Algebra & Number Theory

Volume 15

2021

No. 4

A proof of the Brown–Goodearl conjecture for module-finite weak Hopf algebras

Daniel Rogalski, Robert Won and James J. Zhang

msp

ALGEBRA AND NUMBER THEORY 15:4 (2021)
https://doi.org/10.2140/ant.2021.15.971

A proof of the Brown–Goodearl conjecture for module-finite weak Hopf algebras

Daniel Rogalski, Robert Won and James J. Zhang

Let H be a weak Hopf algebra that is a finitely generated module over its affine center. We show that H has finite self-injective dimension and so the Brown–Goodearl conjecture holds in this special weak Hopf setting.

0. Introduction

In his Seattle lecture in 1997, Brown [1998] posed several open questions about noetherian Hopf algebras which satisfy a polynomial identity. Since then, the homological properties of infinite-dimensional noetherian Hopf algebras have been investigated extensively. While quite a few of the questions in Brown's lecture have been answered, one important question, now called either the Brown–Goodearl question or the Brown–Goodearl conjecture, is still open:

Does every noetherian Hopf algebra have finite injective dimension? (Q1)

Or, asking for a slightly stronger property,

Is every noetherian Hopf algebra Artin–Schelter Gorenstein? (Q2)

(The definition of Artin–Schelter Gorenstein will be recalled in Section 1.) In recent years, the Brown–Goodearl question has been posed in many lectures and survey papers [Brown 1998; 2007; Goodearl 2013], as it is related to the existence of rigid dualizing complexes, and therefore related to the twisted Calabi–Yau property of these Hopf algebras. An affirmative answer to this question has many other consequences, especially in the study of the ring-theoretic properties of noetherian Hopf algebras.

It is natural to ask the Brown–Goodearl question for other classes of noetherian algebras that are similar to Hopf algebras, for example, weak Hopf algebras, braided Hopf algebras, Nichols algebras, and so on. In fact Andruskiewitsch [2004, after Definition 2.1] independently asked the following question: if a Nichols algebra is a domain with finite Gelfand–Kirillov dimension, is it then Artin–Schelter regular (and therefore of finite global dimension)?

In this paper, we consider weak Hopf algebras, which are natural generalizations of Hopf algebras that have applications in conformal field theory, quantum field theory, and the study of operator algebras,

MSC2010: primary 16E10; secondary 16T99, 18D10.
Keywords: weak Hopf algebras, injective dimension, Brown–Goodearl conjecture.

subfactors, tensor categories, and fusion categories. One important fact is that any fusion category is equivalent to the category of modules over a weak Hopf algebra [Hayashi 1999a; Szlachányi 2001], see also [Etingof et al. 2005, Theorem 2.20 and Corollary 2.22]. We refer the reader to Section 5 for the definition of a weak Hopf algebra and a few examples. Much of the existing literature on weak Hopf algebras has focused on the finite-dimensional case. For example, Böhm, Nill, and Szlachányi [Böhm et al. 1999, Theorem 3.11] proved that every finite-dimensional weak Hopf algebra over a field is quasi-Frobenius, or equivalently, has (self-)injective dimension zero (see also [Vecsernyés 2003, Corollary 3.3]). In contrast with the Hopf case, a finite-dimensional weak Hopf algebra may not be Frobenius [Iovanov and Kadison 2010, Proposition 2.5].

The main result of this paper is to affirmatively answer the analog of the Brown–Goodearl question (Q1) for weak Hopf algebras that are module-finite over their affine centers. In fact, our proof works not just for weak Hopf algebras, but for any such algebra whose module category has a monoidal structure with certain basic properties.

In most of our results, we restrict our attention to algebras satisfying the following ring-theoretic hypothesis.

Hypothesis 0.1. *Let A be an algebra with center $Z(A)$. Suppose that A is a finitely generated module over $Z(A)$ and that $Z(A)$ is a finitely generated algebra over the base field $\Bbbk$.*

The favorable homological properties of Hopf algebras seem to arise primarily from the fact that there is an internal tensor product of modules over a Hopf algebra. We abstract this idea here to study the following condition.

Hypothesis 0.2. *Let A be an algebra over $\Bbbk$. Assume that there is a monoidal structure $\overline{\otimes}$ on the category A-*Mod *of left A-modules, where $\overline{\otimes}$ is bilinear on morphisms and biexact, such that every finite dimensional $M \in A$-*Mod *has a left dual M^* in that monoidal category. Assume further that the same hypotheses hold for the category A^{op}-*Mod *of right A-modules.*

We refer the reader to [Etingof et al. 2015] for basic notions concerning monoidal categories; in any case, we will remind the reader of the undefined terms in Section 6. For any weak Hopf algebra A, the category A-Mod has a monoidal structure as in Hypothesis 0.2, where $M \mathbin{\overline{\otimes}} N$ is a subspace of the usual tensor product $M \otimes_{\Bbbk} N$ (see [Böhm et al. 2011; Nill 1998]). If A is a Hopf algebra or a quasibialgebra with antipode, then A also satisfies Hypothesis 0.2 where $\overline{\otimes}$ is the usual tensor product $\otimes_{\Bbbk}$.

For a module M over an algebra, let $\operatorname{GKdim} M$ denote the Gelfand–Kirillov dimension (or GK dimension) of M. (We refer to [Krause and Lenagan 1985] for the definition of GK dimension.) We say an algebra A is *homogeneous* if

$$\operatorname{GKdim} L = \operatorname{GKdim} A$$

for all nonzero left ideals $L \subseteq A$.

Our main result is the following:

Theorem 0.3. *Let A be an algebra over a field $\Bbbk$.*

(1) *Assume Hypotheses 0.1 and 0.2 for A. Then A has finite injective dimension. Further*, *as an algebra*, *A is a finite direct sum of indecomposable noetherian algebras which are Artin–Schelter Gorenstein*, *Auslander Gorenstein*, *Cohen–Macaulay*, *and homogeneous of finite Gelfand–Kirillov dimension equal to their injective dimension.*

(2) *If H is a weak Hopf algebra*, *then H satisfies Hypothesis 0.2. In particular*, *if H also satisfies Hypothesis 0.1 then H satisfies all of the conclusions of part* (1).

We remark that the direct sum of two Artin–Schelter Gorenstein weak Hopf algebras of different injective dimensions is a weak Hopf algebra which is not Artin–Schelter Gorenstein. Hence, this theorem gives the strongest answer to the analog of question (Q2) that is possible in this setting.

While we focus on weak Hopf algebras in this paper, we expect other generalizations of Hopf algebras to satisfy Theorem 0.3(2). As an example, we show this for quasibialgebras with antipode in Theorem 7.1 below. It would be interesting to extend this theory to different kinds of Hopf-like structures.

We also note that if A satisfies Hypothesis 0.2 because it is a weak Hopf algebra or other similar structure, as in Theorem 0.3(2), the coproduct need not respect the direct sum decomposition in Theorem 0.3(1).

Theorem 0.3 has many applications. Below we list some of the consequences of this theorem. Undefined terminology will be reviewed in later sections.

Theorem 0.4. *Assume Hypotheses 0.1 and 0.2 for A. Then A has a quasi-Frobenius artinian quotient ring.*

Theorem 0.5. *Assume Hypotheses 0.1 and 0.2 for A. If A has finite global dimension*, *then A is a direct sum of prime algebras and each summand is homogeneous*, *Artin–Schelter regular*, *Auslander regular*, *and Cohen–Macaulay.*

The following is a version of the Nichols–Zoeller theorem for infinite-dimensional algebras.

Theorem 0.6. *Assume Hypotheses 0.1 and 0.2 for both A_1 and A_2. Let A_1 and A_2 be homogeneous of the same Gelfand–Kirillov dimension. Suppose that A_1 has finite global dimension. If there is an algebra map* (*which is not necessarily a coalgebra map*) *$f : A_1 \to A_2$ such that A_2 is a finitely generated module over A_1 on both sides*, *then A_2 is a projective module over A_1 on both sides.*

Theorem 0.7. *Assume Hypotheses 0.1 and 0.2 for A.*

(1) *A has a rigid Auslander and Cohen–Macaulay dualizing complex which is also an invertible complex of A-bimodules.*

(2) *A has a residue complex.*

(3) *If A is homogeneous*, *then A has a minimal pure injective resolution on both sides.*

The following result concerns the homological properties of A-modules.

Theorem 0.8. *Assume Hypotheses 0.1 and 0.2 for A and let $d =$* GKdim *A. Let M be a nonzero finitely generated left A-module.*

(1) The Auslander–Buchsbaum formula. *If* $\operatorname{projdim} M < \infty$, *then*

$$\operatorname{projdim} M \le d - \operatorname{depth} M.$$

If, further, A is homogeneous, then

$$\operatorname{projdim} M = d - \operatorname{depth} M.$$

(2) Bass's theorem. *If* $\operatorname{injdim} M < \infty$, *then*

$$\operatorname{injdim} M \le d.$$

If, further, A is homogeneous, then

$$\operatorname{injdim} M = d.$$

(3) The no-holes theorem. *Suppose either A is homogeneous or M is indecomposable. For every integer i between* $\operatorname{depth} M$ *and* $\operatorname{injdim} M$, *there is a simple left A-module S such that* $\operatorname{Ext}^i_A(S, M) \neq 0$.

It would be interesting to study in more detail the tensor categories given by the finite-dimensional modules over infinite-dimensional weak Hopf algebras, but we have not attempted to do so here.

We begin, in Section 1, by reviewing the definitions of various homological properties and introducing, for a noetherian algebra A, important homological conditions (L1) and (R1) on the categories of left and right A-modules, respectively. In Section 2, we prove some key lemmas which allow us, in Section 3, to prove that if A is a finite module over its affine center and satisfies (L1) and (R1), then A is a finite direct sum of AS Gorenstein, Auslander Gorenstein, and Cohen–Macaulay algebras. We study the consequences of this result in Section 4 and show that the conclusions of Theorems 0.4–0.8 hold for A. In Section 5, we recall the definition of a weak Hopf algebra and provide some examples. In Section 6, we study the relationship between Hypothesis 0.2 and the conditions (L1) and (R1) and conclude in Section 7 that weak Hopf algebras which are module-finite over their affine centers satisfy (L1) and (R1). We then prove Theorems 0.3–0.8, settling the Brown–Goodearl question for this class of weak Hopf algebras. We conclude in Section 8 by posing some open questions.

1. Preliminaries

We first recall some definitions concerning different homological properties. For an algebra A, let A-Mod denote the category of left A-modules; for $M, N \in A$-Mod we write $\operatorname{Hom}_A(M, N)$ for the space of left module homomorphisms. We identify the category Mod-A of right A-modules with A^{op}-Mod when convenient. In particular, for right modules M and N we write $\operatorname{Hom}_{A^{\mathrm{op}}}(M, N)$ for the space of right A-module homomorphisms.

Definition 1.1 [Levasseur 1992, Definitions 1.2 and 2.1]. Let A be an algebra and M a left A-module.

(1) The *grade number* of M is defined to be

$$j_A(M) := \inf\{i \mid \operatorname{Ext}^i_A(M, A) \neq 0\} \in \mathbb{N} \cup \{+\infty\}.$$

We often write $j(M)$ for $j_A(M)$. Note that $j_A(0) = +\infty$.

(2) We say that M satisfies the *Auslander condition* if for any $q \geq 0$, $j_A(N) \geq q$ for all right A-submodules N of $\mathrm{Ext}_A^q(M, A)$.

(3) We say a noetherian algebra A is *Auslander Gorenstein* (respectively, *Auslander regular*) of dimension n if $\mathrm{injdim}\, A_A = \mathrm{injdim}\, {}_A A = n < \infty$ (respectively, $\mathrm{gldim}\, A = n < \infty$), and every finitely generated left and right A-module satisfies the Auslander condition.

The following notions are defined and used in many papers, for example, [Ajitabh et al. 1998; Wu and Zhang 2003].

Definition 1.2. We say A is *Cohen–Macaulay* (or, *CM* for short) if $\mathrm{GKdim}(A) = d < \infty$ and

$$j(M) + \mathrm{GKdim}(M) = \mathrm{GKdim}(A)$$

for every finitely generated nonzero left (or right) A-module M.

In this paper we will use the following slightly modified version of the Artin–Schelter Gorenstein property.

Definition 1.3. A noetherian algebra A is called *Artin–Schelter Gorenstein* (or *AS Gorenstein*, for short) if the following conditions hold:

(1) A has finite injective dimension $d < \infty$ on both sides.

(2) For every finite-dimensional left A-module S, $\mathrm{Ext}_A^i(S, A) = 0$ for all $i \neq d$ and $\dim \mathrm{Ext}_A^d(S, A) < \infty$.

(3) The analog of part (2) for right A-modules holds.

If, moreover,

(4) A has finite global dimension,

then A is called *Artin–Schelter regular* (or *AS regular*, for short).

Let $A\text{-}\mathrm{Mod}_{\mathrm{fd}}$ denote the category of finite-dimensional left A-modules. The category of finite-dimensional right A-modules will be written $\mathrm{Mod}_{\mathrm{fd}}\text{-}A$ or $A^{\mathrm{op}}\text{-}\mathrm{Mod}_{\mathrm{fd}}$.

Lemma 1.4. *Suppose A is AS Gorenstein of injective dimension d. Then $A\text{-}\mathrm{Mod}_{\mathrm{fd}}$ is contravariant equivalent to $A^{\mathrm{op}}\text{-}\mathrm{Mod}_{\mathrm{fd}}$ via the functor $\mathrm{Ext}_A^d(-, A)$. As a consequence, for each i, both $\mathrm{Ext}_A^i(-, A)$ and $\mathrm{Ext}_{A^{\mathrm{op}}}^i(-, A)$ are exact functors when restricted to finite-dimensional A-modules.*

Proof. By Definition 1.3(2), $\mathrm{Ext}_A^d(-, A)$ is an exact functor on finite-dimensional left A-modules. For every finite-dimensional left A-module S, by Ischebeck's double Ext-spectral sequence [Ajitabh et al. 1998, (0-2)], we have

$$\mathrm{Ext}_{A^{\mathrm{op}}}^d\big(\mathrm{Ext}_A^d(S, A), A\big) \cong S.$$

A similar statement holds for finite-dimensional right A-modules S. The assertion follows, and the consequence is clear. □

Lemma 1.4 implies that if A is AS Gorenstein of dimension d then $\operatorname{Ext}^d_A(S, A)$ is simple right A-module for each finite-dimensional simple left module S, and similarly on the other side. This shows that Definition 1.3 is equivalent to the definition of AS Gorenstein in [Wu and Zhang 2003, Definition 3.1] for any algebra A for which all simple modules are finite-dimensional. In particular, this is the case for the affine noetherian PI algebras of main interest in this paper by [Brown and Goodearl 1997, Proposition 3.1].

Definition 1.5. Let A be a noetherian algebra.

(1) We say that A satisfies (L1) (respectively, (R1)) if, for each $i \geq 0$, the functor $\operatorname{Ext}^i_A(-, A)$ is exact when applied to the category A-$\mathrm{Mod}_{\mathrm{fd}}$ (respectively, $\operatorname{Ext}^i_{A^{\mathrm{op}}}(-, A)$ is exact when applied to $\mathrm{Mod}_{\mathrm{fd}}$-$A$).

(2) We say that A satisfies (L2) (respectively, (R2)) if A satisfies (L1) (respectively, (R1)), and for each $i \geq 0$, for $0 \neq S, T \in A$-$\mathrm{Mod}_{\mathrm{fd}}$, $\operatorname{Ext}^i_A(S, A) = 0$ if and only if $\operatorname{Ext}^i_A(T, A) = 0$ (respectively, for $0 \neq S, T \in \mathrm{Mod}_{\mathrm{fd}}$-$A$, $\operatorname{Ext}^i_{A^{\mathrm{op}}}(S, A) = 0$ if and only if $\operatorname{Ext}^i_{A^{\mathrm{op}}}(T, A) = 0$).

By [Wu and Zhang 2003, Proposition 3.2], if A is affine noetherian PI, then A is AS Gorenstein if and only if A satisfies (L2) (or equivalently, satisfies (R2)). The following lemma follows from algebra decomposition and [Wu and Zhang 2003, Proposition 3.2].

Lemma 1.6. *Suppose that A is an affine noetherian PI algebra. If A is a direct sum of finitely many AS Gorenstein algebras* (*of possibly different dimensions*), *then A satisfies* (L1) *and* (R1).

One of our main goals is to show that the converse of Lemma 1.6 holds under some extra hypotheses. We will need the following easy lemmas about algebra decompositions.

Lemma 1.7. *Let A be an algebra.*

(1) *Let e and e' be two idempotents such that $eA = Ae'$. Then $e = e'$ is a central idempotent.*

(2) *Let I be a two sided ideal of A such that $A = I \oplus B$ as left A-modules and $A = I \oplus C$ as right A-modules. Then $B = C$ and $A = I \oplus B$ as algebras.*

Proof. (1) Since $eA = Ae'$, there are elements $a, b \in A$ such that $e = ae'$ and $e' = eb$. Then

$$e = ae' = ae'e' = ee' = eeb = eb = e'.$$

For every $a \in A$, $ea, ae \in eA = Ae$. Hence $ea = eae = ae$. This shows that e is a central idempotent.

(2) Since $A = I \oplus B$, $1 = e + (1 - e)$, where $e \in I$ and $(1 - e) \in B$. Since $e(1 - e) \in I \cap B = 0$, we have $e = e^2$ and so e is idempotent. Since $Ae \oplus A(1 - e) = A = I \oplus B$, we obtain that $Ae = I$ and $A(1 - e) = B$.

Similarly, there is an idempotent e' such that $I = e'A$. By part (1), $e = e'$ which is central. Therefore $A = eA \oplus (1 - e)A$, where both eA and $(1 - e)A$ are two-sided ideals of A. The assertion follows. □

We say that A is *indecomposable* if A is not isomorphic to a direct sum of two algebras. The following lemma is standard.

Lemma 1.8 [Lam 1991, Proposition 22.2]. *Let A be a noetherian algebra. Then*

$$A = \bigoplus_{i \in I} A_i$$

for a finite set of indecomposable algebras $\{A_i\}_{i \in I}$. Further, this decomposition is unique up to permutation.

2. Some key lemmas

While we primarily consider the GK dimension of modules as our dimension function in this paper, in this section it is convenient to use also the (Gabriel–Rentschler) Krull dimension of a module M, which we denote by $\operatorname{Kdim} M$. Fortunately, if A is finite over its affine center, then by [Wu and Zhang 2003, Lemma 1.2(3)], for all finitely generated left (or right) A-modules M,

$$\operatorname{Kdim} M = \operatorname{GKdim} M.$$

For any such module M, for every s, let $\tau_s(M)$ denote the largest submodule of M with GK dimension which is less than or equal to s. If M is an (A, A)-bimodule which is finitely generated on both sides, then

$$\tau_s({}_A M) = \tau_s(M_A)$$

since GK dimension is symmetric [Krause and Lenagan 1985, Corollary 5.4]. In particular,

$$\tau_s({}_A A) = \tau_s(A_A)$$

which we denote by $\tau_s(A)$.

Let us introduce some temporary notation. Let A-mod denote the category of finitely generated left A-modules. For each integer d, let A-mod_d denote the category of finitely generated left A-modules of Krull dimension no more than d. The following is a key lemma.

Lemma 2.1. *Let A be a noetherian PI complete semilocal algebra and M be a finitely generated left A-module. If $\operatorname{Hom}_A(M, -)$ is exact when applied to A-mod_0, then M is projective.*

Proof. Since A is noetherian and complete semilocal, it is semiperfect in the sense of [Lam 1991, Definition 23.1]. Then by [Lam 1991, Proposition 24.12], every finitely generated A-module has a projective cover. Let P be the projective cover of M and let K be the kernel of the surjective map $P \to M$. Then we have a short exact sequence

$$0 \to K \xrightarrow{f} P \xrightarrow{g} M \to 0.$$

It suffices to show that $K = 0$.

Let $\mathfrak{m}$ be the Jacobson radical of A. Then a finitely generated left A-module N has finite length if and only if $\mathfrak{m}^s N = 0$ for some integer s. Let n be any positive integer. By adjunction,

$$\operatorname{Hom}_{A/\mathfrak{m}^n}(A/\mathfrak{m}^n \otimes_A M, -) \cong \operatorname{Hom}_A(M, -)$$

when applied to modules over $A/\mathfrak{m}^n$. Hence $\mathrm{Hom}_{A/\mathfrak{m}^n}(A/\mathfrak{m}^n \otimes_A M, -)$ is exact when applied to modules in $A/\mathfrak{m}^n\text{-mod}_0 = A/\mathfrak{m}^n\text{-mod}$. Since $A/\mathfrak{m}^n \otimes_A M$ is a finitely generated $A/\mathfrak{m}^n$-module, it follows that $A/\mathfrak{m}^n \otimes_A M$ is a projective module over $A/\mathfrak{m}^n$.

It is easy to check that $A/\mathfrak{m}^n \otimes_A P$ is a projective cover of $A/\mathfrak{m}^n \otimes_A M$. Then

$$A/\mathfrak{m}^n \otimes_A g : \quad A/\mathfrak{m}^n \otimes_A P \to A/\mathfrak{m}^n \otimes_A M$$

is an isomorphism. Therefore

$$A/\mathfrak{m}^n \otimes_A f : \quad A/\mathfrak{m}^n \otimes_A K \to A/\mathfrak{m}^n \otimes_A P$$

is the zero map for all $n \geq 1$. Equivalently, $f(K) \subseteq \mathfrak{m}^n P$ for all n. Since A is PI, by [Goodearl and Warfield 2004, Theorem 9.13], $\bigcap_n \mathfrak{m}^n = 0$. Hence $\bigcap_n \mathfrak{m}^n P = 0$, as P is a finitely generated projective A-module, and consequently, $f(K) = 0$. Since f is monomorphism, $K = 0$ as required. □

Assume that A is finitely generated over its affine center $Z(A)$. Let $\mathfrak{n}$ be a maximal ideal of $Z(A)$. Let $Z_\mathfrak{n}$ denote the completion of the commutative noetherian local ring $Z(A)_\mathfrak{n}$ with respect to its maximal ideal. Then

(1) $Z_\mathfrak{n}$ is noetherian [Atiyah and Macdonald 1969, Theorem 10.26].

(2) $A_\mathfrak{n} := Z_\mathfrak{n} \otimes_{Z(A)} A$ is finitely generated over $Z_\mathfrak{n}$ (but its center could be bigger than $Z_\mathfrak{n}$).

(3) $A_\mathfrak{n}$ is complete semilocal.

(4) The functor $Z_\mathfrak{n} \otimes_{Z(A)} - : A\text{-mod} \to A_\mathfrak{n}\text{-mod}$ is exact [Atiyah and Macdonald 1969, Proposition 10.14].

Every left A-module M can be considered as an $(A, Z(A))$-bimodule. It is well-known that if M and W are left A-modules, then $\mathrm{Ext}^i_A(M, W)$ has a central $(Z(A), Z(A))$-bimodule structure.

Lemma 2.2. *Let A be a finitely generated module over its affine center. Suppose M is a finitely generated left A-module. Then M is projective over A if and only if*

$$M \otimes_{Z(A)} Z_\mathfrak{n} \cong Z_\mathfrak{n} \otimes_{Z(A)} M$$

is projective over $A_\mathfrak{n}$ for all maximal ideals $\mathfrak{n}$ of $Z(A)$.

Proof. Since $M \otimes_{Z(A)} Z_\mathfrak{n} \cong A_\mathfrak{n} \otimes_A M$, one implication is clear. For the other implication, assume that $M \otimes_{Z(A)} Z_\mathfrak{n}$ is projective for all maximal ideals $\mathfrak{n}$ of $Z(A)$. If M is not projective, then there is a finitely generated A-module W such that $\mathrm{Ext}^i_A(M, W) \neq 0$ for some $i > 0$. Let $\mathfrak{n}$ be a maximal ideal of $Z(A)$ such that

$$\mathrm{Ext}^i_A(M, W) \otimes_{Z(A)} Z_\mathfrak{n} \neq 0.$$

By [Yekutieli and Zhang 2003, Lemma 3.7],

$$\mathrm{Ext}^i_{A_\mathfrak{n}}\big(M \otimes_{Z(A)} Z_\mathfrak{n}, W \otimes_{Z(A)} Z_\mathfrak{n}\big) \cong \mathrm{Ext}^i_A(M, W) \otimes_{Z(A)} Z_\mathfrak{n} \neq 0.$$

Hence $M \otimes_{Z(A)} Z_\mathfrak{n}$ is not projective, a contradiction. □

We will use the following result in the analysis of the dualizing complex over A.

Proposition 2.3. *Let A be a finitely generated module over its affine center. Suppose M is a finitely generated left A-module such that $\mathrm{Hom}_A(M, -)$ is exact on finite-dimensional left A-modules. Then M is projective.*

Proof. By Lemma 2.2, it suffices to show that $M_{\mathfrak{n}} := M \otimes_{Z(A)} Z_{\mathfrak{n}}$ is projective for all maximal ideals $\mathfrak{n}$ of $Z(A)$.

Let W be a finite-dimensional left $A_{\mathfrak{n}}$-module. Then

$$W \cong A_{\mathfrak{n}} \otimes_A W \cong W \otimes_{Z(A)} Z_{\mathfrak{n}}.$$

By adjunction, we have

$$\mathrm{Hom}_{A_{\mathfrak{n}}}(M_{\mathfrak{n}}, W) \cong \mathrm{Hom}_A(M, W).$$

Since $Z(A)/\mathfrak{n}$ is finite-dimensional, every finitely generated artinian module over $A_{\mathfrak{n}}$ is finite-dimensional. Hence, by hypothesis $\mathrm{Hom}_{A_{\mathfrak{n}}}(M_{\mathfrak{n}}, -) \cong \mathrm{Hom}_A(M, -)$ is exact when applied to objects in $A_{\mathfrak{n}}\text{-mod}_0$. By Lemma 2.1, $M_{\mathfrak{n}}$ is projective. □

We need one more homological lemma, which depends on the basic properties of Krull dimension for PI algebras.

Lemma 2.4. *Let A be a noetherian PI algebra of finite Krull dimension. Let M be an (A, A)-bimodule which is finitely generated on both sides and let w be a nonnegative integer.*

(1) *Suppose that for all simple left A-modules S, we have $\mathrm{Ext}^s_A(S, M) = 0$ for all $s \le w$. Then, for each integer $d \ge 0$, if N is a finitely generated left A-module with $\mathrm{Kdim}\, N \le d$, we have $\mathrm{Ext}^s_A(N, M) = 0$ for all $s \le w - d$. As a consequence, ${}_AM$ does not contain any nonzero left A-submodules of Krull dimension less than or equal to w.*

(2) *Suppose that for all simple left A-modules S, we have $\mathrm{Ext}^s_A(S, M) = 0$ for all $s > w$. Then for all finitely generated left A-modules N, we have $\mathrm{Ext}^s_A(N, M) = 0$ for all $s > w$. As a consequence, $\mathrm{injdim}({}_AM) \le w$.*

Proof. (1) We prove the assertion by induction on d. When $d = 0$, it follows from exact sequences and the hypothesis that $\mathrm{Ext}^s_A(S, M) = 0$ for all finite length left A-modules S and for all $s \le w$. Since the A-modules of finite length are precisely the A-modules of Krull dimension 0, the result holds for $d = 0$.

Now let $d > 0$ and assume that the assertion $\mathrm{Ext}^s_A(N', M) = 0$ holds for all finitely generated left A-modules N' with $\mathrm{Kdim}\, N' \le d - 1$ and for all $s \le w - (d - 1)$. We wish to show that $\mathrm{Ext}^s_A(N, M) = 0$ for all finitely generated left A-modules N of Krull dimension d and for all $s \le w - d$. By choosing a prime filtration of N, similarly as in [Stafford and Zhang 1994, Lemma 2.1(i,ii)], we may assume that $N = A/\mathfrak{p} =: B$ for some prime ideal $\mathfrak{p}$, and where $\mathrm{Kdim}(B) = d$. If x is a nonzero central element $x \in B$, then x is regular and there is a short exact sequence

$$0 \to B \xrightarrow{r_x} B \to B/(x) \to 0,$$

where r_x denotes right multiplication by x and $B/(x)$ is a left A-module with $\mathrm{Kdim}(B/(x)) \le d-1$. By the induction hypothesis, $\mathrm{Ext}^s_A(B/(x), M) = 0$ for all $s \le w-d+1$. Then, for every $s \le w-d$, the long exact sequence

$$\cdots \to \mathrm{Ext}^s_A(B/(x), M) \to \mathrm{Ext}^s_A(B, M) \xrightarrow{(r_x)^*} \mathrm{Ext}^s_A(B, M) \to \mathrm{Ext}^{s+1}_A(B/(x), M) \to \cdots$$

implies that $(r_x)^* : \mathrm{Ext}^s_A(B, M) \to \mathrm{Ext}^s_A(B, M)$ is an isomorphism. Note that $\mathrm{Ext}^s_A(B, M)$ is a left B-module, and that $(r_x)^*$ is just left multiplication l_x by x [Stafford and Zhang 1994, Lemma 3.4]. Let $Q(B)$ be the total fraction ring of B, obtained by inverting all central nonzero elements x. Since $l_x = (r_x)^*$ is an isomorphism, we can naturally define a left $Q(B)$-action on $\mathrm{Ext}^s_A(B, M)$ by setting $l_{x^{-1}} = (l_x)^{-1}$. So $W := \mathrm{Ext}^s_A(B, M)$ is a $(Q(B), A)$-bimodule. By [Stafford and Zhang 1994, Theorem 3.5], W is finitely generated as a left B-module. Since A is noetherian and M is a noetherian right A-module, computing $W = \mathrm{Ext}^s_A(B, M)$ with a projective resolution of B by finite rank free A-modules shows that W is also a finitely generated right A-module. Thus W is a (B, A)-bimodule, finitely generated on both sides (as well as a $(Q(B), A)$-bimodule, also finitely generated on both sides). By Krull symmetry [Goodearl and Warfield 2004, Theorem 15.15],

$$\mathrm{Kdim}\,{}_B W = \mathrm{Kdim}\, W_A = \mathrm{Kdim}\,{}_{Q(B)} W = 0.$$

Since $\mathrm{Kdim}\, B = d > 0$, there is a nonzero ideal I of B such that $IW = 0$. Since any nonzero ideal in a prime PI ring contains a nonzero central element, $Q(B) = Q(B)I$. Therefore $W = Q(B)W = Q(B)IW = 0$, as desired.

The consequence follows by taking $d = w$ and $s = 0$.

(2) The assertion follows by induction on $d := \mathrm{Kdim}\, N$. The proof is similar to the proof of (1), so it is omitted. □

3. Dualizing complexes and residue complexes

The noncommutative version of a dualizing complex was introduced by Yekutieli [1992] (we also refer the reader to [Yekutieli 2019]). Let $\mathbb{D}^{\mathrm{b}}_{\mathrm{f.g.}}(A\text{-Mod})$ denote the bounded derived category of complexes of left A-modules with finitely generated cohomology modules. Roughly speaking, a dualizing complex over an algebra A is a complex R of A-bimodules, such that the two derived functors $\mathrm{RHom}_A(-, R)$ and $\mathrm{RHom}_{A^{\mathrm{op}}}(-, R)$ induce a duality between the derived categories $\mathbb{D}^{\mathrm{b}}_{\mathrm{f.g.}}(A\text{-Mod})$ and $\mathbb{D}^{\mathrm{b}}_{\mathrm{f.g.}}(A^{\mathrm{op}}\text{-Mod})$. Let $A^e = A \otimes_{\Bbbk} A^{\mathrm{op}}$ denote the enveloping algebra of A.

Definition 3.1. Let A be a noetherian algebra. A complex $R \in \mathbb{D}^{\mathrm{b}}(A^e\text{-Mod})$ is called a *dualizing complex* over A if it satisfies the three conditions below:

(i) R has finite injective dimension on both sides.

(ii) R has finitely generated cohomology modules on both sides.

(iii) The canonical morphisms $A \to \mathrm{RHom}_A(R, R)$ and $A \to \mathrm{RHom}_{A^{\mathrm{op}}}(R, R)$ in $\mathbb{D}(A^e\text{-Mod})$ are both isomorphisms.

We also need a few other definitions related to dualizing complexes.

Definition 3.2. Let A be a noetherian algebra and R a dualizing complex over A.

(1) Let M be a finitely generated left A-module. The *grade* of M (with respect to R) is defined to be

$$j_R(M) = \min\{i \mid \mathrm{Ext}^i_A(M, R) \neq 0\} \in \mathbb{Z} \cup \{+\infty\}.$$

We write the grade of M as $j(M)$ when the choice of R is clear. The grade of a right A-module is defined similarly.

(2) [Yekutieli and Zhang 1999, Definition 2.1]. We say that R has the *Auslander property*, or that R is an *Auslander dualizing complex*, if

(i) for every finitely generated left A-module M, integer q, and right A-submodule $N \subseteq \mathrm{Ext}^q_A(M, R)$, one has $j(N) \geq q$;

(ii) the same holds after exchanging left and right.

(3) [van den Bergh 1997, Definition 8.1]. A dualizing complex R is called *rigid* if there is an isomorphism

$$R \xrightarrow{\cong} \mathrm{RHom}_{A^e}(A, R \otimes R)$$

in $\mathbb{D}(A^e\text{-Mod})$.

(4) [Yekutieli and Zhang 1999, Definition 2.24]. Suppose R is an Auslander dualizing complex over A. We say R is *Cohen–Macaulay* if for every finitely generated left (respectively, right) A-module M,

$$j(M) + \mathrm{GKdim}\, M = 0.$$

(This is a slightly stronger version than [Yekutieli and Zhang 1999, Definition 2.24].)

(5) Suppose that A has finite injective dimension d as a left and right A-module, and that $\mathrm{GKdim}(A) = d$. Then the complex $R = A[d]$ is a dualizing complex for A, and we say that A is *Auslander Gorenstein* if R is an Auslander dualizing complex, and that A is *Cohen–Macaulay* if R is a Cohen–Macaulay dualizing complex.

Since we are working with algebras that are finite over their affine centers, the natural dimension function to use is Gelfand–Kirillov dimension. Recall that an algebra A is called (left) *homogeneous* if $\mathrm{GKdim}\, L = \mathrm{GKdim}\, A$ for all nonzero left ideals $L \subseteq A$. This notion generalizes to A-modules M as defined below.

Definition 3.3. Let A be a noetherian algebra and M be a nonzero left A-module.

(1) [Ajitabh et al. 1998, Definition 0.2]. Suppose $\mathrm{GKdim}\, M = s$. We say M is *s-pure*, if $\mathrm{GKdim}\, N = s$ for all nonzero submodules N of M.

(2) [Ajitabh et al. 1998, Definition 0.3]. Suppose $\operatorname{injdim}_A A = d < \infty$ and let

$$0 \to A \to I^0 \to I^1 \to \cdots \to I^d \to 0$$

be a minimal injective resolution of the left A-module ${}_A A$. We say this resolution is *pure* if each I^i is $(d-i)$-pure.

Definition 3.4 [Yekutieli and Zhang 2003, Definitions 4.3 and 5.1]. A dualizing complex K over A is called a *residual complex* over A if the following conditions are satisfied:

(i) K is Auslander,

(ii) each K^{-q} is an injective module over A on both sides,

(iii) each K^{-q} is q-pure on both sides.

A rigid residual complex K is called a *residue* complex.

When a residue complex K exists, then ${}_A K$ (respectively, K_A) can be viewed as a minimal injective resolution of a rigid dualizing complex ${}_A R$ (respectively, R_A).

Here are some basic facts about dualizing complexes for the algebras A of interest in this paper.

Lemma 3.5. *Let A be a finitely generated module over its affine center. Then the following hold.*

(1) *There is a rigid dualizing complex, denoted by R, over A that is Auslander and Cohen–Macaulay.*

(2) *There is a residue complex, denoted by K, over A that is Auslander and Cohen–Macaulay.*

(3) *Let M be a finitely generated left A-module. If $\operatorname{Ext}^i_A(M, K)$ is nonzero only when $i = -d$, then* $\operatorname{GKdim} M = d$.

Proof. (1) If A is a finitely generated module over its affine center then it admits a noetherian connected filtration (see [Yekutieli and Zhang 2003, Remark 6.4]). Hence [Yekutieli and Zhang 2003, Proposition 6.5] applies, and shows that A has an Auslander rigid dualizing complex R. For a module M, the canonical dimension of M is defined in [Yekutieli and Zhang 2003] to be $\operatorname{Cdim}(M) = -j_R(M)$. The proof of [Yekutieli and Zhang 2003, Proposition 6.5] shows that for finitely generated left and right modules, the canonical dimension is equal to the GK dimension. This implies that R is Cohen–Macaulay.

(2) A has a residue complex K by [Yekutieli and Zhang 2003, Proposition 6.6]. Since a residue complex is rigid, and a rigid dualizing complex is unique up to isomorphism in the derived category [van den Bergh 1997, Proposition 8.2], K is also Auslander and Cohen–Macaulay by part (1).

(3) By definition, $j(M) = -d$, and by the Cohen–Macaulay property (Definition 3.2(4)), $\operatorname{GKdim} M = d$. □

Recall that an algebra A is called indecomposable if it is not possible to write $A = B \oplus C$ as a direct sum of algebras. The next result contains the bulk of the work needed for the proof of our main theorem.

Theorem 3.6. *Let A be a finite module over its affine center. Suppose A is indecomposable and satisfies* (L1) *and* (R1) *of Definition 1.5*(1). *Then the following hold*:

(1) *A is AS Gorenstein of injective dimension* $\mathrm{GKdim}\, A$; *in particular*, A *satisfies conditions* (L2) *and* (R2) *of Definition 1.5*(2).

(2) *A is Auslander Gorenstein and Cohen–Macaulay.*

(3) *A is a homogeneous A-module on both sides.*

Proof. Part (2) follows from part (1) and [Stafford and Zhang 1994, Theorem 1.3]. Part (3) follows from the Cohen–Macaulay property of A. Hence, we only need to prove part (1).

By Lemma 3.5(2), A has a residue complex K which is Auslander and Cohen–Macaulay. Let $d = \mathrm{GKdim}\, A$. Note that by the definition of residue complex, we must have $K^{-i} = 0$ for $i < 0$ and $i > d$.

First, we claim that for all $i \neq d$, $H^{-i}(K) = 0$. Suppose this claim is not true. Then there is $0 \le s < d$ such that $H^{-s}(K) \neq 0$. Choose the smallest such s. Let Ω be the nonzero (A, A)-bimodule $H^{-s}(K)$. By the definition of a dualizing complex, Ω is finitely generated over A on both sides. Let

$$D := \mathrm{RHom}_A(-, K) : \mathbb{D}^{\mathrm{b}}_{\mathrm{f.g.}}(A\text{-Mod}) \to \mathbb{D}^{\mathrm{b}}_{f.g}(\text{Mod-}A)$$

and

$$D^{\mathrm{op}} := \mathrm{RHom}_{A^{\mathrm{op}}}(-, K) : \mathbb{D}^{\mathrm{b}}_{\mathrm{f.g.}}(\text{Mod-}A) \to \mathbb{D}^{\mathrm{b}}_{\mathrm{f.g.}}(A\text{-Mod})$$

be the duality functors in [Yekutieli 1992, Proposition 3.4]. Recall from the definition that each term K^{-i} is an injective, i-pure module on both the left and the right. It follows that if S is a finite-dimensional left A-module then

$$H^i(D(S)) = H^i(\mathrm{RHom}_A(S, K)) = H^i(\mathrm{Hom}_A(S, K)) = \begin{cases} 0, & i \neq 0, \\ \mathrm{Hom}_A(S, K^0), & i = 0, \end{cases}$$

and $H^0(D(S))$ is finitely generated on the right. Since K is Auslander and Cohen–Macaulay, it follows that $H^0(D(S))$ is finite-dimensional. Since the complex $D(S)$ has cohomology in only one degree, we have $D(S) \cong T$ in $\mathbb{D}^{\mathrm{b}}_{\mathrm{f.g.}}(\text{Mod-}A)$ for some finite-dimensional right module $T = \mathrm{Hom}_A(S, K^0) \in \text{Mod-}A$. We have $D^{\mathrm{op}}(D(S)) \cong S$ in $\mathbb{D}^{\mathrm{b}}_{\mathrm{f.g.}}(A\text{-Mod})$ [Yekutieli 1999, Proposition 4.2(2)], and hence also in A-Mod. A similar result holds on the other side, and we conclude that D and D^{op} induce a duality between finite-dimensional left and right A-modules.

Next, let S be any left A-module. Let P be a projective resolution of K (as a complex of left A-modules). Since $H^{-i}(K) = 0$ for all $i \le s$, we can assume that $P^{-s+i} = 0$ for all $i > 0$. Then $\mathrm{Ext}^i_A(K, S) \cong \mathrm{Ext}^i_A(P, S) = 0$ for all $i < s$. Using the fact that $\mathrm{Hom}_A(-, S)$ is left exact, one sees that

$$\mathrm{Ext}^s_A(K, S) \cong \mathrm{Ext}^s_A(P, S) \cong \mathrm{Hom}_A(H^{-s}(P), S) \cong \mathrm{Hom}_A(\Omega, S).$$

Note that we have $A \cong D(D^{\mathrm{op}}(A)) = D(K)$ in Mod-A [Yekutieli 1999, Proposition 4.2(1)]. The dualizing functor D also gives an isomorphism

$$\mathrm{RHom}_A(M, N) \cong \mathrm{RHom}_{A^{\mathrm{op}}}(D(N), D(M)),$$

which is functorial in all $M, N \in \mathbb{D}^{\mathrm{b}}_{\mathrm{f.g.}}(\text{Mod-}A)$ [Yekutieli 1999, Proposition 4.2(2)]. Hence

$$\operatorname{Ext}^i_{A^{\mathrm{op}}}(D(S), A) \cong \operatorname{Ext}^i_{A^{\mathrm{op}}}(D(S), D(K)) \cong \operatorname{Ext}^i_A(K, S) = \begin{cases} 0, & i < s, \\ \operatorname{Hom}_A(\Omega, S), & i = s. \end{cases} \tag{E3.6.1}$$

As a consequence, if T is any finite-dimensional right A-module, then since $T = D(S)$ where $S = D^{\mathrm{op}}(T)$ is a finite-dimensional left module, we have $\operatorname{Ext}^i_{A^{\mathrm{op}}}(T, A) = 0$ for all $i < s$. Now D, considered as a duality from finite-dimensional left A-modules to finite-dimensional right A-modules, is an exact functor. Since the functor $\operatorname{Ext}^s_{A^{\mathrm{op}}}(-, A)$ is exact on finite-dimensional right modules by hypothesis (R1), we see that $\operatorname{Ext}^s_{A^{\mathrm{op}}}(D(-), A)$ is an exact functor from finite-dimensional left A-modules to left A-modules. Now (E3.6.1) is functorial in S and so

$$\operatorname{Ext}^s_{A^{\mathrm{op}}}(D(-), A) \cong \operatorname{Hom}_A(\Omega, -)$$

is exact on finite-dimensional left A-modules. By Proposition 2.3, Ω is projective over A as a left module.

Now because $\Omega \cong H^{-s}(K)$ is projective, where $H^{-i}(K) = 0$ for $i < s$, the complex K is quasi-isomorphic to $K' \oplus \Omega[s]$ as a complex of left A-modules, where $H^{-i}(K') = 0$ for all $i \le s$. Since K is a dualizing complex, we have, as complexes of right A-modules,

$$\begin{aligned} A &\cong \operatorname{RHom}_A(K, K) \cong \operatorname{RHom}_A(K' \oplus \Omega[s], K) \\ &\cong \operatorname{RHom}_A(K', K) \oplus \operatorname{RHom}_A(\Omega[s], K) \\ &\cong \operatorname{RHom}_A(K', K) \oplus \operatorname{Ext}^0_A(\Omega[s], K) \end{aligned}$$

and $\operatorname{Ext}^i_A(\Omega[s], K) = 0$ for all $i \neq 0$. The last assertion is equivalent to

$$\operatorname{Ext}^i_A(\Omega, K) = 0$$

for all $i \neq -s$. By Lemma 3.5(3), $\operatorname{GKdim} \Omega = s$. So we have a right A-module decomposition

$$A \cong A' \oplus V,$$

where $A' = \operatorname{RHom}_A(K', K)$ and $V \cong \operatorname{Ext}^{-s}_A(\Omega, K)$. By the Auslander and Cohen–Macaulay properties of K, $\operatorname{GKdim} V \le s$. Now from the equation $\operatorname{Ext}^i_A(K, S) \cong \operatorname{Ext}^i_A(K', S) \oplus \operatorname{Ext}^{i-s}_A(\Omega, S)$ and (E3.6.1), we obtain for every finite-dimensional left A-module S that $\operatorname{Ext}^i_A(K', S) = 0$ for all $i \le s$. Then for any finite dimensional right A-module T, writing $T \cong D(S)$ for $S = D^{\mathrm{op}}(T)$, for all $i \le s$ we obtain

$$\operatorname{Ext}^i_{A^{\mathrm{op}}}(T, A') = \operatorname{Ext}^i_{A^{\mathrm{op}}}(D(S), A') \cong \operatorname{Ext}^i_{A^{\mathrm{op}}}(D(S), D(K')) \cong \operatorname{Ext}^i_A(K', S) = 0.$$

By the right-sided version of Lemma 2.4(1), A' does not have a submodule of $\operatorname{GKdim} \le s$. Therefore $V = \tau_s(A)$. So we have a canonical decomposition

$$A = A' \oplus \tau_s(A)$$

as right A-modules. By symmetry, there is a decomposition

$$A = A'' \oplus \tau_s(A)$$

as left A-modules. By Lemma 1.7(2), $A = B \oplus \tau_s(A)$ as algebras. This yields a contradiction to the hypothesis that A is indecomposable. This, finally, proves the claim that $H^{-i}(K) = 0$ for $i \neq d$.

We conclude that K is quasi-isomorphic to $\Omega[d]$ for some (A, A)-bimodule Ω, which we have seen is projective as a left A-module. By repeating the above proof, we obtain that Ω is a projective A-module on both sides. This means that K has finite projective dimension on both sides. Further, by (E3.6.1), for every finite-dimensional left A-module S,

$$\mathrm{Ext}^i_{A^{\mathrm{op}}}(D(S), A) = \begin{cases} 0, & i < d, \\ \mathrm{Hom}_A(\Omega, S), & i = d. \end{cases}$$

For every $i > d$,

$$\mathrm{Ext}^i_{A^{\mathrm{op}}}(D(S), A) \cong \mathrm{Ext}^i_A(K, S) \cong \mathrm{Ext}^i_A(\Omega[d], S) \cong \mathrm{Ext}^{i-d}_A(\Omega, S) = 0$$

as Ω is projective as a left A-module. Thus for every finite dimensional right A-module T, we have $\mathrm{Ext}^i_{A^{\mathrm{op}}}(T, A) = 0$ unless $i = d$, in which case $\mathrm{Ext}^d(T, A)$ is a finite dimensional left A-module, as we have seen. Also, by a right-sided version of Lemma 2.4(2), A has finite injective dimension d as a right module. Symmetric arguments prove these facts on the other side. Thus we have proved that A is AS Gorenstein by definition. That A satisfies conditions (L2) and (R2) follows from Lemma 1.4. □

Corollary 3.7. *Let A be a finite module over its affine center. Suppose A satisfies* (L1) *and* (R1) *of Definition 1.5*(1)*. Then A is a finite direct sum $A = \bigoplus_{i \in I} A_i$ of homogeneous AS Gorenstein algebras of injective dimension $\leq$* GKdim A*. Further, for each $i \in I$, A_i satisfies the conclusions of Theorem 3.6.*

Proof. By Lemma 1.8, A is a finite direct sum of indecomposable algebras, $A = \bigoplus_{i \in I} A_i$. It is easy to show that each A_i satisfies (L1) and (R1) of Definition 1.5(1). Since $Z(A) = \bigoplus_{i \in I} Z(A_i)$, it is also easy to see that each A_i is finite over its affine center. The assertion follows by applying Theorem 3.6 to each component A_i. □

Corollary 3.8. *Let A be a finite module over its affine center. Suppose A satisfies* (L1) *and* (R1) *of Definition 1.5*(1)*. Write $A = \bigoplus_{i \in I} A_i$, where each A_i is indecomposable of GK dimension d_i. Then A has a rigid dualizing complex R of the form $\bigoplus_{i \in I} \Omega_i[d_i]$, where each Ω_i is an invertible (A_i, A_i)-bimodule. As a consequence, R is an invertible complex of (A, A)-bimodules.*

Proof. If $R = \bigoplus_{i \in I} \Omega_i[d_i]$ where Ω_i is an invertible A_i-bimodule, then R has inverse $\bigoplus_{i \in I} \Omega_i^{-1}[-d_i]$. So we only need to prove the assertion when A is indecomposable. Let $d = \mathrm{GKdim}\, A$. By the proof of Theorem 3.6, A has a rigid dualizing complex of the form $\Omega[d]$, where Ω is a projective on both sides. We have also seen in Theorem 3.6 that A has finite injective dimension d as a left and right A-module. It is then obvious from the definition that A is also a dualizing complex over A. By [Yekutieli 1999, Theorem 4.5(ii)], $\Omega[d] = \mathrm{RHom}_A(A, \Omega[d])$ is a tilting complex over A, which is a (derived) invertible complex of A-bimodules [Yekutieli 1999, Theorem 1.6]. Since Ω is projective over A on both sides, Ω must be an invertible (A, A)-bimodule. □

4. Some consequences

In this section we give some immediate consequences for algebras that satisfy the conclusions of Corollary 3.7 or, in other words, the following hypothesis.

Hypothesis 4.1. *Let A be a finite direct sum $A = \bigoplus_{i \in I} A_i$ of indecomposable, noetherian, homogeneous, AS Gorenstein, Auslander Gorenstein, Cohen–Macaulay algebras. For each $i \in I$, suppose* $\operatorname{injdim} A_i = \operatorname{GKdim} A_i$.

Lemma 4.2. *Suppose A satisfies Hypothesis 4.1. Then A has a quasi-Frobenius artinian quotient ring.*

Proof. It suffices to show that for each $i \in I$, A_i has a quasi-Frobenius artinian quotient ring. This follows from [Ajitabh et al. 1998, Corollary 6.2]. □

Lemma 4.3. *Suppose $A = \bigoplus_{i \in I} A_i$ satisfies Hypothesis 4.1. If A is a PI algebra of finite global dimension, then each A_i is a prime, homogeneous, AS regular, Auslander regular and Cohen–Macaulay algebra.*

Proof. By definition, A_i is AS regular, Auslander regular and Cohen–Macaulay for each i. By [Stafford and Zhang 1994, Theorem 5.4], A_i is prime. A prime PI algebra of finite GK dimension is homogeneous [Krause and Lenagan 1985, Lemma 10.18]. The assertion follows. □

We also recall:

Lemma 4.4 [Wu and Zhang 2003, Proposition 4.3]. *Let A and B be noetherian, PI, AS Gorenstein algebras of the same injective dimension and let $A \to B$ be an algebra homomorphism such that ${}_AB$ and B_A are finitely generated. Then the following are equivalent*:

(1) $\operatorname{injdim} B_A < \infty$.

(2) $\operatorname{injdim} {}_AB < \infty$.

(3) $\operatorname{projdim} B_A < \infty$.

(4) $\operatorname{projdim} {}_AB < \infty$.

(5) *B_A is projective over A.*

(6) *${}_AB$ is projective over A.*

As a consequence, if A has finite global dimension, then both B_A and ${}_AB$ are projective.

We recall the following definition given in [Wu and Zhang 2003, p. 1059] or [2001, p. 521].

Definition 4.5. Let A be an algebra and M be a left A-module. The *depth* of M is defined to be

$$\operatorname{depth} M := \min\{i \mid \operatorname{Ext}^i_A(S, M) \neq 0 \text{ for some simple } A\text{-module } S\}.$$

Proposition 4.6. *Let A be a PI algebra satisfying Hypothesis 4.1. Let $d = \operatorname{GKdim} A$. Let M be a nonzero finitely generated left* (*or right*) *A-module.*

(1) The Auslander–Buchsbaum formula. *If* $\operatorname{projdim} M < \infty$, *then*

$$\operatorname{projdim} M \leq d - \operatorname{depth} M.$$

If, *further*, A *is homogeneous*, *then*

$$\operatorname{projdim} M = d - \operatorname{depth} M.$$

(2) Bass's theorem. *If* $\operatorname{injdim} M < \infty$, *then*

$$\operatorname{injdim} M \leq d.$$

If, *further*, A *is homogeneous*, *then*

$$\operatorname{injdim} M = d.$$

(3) The no-holes theorem. *Suppose either A is homogeneous or M is indecomposable. Then for every integer i between* $\operatorname{depth} M$ *and* $\operatorname{injdim} M$, *there is a simple left* (*or right*) *A-module S such that* $\operatorname{Ext}^i_A(S, M) \neq 0$.

Proof. First, if A is homogeneous, then A is AS Gorenstein. In this case all statements are given in [Wu and Zhang 2001, Theorem 0.1].

Next, we suppose that A is not homogeneous. As in Hypothesis 4.1, write $A = \bigoplus_{i \in I} A_i$ where each A_i is homogeneous. Write $M = \bigoplus_{i \in I} M_i$ where each M_i is a left A_i-module. By the assertions in the homogeneous case, for each $i \in I$, we have:

(1_i) If $\operatorname{projdim} M_i < \infty$,

$$\operatorname{projdim} M_i = d_i - \operatorname{depth} M_i,$$

where $d_i = \operatorname{injdim} A_i$.

(2_i) If $\operatorname{injdim} M_i < \infty$, then

$$\operatorname{injdim} M_i = d_i.$$

(3_i) For every integer s between $\operatorname{depth} M_i$ and $\operatorname{injdim} M_i$, there is a simple left A_i-module T such that $\operatorname{Ext}^s_{A_i}(T, M_i) \neq 0$.

(1) Suppose that $\operatorname{projdim} M < \infty$. Then

$$\operatorname{projdim} M = \max_{i \in I}\{\operatorname{projdim} M_i\} \quad \text{and} \quad \operatorname{depth} M = \min_{i \in I}\{\operatorname{depth} M_i\}.$$

Choose $j \in I$ such that $\operatorname{projdim} M = \operatorname{projdim} M_j$. Then

$$\operatorname{projdim} M = \operatorname{projdim} M_j = d_j - \operatorname{depth} M_j \leq d - \operatorname{depth} M$$

as desired.

(2) Let $j \in I$ satisfy $\operatorname{injdim} M = \operatorname{injdim} M_j$. Then

$$\operatorname{injdim} M = \operatorname{injdim} M_j = d_j \leq d$$

as desired.

(3) Since we are in the case that A is not homogeneous, M is indecomposable by hypothesis. In this case $M = M_j$ for some j. But then M is an A_j-module and A_j is homogeneous, so the assertion follows by [Wu and Zhang 2001, Theorem 0.1(3)]. □

5. Weak Hopf algebras

Our goal is to apply the main results in Sections 3 and 4 to weak Hopf algebras. In this section, we recall the definition of a weak Hopf algebra and give some examples. Throughout, we use Sweedler notation. If (C, Δ, ϵ) is a coalgebra, then for $c \in C$, we write $\Delta(c) = \sum c_1 \otimes c_2$. When there is no danger of confusion, we suppress the summation notation and simply write $\Delta(c) = c_1 \otimes c_2$.

Definition 5.1 [Böhm et al. 1999; Hayashi 1993; 1999b]. A *weak bialgebra* consists of a 5-tuple $(H, \mu, u, \Delta, \epsilon)$, where (H, μ, u) is a unital associative $\Bbbk$-algebra and (H, Δ, ϵ) is a counital coassociative $\Bbbk$-coalgebra, satisfying:

(a) Multiplicativity of comultiplication:
$$\Delta(ab) = \Delta(a)\Delta(b)$$
for all $a, b \in H$.

(b) Weak comultiplicativity of the unit:
$$\Delta^2(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1).$$

(c) Weak multiplicativity of the counit:
$$\epsilon(abc) = \epsilon(ab_1)\epsilon(b_2c) = \epsilon(ab_2)\epsilon(b_1c)$$
for all $a, b, c \in H$.

We do not assume that H is finite-dimensional over $\Bbbk$.

Because the coproduct does not necessarily preserve the unit, applying the usual sumless Sweedler notation we write $\Delta(1) = 1_1 \otimes 1_2$. A weak bialgebra is a bialgebra if and only if $\Delta(1) = 1 \otimes 1$, if and only if $\epsilon(ab) = \epsilon(a)\epsilon(b)$ for all $a, b \in H$. For a weak bialgebra H, the *source counital map* $\epsilon_s : H \to H$ is defined by $\epsilon_s(h) = 1_1\epsilon(h1_2)$ for all $h \in H$. The *source counital subalgebra* H_s is defined to be the image of ϵ_s. Similarly, the *target counital map* $\epsilon_t : H \to H$ is defined by $\epsilon_t(h) = \epsilon(1_1h)1_2$ for all $h \in H$, and the *target counital subalgebra* is $H_t = \epsilon_t(H)$. The counital subalgebras are finite-dimensional, separable, coideal subalgebras of H that commute with each other [Nikshych and Vainerman 2002, Propositions 2.2.2 and 2.3.4].

Definition 5.2 [Böhm et al. 1999; Hayashi 1993; 1999b]. Let H be a weak bialgebra.

(1) H is a *weak Hopf algebra* if there exists an algebra antihomomorphism $S : H \to H$ called the *antipode* satisfying, for all $a \in H$:
$$S(a_1)a_2 = \epsilon_s(a), \quad a_1S(a_2) = \epsilon_t(a), \quad S(a_1)a_2S(a_3) = S(a).$$

(2) A *morphism* between weak Hopf algebras H_1 and H_2 with antipodes S_1 and S_2 is a map $f : H_1 \to H_2$ which is a unital algebra homomorphism and a counital coalgebra homomorphism satisfying $f \circ S_1 = S_2 \circ f$.

There are many examples of finite-dimensional weak Hopf algebras in the literature, see for example, [Böhm et al. 1999; 2011; Hayashi 1993; 1999a; 1999b; Nikshych and Vainerman 2002; Szlachányi 2001]. We now provide several examples of weak Hopf algebras.

Example 5.3. (1) Every Hopf algebra is a weak Hopf algebra.

(2) If H_1 and H_2 are weak Hopf algebras, then so are $H_1 \oplus H_2$ and $H_1 \otimes H_2$, with their usual algebra and coalgebra structures.

(3) For every positive integer n, the matrix algebra $M_n(\Bbbk)$ is a weak Hopf algebra, which is a special case of a groupoid algebra. As a consequence of part (2), if H is a weak Hopf algebra, so is $M_n(H)$.

In contrast to the examples above, note that if H_1 and H_2 are Hopf algebras, then $H_1 \oplus H_2$ and $M_n(H_1)$ (for $n > 1$) are not Hopf algebras.

The *face algebras* defined by Hayashi [1993] are a special class of weak Hopf algebras. Other examples include groupoid algebras and their duals, Temperley–Lieb algebras, and quantum transformation groupoids (see [Nikshych and Vainerman 2002]). The focus of this paper is on infinite-dimensional weak Hopf algebras (of finite GK dimension). In addition to the Hopf algebras of finite positive GK dimension, we also provide the following examples and constructions.

Example 5.4. Suppose that $(W, \mu_W, u_W, \Delta_W, \epsilon_W, S_W)$ is a weak Hopf algebra and that σ is a weak Hopf algebra automorphism of W. Then the group $\mathbb{Z} = \langle a \rangle$ acts on W via σ, so W is a $\Bbbk\mathbb{Z}$-module algebra. Hence, we may form the smash product $H = W \# \Bbbk\mathbb{Z}$.

As a vector space, $H = W \otimes \Bbbk\mathbb{Z}$. As an algebra,

$$(w \otimes a^m)(v \otimes a^n) = w\sigma^m(v) \otimes a^{m+n}$$

for $w, v \in W$ and $m, n \in \mathbb{Z}$. The unit of H is given by $1_H = 1_W \otimes a^0$. As a coalgebra, define $\Delta(w \otimes a^m) = (w_1 \otimes a^m) \otimes (w_2 \otimes a^m)$ and $\epsilon(w \otimes a^m) = \epsilon_W(w)\epsilon_{k\mathbb{Z}}(a^m) = \epsilon_W(w)$ for all $w \in W$ and $m \in \mathbb{Z}$. We claim that this makes H a weak bialgebra.

Note that since σ is a weak Hopf algebra automorphism of W, we have that for all $w \in W$, $\epsilon(\sigma(w)) = \epsilon(w)$ and $\sigma(w)_1 \otimes \sigma(w)_2 = \sigma(w_1) \otimes \sigma(w_2)$. First, Δ is multiplicative as

$$\begin{aligned}
\Delta\big((w \otimes a^m)(v \otimes a^n)\big) &= \Delta(w\sigma^m(v) \otimes a^{m+n}) \\
&= \big((w\sigma^m(v))_1 \otimes a^{m+n}\big) \otimes \big((w\sigma^m(v))_2 \otimes a^{m+n}\big) \\
&= \big(w_1\sigma^m(v)_1 \otimes a^{m+n}\big) \otimes \big(w_2\sigma^m(v)_2 \otimes a^{m+n}\big) \\
&= \big(w_1\sigma^m(v_1) \otimes a^{m+n}\big) \otimes \big(w_2\sigma^m(v_2) \otimes a^{m+n}\big) \\
&= \big((w_1 \otimes a^m) \otimes (w_2 \otimes a^m)\big)\big((v_1 \otimes a^n) \otimes (v_2 \otimes a^n)\big) \\
&= \Delta(w \otimes a^m)\Delta(v \otimes a^n).
\end{aligned}$$

We also have that the unit is weakly comultiplicative, since

$$\begin{aligned}\Delta^2(1_H) &= (1_1\otimes a^0)\otimes(1_2\otimes a^0)\otimes(1_3\otimes a^0)\\ &= \big((1_1\otimes a^0)\otimes(1_2\otimes a^0)\otimes(1_W\otimes a^0)\big)\big((1_W\otimes a^0)\otimes(1'_1\otimes a^0)\otimes(1'_2\otimes a^0)\big)\\ &= (\Delta(1_H)\otimes 1_H)(1_H\otimes\Delta(1_H)),\end{aligned}$$

and similarly for the other weak comultiplicativity axiom.

Finally, to see that ϵ is weakly multiplicative, we have

$$\begin{aligned}\epsilon\big((u\otimes a^l)(v\otimes a^m)(w\otimes a^n)\big) &= \epsilon\big(u\sigma^l(v)\sigma^{l+m}(w)\otimes a^{l+m+n}\big)\\ &= \epsilon_W\big(u\sigma^l(v)\sigma^{l+m}(w)\big)\\ &= \epsilon_W(u\sigma^l(v)_1)\epsilon_W\big(\sigma^l(v)_2\sigma^{l+m}(w)\big)\\ &= \epsilon_W(u\sigma^l(v)_1)\epsilon_W\big(\sigma^l(v_2\sigma^m(w))\big)\\ &= \epsilon_W(u\sigma^l(v_1))\epsilon_W(v_2\sigma^m(w))\\ &= \epsilon\big((u\otimes a^l)(v_1\otimes a^m)\big)\epsilon\big((v_2\otimes a^m)(w\otimes a^n)\big)\\ &= \epsilon\big((u\otimes a^l)(v\otimes a^m)_1\big)\epsilon\big((v\otimes a^m)_2(w\otimes a^n)\big),\end{aligned}$$

and similarly for the other weak multiplicativity axiom. Thus H is a weak bialgebra, as claimed.

It is now easy to check that we can give H the structure of a weak Hopf algebra by defining the antipode as

$$S(w\otimes a^m) = (1\otimes S_{k\mathbb{Z}}(a^m))(S_W(w)\otimes a^0) = (1\otimes a^{-m})(S_W(w)\otimes a^0) = \sigma^{-m}(S_W(w))\otimes a^{-m}.$$

As an algebra, H is isomorphic to the skew Laurent ring $W[t^{\pm 1};\sigma]$, where $w\in W$ corresponds to $w\otimes a^0$ and t corresponds to $1\otimes a$. Hence, when W is finite-dimensional, H is a weak Hopf algebra of GK dimension 1. Under this isomorphism, $\Delta(t) = \Delta(1)(t\otimes t)$, so t is group-like in the sense of [Nikshych and Vainerman 2002].

Example 5.5. Let W be a weak Hopf algebra. Let $\sigma : W\to W$ be an algebra automorphism and assume that (i) there is an algebra homomorphism $\chi : W\to\Bbbk$ such that $\sigma(w)=\sum\chi(w_1)w_2=\sum w_1\chi(w_2)$ for all $w\in W$, that is, σ is both a left and right winding homomorphism of some character χ; and (ii) the antipode S of W satisfies $S=\sigma S\sigma$.

Then by [Lomp et al. 2019, Theorem 4.4], there is a unique weak Hopf algebra structure on the Ore extension $H=W[t;\sigma]$ such that Δ_H, ϵ_H, and S_H restrict on W to Δ_W, ϵ_W, and S_W respectively and t is primitive in the weak sense, in other words, $\Delta(t)=\Delta(1)(1\otimes t+t\otimes 1)$, $\epsilon(t)=0$, and $S(t)=-t$.

Note that if W is finite-dimensional, then H has GK dimension 1.

Remark 5.6. In fact, Lomp, Sant'Ana, and Leite dos Santos [Lomp et al. 2019, Theorem 4.4] give necessary and sufficient conditions for the existence of a weak Hopf algebra structure on a more general Ore extension $W[t;\sigma,\delta]$ which extends the weak Hopf algebra structure on W, under the assumption that

t is weakly skew-primitive. When W is finite-dimensional, these Ore extensions give additional examples of weak Hopf algebras of GK dimension 1 which generalize the previous example.

Applying the constructions in the previous examples repeatedly, we can obtain many different weak Hopf algebras of positive GK dimension.

6. An analog of (L1) and (R1) for monoidal categories

It is well-known that if H is a weak Hopf algebra, then there is a monoidal product endowing H-Mod with the structure of a monoidal category. In this section, we study analogs of (L1) and (R1) in the more general setting of monoidal categories, and then apply these results to the special case of modules over a weak Hopf algebra in the next section. The reader can find the basic definitions of monoidal categories and related concepts in [Etingof et al. 2015].

Definition 6.1. Let $\mathcal{C}$ be a $\Bbbk$-linear abelian category. We say that $\mathcal{C}$ satisfies (C1) if, for all projective objects $P \in \mathcal{C}$ and all $i \geq 0$, the functor $\operatorname{Ext}^i_{\mathcal{C}}(-, P)$ is exact on the subcategory of objects of finite length in $\mathcal{C}$.

In this definition Ext means Yoneda Ext, though in our main intended application where $\mathcal{C} = H$-Mod for an algebra H, the Ext functors can be computed with projective or injective resolutions, as usual. We want to show that (C1) follows from quite general hypotheses when $\mathcal{C}$ is a monoidal category. For the rest of this section, suppose that the $\Bbbk$-linear abelian category $\mathcal{C}$ is also a monoidal category, with bilinear, biexact tensor product denoted $\overline{\otimes}$. Let $\mathbb{1}$ denote the unit object of $\mathcal{C}$.

Definition 6.2. Suppose that for some object $M \in \mathcal{C}$, the functor $- \,\overline{\otimes}\, M$ has a right adjoint. We write $\overline{\operatorname{Hom}}(M, -)$ for the right adjoint. By definition, for all $P, N \in \mathcal{C}$ there is an adjoint isomorphism

$$\Phi_{P,M,N} : \operatorname{Hom}_{\mathcal{C}}(P \,\overline{\otimes}\, M, N) \to \operatorname{Hom}_{\mathcal{C}}(P, \overline{\operatorname{Hom}}(M, N))$$

which is natural in P and N. The object $\overline{\operatorname{Hom}}(M, N)$ is called the *internal Hom* from M to N.

Another way of describing the definition above is as follows: $\overline{\operatorname{Hom}}(M, N) \in \mathcal{C}$ is the object representing the functor $\operatorname{Hom}_{\mathcal{C}}(- \,\overline{\otimes}\, M, N)$, when that functor is representable. If $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ such that $- \,\overline{\otimes}\, M$ has a right adjoint for all $M \in \mathcal{D}$, then $\overline{\operatorname{Hom}}(M, N)$ is defined for all $M \in \mathcal{D}$, $N \in \mathcal{C}$. It is easy to see in this case that $\overline{\operatorname{Hom}}(-, -)$ is a bifunctor $\mathcal{D} \times \mathcal{C} \to \mathcal{C}$, which is contravariant in the first coordinate.

We have the following version of Freyd's adjoint functor theorem.

Lemma 6.3. *Assume that $\mathcal{C}$ is cocomplete, in other words that $\mathcal{C}$ has (small) direct sums, and that $\mathcal{C}$ has a generator. Then the functor $- \,\overline{\otimes}\, M : \mathcal{C} \to \mathcal{C}$ has a right adjoint if and only if it commutes with all (small) direct sums in $\mathcal{C}$, for all $M \in \mathcal{C}$.*

Recall the definition of a left dual V^* of an object $V \in \mathcal{C}$ [Etingof et al. 2015, Definition 2.10.1].

Lemma 6.4. *Let $\mathcal{C}$ be as above. Suppose that $V \in \mathcal{C}$ has a left dual V^*.*

(1) *The functor* $-\mathbin{\overline{\otimes}} V^*$ *is a right adjoint of* $-\mathbin{\overline{\otimes}} V$. *In particular, the internal* Hom, $\overline{\operatorname{Hom}}(V, N)$, *exists for all* $N \in \mathcal{C}$, *and* $\overline{\operatorname{Hom}}(V, N) \cong N \mathbin{\overline{\otimes}} V^*$.

(2) *The functor* $\overline{\operatorname{Hom}}(V, -)$ *is exact.*

(3) *For any projective object* $P \in \mathcal{C}$, *the object* $P \mathbin{\overline{\otimes}} V$ *is also projective.*

Proof. (1) See [Etingof et al. 2015, Proposition 2.10.8].

(2) This follows from part (1), since $\overline{\operatorname{Hom}}(V, -) \cong - \mathbin{\overline{\otimes}} V^*$ and $\overline{\otimes}$ is biexact.

(3) This is the same proof as in [Etingof et al. 2015, Proposition 4.2.12]. We have an isomorphism of functors $\operatorname{Hom}_{\mathcal{C}}(P \mathbin{\overline{\otimes}} V, -) \cong \operatorname{Hom}_{\mathcal{C}}(P, - \mathbin{\overline{\otimes}} V^*)$ by part (1). Since $\overline{\otimes}$ is biexact and P is projective, the second functor is exact. So the first functor is exact, implying that $P \mathbin{\overline{\otimes}} V$ is projective. □

We now get the following adjunction at the level of Ext.

Lemma 6.5. *Suppose that* $\mathcal{C}$ *has enough projectives. Let* $\mathcal{D}$ *be a full subcategory of* $\mathcal{C}$ *such that every* $V \in \mathcal{D}$ *has a left dual* V^*. *Then for all* $M, N \in \mathcal{C}$ *and* $V \in \mathcal{D}$ *there is a vector space isomorphism*

$$\operatorname{Ext}^i_{\mathcal{C}}(M, \overline{\operatorname{Hom}}(V, N)) \cong \operatorname{Ext}^i_{\mathcal{C}}(M \mathbin{\overline{\otimes}} V, N),$$

which is natural in M, V, *and* N.

Proof. This is similar to the proof in [Brown and Goodearl 1997, Proposition 1.3]. Let $P^\bullet$ be a projective resolution of M. By Lemma 6.4(3), the complex $P^\bullet \mathbin{\overline{\otimes}} V$ consists of projectives, and since $- \mathbin{\overline{\otimes}} V$ is exact, it forms a projective resolution of $M \mathbin{\overline{\otimes}} V$. Then $\operatorname{Hom}_{\mathcal{C}}(P^\bullet \mathbin{\overline{\otimes}} V, N)$ has homology groups $\operatorname{Ext}^i_{\mathcal{C}}(M \mathbin{\overline{\otimes}} V, N)$. On the other hand, using that $- \mathbin{\overline{\otimes}} V$ has a right adjoint by Lemma 6.4(1), this complex is isomorphic to $\operatorname{Hom}_{\mathcal{C}}(P^\bullet, \overline{\operatorname{Hom}}(V, N))$, which has homology groups $\operatorname{Ext}^i_{\mathcal{C}}(M, \overline{\operatorname{Hom}}(V, N))$. The naturality is easy to check. □

The main result of this section is the following version of [Brown and Goodearl 1997, Lemma 1.11; Wu and Zhang 2003, Lemma 3.4].

Proposition 6.6. *let* $\mathcal{C}$ *be as above. Suppose that* $\mathcal{C}$ *has enough projectives. Let* $\mathcal{D}$ *be a full abelian subcategory of* $\mathcal{C}$ *such that every* $V \in \mathcal{D}$ *has a left dual* $V^* \in \mathcal{D}$.

(1) *Suppose* $Q \in \mathcal{C}$ *is projective and let* $V \in \mathcal{D}$. *Then* $P = Q \mathbin{\overline{\otimes}} V^*$ *is projective and for any* $i \geq 0$, *there is an isomorphism* $\operatorname{Ext}^i_{\mathcal{C}}(V, Q) \cong \operatorname{Ext}^i_{\mathcal{C}}(\mathbb{1}, Q \mathbin{\overline{\otimes}} V^*)$, *which is natural in* $V \in \mathcal{D}$.

(2) *For any projective object* $Q \in \mathcal{C}$ *and any* $i \geq 0$, *the functor* $\operatorname{Ext}^i_{\mathcal{C}}(-, Q)$ *is a contravariant exact functor* $\mathcal{D} \to \Bbbk\text{-Mod}$.

Proof. (1) By Lemma 6.4(1), there is an isomorphism $Q \mathbin{\overline{\otimes}} V^* \to \overline{\operatorname{Hom}}(V, Q)$, which it is straightforward to check is natural in $V \in \mathcal{D}$. Moreover, by Lemma 6.4(3), $P = Q \mathbin{\overline{\otimes}} V^*$ is projective. Now by the adjoint isomorphism in Lemma 6.5, we see that

$$\operatorname{Ext}^i_{\mathcal{C}}(\mathbb{1}, Q \mathbin{\overline{\otimes}} V^*) \cong \operatorname{Ext}^i_{\mathcal{C}}(\mathbb{1}, \overline{\operatorname{Hom}}(V, Q)) \cong \operatorname{Ext}^i_{\mathcal{C}}(\mathbb{1} \mathbin{\overline{\otimes}} V, Q) \cong \operatorname{Ext}^i_{\mathcal{C}}(V, Q),$$

where all of the displayed isomorphisms are natural in $V \in \mathcal{D}$.

(2) Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an exact sequence in $\mathcal{D}$. By assumption, V_1, V_2, and V_3 all have left duals. Note that the functor which assigns to an object its left dual is an exact functor, by the same proof as in [Etingof et al. 2015, Proposition 4.2.9]. We therefore have an exact sequence $0 \to V_3^* \to V_2^* \to V_1^* \to 0$ of objects in $\mathcal{D}$. Since $\overline{\otimes}$ is biexact, $0 \to Q \,\overline{\otimes}\, V_3^* \to Q \,\overline{\otimes}\, V_2^* \to Q \,\overline{\otimes}\, V_1^* \to 0$ is an exact sequence as well. As the terms of this sequence are all projective by Lemma 6.4, the sequence is split. Then setting $P_i = Q \,\overline{\otimes}\, V_i^*$, we have

$$0 \to \mathrm{Ext}^i_{\mathcal{C}}(\mathbb{1}, P_3) \to \mathrm{Ext}^i_{\mathcal{C}}(\mathbb{1}, P_2) \to \mathrm{Ext}^i_{\mathcal{C}}(\mathbb{1}, P_1) \to 0$$

is exact, since $\mathrm{Ext}^i_{\mathcal{C}}$ commutes with finite direct sums in the second coordinate. By the isomorphism in part (1),

$$0 \to \mathrm{Ext}^i_{\mathcal{C}}(V_3, Q) \to \mathrm{Ext}^i_{\mathcal{C}}(V_2, Q) \to \mathrm{Ext}^i_{\mathcal{C}}(V_1, Q) \to 0$$

is exact, as required. □

Corollary 6.7. *Suppose that $\mathcal{C}$ is a $\Bbbk$-linear abelian monoidal category with bilinear, biexact tensor product $\otimes$. Suppose that $\mathcal{C}$ has enough projectives. Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ consisting of finite length objects, and suppose that every $V \in \mathcal{D}$ has a left dual $V^* \in \mathcal{D}$. Then $\mathcal{C}$ satisfies the condition* (C1) *of Definition 6.1.*

7. Proofs of Theorems 0.3–0.8

Let H be a weak Hopf algebra. The monoidal product on H-Mod is defined as follows. For $M, N \in H$-Mod, $M \otimes_k N$ is a (nonunital) left H-module via the action $h \cdot (m \otimes n) = \sum h_1 m \otimes h_2 n$. Then one defines

$$M \,\overline{\otimes}\, N := \Delta(1)(M \otimes N) = \{1_1 m \otimes 1_2 n \mid m \in M, n \in N\},$$

which is a unital submodule of $M \otimes N$ and thus in H-Mod. This makes H-Mod into a monoidal category [Nikshych and Vainerman 2002, Section 5.1].

There is also another way to describe this product which is often convenient. Recall the counital subalgebra $H_t = \epsilon_t(H)$, where $\epsilon_t(h) = \epsilon(1_1 h)1_2$. Let us define also $\overline{\epsilon_s}(h) = 1_1 \epsilon(1_2 h)$. If $M \in H$-Mod, then of course M is also a left H_t-module by restriction. The module M also has a right H_t-structure, where for $m \in M$, $h \in H_t$ we define $m \cdot h := \overline{\epsilon_s}(h)m$. Under these two actions, M becomes an (H_t, H_t)-bimodule. Then we can also identify $M \,\overline{\otimes}\, N$ with $M \otimes_{H_t} N$, with the same formula for the left H-action, that is $h \cdot (m \otimes n) = h_1 m \otimes h_2 n$ [Böhm et al. 2011, Theorem 2.4].

We are now ready to prove the main results given in the Introduction.

Proof of Theorem 0.3. (1) Let A be an algebra which satisfies Hypotheses 0.1 and 0.2. Since A is a finitely generated module over its affine center, by [Brown and Goodearl 1997, Proposition 3.1], every finite length H-module is finite-dimensional over $\Bbbk$. Since A satisfies Hypothesis 0.2, by Corollary 6.7, both A-Mod and A^{op}-Mod satisfy (C1) of Definition 6.1 and so A satisfies (L1) and (R1). The assertions follow from Corollary 3.7.

(2) Let H be a weak Hopf algebra. Hypothesis 0.2 for H follows from well-known results, though we will briefly review the details. The existing references sometimes assume a weak Hopf algebra is finite-dimensional, but since we only claim that finite-dimensional objects have left duals, there is no significant change to the proofs.

The category H-Mod is obviously abelian, and it is monoidal with the operation $\overline{\otimes}$ defined at the beginning of this section. The unit object is $\mathbb{1} = H_t$, which is a left H-module with action $h \cdot z = \epsilon_t(hz)$ for $h \in H, z \in H_t$ [Nikshych and Vainerman 2002, Lemma 5.1.1]. By the description of $\overline{\otimes}$ as $\otimes_{H_t}$, where each $M \in H$-Mod has a canonical (H_t, H_t)-bimodule structure, since H_t is semisimple, it is clear that $\overline{\otimes}$ is bilinear on morphisms and biexact. If $M \in H$-Mod is finite-dimensional, it has a left dual $M^* = \mathrm{Hom}_{\Bbbk}(M, \Bbbk)$ with H-module action $[h \cdot \phi](m) = \phi(S(h)m)$, using the antipode S of H [Nikshych and Vainerman 2002, Lemma 5.1.2]. Note that we have not assumed that S is bijective, and the proof of [Nikshych and Vainerman 2002, Lemma 5.1.2] uses S^{-1}. But in fact S^{-1} is only applied in the proof to elements of H_s, and for any weak Hopf algebra the antipode S gives a bijection from H_t to H_s [Böhm et al. 1999, Theorem 2.10], so that $S^{-1}|_{H_s}$ makes sense.

This shows that the category H-Mod has the properties needed in Hypothesis 0.2. Since $(H^{\mathrm{op,\,cop}}, S)$ is also a weak Hopf algebra (see [Nikshych and Vainerman 2002, Remark 2.4.1]), the category Mod-H also has these properties, so that Hypothesis 0.2 holds. □

Proof of Theorem 0.4. By Theorem 0.3, A satisfies Hypothesis 4.1. The assertion follows from Lemma 4.2. □

Proof of Theorem 0.5. By Theorem 0.3, A satisfies Hypothesis 4.1. The assertion follows from Lemma 4.3. □

Proof of Theorem 0.6. By Theorem 0.3, A satisfies Hypothesis 4.1. The assertion follows from Lemma 4.4. □

Proof of Theorem 0.7. (1) The assertion follows from Lemma 3.5(1) and Corollary 3.8.

(2) This follows from Lemma 3.5(2).

(3) By Lemma 3.5 and Corollary 3.8, A has a residue complex K that has nonzero cohomology only at $H^{-d}(K) = \Omega$, where $d = \mathrm{GKdim}\, A$. By Corollary 3.8, Ω is an invertible A-bimodule. It follows from the definition of residue complex that Ω has a minimal pure injective resolution on both sides. By tensoring with Ω^{-1} we obtain that A has a minimal pure injective resolution on both sides. □

Proof of Theorem 0.8. By Theorem 0.3, A satisfies Hypothesis 4.1. The assertion follows from Proposition 4.6. □

In this paper we have focused on the class of weak Hopf algebras, which seems to be an especially fertile ground for generalizing the homological theory of infinite dimensional Hopf algebras. The conditions in Hypothesis 0.2 are quite weak, however, and so we expect other generalizations of Hopf algebras to satisfy the analog of Theorem 0.3(2). We have not attempted to catalog all of the structures for which this theory applies, but mention one such further example here.

A *quasibialgebra* is a generalization of a bialgebra for which the coproduct is not coassociative, but satisfies a weaker form of coassociativity up to twisting by a unit. There is a natural notion of antipode S for a quasibialgebra. Such algebras arise naturally in the theory of tensor categories; we refer to [Etingof et al. 2015, Sections 5.13–5.15] for the definition and some basic properties. Traditionally, the term quasi-Hopf algebra is reserved for quasibialgebras with invertible antipode.

Given a quasibialgebra H, the category H-Mod is monoidal, where for $M, N \in H$-Mod we have $M \mathbin{\overline{\otimes}} N = M \otimes_{\Bbbk} N$ with action $h \cdot (m \otimes n) = h_1 m \otimes h_2 n$, similarly as for Hopf algebras. In particular, it is clear that $\overline{\otimes}$ is bilinear on morphisms and biexact. The existence of an antipode S implies that every finite dimensional $M \in H$-Mod has a left dual M^*, with H-action given by the same formula as in the Hopf case [Etingof et al. 2015, p. 113]. Analogous results hold for the category of right modules, since $H^{\mathrm{op,\,cop}}$ is also a quasi-Hopf algebra with antipode S. Thus we have the following result.

Theorem 7.1. *Let H be a quasibialgebra with antipode. Then H satisfies Hypothesis 0.2, and hence if H is also finite over an affine center then H satisfies all of the conclusions of Theorem 0.3*(1).

8. Further questions

We conclude by posing some further questions for weak Hopf algebras. For open questions in the Hopf case, we refer the reader to [Brown 1998; Brown and Zhang 2008] and the survey papers [Brown 2007; Goodearl 2013].

The main result of this paper (Theorem 0.3) is a proof of the Brown–Goodearl conjecture for weak Hopf algebras that satisfy Hypothesis 0.1. It is natural to ask if the Brown–Goodearl conjecture holds for weak Hopf algebras satisfying weaker hypotheses. In particular, we ask:

Question 8.1 (the Brown–Goodearl question for weak Hopf algebras). Let H be a noetherian weak Hopf algebra. Does H have finite injective dimension? What if we assume further that H is affine and PI?

Again motivated by Theorem 0.3, we ask:

Question 8.2 (decomposition question). Suppose H is a noetherian weak Hopf algebra with finite Gelfand–Kirillov dimension d. Is then H a direct sum of homogeneous weak Hopf subalgebras H_i of Gelfand–Kirillov dimension i for integers $0 \le i \le d$?

Wu and Zhang [2003] proved that noetherian affine PI Hopf algebras have artinian quotient rings. As a corollary of [Skryabin 2006, Theorem A], Skryabin deduced that these Hopf algebras have a bijective antipode. In light of Theorem 0.4, it is therefore natural to ask:

Question 8.3 (Skryabin's question for weak Hopf algebras). Does every noetherian weak Hopf algebra H have a bijective antipode? What if we assume further that H is affine and PI?

Acknowledgments

The authors thank referees for their careful reading and valuable comments. R. Won was partially supported by an AMS–Simons Travel Grant. J. J. Zhang was partially supported by the US National Science Foundation (Nos. DMS-1700825 and DMS-2001015).

References

[Ajitabh et al. 1998] K. Ajitabh, S. P. Smith, and J. J. Zhang, "Auslander–Gorenstein rings", *Comm. Algebra* **26**:7 (1998), 2159–2180. MR Zbl

[Andruskiewitsch 2004] N. Andruskiewitsch, "Some remarks on Nichols algebras", pp. 35–45 in *Hopf algebras* (Chicago, 2002), edited by J. Bergen et al., Lect. Notes Pure Appl. Math. **237**, Dekker, New York, 2004. MR Zbl

[Atiyah and Macdonald 1969] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, MA, 1969. MR Zbl

[van den Bergh 1997] M. van den Bergh, "Existence theorems for dualizing complexes over non-commutative graded and filtered rings", *J. Algebra* **195**:2 (1997), 662–679. MR Zbl

[Böhm et al. 1999] G. Böhm, F. Nill, and K. Szlachányi, "Weak Hopf algebras, I: Integral theory and C^*-structure", *J. Algebra* **221**:2 (1999), 385–438. MR

[Böhm et al. 2011] G. Böhm, S. Caenepeel, and K. Janssen, "Weak bialgebras and monoidal categories", *Comm. Algebra* **39**:12 (2011), 4584–4607. MR Zbl

[Brown 1998] K. A. Brown, "Representation theory of Noetherian Hopf algebras satisfying a polynomial identity", pp. 49–79 in *Trends in the representation theory of finite-dimensional algebras* (Seattle, 1997), edited by E. L. Green and B. Huisgen-Zimmermann, Contemp. Math. **229**, Amer. Math. Soc., Providence, RI, 1998. MR Zbl

[Brown 2007] K. A. Brown, "Noetherian Hopf algebras", *Turkish J. Math.* **31**:suppl. (2007), 7–23. MR Zbl

[Brown and Goodearl 1997] K. A. Brown and K. R. Goodearl, "Homological aspects of Noetherian PI Hopf algebras of irreducible modules and maximal dimension", *J. Algebra* **198**:1 (1997), 240–265. MR Zbl

[Brown and Zhang 2008] K. A. Brown and J. J. Zhang, "Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras", *J. Algebra* **320**:5 (2008), 1814–1850. MR Zbl

[Etingof et al. 2005] P. Etingof, D. Nikshych, and V. Ostrik, "On fusion categories", *Ann. of Math.* (2) **162**:2 (2005), 581–642. MR Zbl

[Etingof et al. 2015] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, Math. Surv. Monogr. **205**, Amer. Math. Soc., Providence, RI, 2015. MR Zbl

[Goodearl 2013] K. R. Goodearl, "Noetherian Hopf algebras", *Glasg. Math. J.* **55**:A (2013), 75–87. MR Zbl

[Goodearl and Warfield 2004] K. R. Goodearl and R. B. Warfield, Jr., *An introduction to noncommutative Noetherian rings*, 2nd ed., Lond. Math. Soc. Stud. Texts **61**, Cambridge Univ. Press, 2004. MR Zbl

[Hayashi 1993] T. Hayashi, "Quantum group symmetry of partition functions of IRF models and its application to Jones' index theory", *Comm. Math. Phys.* **157**:2 (1993), 331–345. MR Zbl

[Hayashi 1999a] T. Hayashi, "A canonical Tannaka duality for finite seimisimple tensor categories", preprint, 1999. arXiv

[Hayashi 1999b] T. Hayashi, "Face algebras and unitarity of $\mathrm{SU}(N)_L$-TQFT", *Comm. Math. Phys.* **203**:1 (1999), 211–247. MR Zbl

[Iovanov and Kadison 2010] M. C. Iovanov and L. Kadison, "When weak Hopf algebras are Frobenius", *Proc. Amer. Math. Soc.* **138**:3 (2010), 837–845. MR Zbl

[Krause and Lenagan 1985] G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand–Kirillov dimension*, Res. Notes Math. **116**, Pitman, Boston, 1985. MR Zbl

[Lam 1991] T. Y. Lam, *A first course in noncommutative rings*, Grad. Texts Math. **131**, Springer, 1991. MR Zbl

[Levasseur 1992] T. Levasseur, "Some properties of noncommutative regular graded rings", *Glasgow Math. J.* **34**:3 (1992), 277–300. MR Zbl

[Lomp et al. 2019] C. Lomp, A. Sant'Ana, and R. Leite dos Santos, "Panov's theorem for weak Hopf algebras", pp. 277–292 in *Rings, modules and codes* (Lens, France, 2017), edited by A. Leroy et al., Contemp. Math. **727**, Amer. Math. Soc., Providence, RI, 2019. MR Zbl

[Nikshych and Vainerman 2002] D. Nikshych and L. Vainerman, "Finite quantum groupoids and their applications", pp. 211–262 in *New directions in Hopf algebras* (Berkeley, CA, 1999-2000), edited by S. Montgomery and H.-J. Schneider, Math. Sci. Res. Inst. Publ. **43**, Cambridge Univ. Press, 2002. MR Zbl

[Nill 1998] F. Nill, "Axioms for weak bialgebras", preprint, 1998. arXiv

[Skryabin 2006] S. Skryabin, "New results on the bijectivity of antipode of a Hopf algebra", *J. Algebra* **306**:2 (2006), 622–633. MR Zbl

[Stafford and Zhang 1994] J. T. Stafford and J. J. Zhang, "Homological properties of (graded) Noetherian PI rings", *J. Algebra* **168**:3 (1994), 988–1026. MR Zbl

[Szlachányi 2001] K. Szlachányi, "Finite quantum groupoids and inclusions of finite type", pp. 393–407 in *Mathematical physics in mathematics and physics* (Siena, Italy, 2000), edited by R. Longo, Fields Inst. Commun. **30**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl

[Vecsernyés 2003] P. Vecsernyés, "Larson–Sweedler theorem and the role of grouplike elements in weak Hopf algebras", *J. Algebra* **270**:2 (2003), 471–520. MR Zbl

[Wu and Zhang 2001] Q.-S. Wu and J. J. Zhang, "Homological identities for noncommutative rings", *J. Algebra* **242**:2 (2001), 516–535. MR Zbl

[Wu and Zhang 2003] Q.-S. Wu and J. J. Zhang, "Noetherian PI Hopf algebras are Gorenstein", *Trans. Amer. Math. Soc.* **355**:3 (2003), 1043–1066. MR Zbl

[Yekutieli 1992] A. Yekutieli, "Dualizing complexes over noncommutative graded algebras", *J. Algebra* **153**:1 (1992), 41–84. MR Zbl

[Yekutieli 1999] A. Yekutieli, "Dualizing complexes, Morita equivalence and the derived Picard group of a ring", *J. Lond. Math. Soc.* (2) **60**:3 (1999), 723–746. MR Zbl

[Yekutieli 2019] A. Yekutieli, *Derived categories*, Cambridge Stud. Adv. Math. **183**, Cambridge Univ. Press, 2019. MR Zbl

[Yekutieli and Zhang 1999] A. Yekutieli and J. J. Zhang, "Rings with Auslander dualizing complexes", *J. Algebra* **213**:1 (1999), 1–51. MR Zbl

[Yekutieli and Zhang 2003] A. Yekutieli and J. J. Zhang, "Residue complexes over noncommutative rings", *J. Algebra* **259**:2 (2003), 451–493. MR Zbl

Communicated by Bjorn Poonen
Received 2020-01-05 Revised 2020-06-19 Accepted 2020-10-11

drogalski@ucsd.edu — *Department of Mathematics, University of California, San Diego, La Jolla, CA, United States*

robwon@gmail.com — *Department of Mathematics, George Washington University, Washington, DC, United States*

zhang@math.washington.edu — *Department of Mathematics, University of Washington, Seattle, WA, United States*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2021 is US $415/year for the electronic version, and $620/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 15 No. 4 2021