalize a result of [LWZ, h eorem ¨...].

Theorem 1.3 *Suppose A is strongly Hopfian (Definition ...). and the center of A is artinian. Then A is Morita cancellative.*

Note that left (or right) noetherian algebras and locally ÿnite $\mathbb{N}$-graded algebras are strongly Hopÿan [Example . þ]. So h eorem Ëq covers a large class of algebras. h e following are consequences of the above theorem; see also [LWZ, Corollary ¨.q and h eorem ¨.] for comparison.

Theorem 1.4 *Let A be a left (or right) noetherian algebra such that its center is artinian. Then A is Morita cancellative. As a consequence, every finite dimensional algebra over a base field $\Bbbk$ is Morita cancellative.*

For non-noetherian algebras we have the following theorem.

Theorem 1.5 *For every finite quiver Q, the path algebra $\Bbbk Q$ is Morita cancellative.*

Recall from [BZË, h eorem ¨.þ] that, if A is an aõ ne domain of GK-dimension two over an algebraically closed ÿeld of characteristic zero and A is not commutative, then A is cancellative. It is well-known that, in contrast, noncommutative aõ ne prime (non-domain) algebras of GK-dimension two need not be cancellative [LWZ, Example Ëq(þ)] and that commutative aõ ne domains of GK-dimension two need not be cancellative, by examples of Hochster [Ho] and Danielewski [Da]; see Example . .þ(i) and (ii). For GK-dimension one, a classical result of Abhyankar, Eakin, and Heinzer [AEH, h eorem q.q] says that every aõ ne commutative domain of GK-dimension one is cancellative. Recently, it was proved that every aõ ne prime $\Bbbk$-algebra of GK-dimension one is cancellative. Next we add another result in low GK-dimension.

Theorem 1.6 *Let $\Bbbk$ be algebraically closed. Then every affine prime $\Bbbk$-algebra of GK-dimension one is Morita cancellative.*

We are mainly dealing with the Morita cancellation property in this paper, but occasionally, we have some results concerning the derived cancellation property, such as the next result.

Theorem 1.7 (Corollary Ç. .). *Let Z be a commutative domain. Then Z is cancellative if and only if Z is Morita cancellative, if and only if Z is derived cancellative.*

A question in [LWZ, Question þ. (q)] asks if the Sklyanin algebras are cancellative. We partially answer this question.

Corollary 1.8 (Example B.Ë̇ (.).) *Let A be a non-PI Sklyanin algebra of global dimension three. Then A is both cancellative and Morita cancellative.*

h e paper is organized as follows. Section . .contains deÿnitions, known examples, and preliminaries. In Sections q and , we introduce the Morita version of the retractable and detectable properties. In Section þ, we prove h eorems Ëq and Ë. . h eorems ËB and Ë8 are proved in Section B and Section 8, respectively. h e derived cancellation property is briefly studied in Section Ç Section Ç also contains some comments, remarks, and examples.

2 Definitions and Preliminaries

Some deÿnitions and examples are copied from [BZË, LWZ]. First, we recall a classical deÿnition. Let $A[t]$ (or $A[s]$) be the polynomial algebra over A by adding one central indeterminate.

Definition 2.1 Let A be an algebra.

(i) We call A *cancellative* if any algebra isomorphism $A[t] \cong B[s]$ implies that $A \cong B$.

(ii) We call A *strongly cancellative* if, for each $n \geq$ Ë any algebra isomorphism

$$A[t_1, \dots, t_n] \cong B[s_1, \dots, s_n]$$

implies that $A \cong B$.

The following are two new cancellation properties that we will study in this paper.

Definition 2.2 Let A be an algebra.

(i) We call A *m-cancellative* if any equivalence of abelian categories $M(A[t]) \cong M(B[s])$ implies that $M(A) \cong M(B)$.

(ii) We call A *strongly m-cancellative* if, for each $n \geq 1$, any equivalence of abelian categories

$$M(A[t_1, \dots, t_n]) \cong M(B[s_1, \dots, s_n])$$

implies that $M(A) \cong M(B)$.
The letter m here stands for the word "Morita".

Definition 2.3 Let A be an algebra.

(i) We call A *d-cancellative* if any equivalence of triangulated categories

$$D(A[t]) \cong D(B[s])$$

implies that $D(A) \cong D(B)$.

(ii) We call A *strongly d-cancellative* if, for each $n \geq 1$, any equivalence of triangulated categories

$$D(A[t_1, \dots, t_n]) \cong D(B[s_1, \dots, s_n])$$

implies that $D(A) \cong D(B)$.
The letter d here stands for the word "derived".

Let $A[\underline{t}]$ denote the polynomial algebra $A[t_1, \dots, t_n]$ and $A[\underline{s}]$ the polynomial algebra $A[s_1, \dots, s_n]$ for an integer n (that is not specified) when no confusion occurs.

Lemma 2.4 *Let A be a commutative algebra that is not (strongly) cancellative. Let B be an algebra with center $Z(B) = \Bbbk$. Then $A \otimes B$ is neither (strongly) cancellative, nor (strongly) m-cancellative, nor (strongly) d-cancellative.*

Proof Since A is not (strongly) cancellative, there is a commutative algebra C such that A is not isomorphic to C, but $A[t_1, \dots, t_n] \cong C[s_1, \dots, s_n]$ for $n = 1$ (or some $n \geq 1$). Then $A \otimes B[\underline{t}] \cong C \otimes B[\underline{s}]$. As a consequence, we obtain that

$$M\big(A \otimes B[\underline{t}]\big) \cong M\big(C \otimes B[\underline{s}]\big) \quad \text{and} \quad D\big(A \otimes B[\underline{t}]\big) \cong D\big(C \otimes B[\underline{s}]\big).$$

Since the center $Z(A \otimes B) = A$ is not isomorphic to $Z(C \otimes B) = C$, we obtain that $M(A \otimes B) \ncong M(C \otimes B)$ and that $D(A \otimes B) \ncong D(C \otimes B)$. Therefore, the assertions follow. ∎

Next we give some precise examples of non-cancellative commutative algebras. The above lemma gives an easy way of producing non-cancellative noncommutative algebras.

Example 2.5 (i) Let $\Bbbk$ be the field of real numbers $\mathbb{R}$. Hochster showed that $\Bbbk[P, Q, X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$ is not cancellative [Ho].

(ii) The following example is due to Danielewski [Da]. Let $n \geq 1$ and let B_n be the coordinate ring of the surface $x^n y = z^2 - 1$ over $\Bbbk := \mathbb{C}$. Then $B_i \ncong B_j$ if $i \neq j$, but $B_i[t] \cong B_j[s]$ for all $i, j \geq 1$; see [Fi, Wi] for more details. Therefore, all the B_n's are not cancellative.

(iii) Suppose char $\Bbbk > 0$. Gupta showed that $\Bbbk[x_1, \dots, x_n]$ is not cancellative for every $n \geq 3$ [Gu1, Gu..].

As a consequence of Lemma ... (by taking $B = \Bbbk$), the algebras above are neither m-cancellative nor d-cancellative.

We also need to recall higher derivations and Makar–Limanov invariants.

Definition 2.6 Let A be an algebra.

(i) [HS] A *higher derivation* (or *Hasse–Schmidt derivation*) on A is a sequence of $\Bbbk$-linear endomorphisms $\partial := \{\partial_i\}_{i=0}^{\infty}$ such that:

$$\partial_0 = \mathrm{id}_A \quad \text{and} \quad \partial_n(ab) = \sum_{i=0}^{n} \partial_i(a)\partial_{n-i}(b)$$

for all $a, b \in A$ and all $n \geq 0$. The collection of all higher derivations on A is denoted by $\mathrm{Der}^H(A)$.

(ii) A higher derivation is called *locally nilpotent* if

(a) given any $a \in A$ there exists $n \geq 1$ such that $\partial_i(a) = 0$ for all $i \geq n$,

(b) the map

$$G_{\partial,t} : A[t] \longrightarrow A[t]$$

defined by

$$a \longmapsto \sum_{i=0}^{\infty} \partial_i(a)t^i \text{ for all } a \in A \text{ and } \ t \longmapsto t$$

is an algebra automorphism of $A[t]$.

(iii) For any $\partial \in \mathrm{Der}^H(A)$, the kernel of ∂ is defined to be

$$\ker \partial = \bigcap_{i \geq 1} \ker \partial_i.$$

(iv) The set of locally nilpotent higher derivations is denoted by $\mathrm{LND}^H(A)$. Given a nonzero element $d \in A$, let

$$\mathrm{LND}_d^H(A) = \{\partial \in \mathrm{LND}^H(A) \mid d \in \ker \partial\}.$$

Note that (a) in part (ii) of the above definition implies that the map $G_{\partial,t}$ defined in (b) is an algebra endomorphism. It is not clear to us whether $G_{\partial,t}$ is automatically an automorphism. However, by [BZ1, Lemma ...(..)], when ∂ is an iterative higher derivation, $G_{\partial,t}$ is automatically an automorphism.

It is easy to see that $1 \in \ker \partial$ for all higher derivations ∂. Hence, $\mathrm{LND}_1^H(A) = \mathrm{LND}^H(A)$. We generalize the original definition of the Makar–Limanov invariant [Mak].

Definition 2.7 Let A be an algebra and d a nonzero element in A.

(i) The *Makar-Limanov$_d^H$ invariant* of A is defined to be

(EË8Ë) $$ML_d^H(A) := \bigcap_{\delta \in \mathrm{LND}_d^H(A)} \ker(\delta).$$

(ii) We say that A is LND_d^H*-rigid* if $ML_d^H(A) = A$.
(iii) A is called *strongly* LND_d^H*-rigid* if $ML_d^H(A[t_1, \dots, t_n]) = A$, for all $n \geq$ Ë.
(iv) h e *Makar–Limanov*$_d^H$ *center* of A is deÿned to be

$$ML_{d,Z}^H(A) = ML_d^H(A) \cap Z(A).$$

(v) A is called *strongly* $\mathrm{LND}_{d,Z}^H$*-rigid* if $ML_{d,Z}^H(A[t_1, \dots, t_n]) = Z(A)$, for all $n \geq$ Ë.

3 Morita Invariant Properties and the $\mathcal{P}$-discriminant

In this section we will recall some well-known facts about Morita equivalence. Two algebras A and B are Morita equivalent if their right module categories $M(A)$ and $M(B)$ are equivalent. We list some properties concerning Morita theory.

Lemma 3.1 ([AF, Ch. B]) *Let A and B be two algebras that are Morita equivalent.*

(i) *There is an* (A,B)*-bimodule* Ä *that is invertible, namely,* Ä $\otimes_B$ Ä$^\vee \cong A$ *and* Ä$^\vee \otimes_A$ Ä $\cong B$ *as bimodules, where* Ä$^\vee := \mathrm{Hom}_B($Ä$_B, B_B)$.

(ii) *The bimodule* Ä *induces naturally algebra isomorphisms* $A \cong \mathrm{End}($Ä$_B)$ *and* $B^{op} \cong \mathrm{End}({}_A$Ä$)$.

(iii) *Further,* $Z(A) \cong \mathrm{Hom}_{(A,B)}($Ä, Ä$) \cong Z(B)$, *which induces an isomorphism*

(E..Ë.Ë) $$\omega : Z(A) \longrightarrow Z(B)$$

such that, for each $x \in Z(A)$, *the left multiplication of* x *on* Ä *equals the right multiplication of* $\omega(x)$ *on* Ä.

(iv) *By using* ω *to identify the center* $Z = Z(A)$ *of A with the center of B, both A and B are central Z-algebras. In this case, both* Ä *and* Ä$^\vee$ *are central Z-modules.*

(v) *Let* ω *be given as in* (E..Ë.Ë). *Then, for any ideal I of* $Z(A)$, A/IA *and* $B/\omega(I)B$ *are Morita equivalent.*

(vi) [AF, Ex.™, p....B] *Let* A, B, T *be K-algebra for some commutative ring K. Then* $A \otimes_K T$ *and* $B \otimes_K T$ *are Morita equivalent.*

Morita equivalences have been studied extensively for decades. A ring theoretic property is called a *Morita invariant* if it is preserved by Morita equivalences.

Example 3.2 h e following properties are Morita invariants:

(i) being simple (resp., semisimple);
(ii) being right (or left) noetherian, right (or left) artinian;
(iii) having global dimension d (Krull dimension d, GK-dimension d, etc);
(iv) being a full matrix algebra $M_n(\mathbb{k})$ for some n, when $\mathbb{k}$ is algebraically closed;
(v) being an Azumaya algebra [Sc, h eorem];
(vi) being quasi-Frobenius;
(vii) being prime, semiprime, right (or left) primitive, semiprimitive;
(viii) being semilocal;
(ix) being primitive, but not simple;
(x) being noetherian, but not artinian;
(xi) the center being $\mathbb{k}$;
(xii) being projective over its center.

Let R be a commutative algebra, $\operatorname{Spec} R$ be the prime spectrum of R and $\operatorname{MaxSpec}(R) := \{\mathfrak{m} \mid \mathfrak{m} \text{ is a maximal ideal of } R\}$ be the maximal spectrum of R. For any $S \subseteq \operatorname{Spec} R$, $I(S)$ is the ideal of R vanishing on S, namely,

$$I(S) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}.$$

For any algebra A, $A^{\times}$ denotes the set of invertible elements in A.

A property $\mathcal{P}$ considered in the following means a property defined on a class of algebras that is an invariant under algebra isomorphisms.

Definition 3.3 Let A be an algebra, $Z = Z(A)$ be the center of A. Let $\mathcal{P}$ be a property defined for $\Bbbk$-algebras (not necessarily a Morita invariant).

(i) The $\mathcal{P}$-*locus* of A is defined to be

$$L_{\mathcal{P}}(A) := \{\mathfrak{m} \in \operatorname{MaxSpec}(Z) \mid A/\mathfrak{m}A \text{ has property } \mathcal{P}\}.$$

(ii) The $\mathcal{P}$-*discriminant set* of A is defined to be

$$D_{\mathcal{P}}(A) := \operatorname{MaxSpec}(Z) \setminus L_{\mathcal{P}}(A).$$

(iii) The $\mathcal{P}$-*discriminant ideal* of A is defined to be

$$I_{\mathcal{P}}(A) := I(D_{\mathcal{P}}(A)) \subseteq Z.$$

(iv) If $I_{\mathcal{P}}(A)$ is a principal ideal of Z generated by $d \in Z$, then d is called the $\mathcal{P}$-*discriminant* of A, denoted by $d_{\mathcal{P}}(A)$. In this case $d_{\mathcal{P}}(A)$ is unique up to an element in $Z^{\times}$.

(v) Let $\mathcal{C}$ be a class of algebras over $\Bbbk$. We say that $\mathcal{P}$ is $\mathcal{C}$-*stable* if for every algebra A in $\mathcal{C}$ and every $n \geq 1$,

$$I_{\mathcal{P}}\big(A \otimes \Bbbk[t_1, \ldots, t_n]\big) = I_{\mathcal{P}}(A) \otimes \Bbbk[t_1, \ldots, t_n]$$

as an ideal of $Z \otimes \Bbbk[t_1, \ldots, t_n]$. If $\mathcal{C}$ is a singleton $\{A\}$, we simply call $\mathcal{P}$ A-*stable*. If $\mathcal{C}$ is the whole collection of $\Bbbk$-algebras with the center affine over $\Bbbk$, we simply call $\mathcal{P}$ *stable*.

In general, neither $L_{\mathcal{P}}(A)$ nor $D_{\mathcal{P}}(A)$ is a subscheme of $\operatorname{Spec} Z(A)$.

Example 3.4 Suppose $\Bbbk = \mathbb{C}$. Let A be the universal enveloping algebra of the simple Lie algebra sl_2. It is well known that $Z(A) = \Bbbk[Q]$, where $Q = 2(ef + fe) + h^2$.

Let $\mathcal{S}$ be the property of being simple. Then $D_{\mathcal{S}}(A)$ is the set of integer points of the form $\{n^2 + 2n \mid n \in \mathbb{N}\}$ inside the $\operatorname{MaxSpec} \Bbbk[Q]$; see [Di] or [Sm, p. ™Ç]. In this case, the $\mathcal{S}$-discriminant ideal of A is the zero ideal of $\Bbbk[Q]$ and the $\mathcal{S}$-discriminant of A is the element $0 \in \Bbbk[Q]$.

Note from [Di] or [Sm, p. ™Ç] that for each $c = n^2 + 2n$, $A/(Q - c)A$ has a unique proper two-sided ideal M_c and M_c is of codimension $(n+1)^2$. Let $\mathcal{P}_n$ be the property of not having a factor ring isomorphic to the matrix algebra $M_{n+1}(\Bbbk)$. Then $D_{\mathcal{P}_n}(A)$ is the singleton $\{n^2 + 2n\}$, as a subset of $D_{\mathcal{S}}(A)$. As a consequence, the $\mathcal{P}_n$-discriminant ideal of A is $(Q - (n^2 + 2n)) \subseteq \Bbbk[Q]$ and the $\mathcal{P}_n$-discriminant of A is the element $Q - (n^2 + 2n) \in \Bbbk[Q]$.

It is clear that $\mathcal{S}$ is a Morita invariant, but $\mathcal{P}_n$ is not for each fixed n.

Lemma 3.5 *Let $\mathcal{P}$ be a property.*

(i) *Suppose $\phi : A \to B$ is an isomorphism. Then ϕ preserves the following:*

(a) *$\mathcal{P}$-locus;*
(b) *$\mathcal{P}$-discriminant set;*
(c) *$\mathcal{P}$-discriminant ideal;*
(d) *$\mathcal{P}$-discriminant (if it exists).*

(ii) *Suppose that $\mathcal{P}$ is a Morita invariant and that A and B are Morita equivalent. Then the algebra map ω in (E..Ë.Ë) preserves the following:*

(a) $\mathcal{P}$-locus;
(b) $\mathcal{P}$-discriminant set;
(c) $\mathcal{P}$-discriminant ideal;
(d) $\mathcal{P}$-discriminant (if it exists).

Proof (i) h is is clear.

(ii) h is follows from the deÿnition, Lemma q.Ë(v) and the hypothesis that $\mathcal{P}$ is a Morita invariant. ∎

In this and the next sections we study two properties that are closely related to the m-cancellative property. h e retractable property was introduced in [LWZ, Deÿnitions ..Ë and ..þ]. Next we generalize Z-retractability to the Morita setting.

Definition 3.6 Let A be an algebra.

(i) [LWZ, Deÿnition ..þ(Ë)] We call A *Z-retractable*, if for any algebra B, an algebra isomorphism $\phi : A[t] \cong B[s]$ implies that $\phi(Z(A)) = Z(B)$.

(ii) [LWZ, Deÿnition ..þ(..)] We call A *strongly Z-retractable*, if for any algebra B and integer $n \geq$ Ë, an algebra isomorphism $\phi : A[t_1, \ldots, t_n] \cong B[s_1, \ldots, s_n]$ implies that $\phi(Z(A)) = Z(B)$.

(iii) We call A *m-Z-retractable* if, for any algebra B, an equivalence of categories $M(A[t]) \cong M(B[s])$ implies that $\omega(Z(A)) = Z(B)$, where $\omega : Z(A)[t] \to Z(B)[s]$ is given as in (E..Ë.Ë).

(iv) We call A *strongly m-Z-retractable* if, for any algebra B and $n \geq$ Ë, an equivalence of categories $M(A[t_1, \ldots, t_n]) \cong M(B[s_1, \ldots, s_n])$ implies that $\omega(Z(A)) = Z(B)$, where $\omega : Z(A)[t_1, \ldots, t_n] \to Z(B)[s_1, \ldots, s_n]$ is given as in (E..Ë.Ë).

h e following proposition is similar to [LWZ, Lemma ..B].

Proposition 3.7 *Let A be an algebra whose center $Z := Z(A)$ is an affine domain. Let $\mathcal{P}$ be a stable Morita invariant property (resp., stable property) and assume that the $\mathcal{P}$-discriminant of A, denoted by d, exists.*

(i) *Suppose $ML_d^H(Z[t]) = Z$. Then A is m-Z-retractable (resp., Z-retractable).*

(ii) *Suppose that Z is strongly LND_d^H-rigid. Then A is strongly m-Z-retractable (resp., strongly Z-retractable).*

Proof h e proofs of (i) and (ii) are similar, so we prove only (ii). We only work on the strongly m-Z-retractable version; the strongly Z-retractable version is similar.

Suppose that $A[t_1,\dots,t_n]$ is Morita equivalent to $B[s_1,\dots,s_n]$ for some algebra B and for some $n \geq$ Ë.Let $\omega : Z\otimes \Bbbk[\underline{t}] \to Z(B)\otimes\Bbbk[\underline{s}]$ be the map given in (E..Ë.Þ). Since $\mathcal{P}$ is stable, $d_{\mathcal{P}}(A[\underline{t}]) = d\otimes$ Ë, where Ëis the identity element of the polynomial ring $\Bbbk[\underline{t}]$. In other words, the principal ideal ($d\otimes$ Ë) is the $\mathcal{P}$-discriminant ideal of $A[\underline{t}]$. Since ω preserves the discriminant ideal [Lemma q.þ(.c)] and $\mathcal{P}$ is stable, we obtain that

$$\text{(E..8Ë)}\quad \omega\big((d\otimes \text{Ë})\big) = \omega\big((d)\otimes\Bbbk[\underline{t}]\big) = \omega\big(I_{\mathcal{P}}(A[\underline{t}])\big) = I_{\mathcal{P}}\big(B[\underline{s}]\big) = I_{\mathcal{P}}(B)\otimes\Bbbk[\underline{s}].$$

As a consequence, $I_{\mathcal{P}}(B)$ is a principal ideal, denoted by (d'), where d' is the $\mathcal{P}$-discriminant of B. Equation (E..8Ë) implies that

$$\omega(d\otimes \text{Ë}) =_{Z(B[\underline{s}])^\times} d'\otimes \text{Ë}',$$

where Ë$'$ is the identity element of the polynomial ring $\Bbbk[\underline{s}]$. Since $Z(B)$ is a domain, $Z(B[\underline{s}])^\times = Z(B)^\times$. Hence ω maps d to d' up to a scalar in $Z(B)^\times$.

Now consider the map $\omega : Z\otimes\Bbbk[\underline{t}] \to Z(B)\otimes\Bbbk[\underline{s}]$ again. Since ω maps d to d', by the strongly LND^H_d-rigidity of Z, we have

$$\omega(Z) = \omega\big(ML^H_d(Z\otimes\Bbbk[\underline{t}])\big) = ML^H_{d'}\big(Z(B)\otimes\Bbbk[\underline{s}]\big) \subseteq Z(B),$$

where the last $\subseteq$ follows from the computation given in [BZË, Example ...]. h is means that the isomorphism ω induces an algebra map from Z to $Z(B)$. Let Z' be the subalgebra $\omega^{-1}(Z(B)) \subset Z[\underline{t}]$. h en Z' contains Z, which is considered as the degree zero part of the algebra $Z[\underline{t}]$, and we have

$$\begin{aligned}\operatorname{GKdim} Z' &= \operatorname{GKdim} Z(B) = \operatorname{GKdim} Z(B)[\underline{s}] - n = \operatorname{GKdim} Z[\underline{t}] - n\\ &= \operatorname{GKdim} Z.\end{aligned}$$

By [BZË, Lemma q..], $Z' = Z$. h erefore, $\omega(Z) = Z(B)$ as required. ■

h e rest of this section follows closely [LWZ, Section ..]. By [BZË, Section þ], eòectiveness (and the dominating property) of the discriminant controls LND^H-rigidity. We now recall the deÿnition of the eòectiveness of an element. An algebra is called *PI* if it satisÿes a polynomial identity.

Next we will use ÿltered algebras and associated graded algebras; see [YZ..., Section Ë] for more details. By a ÿltration of a $\Bbbk$-algebra A, we mean an ascending ÿltration $F := \{F_iA\}_{i\geq 0}$ of vector spaces such that Ë $\in F_0A$ and $F_iAF_jA \subseteq F_{i+j}A$ for all $i,j \geq$ ¨. We assume that F is (separated and) exhaustive. By [YZ..., Lemma Ë.Ë], giving a ÿltration on an algebra A is equivalent to giving a degree on the set of generators of A.

Definition 3.8 ([BZË, Deÿnition þ.Ë]) Let A be a domain and suppose that $Y = \bigoplus_{i=1}^n \Bbbk x_i$ generates A as an algebra. An element ¨ $\neq f \in A$ is called *effective* if the following conditions hold.

(i) h ere is an $\mathbb{N}$-ÿltration $\{F_iA\}_{i\geq 0}$ on A such that the associated graded ring $\operatorname{gr} A$ is a domain (one possible ÿltration is the trivial ÿltration $F_0A = A$). With this ÿltration we deÿne the degree of elements in A, denoted by $\deg_A$.

(ii) For every testing $\mathbb{N}$-ÿltered PI algebra T with $\operatorname{gr} T$ being an $\mathbb{N}$-graded domain and for every testing subset $\{y_1,\dots,y_n\} \subset T$ satisfying the following:

(a) it is linearly independent in the quotient $\Bbbk$-module $T/\Bbbk Ë_T$, and
(b) $\deg_T y_i \geq \deg_A x_i$ for all i and $\deg_T y_{i_0} > \deg_A x_{i_0}$ for some i_0,

then there is a presentation of f of the form $f(x_1, \ldots, x_n)$ in the free algebra $\Bbbk\langle x_1, \ldots, x_n\rangle$, such that either

$$f(y_1, \ldots, y_n) = ¨ \quad \text{or} \quad \deg_T f(y_1, \ldots, y_n) > \deg_A f.$$

Here is an easy example.

Example 3.9 ([LWZ, Example ..Ç]) Every non-invertible nonzero element in $\Bbbk[t]$ is eòective in $\Bbbk[t]$.

Other examples of eòective elements are given in [BZË, Section þ]. h ere is another concept, called "dominating"; see [BZË, Deÿnition . þ] or [CPWZË, Deÿnition ..Ë(.)], that is similar to eòectiveness. Both of these properties control LND^H-rigidity. h e following result is similar to [BZË, h eorem þ...] and [LWZ, h eorem ..™].

Theorem 3.10 *If d is an effective (resp., dominating) element in an affine commutative domain Z, then Z is strongly LND^H_d-rigid.*

Proof Since the proofs for the "eòective" case and the "dominating" case are very similar, we prove only the "eòective" case.

Suppose Z is generated by $\{x_j\}_{j=1}^m$. Let $\partial \in \mathrm{LND}^H_d(Z[t_1, \ldots, t_n])$ and $G := G_{\partial,t} \in \mathrm{Aut}_{\Bbbk[t]}(Z[t_1, \ldots, t_n][t])$ as in Deÿnition ..B(.). h en, for each j,

$$G(x_j) = x_j + \sum_{i \geq 1} t^i \partial_i(x_j).$$

Since $d \in \ker \partial$, by deÿnition,

$$G(d) = d. \tag{E..Ë¨.Ë}$$

Recall from Deÿnition q.Ç that, when d is eòective, Z is a ÿltered algebra with $\deg_Z$ is deÿned as in [YZ..,Lemma Ë.Ì]. It is clear that $Z' := Z[t_1, \ldots, t_n]$ is a ÿltered algebra with $\deg_{Z'} z = \deg_Z z$ for all $z \in Z$ and $\deg_{Z'} t_s = Ë$ for $s = Ë, \ldots, n$. We take the test algebra T to be $Z[t_1, \ldots, t_n][t] = Z'[t]$, where the ÿltration on T is determined by $\deg_T(z) = \deg_Z(z)$ for all $z \in Z$, $\deg_T t_s = Ë$ for $s = Ë, \ldots, n$, and $\deg_T t = \alpha$, where

$$\alpha > \sup\{\deg_{Z'} \partial_i(x_j) \mid j = Ë, \ldots, m, i \geq ¨\}.$$

Now set $y_j = G(x_j) \in T$. By the choice of α, we have that

(a) $\deg_T y_j \geq \deg_Z x_j$, and that
(b) $\deg_T y_j = \deg_Z x_j$ if and only if $y_j = x_j$.

Let $f(x_1, \ldots, x_m)$ be some polynomial presentation of d as in Deÿnition q.Ç. If $G(x_j) \neq x_j$ for some j, by the eòectiveness of d as in Deÿnition q.Ç, $f(y_1, \ldots, y_m) = ¨$ or $\deg_T f(y_1, \ldots, y_m) > \deg_Z d = \deg_T d$. So $f(y_1, \ldots, y_m) \neq_{Z^\times} d$. But $f(y_1, \ldots, y_m) = G(d) =_{Z^\times} d$ by (E..Ë¨.Ë), a contradiction. h erefore, $G(x_j) = x_j$ for all j. As a consequence, $\partial_i(x_j) = ¨$ for all $i \geq Ë$, or equivalently, $x_j \in \ker \partial$. Since Z is generated by x_j's, $Z \subset \ker \partial$. h us, $Z \subseteq ML^H_d(Z[t_1, \ldots, t_n])$. It is clear that $Z \supseteq ML^H_d(Z[t_1, \ldots, t_n])$; see [BZË, Example ...]. h erefore, $Z = ML^H_d(Z[t_1, \ldots, t_n])$, as required. ■

h e following corollary will be used several times.

Corollary 3.11 *Let A be an algebra such that the center of A is $\Bbbk[x]$. Let $\mathcal{P}$ be a stable Morita invariant property (resp., stable property) such that the $\mathcal{P}$-discriminant of A, denoted by d, is a nonzero non-invertible element in $Z(A) = \Bbbk[x]$. Then $Z(A)$ is strongly* LND_d^H*-rigid and A is strongly m-Z-retractable (resp., strongly Z-retractable).*

Proof By Example q.™, d is an eòective element in $Z(A)$. By h eorem q.Ë˙, $Z(A)$ is strongly LND_d^H-rigid. By Proposition q.8(ii), A is strongly m-Z-retractable (resp., strongly Z-retractable). ■

4 Morita Detectability

First, we recall the detectability introduced in [LWZ]. If B is a subring of C and $f_1, \dots, f_m$ are elements of C, then the subring generated by B and the subset $\{f_1, \dots, f_m\}$ is denoted by $B\{f_1, \dots, f_m\}$.

Definition 4.1 ([LWZ, Deÿnition q.Ë]) Let A be an algebra.

(i) We call A *detectable* if any algebra isomorphism $\phi : A[t] \cong B[s]$ implies that $B[s] = B\{\phi(t)\}$, or equivalently, $s \in B\{\phi(t)\}$.

(ii) We call A *strongly detectable* if for each integer $n \geq$ Ë, any algebra isomorphism

$$\phi : A[t_1, \dots, t_n] \cong B[s_1, \dots, s_n]$$

implies that $B[s_1, \dots, s_n] = B\{\phi(t_1), \dots, \phi(t_n)\}$, or equivalently, for each $i =$ Ë$, \dots, n$, $s_i \in B\{\phi(t_1), \dots, \phi(t_n)\}$.

In the above deÿnition, we do not assume that $\phi(t) = s$. Every strongly detectable algebra is detectable. h e polynomial ring $\Bbbk[x]$ is cancellative, but not detectable. By [LWZ, Lemma q..], if A is Z-retractable in the sense of [LWZ, Deÿnition ..þ], then it is detectable. We ÿrst recall a deÿnition from [LWZ, Deÿnition q.].

Definition 4.2 ([LWZ, Deÿnition q.]) Let A be an algebra over $\Bbbk$.

(i) We say A is *Hopfian* if every $\Bbbk$-algebra epimorphism from A to itself is an automorphism.

(ii) We say A is *strongly Hopfian* if $A[t_1, \dots, t_n]$ is Hopÿan for every $n \geq$ ¨.

By [LWZ, Lemma q.B], if A is detectable and strongly Hopÿan, then A is cancellative. We will generalize these facts in the Morita setting. In the following deÿnition, we use ω^{-1} instead of ω for technical reasons.

Definition 4.3 Let A be an algebra. Let ω be the map given in (E..Ë.Þ) when in a Morita context.

(i) We call A *m-detectable* if any equivalence of categories $M(A[t]) \cong M(B[s])$ implies that

$$A[t] = A\{\omega^{-1}(s)\},$$

or equivalently, $t \in A\{\omega^{-1}(s)\}$.

(ii) We call A *strongly m-detectable* if for each $n \geq$ Ë, any equivalence of categories $M(A[t_1,\dots,t_n]) \cong M(B[s_1,\dots,s_n])$ implies that

$$A[t_1,\dots,t_n] = A\{\omega^{-1}(s_1),\dots,\omega^{-1}(s_n)\},$$

or equivalently, $t_i \in A\{\omega^{-1}(s_1),\dots,\omega^{-1}(s_n)\}$ for $i =$ Ë, $\dots, n$.

h e following result is analogous to [LWZ, Lemma q..].

Lemma 4.4 *If A is m-Z-retractable (resp., strongly m-Z-retractable), then it is m-detectable (resp., strongly m-detectable).*

Proof We show only the "strongly" version.

Suppose that A is strongly m-Z-retractable. Let B be any algebra such that the abelian categories $M(A[\underline{t}])$ and $M(B[\underline{s}])$ are equivalent. Since A is strongly m-Z-retractable, the map $\omega : Z(A)[\underline{t}] \to Z(B)[\underline{s}]$ in (E..Ë.Ë) restricts to an algebra isomorphism $Z(A) \to Z(B)$. Write $\phi = \omega^{-1}$ and $f_i := \phi(s_i)$ for $i =$ Ë, $\dots, n$. h en

$$\begin{aligned} Z(A)\{f_1,\dots,f_n\} &= \phi(Z(B))\{\phi(s_1),\dots,\phi(s_n)\} \\ &= \phi(Z(B)\{s_1,\dots,s_n\}) \\ &= \phi(Z(B)[\underline{s}]) = Z(A)[\underline{t}]. \end{aligned}$$

h en, for every i, $t_i \in Z(A)[\underline{t}] = Z(A)\{f_1,\dots,f_n\} \subseteq A\{f_1,\dots,f_n\}$, as desired. ∎

Next we show that m-detectability implies m-cancellative property under some mild conditions.

Example 4.5 ([LWZ, Lemma q.þ]) h e following algebras are strongly Hopÿan:

(i) left or right noetherian algebras;
(ii) ÿnitely generated locally ÿnite $\mathbb{N}$-graded algebras;
(iii) prime aõ ne $\Bbbk$-algebras satisfying a polynomial identity.

Lemma 4.6 *Suppose A is strongly Hopfian.*

(i) *If A is m-detectable, then A is m-cancellative and cancellative.*

(ii) *If A is strongly m-detectable, then A is strongly m-cancellative and strongly cancellative.*

Proof We prove only (ii).

First, we consider the Morita version. Suppose that $A[\underline{t}]$ and $B[\underline{s}]$ are Morita equivalent and $\omega : Z(A)[\underline{t}] \to Z(B)[\underline{s}]$ is the algebra isomorphism given as in (E..Ë.Ë). Write $\phi = \omega^{-1}$ and $f_i = \phi(s_i)$ for $i =$ Ë, $\dots, n$. h en f_i are central elements in $A[\underline{t}]$. h us, $A\{f_1,\dots,f_n\}$ is a homomorphic image of $A[t_1,\dots,t_n]$ by sending $t_i \mapsto f_i$. Since A is strongly m-detectable, $A\{f_1,\dots,f_n\} = A[\underline{t}]$. h en we have an algebra homomorphism

(Eq.B.Ë) $$A[\underline{t}] \xrightarrow{\pi} A\{f_1,\dots,f_n\} \xrightarrow{=} A[\underline{t}].$$

Since A is strongly Hopÿan, $A[\underline{t}]$ is Hopÿan. Now (Eq.B.Ë) implies that π is an isomorphism. As a consequence, $A\{f_1,\dots,f_n\} = A[f_1,\dots,f_n]$ viewing f_i as central indeterminates in $A[f_1,\dots,f_n]$. As a consequence, $A[\underline{t}] = A[\underline{f}]$. Going back to the map

$$\omega : Z(A[\underline{t}]) = Z(A[\underline{f}]) \longrightarrow Z(B[\underline{s}]),$$

one sees that ω maps f_i to s_i for $i =$ Ë$, \dots, n$. Let J be the ideal of $Z(A[\underline{t}])$ generated by $\{f_i\}_{i=1}^n$ and J' be the ideal of $Z(B[\underline{s}])$ generated by $\{s_i\}_{i=1}^n$. h en $J' = \omega(J)$. By Lemma q.Ë(v), the algebra A (which is isomorphic to $A[\underline{t}]/JA$) is Morita equivalent to B (which is isomorphic to $B[\underline{s}]/J'B$). h e assertion follows.

Next we consider the "cancellative" version. Suppose that $\omega' : A[\underline{t}] \to B[\underline{s}]$ is an isomorphism that restricts to an isomorphism between the centers $\omega : Z(A)[\underline{t}] \to Z(B)[\underline{s}]$. h en ω' induces a (trivial) Morita equivalence, and ω is the map given in (E..Ë.Ë). Re-use the notation introduced in the above proof. h e above proof shows that $A[\underline{t}] = A[\underline{f}]$, where $f_i = \omega^{-1}(s_i)$ for all i. h erefore, ω' induces an isomorphism

$$A \cong A[\underline{f}]/(\{f_i\}_{i=1}^n) \xrightarrow{\overline{\omega'}} B[\underline{s}]/(\{s_i\}_{i=1}^n) \cong B,$$

as desired. ∎

For the rest of this section we study more properties concerning m-detectability.

Lemma 4.7 *Let A be an algebra with center Z. Suppose Z is (strongly) cancellative.*

(i) *If Z is (strongly) detectable, then A is (strongly) m-detectable.*

(ii) *Z is (strongly) detectable if and only if it is (strongly) m-detectable.*

Proof Following the pattern before, we prove only the "strongly" version.

(i) Suppose B is an algebra such that $A[\underline{t}]$ and $B[\underline{s}]$ are Morita equivalent. Let $\omega : Z[\underline{t}] \to Z(B)[\underline{s}]$ be the algebra isomorphism given in (E..Ë.Ë). Since Z is strongly cancellative, one has that $Z(B) \cong Z$. Now we have an isomorphism $\omega^{-1} : Z(B)[\underline{s}] \cong Z[\underline{t}]$. Since $Z(B)$ (or Z) is strongly detectable, $t_i \in Z\{\omega^{-1}(s_1),\dots,\omega^{-1}(s_n)\}$ for all i. h us, $t_i \in A\{\omega^{-1}(s_1),\dots,\omega^{-1}(s_n)\}$ for all i. h is means that A is strongly m-detectable.

(ii) One direction is part (i). For the other direction, assume that Z is strongly m-detectable. Consider any algebra isomorphism $\phi : Z[\underline{t}] \to B[\underline{s}]$. It is clear that B is commutative and $B \cong Z$, since Z is strongly cancellative. h en ϕ induces a (trivial) Morita equivalent, and the map ω in (E..Ë.Ë) is just ϕ. Now the strong m-detectability of Z implies that Z is strongly detectable. ∎

h e next result is similar to [LWZ, Proposition q.Ë].

Proposition 4.8 *If the center Z of A is an affine domain of GK-dimension one that is not isomorphic to $\Bbbk'[x]$ for some field extension $\Bbbk' \supseteq \Bbbk$, then A is strongly m-detectable.*

Proof By [AEH, h eorem q.q], Z is strongly retractable and cancellative. As a consequence, Z is a strongly m-Z-retractable. By Lemma . , A is strongly m-detectable. ∎

5 Proofs of Theorems 1.3 and 1.4

In this section we will use the results in the previous sections to show some classes of algebras are m-cancellative. We ÿrst prove h eorem Ë. .

Theorem 5.1 *If A is left (or right) noetherian, and the center of A is artinian, then A is strongly m-detectable. As a consequence, A is strongly m-cancellative.*

Proof Let Z be the center of A. h en Z is artinian by hypothesis. By [LWZ, h eorem .Ë], Z is strongly detectable and strongly cancellative. By Lemma . 8(i), A is strongly m-detectable. By Example . þ(i), A is strongly Hopÿan. h e consequence follows from Lemma . B(ii). ∎

h eorem Ë. is a special case of h eorem þ.Ë

Theorem 5.2 *Let A be an algebra with strongly cancellative center Z. Suppose J is the prime radical of Z such that* (a) *J is nilpotent and* (b) *Z/J is a finite direct sum of fields. Then the following hold.*

(i) *A is strongly m-detectable.*
(ii) *If further A is strongly Hopfian, then A is strongly m-cancellative.*

Proof (i) By the proof of [LWZ, h eorem . .], Z is strongly detectable. By Lemma . 8, A is strongly m-detectable.

(ii) Follows from Lemma . B and part (Ë). ∎

Next is h eorem Ëq.

Corollary 5.3 *Suppose A is strongly Hopfian and the center of A is artinian. Then A is strongly m-detectable and strongly m-cancellative.*

Proof Let Z be the center of A. By [LWZ, h eorem .Ë], Z is strongly detectable and strongly cancellative. Since Z is artinian, it satisÿes conditions (a) and (b) in h eorem þ. . . h e assertion follows by h eorem þ. . . ∎

6 Proof of Theorem 1.6

We assume in this section that $\Bbbk$ is algebraically closed. Under this hypothesis, a $\mathcal{P}$-discriminant ideal has the following nice property. h is is one of the reasons we need the above hypothesis.

Lemma 6.1 *Let $\mathcal{P}$ be a property. Then $\mathcal{P}$ is stable.*

Proof Let Z be the center of A. By Deÿnition q.q(v), we may assume that Z is aõ ne and write it as $\Bbbk[z_1,\dots,z_m]/(R)$, where $\{z_1,\dots,z_m\}$ is a generating set of Z and R is a set of relations. Every maximal ideal of Z is of the form $(z_i-\alpha_i) := (z_1-\alpha_1,\dots,z_m-\alpha_m)$, where $\alpha_i \in \Bbbk$ for all i. Every maximal ideal of $Z[\underline{t}]$ is of the form

$$(z_i-\alpha_i, t_j-\beta_j) := (z_1-\alpha_1,\dots,z_m-\alpha_m, t_1-\beta_1,\dots,t_n-\beta_n),$$

where $\alpha_i, \beta_j \in \Bbbk$. h e natural embedding $Z \to Z[\underline{t}]$ induces a projection

$$\pi : \operatorname{MaxSpec}(Z[\underline{t}]) \longrightarrow \operatorname{MaxSpec} Z$$

by sending $\mathfrak{m} := (z_i-\alpha_i, t_j-\beta_j)$ to $\pi(\mathfrak{m}) := (z_i-\alpha_i)$.

Let $D_{\mathcal{P}}(A)$ be the $\mathcal{P}$-discriminant set of A. A maximal ideal $\mathfrak{m}$ is in $D_{\mathcal{P}}(A[\underline{t}])$ if and only if $A[\underline{t}]/\mathfrak{m}A[\underline{t}]$ does not have property $\mathcal{P}$. Since

$$A[\underline{t}]/\mathfrak{m}A[\underline{t}] \cong A/\pi(\mathfrak{m})A,$$

$\mathfrak{m} \in D_{\mathcal{P}}(A[\underline{t}])$ if and only if $\pi(\mathfrak{m}) \in D_{\mathcal{P}}(A)$. h is implies that $D_{\mathcal{P}}(A[\underline{t}]) = D_{\mathcal{P}}(A) \times \mathbb{A}^n$. As a consequence,

$$I_{\mathcal{P}}(A[\underline{t}]) = \bigcap_{\mathfrak{m} \in D_{\mathcal{P}}(A[\underline{t}])} \mathfrak{m} = \Big(\bigcap_{p \in D_{\mathcal{P}}(A)} p \Big) \otimes \Bbbk[\underline{t}] = I_{\mathcal{P}}(A) \otimes \Bbbk[\underline{t}].$$

h erefore, $\mathcal{P}$ is stable by Deÿnition q.q(v). ∎

Let A be an algebra with the center Z being a domain. Let $\tau(A/Z)$ be the ideal of A consisting of elements in A that are annihilated by some nonzero element in Z. Deÿne the *annihilator ideal* of Z to be

$$\kappa(A/Z) = \{z \in Z \mid z(\tau(A/Z)) = \text{¨}\}.$$

Lemma 6.2 *Retain the notation as above.*

(i) *κ is stable in the sense that $\kappa(A[\underline{t}]/Z[\underline{t}]) = \kappa(A/Z) \otimes \Bbbk[\underline{t}]$.*

(ii) *If A and B are Morita equivalent, then ω maps $\kappa(A/Z)$ to $\kappa(B/Z(B))$ bijectively.*

(iii) *If A is left noetherian and suppose the center Z is a domain, then $\tau(A/Z) \neq$ ¨ if and only if $\kappa(A/Z)$ is a proper ideal, neither Z nor 0.*

Proof h is is easy to check, so details are omitted. ∎

Lemma 6.3 *Suppose A is a finitely generated module over its center Z and Z is a domain. If A is prime, then $\tau(A/Z) =$ ¨.*

Proof h is is easy to check, so details are omitted. ∎

Proposition 6.4 *Let A be left noetherian such that the center Z is an affine domain of GK-dimension one.*

(i) *If Z is not $\Bbbk[x]$, then A is strongly m-Z-retractable, m-detectable, and m-cancellative.*

(ii) *If $Z = \Bbbk[x]$ and $\tau(A/Z) \neq$ ¨, then A is strongly m-Z-retractable, m-detectable, and m-cancellative.*

Proof (i) By [AEH, h eorem q.q and Corollary q.], Z is strongly retractable. By Deÿnition q.B(iii), A is strongly m-Z-retractable. By Lemma . , A is strongly m-detectable. Since A is left noetherian, by Lemma . B(ii), A is strongly m-cancellative.

(ii) Since A is left noetherian and $\tau(A/Z) \neq$ ¨, $\kappa(A/Z)$ is a nonzero proper ideal of $\Bbbk[x]$ by Lemma B. . (iii). So there is a nonzero non-invertible element $f \in \Bbbk[x]$ such that $\kappa(A/Z) = (f)$. By Lemma B. . (i) and (ii), κ is a stable Morita invariant property. By replacing $\mathcal{P}$ by κ, Corollary q.ËË implies that A is strongly m-Z-retractable. h e rest of the proof is similar to the proof of part (i). ∎

For the rest of the section we consider the case when $Z = \Bbbk[x]$ and $\tau(A/Z) = $ ¨, or more precisely, when A is aõ ne prime PI of GK-dimension one with $Z = \Bbbk[x]$. We need to recall some concepts.

Let A be an aõ ne prime algebra of GK-dimension one. By a result of Small and Warÿeld [SW], A is a ÿnitely generated module over its aõ ne center. As a consequence, A is noetherian.

Let R be a commutative algebra, an R-algebra A is called *Azumaya* if A is a ÿnitely generated faithful projective R-module and the canonical morphism

(Eþ. .Ë) $$A \otimes_R A^{op} \longrightarrow \mathrm{End}_R(A)$$

is an isomorphism. By [DeI, h eorem q.], A is Azumaya if and only if A is a central separable algebra over R. Since we assume that $\Bbbk$ is algebraically closed, we have the following equivalent deÿnition.

Definition 6.5 ([BY, Introduction]) Let A be an aõ ne prime $\Bbbk$-algebra which is a ÿnitely generated module over its aõ ne center $Z(A)$. Let n be the PI-degree of A, which is also the maximal possible $\Bbbk$-dimension of irreducible A-modules.

(i) h e *Azumaya locus* of A, denoted by $\mathcal{A}(A)$, is the dense open subset of MaxSpec $Z(A)$ which parametrizes the irreducible A-modules of maximal $\Bbbk$-dimension. In other words, $\mathfrak{m} \in \mathcal{A}(A)$ if and only if $\mathfrak{m}A$ is the annihilator in A of an irreducible A-module V with $\dim V = n$, if and only if $A/\mathfrak{m}A \cong M_n(\Bbbk)$.

(ii) If $\mathcal{A}(A) = \mathrm{MaxSpec}\, Z(A)$, A is called *Azumaya*.

We can relate the Azumaya locus with the "simple"-locus. Let $\mathcal{S}$ be the property of being simple.

Lemma 6.6 *Assume that A is free over its affine center Z.*

(i) *$A[\underline{t}]$ is free over $Z[\underline{t}]$.*

(ii) *$\mathcal{A}(A) = L_{\mathcal{S}}(A)$, where the latter is defined in Definition* q.q(i).

Proof (i) is obvious.

(ii) Since A is free over Z of rank n^2, $A/\mathfrak{m}A$ is isomorphic to $M_n(\Bbbk)$ if and only if $A/\mathfrak{m}A$ is simple. h e assertion follows. ∎

Proposition 6.7 *Suppose that A is an affine prime algebra of GK-dimension one with center $\Bbbk[x]$.*

(i) *If A is not Azumaya, then A is strongly m-Z-retractable, m-detectable, and m-cancellative.*

(ii) *If A is Azumaya, then A is strongly m-cancellative.*

Proof (i) Since the Azumaya locus is open and dense, the non-Azumaya locus of A is a proper nonzero ideal of $Z = \Bbbk[x]$, which is principal. Since A is prime, $\tau(A/Z) = $ ¨ and whence A is projective and then free over Z. By Lemma B.B(ii), the Azumaya locus of $A[\underline{t}]$ agrees with the $\mathcal{S}$-locus of $A[\underline{t}]$. Hence, $\mathcal{S}$ is a stable Morita invariant property such that the $\mathcal{S}$-discriminant is a nonzero non-invertible element in Z. By Corollary q.ËĘ A is strongly m-Z-retractable. h e rest of the proof follows from the proof of Proposition B. (i).

(ii) Since A is Azumaya, by [LWZ, Lemma .™(q)], $A = M_n(\Bbbk[x])$ for some integer $n \geq$ Ë.If $A[\underline{t}]$ is Morita equivalent to $B[\underline{s}]$, then $Z(A)[\underline{t}] \cong Z(B)[\underline{s}]$. Since $Z(A) = \Bbbk[x]$ is strongly cancellative, $Z(B)$ is also isomorphic to $\Bbbk[x]$. If B is not Azumaya, it follows from part (i) that A and B are Morita equivalent. If B is Azumaya, then by [LWZ, Lemma .™(q)], B is a matrix algebra $M_{n'}(\Bbbk[x])$ for some $n' \geq$ Ë, which is also Morita equivalent to A. h erefore, A is strongly m-cancellative. ∎

Now we are ready to prove h eorem ËB.

Theorem 6.8 *Let A be an affine prime algebra of GK-dimension one.*

(i) *A is strongly m-cancellative.*

(ii) *If either $Z(A) \neq \Bbbk[x]$ or A is not Azumaya, then A is strongly m-Z-retractable and m-detectable.*

Proof Since we assume that $\Bbbk$ is algebraically closed in this section, by [LWZ, Lemma .™], there are three cases to consider.

Case Ë: $Z(A) \ncong \Bbbk[x]$.

Case .:.$Z(A) \cong \Bbbk[x]$ and A is not Azumaya.

Case q: $Z(A) \cong \Bbbk[x]$ and A is Azumaya.

Applying Proposition B. (i) in Case Ë, Proposition B.8(i) in Case . . and Proposition B.8(ii) in Case q, the assertion follows. ∎

It is clear that h eorem ËB is an immediate consequence of h eorem B.Ç. As far as we know there are no examples of algebras with the center being an aõ ne domain of GK-dimension one that are not m-cancellative. h erefore, we ask the following question.

Question 6.9 Let A be a left noetherian algebra such that $Z(A)$ is an aõ ne domain of GK-dimension one. h en is A m-cancellative?

We ÿnish this section with some examples of non-PI algebras that are strongly (m-)cancellative.

Example 6.10 Let Z denote the center of the given algebra A. Assume that $\Bbbk$ has characteristic zero.

(i) Let A be the homogenization of the ÿrst Weyl algebra that is generated by x, y, t subject to the relations

$$[x,t] = [y,t] = \text{¨}, [x,y] = t^2.$$

It is well known that the center of A is $\Bbbk[t]$. Let $\mathcal{S}$ be the property of being simple. Since $\mathfrak{m} := (t - \text{¨})$ is the only maximal ideal of $\Bbbk[t]$ such that $A/\mathfrak{m}A$ is not simple, the $\mathcal{S}$-discriminant $d_{\mathcal{S}}(A)$ exists and equals t. By Corollary q.ËË, A is strongly m-Z-retractable. By Lemma . , A is strongly m-detectable. By Lemma . B(ii), A is both strongly cancellative and strongly m-cancellative.

(ii) Let A be a non-PI quadratic Sklyanin algebra of global dimension q. It is well known that the center of A is $\Bbbk[g]$ where $g \in A$ has degree q. We claim that $A/(g-\alpha)$ is simple if and only if $\alpha \neq$ ¨. If $\alpha =$ ¨, then $A/(g)$ is connected graded which is not simple. Now assume that $\alpha \neq$ ¨. It is well known that $(A[g^{-1}])_0$ is simple. Let T be the qrd Veronese subring of A. h en $(T[g^{-1}])_0 \cong (A[g^{-1}])_0$ is simple. Now

$$T/(g-\alpha) \cong T/(\alpha^{-1}g - \text{Ë}) \cong \left(T[(\alpha^{-1}g)^{-1}]\right)_0 = \left(T[g^{-1}]\right)_0 \cong \left(A[g^{-1}]\right)_0,$$

where the second $\cong$ is due to [RSS, Lemma ..Ë]. It is clear that $A/(g-\alpha)$ contains $T/(g-\alpha)$. Since $T/(g-\alpha)$ is simple and hence has no ÿnite dimensional modules, $A/(g-\alpha)$ does not have ÿnite dimensional modules. Since the algebra $A/(g-\alpha)$ is aõ ne of GK-dimension two, it must be simple. So we proved the claim.

h e claim implies that the $\mathcal{S}$-discriminant $d_{\mathcal{S}}(A)$ exists and equals $g \in \Bbbk[g]$. Following the last part of the above example, A is both strongly cancellative and strongly m-cancellative.

Example 6.11 Suppose char $\Bbbk =$ ¨. Let A be the universal enveloping algebra of the simple Lie algebra sl_2. By Example q. , the center of A is $\Bbbk[Q]$, where Q is the Casimir element. In this example, we will consider two diòerent properties.

Let $\mathcal{W}$ be the property of not having a factor ring isomorphic to $M_{n+1}(\Bbbk)$ (for a ÿxed integer n). h en $d_{\mathcal{W}}(A) = Q - (n^2 + .n)$, which is a nonzero non-invertible element in $\Bbbk[Q]$. By Corollary q.ËĘ A is strongly Z-retractable. By [LWZ, Lemma q...], A is strongly detectable, and by [LWZ, Lemma q.B(..)], A is strongly cancellative.

Next we show that A is strongly m-cancellative by using a Morita invariant property. Let $\mathcal{H}$ be the property that $HH_3(R) =$ ¨, where $HH_i(R)$ denotes the i-th Hochschild homology of an algebra R. By [We, h eorem ™.þ.B], the Hochschild homology is Morita invariant. Hence $\mathcal{H}$ is Morita invariant. We claim that the discriminant $d_{\mathcal{H}}(A)$ is $Q + \frac{1}{4}$. h is claim is equivalent to the following assertions:

(a) $HH_3(A/(Q-\lambda)) =$ ¨ for all $\lambda \neq -\frac{1}{4}$;
(b) $HH_3(A/(Q+\frac{1}{4})) \neq$ ¨ (this is the case when $\lambda = -\frac{1}{4}$).

Let $B_\lambda = A/(Q-\lambda)$. h en B_λ agrees with the algebra B_λ in [FSS, Example ..q]. By [FSS, Example ..q], B_λ is a generalized Weyl algebra with $\sigma(h) = h -$ Ë, $a = \lambda - h(h+$Ë$)$. Hence, B_λ satisÿes the hypotheses of [FSS, h eorem ..Ë]. If $\lambda \neq -\frac{1}{4}$, then $a'(h)$ and $a(h)$ are coprime. By [FSS, h eorem ..Ë(Ë)], $HH_3(B_\lambda) =$ ¨, which is part (a). If $\lambda = -\frac{1}{4}$, then the common divisor of $a'(h)$ and $a(h)$ is $a'(h)$, which has degree Ë. By [FSS, h eorem ..Ë(.)], $HH_3(B_\lambda) = \Bbbk$, which is part (b). h erefore, we proved the claim. By Corollary q.ËĘ A is strongly m-Z-retractable. By Lemma . , A is strongly m-detectable. By Lemma . B(ii), A is both strongly cancellative and strongly m-cancellative.

Remark 6.12 (i) h e second half of Example B.ËË shows that using a Morita invariant property results a better conclusion.

(ii) Another consequence of the discussion in Example B.ËË is the following. If σ is an algebra automorphism of $A := U(sl_2)$, then $\sigma(Q) = Q$. Further, for every locally nilpotent derivation $\partial \in \mathrm{LND}(A)$, we have $\partial(Q) =$ ¨. h is could be a useful fact to use in calculating the automorphism group $\mathrm{Aut}(A)$. According to [CL, Section q...], the full automorphism group of A is still unknown. A result of Joseph [Jo] says that $\mathrm{Aut}(A)$ contains a wild automorphism. h e automorphism of $A/(Q-\alpha)A$ was computed in [Di] when $\alpha \neq n^2 + .n$ for all $n \in \mathbb{N}$.

7 Proof of Theorem 1.5

In this section we prove h eorem Ëþ. We refer to [ASS] for basic deÿnitions of quivers and their path algebra. Let C_n be the cyclic quiver with n vertices and n arrow connecting these vertices in one oriented direction. In representation theory of ÿnite dimensional algebras, quiver C_n is also called *type* $\widetilde{A}_{n-1}$. Let ¨, Ë, . . . , n − Ëbe the vertices of C_n, and $a_i : i \to i +$ Ë(in $\mathbb{Z}/(n)$) be the arrows in C_n. h en $w := \sum_{i=0}^{n-1} a_i a_{i+1} \cdots a_{i+n-1}$ is a central element in $\Bbbk C_n$. By [LWZ, Lemma .], we have the following result concerning the center of the path algebra $\Bbbk Q$ when Q is connected:

(EB.¨.Ë)
$$Z(\Bbbk Q) = \begin{cases} \Bbbk & \text{if } Q \text{ has no arrow,} \\ \Bbbk[x] & \text{if } Q = C_1 \text{ or equivalently } \Bbbk Q = \Bbbk[x], \\ \Bbbk[w] & \text{if } Q = C_n \text{ for } n \geq \text{..,}. \\ \Bbbk & \text{otherwise.} \end{cases}$$

Similar to [LWZ, Lemma q.ËÞ, we have the following lemma, whose proof is omitted.

Lemma 7.1 *Let $\Bbbk'$ be a field extension of $\Bbbk$. If $A \otimes_{\Bbbk} \Bbbk'$ is (strongly) m-detectable as an algebra over $\Bbbk'$, then A is (strongly) m-detectable as an algebra over $\Bbbk$.*

Lemma 7.2 *Let $Q = C_n$ for $n \geq$ ... Then $\Bbbk Q$ is strongly m-detectable and strongly m-cancellative.*

Proof By [LWZ, Lemma . þ], $\Bbbk C_n$ is prime of GK-dimension one while not being Azumaya. If $\Bbbk$ is algebraically closed, the assertion is a special case of h eorem B.Ç(ii). If $\Bbbk$ is not algebraically closed, let $\Bbbk'$ be the closure of $\Bbbk$. By h eorem B.Ç(ii), $\Bbbk' Q$ is strongly m-detectable over $\Bbbk'$. By Lemma 8Ë, $\Bbbk Q$ is strongly m-detectable over $\Bbbk$, and then strongly m-cancellative by Lemmas . þ(ii) and . B(ii). ■

We need another lemma before proving the main result of this section. h e ideas of the proof are similar to the proof of [LWZ, Lemma . B], so the proof is omitted.

Lemma 7.3 *Let A and B be two algebras.*

(i) *If A and B are (strongly) m-cancellative, so is $A \oplus B$.*
(ii) *If A and B are (strongly) m-retractable, so is $A \oplus B$.*
(iii) *If A and B are (strongly) m-detectable, so is $A \oplus B$.*

Now we are ready to prove h eorem Ëþ.

Theorem 7.4 *Let Q be a finite quiver and let A be the path algebra $\Bbbk Q$. Then A is strongly m-cancellative. If, further, Q has no connected component being C_1, then A is strongly m-detectable.*

Proof By Lemma 8q, we can assume that Q is connected.

If $Q = C_1$, then $A = \Bbbk[x]$ and the assertion follows from Proposition B.8(ii).

If $Q = C_n$, then this is Lemma 8..(ii).

If $Q \neq C_n$ for any $n \geq$ Ë, then by (EB.¨.Ë), the center of A is $\Bbbk$. By h eorem þ..(i), A is strongly m-detectable. Since A is $\mathbb{N}$-graded and locally ÿnite, it is strongly Hopÿan by Example . þ(ii). By h eorem þ..(ii), A is strongly m-cancellative. h is completes the proof. ■

h eorem Ëþ is clearly a consequence of the above theorem.

8 Comments, Questions, and Examples

One of the remaining questions in this project is to understand whether the cancellation property is equivalent to the m-cancellation property (as well as the d-cancellation property). We will make some comments about it in this section.

First, we will show that three cancellation properties are equivalent for commutative algebras. h e next result was proved in [YZË] using slightly diòerent wording.

Proposition 8.1 ([YZË, Proposition þ.Ë]) *Suppose that A is an Azumaya algebra over its center Z and that* Spec Z *is connected. If $D(A)$ and $D(B)$ are triangulated equivalent for another algebra B, then A and B are Morita equivalent.*

Note that the Brauer group of a commutative algebra R, denoted by $Br(R)$, is the set of Morita-type-equivalence classes of Azumaya algebras over R; in other words, $Br(R)$ classiÿes Azumaya algebras over R up to an equivalence relation [AG]. See [Sc] for some discussion about the Brauer group. One immediate consequence is the following corollary.

Corollary 8.2 *Suppose Z is a commutative algebra with* Spec Z *connected. Then the following are equivalent.*

(i) *Z is (strongly) cancellative.*
(ii) *Z is (strongly) m-cancellative.*
(iii) *Z is (strongly) d-cancellative.*

Proof By Proposition ÇË, it remains to show that (i) and (ii) are equivalent. By Lemma ... , part (i) follows from part (ii). Now we show that part (ii) is a consequence of part (i).

Suppose A is an algebra such that $Z[\underline{t}]$ is Morita equivalent to $A[\underline{s}]$. By the map ω in (E..Ë.Ë), we obtain that $Z[\underline{t}]$ is isomorphic to $Z(A)[\underline{s}]$. Since Z is (strongly) cancellative, $Z(A) \cong Z$. Let us identify $Z(A)$ with Z. Since $Z[\underline{t}]$ is Morita equivalent to $A[\underline{s}]$, $A[\underline{s}]$ is Morita equivalent to its center, which is $Z[\underline{s}]$. h en the Brauer-class $[A[\underline{s}]]$ as an element in $Br(Z[\underline{s}])$ is trivial by [AG, Proposition þ.q]. Since the natural map $Br(Z) \to Br(Z[\underline{s}])$ is injective, the Brauer-class $[A]$ as an element in $Br(Z)$ is trivial. By [Sc, h eorem] or [Ne, Proposition .Ë], A is Morita equivalent to Z, as required. ■

Corollary 8.3 *Let Z be a (strongly) detectable commutative algebra such that* Spec Z *is connected. If A is an Azumaya algebra over Z that is strongly Hopfian, then A is both (strongly) m-cancellative and (strongly) d-cancellative.*

Proof By Proposition ÇË, we need to show only the claim that A is (strongly) m-cancellative. Since A is strongly Hopÿan, the claim follows from Lemmas . B(ii) and . 8(i). ∎

h e next example is similar to [LWZ, Example q.q].

Example 8.4 Let $A = \Bbbk[x, y]/(x^2 = y^2 = xy = \ddot{\ })$. By h eorem þ.Ë, A is strongly m-detectable. By [LWZ, Example q.q] and Corollary Ç. ., the commutative algebra A is neither retractable nor m-retractable.

For non-Azumaya (noncommutative) algebras, there is no general approach to relating the m-cancellation property with the d-cancellation property. However, most of cancellative algebras veriÿed by using the discriminant method in [BZË] are m-cancellative, as we will see next.

Since most of algebras that we are interested in are strongly Hopÿan, to show that an algebra is m-cancellative, it suõce s to show that it is m-detectable [Lemma . B(i)]. By Lemma 8Ë, under some mild hypotheses, we can assume the base ÿeld $\Bbbk$ is algebraically closed. For simplicity, we assume that $\Bbbk$ is algebraically closed of characteristic zero for the rest of this section.

Let I be an ideal of a commutative algebra R. h en the *radical* of I is deÿned to be

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} R,\, I \subseteq \mathfrak{p}} \mathfrak{p}.$$

h e *standard trace* $\mathrm{tr}_{\mathrm{st}}$ deÿned in [BY, Sect. . .Ë(.).] agrees with the *regular trace* $\mathrm{tr}_{\mathrm{reg}}$ deÿned in [CPWZ. ., p. 8þÇ]. So we take $\mathrm{tr} = \mathrm{tr}_{\mathrm{st}} = tr_{\mathrm{reg}}$ in this paper.

Proposition 8.5 *Let A be a prime algebra that is finitely generated as a module over its center Z and let v be the rank of A over Z. Let $D \subseteq Z$ be either the v-discriminant ideal $D_v(B{:}tr)$ in the sense of* [CPWZ. ., Deÿnition Ë.Ë(.)] *or the modified v-discriminant ideal $MD_v(B{:}tr)$ in the sense of* [CPWZ. ., Deÿnition Ë. (. .)]. *Suppose that*

(i) *the center Z is an affine domain and the standard trace* tr *maps A to Z;*
(ii) *$\sqrt{D}$ is a principal ideal of Z generated by an element f;*
(iii) *f is an effective (resp., dominating) element in Z.*

Then the following hold.

(a) *A is strongly m-Z-retractable.*
(b) *A is strongly Z-retractable.*
(c) *A is strongly m-detectable.*
(d) *A is strongly m-cancellative.*
(e) *A is strongly cancellative.*

Proof Since we assume that $\Bbbk$ is algebraically closed of characteristic zero, we can apply [BY, Main h eorem] by taking the standard trace. By [BY, Main h eorem], we have

$$\mathcal{V}(D) = \operatorname{MaxSpec}(Z) \setminus \mathcal{A}(A),$$

where $\mathcal{V}(D)$ is the zero-set of D. By Lemma B.B(ii), $\mathcal{A}(A) = L_{\mathcal{S}}(A)$, where $\mathcal{S}$ denotes the property of being simple. h us, the $\mathcal{S}$-discriminant set of A is equal to $\mathcal{V}(D)$.

As a consequence, the $\mathcal{S}$-discriminant ideal of A is equal to $I(\mathcal{V}(D))$, which is $\sqrt{D}$. By hypothesis (ii), we obtain that the $\mathcal{S}$-discriminant ideal of A is a principal ideal of Z generated by an element f. Since f is effective (resp., dominating), Z is strongly LND^H_f-rigid by Theorem q.Ë. Since $\mathcal{S}$ is a stable Morita invariant property [Lemma B.Ë], by Proposition q.8(ii), A is both strongly m-Z-retractable and strongly Z-retractable. Thus, we proved parts (a) and (b). Note that part (c) follows from part (a) and Lemma . . Since A is noetherian, it is strongly Hopfian [Example . þ(Ë)]. Parts (d) and (e) follow from part (c) and Lemma . B(ii). ∎

The next example is similar to [LWZ, Example þ.Ë].

Example 8.6 Let R be an affine commutative domain and let f be a product of a set of generating elements of R. Let

$$A = \begin{pmatrix} R & fR \\ R & R \end{pmatrix}.$$

It is easy to check that the (modified) - discriminant of A over its center R is the ideal generated by $-f^2$. Clearly, the radical of $(-f^2)$ is the principal ideal (f). By the above proposition, A is strongly m-Z-retractable, m-detectable, m-cancellative, and cancellative.

Other precise examples follow, but we omit some details. See also [BZË, Example . Ç].

Example 8.7 The following algebras are m-cancellative by verifying the hypotheses of Proposition Çþ:

(i) skew polynomial rings $\Bbbk_q[x_1, \ldots, x_n]$ when n is an even number and Ë$\neq q$ is a root of unity;
(ii) $\Bbbk\langle x, y\rangle/(x^2y - yx^2, y^2x + xy^2)$;
(iii) quantum Weyl algebra $\Bbbk\langle x, y\rangle/(yx - qxy -$ Ë$)$, where Ë$\neq q$ is a root of unity;
(iv) every finite tensor product of algebras of the form (i), (ii), and (iii).

Acknowledgments The authors thank Daniel Rogalski for useful conversations on the subject and for providing the reference [RSS], Ellen Kirkman for reading an earlier version of the paper and for her useful comments, and the referees for their careful reading and valuable comments.

References

[AEH] S. Abhyankar, P. Eakin, and W. Heinzer, *On the uniqueness of the coefficient ring in a polynomial ring*. J. Algebra 23(1972), 310–342. https://doi.org/10.1016/0021-8693(72)90134-2.

[AF] F. W. Anderson and K. R. Fuller, *Rings and categories of modules*. Graduate Texts in Mathematics, 13, Springer-Verlag, New York-Heidelberg, 1974.

[ASS] I. Assem, D. Simson, and A. Skowroński, *Elements of the representation theory of associative algebras*. Vol. 1. Techniques of representation theory. London Mathematical Society Student Texts, 65, Cambridge University Press, Cambridge, 2006. https://doi.org/10.1017/CBO9780511614309.

[AG] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*. Trans. Amer. Math. Soc. 97(1960), 367–409. https://doi.org/10.2307/1993378.

[BZ1] J. Bell and J. J. Zhang, *Zariski cancellation problem for noncommutative algebras*. Selecta Math. (N.S.) 23(2017), no. 3, 1709–1737. https://doi.org/10.1007/s00029-017-0317-7.

[BZ2] J. Bell and J. J. Zhang, *An isomorphism lemma for graded rings*. Proc. Amer. Math. Soc. 145(2017), no. 3, 989–994. https://doi.org/10.1090/proc/13276.

[BY] K. A. Brown and M. T. Yakimov, *Azumaya loci and discriminant ideals of PI algebras*. Adv. Math. 340(2018), 1219–1255. https://doi.org/10.1016/j.aim.2018.10.024.

[CL] P. A. A. B. Carvalho and S. A. Lopes, *Automorphisms of generalized down-up algebras*. Comm. Algebra 37(2009), no. 5, 1622–1646. https://doi.org/10.1080/00927870802209987.

[CPWZ1] S. Ceken, J. Palmieri, Y.-H. Wang, and J. J. Zhang, *The discriminant controls automorphism groups of noncommutative algebras*. Adv. Math. 269(2015), 551–584. https://doi.org/10.1016/j.aim.2014.10.018.

[CPWZ2] S. Ceken, J. Palmieri, Y.-H. Wang, and J. J. Zhang, *The discriminant criterion and the automorphism groups of quantized algebras*. Adv. Math. 286(2016), 754–801. https://doi.org/10.1016/j.aim.2015.09.024.

[CYZ1] K. Chan, A. Young, and J. J. Zhang, *Discriminant formulas and applications*. Algebra Number Theory 10(2016), no. 3, 557–596. https://doi.org/10.2140/ant.2016.10.557.

[CYZ2] K. Chan, A. Young, and J. J. Zhang, *Discriminants and automorphism groups of Veronese subrings of skew polynomial rings*. Math. Z. 288(2018), no. 3–4, 1395–1420. https://doi.org/10.1007/s00209-017-1939-3.

[Da] W. Danielewski, *On the cancellation problem and automorphism groups of affine algebraic varieties*. Preprint, 1989, 8 pages, Warsaw.

[DeI] F. DeMeyer and E. Ingraham, *Separable algebras over commutative rings*. Lecture Notes in Mathematics, 181, Springer-Verlag, Berlin-New York, 1971.

[Di] J. Dixmier, *Quotients simples de l'algébre enveloppante de* $\mathfrak{sl}_2$. J. Algebra 24(1973), 551–564. https://doi.org/10.1016/0021-8693(73)90127-0.

[EH] P. Eakin and W. Heinzer, *A cancellation problem for rings*. In: *Conference on Commutative Algebra (Univ. Kansas, Lawrence, Kan., 1972)*, Lecture Notes in Mathematics, 311, Springer, Berlin, 1973, pp. 61–77.

[FSS] M. A. Farinati, A. Solotar, and M. Suárez-Álvarez, *Hochschild homology and cohomology of generalized Weyl algebras*. Ann. Inst. Fourier (Grenoble) 53(2003), no. 2, 465–488.

[Fi] K.-H. Fieseler, *On complex affine surfaces with* $\mathbb{C}^+$*-action*. Comment. Math. Helv. 69(1994), 5–27. https://doi.org/10.1007/BF02564471.

[Fu] T. Fujita, *On Zariski problem*. Proc. Japan Acad. Ser. A Math. Sci. 55(1979), 106–110.

[Ga] J. Gaddis, *The isomorphism problem for quantum affine spaces, homogenized quantized Weyl algebras, and quantum matrix algebras*. J. Pure Appl. Algebra 221(2017), no. 10, 2511–2524. https://doi.org/10.1016/j.jpaa.2016.12.036.

[GKM] J. Gaddis, E. Kirkman, and W. F. Moore, *On the discriminant of twisted tensor products*. J. Algebra 477(2017), 29–55. https://doi.org/10.1016/j.jalgebra.2016.12.019.

[GWY] J. Gaddis, R. Won, and D. Yee, *Discriminants of Taft algebra smash products and applications*. Algebr. Represent. Theory, to appear. https://doi.org/10.1007/s10468-018-9798-0.

[Gu1] N. Gupta, *On the cancellation problem for the affine space* $\mathbb{A}^3$ *in characteristic* p. Inventiones Math. 195(2014), no. 1, 279–288. https://doi.org/10.1007/s00222-013-0455-2.

[Gu2] N. Gupta, *On Zariski's cancellation problem in positive characteristic*. Adv. Math. 264(2014), 296–307. https://doi.org/10.1016/j.aim.2014.07.012.

[Gu3] N. Gupta, *A survey on Zariski cancellation problem*. Indian J. Pure Appl. Math. 46(2015), no. 6, 865–877. https://doi.org/10.1007/s13226-015-0154-3.

[HS] H. Hasse and F. K. Schmidt, *Noch eine Begründung der Theorie der höheren Differrentialquotienten in einem algebraischen Funktionenkorper einer Unbestimmten*. J. Reine Angew. Math. 177(1937), 215–237. https://doi.org/10.1515/crll.1937.177.215.

[Ho] M. Hochster, *Non-uniqueness of the ring of coefficients in a polynomial ring*. Proc. Amer. Math. Soc. 34(1972), 81–82. https://doi.org/10.2307/2037901.

[Jo] A. Joseph, *A wild automorphism of* $U(sl(2))$. Math. Proc. Cambridge Philos. Soc. 80(1976), no. 1, 61–65. https://doi.org/10.1017/S030500410005266X.

[Kr] H. Kraft, *Challenging problems on affine n-space*. Séminaire Bourbaki, 1994/95, Astérisque 237(1996), Exp. No. 802, 5, 295–317.

[LY] J. Levitt and M. Yakimov, *Weyl algebras at roots of unity*. Israel J. Math. 225(2018), no. 2, 681–719. https://doi.org/10.1007/s11856-018-1675-3.

[LWZ] O. Lezama, Y.-H. Wang, and J. J. Zhang, *Zariski cancellation problem for non-domain noncommutative algebras*. Math. Z., to appear. https://doi.org/10.1007/s00209-018-2153-7.

[LMZ] J.-F. Lü, X.-F. Mao, and J. J. Zhang, *Nakayama automorphism and applications*. Trans. Amer. Math. Soc. 369(2017), no. 4, 2425–2460. https://doi.org/10.1090/tran/6718.

[Mak] L. Makar-Limanov, *On the hypersurface* $x + x^2y + z^2 + t^3 = 0$ *in* $\mathbf{C}^4$ *or a* $\mathbf{C}^3$*-like threefold which is not* $\mathbf{C}^3$. Israel J. Math. 96(1996), part B, 419–429. https://doi.org/10.1007/BF02937314.

[MS] M. Miyanishi and T. Sugie, *Affine surfaces containing cylinderlike open sets*. J. Math. Kyoto Univ. 20(1980), 11–42. https://doi.org/10.1215/kjm/1250522319.

[Ne] C. Negron, *The derived Picard group of an affine Azumaya algebra*. Selecta Math. (N.S.) 23(2017), no. 2, 1449–1468. https://doi.org/10.1007/s00029-016-0249-7.

[NTY] B. Nguyen, K. Trampel, and M. Yakimov, *Noncommutative discriminants via Poisson primes*. Adv. Math. 322(2017), 269–307. https://doi.org/10.1016/j.aim.2017.10.018.

[RSS] D. Rogalski, S. J. Sierra, and J. T. Stafford, *Algebras in which every subalgebra is Noetherian*. Proc. Amer. Math. Soc. 142(2014), no. 9, 2983–2990. https://doi.org/10.1090/S0002-9939-2014-12052-1.

[Ru] P. Russell, *On affine-ruled rational surfaces*. Math. Ann. 255(1981), 287–302. https://doi.org/10.1007/BF01450704.

[Sc] S. Schack, *Bimodules, the Brauer group, Morita equivalence, and cohomology*. J. Pure Appl. Algebra 80(1992), no. 3, 315–325. https://doi.org/10.1016/0022-4049(92)90149-A.

[SW] L. W. Small and R. B. Warfield Jr., *Prime affine algebras of Gel'fand-Kirillov dimension one*. J. Algebra 91(1984), no. 2, 386–389. https://doi.org/10.1016/0021-8693(84)90110-8.

[Sm] S. P. Smith, *The primitive factor rings of the enveloping algebra of* $sl(2, \mathbb{C})$. J. London Math. Soc. (2) 24(1981), no. 1, 97–108. https://doi.org/10.1112/jlms/s2-24.1.97.

[Su] M. Suzuki, *Propriétés topologiques des polynomes de deux variables complex et automorphisms algébriques de l'espace C2*. J. Math. Soc. Japan 26(1974), 241–257. https://doi.org/10.2969/jmsj/02620241.

[Ta1] X. Tang, *Automorphisms for some symmetric multiparameter quantized Weyl algebras and their localizations*. Algebra Colloq. 24(2017), no. 3, 419–438. https://doi.org/10.1142/S100538671700027X.

[Ta2] X. Tang, *The automorphism groups for a family of generalized Weyl algebras*. J. Algebra Appl. 18(2018), 1850142. https://doi.org/10.1142/S0219498818501426.

[WZ] Y.-H. Wang and J. J. Zhang, *Discriminants of noncommutative algebras and their applications (Chinese)*. Sci. China Math. 48(2018), 1615–1630. https://doi.org/10.1360/N012017-00263.

[We] C. A. Weibel, *An introduction to homological algebra*. Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, Cambridge, 1994. https://doi.org/10.1017/CBO9781139644136.

[Wi] J. Wilkens, *On the cancellation problem for surfaces*. C. R. Acad. Sci. Paris Sér. I Math. 326(1998), 1111–1116. https://doi.org/10.1016/S0764-4442(98)80071-2.

[YZ1] A. Yekutieli and J. J. Zhang, *Dualizing complexes and tilting complexes over simple rings*. J. Algebra 256(2002), no. 2, 556–567. https://doi.org/10.1016/S0021-8693(02)00005-4.

[YZ2] A. Yekutieli and J. J. Zhang, *Dualizing complexes and perverse modules over differential algebras*. Compos. Math. 141(2005), no. 3, 620–654. https://doi.org/10.1112/S0010437X04001307.

School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China
e-mail: dmlu@zju.edu.cn

School of Mathematical Sciences, Fudan University, Shanghai 200433, China
e-mail: qswu@fudan.edu.cn

Department of Mathematics, Box 354350, University of Washington, Seattle, Washington 98195, USA
e-mail: zhang@math.washington.edu