



# Geometry of Twisted Kähler–Einstein Metrics and Collapsing

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**Abstract:** We prove that the twisted Kähler–Einstein metrics that arise on the base of certain holomorphic fiber space with Calabi–Yau fibers have conical-type singularities along the discriminant locus. These fiber spaces arise naturally when studying the collapsing of Ricci-flat Kähler metrics on Calabi–Yau manifolds, and of the Kähler–Ricci flow on compact Kähler manifolds with semiample canonical bundle and intermediate Kodaira dimension. Our results allow us to understand their collapsed Gromov–Hausdorff limits when the base is smooth and the discriminant has simple normal crossings.

## 1. Introduction

In this paper we are concerned with the study of canonical metrics  $\omega$  which exist on the base of certain holomorphic fiber space  $f : M^m \rightarrow N^n$  ( $n < m$ ) with Calabi–Yau fibers, away from the discriminant locus, and which satisfy a twisted Kähler–Einstein equation of the form

$$\mathrm{Ric}(\omega) = \lambda\omega + \omega_{\mathrm{WP}}, \quad (1.1)$$

where  $\lambda$  will equal 0 or  $-1$  (depending on our setup, to be described below) and  $\omega_{\mathrm{WP}}$  is a semipositive definite Weil–Petersson form which encodes the variation of complex structure of the Calabi–Yau fibers of  $f$ . These metrics were constructed in the work of Song–Tian [64], and since then metrics satisfying (1.1) have been shown to arise as collapsing limits of higher-dimensional Calabi–Yau manifolds (starting from [78] and more recently in [10, 19, 27, 28, 32, 33, 48, 49, 57, 64, 65, 67, 69, 83, 85, 86, 95]) as well as of long-time solutions of the Kähler–Ricci flow on compact Kähler manifolds with semiample canonical bundle and intermediate Kodaira dimension (starting from [64] and more recently in [18, 65, 67, 76, 82, 83, 85, 95]).

When  $M$  is holomorphic symplectic and  $f$  is a holomorphic Lagrangian fibration, these twisted Kähler–Einstein metrics coincide with the McLean–Hitchin metrics of

certain semi-flat hyperkähler structures that appear in the work of McLean [51] and Hitchin [34, 35], in which case these metrics are in fact special Kähler in the sense of [21]. For example, Gross-Wilson showed in [29] that the collapsing limits of Ricci-flat Kähler metrics on elliptically fibered K3 surfaces with  $I_1$  singular fibers are McLean metrics of semi-flat hyperkähler Lagrangian fibrations, and later Song-Tian proved in [64] that the limits satisfy the twisted Kähler–Einstein Equation (1.1).

As mentioned earlier, the canonical metrics  $\omega$  are in general singular along the discriminant locus of  $f$ . The nature of their singularities is intimately related to the aforementioned collapsing problems, but up to now the only situation when the singularities of  $\omega$  were completely understood is when  $\dim N = 1$  [28, 31, 95]. In this paper, we solve this problem in all dimensions by showing roughly speaking that  $\omega$  has conical type singularities, and we derive consequences for these collapsing problems.

Let us now describe our results more precisely. We will work in the two above-mentioned settings separately.

*1.1. The Calabi–Yau Setting.* In this setting we assume that  $M^m$  is a projective Calabi–Yau manifold, that is a projective manifold with  $K_M \cong \mathcal{O}_M$ , and we fix  $\Omega$  a never-vanishing holomorphic  $m$ -form on  $M$ . We suppose we have a fiber space  $f : M \rightarrow N$  (i.e. a surjective holomorphic map with connected fibers) over a normal projective variety  $N^n$ . These fiber spaces arise for example when we have a semiample line bundle  $L$  on  $M$  and  $f$  is its associated fiber space [47, 2.1.27]. We let  $D$  be the closed analytic subvariety of  $N$  which is the union of the singular locus of  $N$  together with the critical values of  $f$  on the regular locus of  $N$ . We will refer somewhat informally to  $D$  as the discriminant locus of  $f$  and to  $S = f^{-1}(D)$  as the locus of singular fibers of  $f$ , and by construction we have that  $f : M \setminus S \rightarrow N \setminus D$  is a proper holomorphic submersion with Calabi–Yau fibers. We can then consider the fiber integral of the Calabi–Yau volume form  $(-1)^{\frac{m^2}{2}} \Omega \wedge \overline{\Omega}$  via  $f|_{M \setminus S}$  and obtain a smooth positive volume form  $f_*((-1)^{\frac{m^2}{2}} \Omega \wedge \overline{\Omega})$  on  $N_0 := N \setminus D$ . We fix a Kähler metric  $\omega_N$  on  $N$  (in the sense of analytic spaces [52] if  $N$  is not smooth). In [11, 17, 65, 78] it is shown that there is a continuous  $\omega_N$ -psh function  $\varphi$  on  $N$ , smooth on  $N_0$ , such that  $\omega := \omega_N + \sqrt{-1} \partial \bar{\partial} \varphi$  is a Kähler metric on  $N_0$  which satisfies the twisted Kähler–Einstein equation (1.1) with  $\lambda = 0$ , where  $\omega_{\text{WP}} \geq 0$  is the Weil-Petersson form described in [65, 78].

An outstanding problem is to understand the behavior and singularities of the twisted Kähler–Einstein metric  $\omega$  near the discriminant locus  $D$ . As explained below, this problem is intimately linked with the collapsing behavior of certain Calabi–Yau metrics on  $M$ . The only general results on this problem are when  $N$  is a Riemann surface, in which case the precise behavior of  $\omega$  was calculated in [31, §3] (see also [28, 64, 95, 96]), and when  $M$  is hyperkähler [86].

In general, denote by  $N^{\text{reg}}$  the regular locus of  $N$  (whose complement has codimension at least 2) and write  $D = D^{(1)} \cup D^{(2)}$  where  $D^{(1)}$  is the union of all codimension 1 irreducible components of  $D$  and  $\dim D^{(2)} \leq n - 2$ . Let  $D^{\text{snc}} \subset D^{(1)} \cap N^{\text{reg}}$  be the snc locus of  $D^{(1)}$ , so that we can write  $D = D^{\text{snc}} \cup D'$  where  $D'$  is a closed analytic subset of codimension at least 2 which contains  $D^{(2)}$  as well as the singularities of  $N$ .

Write  $\{D_i\}_{i=1}^\mu$  for the irreducible components of  $D^{\text{snc}}$  and for each  $1 \leq i \leq \mu$  fix a defining section  $s_i$  for the line bundle corresponding to  $D_i$  and a Hermitian metric  $h_i$  on this line bundle. Since the divisor  $D^{\text{snc}}$  on  $N^{\text{reg}} \setminus D'$  has simple normal crossings support, there is a well-established notion of Kähler metrics with  $\omega_{\text{cone}}$  with conical singularities

along  $D^{\text{snc}}$ , with given cone angles  $2\pi\alpha_i$  along each  $D_i$  ( $0 < \alpha_i \leq 1$ ), as described in Sect. 2. Our first main result is then the following:

**Theorem 1.1.** *In the Calabi–Yau setting,  $\omega$  extends to a smooth Kähler metric across  $D^{(2)} \cap N^{\text{reg}}$ . Furthermore, there exists a conical metric  $\omega_{\text{cone}}$  with conical singularities along  $D^{\text{snc}}$  (and cone angles in  $2\pi\mathbb{Q} \cap (0, 2\pi)$ ) such that for any  $x \in D^{\text{snc}}$  there is an open set  $x \in U \subset N^{\text{reg}}$  and constants  $A, C > 0$  and on  $U \setminus D^{\text{snc}}$  we have*

$$C^{-1} \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^{-A \max(n-2, 0)} \omega_{\text{cone}} \leq \omega \leq C \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^A \omega_{\text{cone}}. \quad (1.2)$$

This is the direct generalization of the statement that is proved in [28, 31, 64, 95, 96] when  $N$  is a Riemann surface, and it strengthens the estimate obtained in [86] when  $M$  is hyperkähler, by giving a precise quasi-isometric description of  $\omega$  on near any point of  $D^{\text{snc}}$ . The logarithmic terms that appear in (1.2) are negligible compared to the “poles” of  $\omega_{\text{cone}}$ , and this will be important in our applications to collapsing below.

In the special case when  $N$  is smooth and  $D^{(1)}$  has simple normal crossings, we are able to show that the better estimate

$$C^{-1} \omega_{\text{cone}} \leq \omega \leq C \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^A \omega_{\text{cone}}, \quad (1.3)$$

holds globally on  $N \setminus D^{(1)}$ .

**1.2. Applications to Collapsing.** In the same Calabi–Yau setting, we can fix Kähler metrics  $\omega_M, \omega_N$  on  $M$  and  $N$  and  $t \geq 0$ , and then we let  $\tilde{\omega}_t = f^* \omega_N + e^{-t} \omega_M + \sqrt{-1} \partial \bar{\partial} \varphi_t$  be the unique Ricci-flat Kähler metric on  $M$  cohomologous to  $f^* \omega_N + e^{-t} \omega_M$ , given by Yau’s Theorem [91], which necessarily satisfies

$$\tilde{\omega}_t^m = (f^* \omega_N + e^{-t} \omega_M + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m = c_t e^{-(m-n)t} (-1)^{\frac{m^2}{2}} \Omega \wedge \bar{\Omega}, \quad \sup_M \varphi_t = 0, \quad (1.4)$$

where the constants  $c_t$  converge to a positive limit as  $t \rightarrow \infty$ . A very natural question is then to understand the behavior of the Ricci-flat metrics  $\tilde{\omega}_t$  as  $t \rightarrow \infty$ , and this has been much studied in recent years, see e.g. [10, 27–29, 32, 33, 48, 49, 57, 64, 65, 67, 78, 83, 85, 86] and references therein.

In particular, in [78] the second-named author proved that there is a twisted Kähler–Einstein metric  $\omega = \omega_N + \sqrt{-1} \partial \bar{\partial} \varphi$  on  $N_0 = N \setminus D$  which satisfies (1.1) with  $\lambda = 0$ , such that away from the singular fibers  $S = f^{-1}(D)$  we have convergence of  $\tilde{\omega}_t$  to  $f^* \omega$  as  $t \rightarrow \infty$  in the sense that  $\varphi_t \rightarrow f^* \varphi$  in  $C_{\text{loc}}^{1, \alpha}(M \setminus S)$  for all  $0 < \alpha < 1$ . This was later improved in [83] to locally uniform convergence of  $\tilde{\omega}_t$  to  $f^* \omega$ , and to  $C_{\text{loc}}^{\alpha}(M \setminus S)$  convergence for all  $0 < \alpha < 1$  in [33]. It is conjectured [79, 80] that in fact  $\tilde{\omega}_t \rightarrow f^* \omega$  in  $C_{\text{loc}}^{\infty}(M \setminus S)$ , but this is only known in general when the generic fibers of  $f$  are tori (or finite quotients) by [27, 32, 85] or when  $f$  is isotrivial [33].

In this paper we focus on the question of understanding the structure of the global Gromov–Hausdorff limit of  $(M, \tilde{\omega}_t)$ . The following was conjectured by the second-named author [79, 80], and is directly inspired by a similar conjecture by Gross–Wilson,

Kontsevich–Soibelman and Todorov [29, 44, 45] in the “mirror” setting of large complex structure limits:

**Conjecture 1.2.** *Let  $(X, d_X)$  denote the metric completion of  $(N_0, \omega)$ , and  $S_X = X \setminus N_0$ . Then*

- (a)  *$(X, d_X)$  is a compact metric space and  $S_X$  has real Hausdorff codimension at least 2*
- (b) *The Ricci-flat manifolds  $(M, \tilde{\omega}_t)$  converge to  $(X, d_X)$  in the Gromov–Hausdorff topology as  $t \rightarrow 0$*
- (c)  *$X$  is homeomorphic to  $N$ .*

The first result in this direction was the work of Gross–Wilson [29] for elliptically fibered  $K3$  with only  $I_1$  singular fibers, where  $\tilde{\omega}_t$  is constructed via a precise gluing construction, which in particular proves Conjecture 1.2. The gluing strategy has been recently carried out for Calabi–Yau 3-folds  $M$  with  $f$  a Lefschetz  $K3$  fibration [49], and even more recently for elliptically fibered  $K3$  surfaces with arbitrary singular fibers [10] (with some restrictions on  $[\omega_M]$ ), but it seems completely unclear how to implement a gluing strategy in general due to the vastly different possible types of singular fibers and nontrivial geometry of the discriminant locus.

A different approach to this conjecture, initiated in [78], is to work directly with the family of complex Monge–Ampère equations (1.4), and prove suitable estimates for the solutions  $\varphi_t$ . In this way, Conjecture 1.2 was completely proved when  $N$  is a Riemann surface [28], or when  $M$  is hyperkähler [86]. Furthermore, it was shown in [86] building upon our earlier work in [27, 28] that parts (a) and (b) of Conjecture 1.2 can be proved provided one can understand the blowup behavior of  $\omega$  near  $D$  (see Sect. 6 for a precise statement), and one does not need to consider the actual Ricci-flat metrics  $\tilde{\omega}_t$  themselves anymore.

Very recently, in [67] part (b) of Conjecture 1.2 was proved in general, and part (c) if  $N$  has at worst orbifold singularities, following a different path that in particular does not give pointwise bounds for  $\omega$  near  $D$ . As a corollary of our main Theorem 1.1, which is completely independent of the results in [67], we are able to settle the conjecture when the codimension 1 part of  $D$  is a divisor with simple normal crossings:

**Corollary 1.3.** *In the Calabi–Yau setting, suppose that  $N$  is smooth and  $D^{(1)}$  has simple normal crossings. Then Conjecture 1.2 holds.*

In the special case when  $N$  is a Riemann surface, this corollary is proved in our earlier work [28]. Examples where this corollary applies with higher-dimensional base  $N$  can be obtained by taking  $M$  to be one of the Calabi–Yau 3-folds constructed by Schoen [63] as fiber products of two rational elliptic surfaces, which admit elliptic fibrations with base  $N$  one of the rational elliptic surfaces, and in some cases have smooth connected discriminant locus.

In corollary 1.3 we allow the possibility that  $D^{(1)} = \emptyset$ , i.e. that  $D$  has codimension 2 or more. However this situation is very special, as it implies that the map  $f$  is isotrivial (see Proposition 3.2). In general, as stated in Theorem 1.1,  $\omega$  extends smoothly across  $D^{(2)} \cap N^{\text{reg}}$ , so all the singularities of  $\omega$  are along the divisorial part  $D^{(1)}$  and along  $N^{\text{sing}}$ .

**1.3. The Kähler–Ricci Flow Setting.** We now discuss the second setting, following Song–Tian [64, 65]. In this setting  $M^m$  is a compact Kähler manifold with  $K_M$  semiample

and with  $0 < \kappa(M) < m$  (call  $n = \kappa(M)$ ). Using sections of  $\ell K_M$  for  $\ell$  sufficiently divisible, we obtain a holomorphic map  $f : M \rightarrow \mathbb{P}^r$  with image  $N \subset \mathbb{P}^r$  a normal projective variety of dimension  $n$  (the canonical model of  $M$ ). The map  $f : M \rightarrow N$  has connected fibers, and the generic fiber is a Calabi–Yau  $(m - n)$ -fold. Write  $\omega_N = \frac{1}{\ell} \omega_{FS}|_N$ , so that  $f^* \omega_N$  belongs to  $c_1(K_M)$ . Fix also a basis  $\{s_i\}$  of  $H^0(M, \ell K_M)$ , which define the map  $f$ , and let

$$\mathcal{M} = \left( (-1)^{\frac{\ell m^2}{2}} \sum_i s_i \wedge \bar{s}_i \right)^{\frac{1}{\ell}},$$

which is a smooth positive volume form on  $M$  which satisfies

$$\mathrm{Ric}(\mathcal{M}) \stackrel{\mathrm{loc}}{:=} -\sqrt{-1} \partial \bar{\partial} \log \mathcal{M} = -f^* \omega_N.$$

If  $D \subset N$  denotes the singular locus of  $N$  together with the critical values of  $f$  on the smooth part of  $N$ , and  $S = f^{-1}(D)$ , then  $f : M \setminus S \rightarrow N_0 := N \setminus D$  is a proper holomorphic submersion and  $f_* \mathcal{M}$  is a smooth positive volume form on  $N_0$ . It is known [11, 17, 65, 78] that we can solve

$$(\omega_N + \sqrt{-1} \partial \bar{\partial} \varphi)^m = e^\varphi f_* \mathcal{M},$$

with  $\varphi \in C^0(N)$ ,  $\varphi$  is  $\omega_N$ -psh and smooth on  $N_0$ , where  $\omega := \omega_N + \sqrt{-1} \partial \bar{\partial} \varphi$  is a Kähler metric that satisfies the twisted Kähler–Einstein equation (1.1) with  $\lambda = -1$ . By [64, 65, 83] we know that if  $\omega_0$  is any Kähler metric on  $M$  and  $\omega(t)$  is its evolution by the Kähler–Ricci flow

$$\frac{\partial}{\partial t} \omega(t) = -\mathrm{Ric}(\omega(t)) - \omega(t), \quad \omega(0) = \omega_0,$$

then  $\omega(t)$  exists for all  $t \geq 0$  and as  $t \rightarrow \infty$  we have that  $\omega(t) \rightarrow f^* \omega$  in  $C_{\mathrm{loc}}^0(M \setminus S)$ .

One then has the completely analogous conjecture to Conjecture 1.2 for the collapsed Gromov–Hausdorff limit of  $(M, \omega(t))$ , but even less is known about it in general, with the only case that has been settled so far is when  $\kappa(M) = 1$  and the general fiber of  $f$  is a torus (or finite quotient), by [67, 76].

Our result in this setting is again that the twisted Kähler–Einstein metric  $\omega$  on  $N_0$  has conical-type singularities along  $D^{\mathrm{snc}}$ :

**Theorem 1.4.** *In the Kähler–Ricci flow setting, the same results as in Theorem 1.1 hold for  $\omega$ , namely  $\omega$  extends to a smooth Kähler metric across  $D^{(2)} \cap N^{\mathrm{reg}}$ , it satisfies (1.2) near every point of  $D^{\mathrm{snc}}$ , and when  $N$  is smooth and  $D^{(1)}$  has simple normal crossings the estimate (1.3) holds globally.*

Lastly, we show how this result can be used to prove the analog of Conjecture 1.2 in this setting, in the case when  $N$  is smooth and  $D^{(1)}$  is snc (so that the stronger estimate (1.3) holds), and assuming also a locally uniform Ricci curvature bound away from the singular fibers of  $f$ , as in [67, 76]. This generalizes the results in [67, 76] which also assumed that  $\dim N = 1$  and used the analog of (1.3), proved in [28, 31, 95].

**Theorem 1.5.** *In the Kähler–Ricci flow setting, suppose that the Ricci curvature of  $\omega(t)$  remains locally uniformly bounded on  $M \setminus S$ , and that  $N$  is smooth and  $D^{(1)}$  has simple normal crossings. Then  $(X, \omega(t))$  has uniformly bounded diameter and converges in the Gromov–Hausdorff topology to the metric completion of  $(N_0, \omega)$ , which is homeomorphic to  $N$ .*

It follows from [20, 27, 32, 85] (see the exposition in [82, Theorem 5.24]) that the assumption on the Ricci curvature of  $\omega(t)$  always holds the the smooth fibers of  $f$  are tori or finite quotients of tori, and it is expected to hold in general [75, Conjecture 4.7].

*1.4. Strategy of Proof.* The strategy of proof is the following. First, we pass to a resolution of singularities  $\pi : \tilde{N} \rightarrow N$  such that  $\tilde{N}$  is smooth and  $\pi^{-1}(D) = E$  is a divisor with simple normal crossings, and  $\pi$  is an isomorphism over the locus where  $N$  is regular and  $D^{(1)}$  is a simple normal crossings divisor. In particular, we may choose  $\pi = \text{Id}$  in the case when  $N$  is smooth and  $D^{(1)}$  has simple normal crossings. Then we prove in Sect. 2 that the volume form of  $\pi^*\omega$  satisfies the estimate

$$C^{-1}H\omega_{\text{cone}}^n \leq (\pi^*\omega)^n \leq CH \left(1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i}\right)^A \omega_{\text{cone}}^n, \quad (1.5)$$

where  $H$  is a smooth nonnegative function which is bounded above and may go to zero near the exceptional components of  $E$ . Estimate (1.5) is obtained using Hodge theory, like in [28] where a weaker estimate was obtained when  $\dim N = 1$ . Very recently, similar estimates were obtained in more general settings in [40, 72] using different ideas.

The main task is then to deduce a pointwise bound for  $\omega$  of the form (1.2) from (1.5). First, in Sect. 3 we show that  $\omega$  extends smoothly across the analytic subset  $D^{(2)} \cap N^{\text{reg}}$ , which has codimension at least 2, so we it suffices to understand the behavior of  $\omega$  near the divisor  $D^{(1)}$ .

Next, if  $N$  is smooth and  $D^{(1)}$  has simple normal crossings (so that  $\pi = \text{Id}$ ), we show in Sect. 4 that (1.3) holds, using estimates for Monge-Ampère equations with conical singularities as developed in [5, 6, 13, 30, 37].

In the general case, the resolution  $\pi$  will be nontrivial, which introduces the major issue that the class  $[\pi^*\omega_N]$  where the form  $\pi^*\omega$  lives is not Kähler anymore, and nothing is known about “conical” metrics in such classes. In Sect. 5, we show that on  $\tilde{N} \setminus E$  we have the “Tsuji-type” [87] estimate

$$\pi^*\omega \leq \frac{C}{|s_F|^{2A}} \left(1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i}\right)^A \omega_{\text{cone}}, \quad (1.6)$$

where  $F \subset E$  denotes the exceptional components of  $E$ . This, together with (1.5), easily implies the estimate (1.2) since  $\pi$  is an isomorphism where  $N$  is smooth and  $D^{(1)}$  is snc. This proves the main Theorem 1.1, and the application to collapsing of Ricci-flat metrics in Corollary 1.3 then follows from this and our earlier work [86] (extending ideas that we introduced in [28] when  $\dim N = 1$ ), as explained in Sect. 6. Indeed there we showed that to prove parts (a) and (b) of Conjecture 1.2 in general it suffices to show the estimate

$$\pi^*\omega \leq C \left(1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i}\right)^A \omega_{\text{cone}}, \quad (1.7)$$

and that part (c) also follows from this provided that  $\pi = \text{Id}$ . Of course, when  $\pi = \text{Id}$  the estimates (1.6) and (1.7) are identical, but it remains an open problem to show that (1.7) holds in general when  $\pi$  is nontrivial (which would prove part (a) of Conjecture 1.2).

Lastly, in Sect. 7 we extend these arguments to the Kähler–Ricci flow setting, proving Theorem 1.4. The arguments are very similar to the Calabi–Yau setting, with the only main difference being that now  $K_M$  is replaced by  $\ell K_M$  (which instead of being trivial is semiample). We show via a standard ramified covering trick that estimates analogous to (1.5) also hold in this setting, by reducing to the case when  $\ell = 1$  which is treated in Sect. 2. Once this is established, the rest of the arguments follow the same path as above. We then deduce Theorem 1.5 from this by extending the arguments in [67, 76].

## 2. Estimates on the Volume Form

In this section we use Hodge theory to derive asymptotics for the pushforward of the holomorphic volume form on a Calabi–Yau manifold which is the total space of a fiber space over a lower-dimensional base. There is by now a large literature on related questions, including for example [2–4, 7–9, 15, 16, 23, 27, 38, 40, 41, 53–55, 59, 62, 71, 72, 86, 88, 89, 93, 95], with the very recent papers [40, 72] explicitly focusing on the case when  $\dim N \geq 2$ . Here we decided to follow closely our previous work [28, Section 2] where we proved the upper bound in (2.1) in the case when  $\dim N = 1$ . Our arguments generalize this to bases of arbitrary dimensions, and also provide the lower bound in (2.1) which crucially matches the upper bound up to logarithmic error terms. We note that a more refined analysis can in principle be carried out as in [7, 8, 41] to obtain precisely matching upper and lower bounds, with a more complicated logarithmic behavior. However, since for our purposes the cruder estimate (2.1) suffices, we have decided not to pursue this.

The setup in this section will be the Calabi–Yau setting described in the introduction, namely we let  $M$  be an  $m$ -dimensional projective Calabi–Yau manifold ( $K_M \cong \mathcal{O}_M$ ) with holomorphic  $m$ -form  $\Omega$ ,  $N$  a normal projective  $n$ -dimensional variety,  $0 < n < m$ , and assume that we have a surjective holomorphic map  $f : M \rightarrow N$  with connected fibers, and let  $D \subset N$  be the discriminant locus of the map  $f$ , i.e. the union of the singular locus of  $N$  together with the critical values of  $f$  on  $N^{\text{reg}}$ . As in the introduction, we write  $D = D^{(1)} \cup D^{(2)}$  so that  $D^{(1)}$  is a divisor and  $D^{(2)}$  has codimension at least 2.

By Hironaka’s resolution of singularities, there is a birational morphism  $\pi : \tilde{N} \rightarrow N$  such that  $\tilde{N}$  is nonsingular and  $E = \pi^{-1}(D)$  is a simple normal crossings divisor, and such that  $\pi : \tilde{N} \setminus E \rightarrow N \setminus D$  is an isomorphism. In fact, it is known (see e.g. [73]) that we can choose  $\pi$  so that it is an isomorphism over the locus where  $N$  is regular and  $D^{(1)}$  has simple normal crossings. This locus is Zariski open in  $N$ , with complement of codimension 2.

Let  $\phi : \tilde{M} \rightarrow M \times_N \tilde{N}$  be a birational morphism, inducing a map  $\tilde{f} : \tilde{M} \rightarrow \tilde{N}$  such that

- (1)  $\tilde{M}$  is nonsingular
- (2)  $\tilde{f}^{-1}(E)$  is simple normal crossings in  $\tilde{M}$
- (3)  $E$  is the discriminant locus of  $\tilde{f}$ .

Again, this can be accomplished by Hironaka’s theorem.

From these, it follows by elementary arguments (see e.g. [55, Setup 2.3, Lemma 2.5]) that there is a Zariski open subset  $\tilde{N}^o \subset \tilde{N}$  with  $\tilde{N} \setminus E \subsetneq \tilde{N}^o \subset \tilde{N} \setminus \text{Sing}(E)$ , such that  $\tilde{f}^{-1}(\tilde{N}^o) \rightarrow \tilde{N}^o$  is a normal crossings morphism, i.e., locally one can find coordinates  $(z_1, \dots, z_m)$  on  $\tilde{f}^{-1}(\tilde{N}^o)$  and coordinates  $(y_1, \dots, y_n)$  on  $\tilde{N}^o$  such that  $\tilde{f}$  is given by



$$(z_1, \dots, z_m) \mapsto \left( \prod_{i=1}^{m-n+1} z_i^{d_i}, z_{m-n+2}, \dots, z_m \right).$$

Note that the fibres may be non-reduced, and  $d_i$  are multiplicities of irreducible components of  $\tilde{f}^{-1}(E)$ .

Given a point  $y_0 \in E$  let  $U$  be an open neighborhood of  $y_0$  in  $\tilde{N}$  with  $U \cong \Delta^n$  (the unit polydisc) and  $U \setminus E \cong (\Delta^*)^\ell \times \Delta^{n-\ell}$ , for some  $1 \leq \ell \leq n$ , and with local holomorphic coordinates  $(y_1, \dots, y_n)$  on  $U$  (centered at  $y_0$ ). If we write  $\Omega$  also for the pullback of  $\Omega$  to  $\tilde{M}$ , then we can define a real nonnegative function  $\varphi_U$  on  $U \setminus D$  by

$$\tilde{f}_*(\Omega \wedge \bar{\Omega}) = \varphi_U dy_1 \wedge \dots \wedge dy_n \wedge d\bar{y}_1 \wedge \dots \wedge d\bar{y}_n.$$

The main result of this section is then:

**Theorem 2.1.** *After possibly shrinking  $U$ , we have*

$$C^{-1} \prod_{i=1}^{\ell} |y_i|^{-2(1-\alpha_i)} \leq \varphi_U(y) \leq C \prod_{i=1}^{\ell} |y_i|^{-2(1-\alpha_i)} \left( 1 - \sum_{i=1}^{\ell} \log |y_i| \right)^d, \quad (2.1)$$

for some non-negative integer  $d$  and rational numbers  $\alpha_i \in \mathbb{Q}_{>0}$ , and for all  $y \in U \setminus E$ .

Note that we do not assert that necessarily  $\alpha_i \leq 1$ , so some of the exponents  $-2(1 - \alpha_i)$  may be positive.

*Remark 2.2.* As is well-known (cf. [4, 95]) from the proof it follows that when  $\dim N = 1$  the rational number  $\alpha$  equals the log canonical threshold of the fiber  $f^{-1}(0)$ , an algebro-geometric invariant. See [40] for a similar interpretation in the general case.

*Proof.* Let  $n_0 \in U \setminus E$  be a basepoint and let

$$T_i : H^{m-n}(\tilde{f}^{-1}(n_0), \mathbb{C}) \rightarrow H^{m-n}(\tilde{f}^{-1}(n_0), \mathbb{C})$$

be the monodromy operator for a loop based at  $n_0$  around the  $i$ th copy of  $\Delta^*$ ,  $1 \leq i \leq \ell$ . By the Monodromy Theorem (see e.g., the appendix of [46], or [25, Chap.II, Application 17])  $T_i$  is quasi-unipotent with  $(T_i^{m_i} - I)^{d_i} = 0$  for some positive integers  $d_i$  and  $m_i$ , and  $m_i$  is the least common multiple of the multiplicities of the irreducible components over a general point of  $\{y_i = 0\}$ . Set also  $m_i = 1$  for  $\ell < i \leq n$ . Let  $\tilde{U} = \Delta^n$  with coordinate  $(w_1, \dots, w_n)$ , and let  $\mu : \tilde{U} \rightarrow U$  be given by  $\mu(w_1, \dots, w_n) = (w_1^{m_1}, \dots, w_n^{m_n})$ .

Pull-back and normalize the family  $\tilde{M} \rightarrow \tilde{N}$  via the composition  $\tilde{U} \xrightarrow{\mu} U \hookrightarrow \tilde{N}$ , to obtain a family  $\tilde{f} : \tilde{M} \rightarrow \tilde{U}$ . This has discriminant locus  $\tilde{D} = \mu^{-1}(U \cap E)$  and  $\tilde{f}$  now has the property that the monodromy around a loop in  $\tilde{U}^o := \tilde{U} \setminus \tilde{D} \cong (\Delta^*)^\ell \times \Delta^{n-\ell}$  is unipotent.

Now the trivial vector bundle  $\mathcal{H}^{m-n} = (R^{m-n} \tilde{f}_* \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{U}^o}$  on  $\tilde{U}^o$  comes with the Gauss-Manin connection, whose flat sections are sections of  $R^{m-n} \tilde{f}_* \mathbb{C}$ . It is standard that this vector bundle has a canonical extension to  $\tilde{U}$ , (see e.g., [25], Chapter IV or [62]) constructed as follows. Choosing a basepoint  $t_0 \in \tilde{U}^o$ , let  $e_1, \dots, e_s$  be a basis for  $H^{m-n}(\tilde{f}^{-1}(t_0))$ . These extend to multi-valued flat sections of  $\mathcal{H}^{m-n}$ , which we write as  $e_i(w)$ . However,

$$\sigma_i(w) := \exp \left( - \sum_{j=1}^{\ell} N_j \frac{\log w_j}{2\pi\sqrt{-1}} \right) e_i(w)$$



with  $N_j = \log T_j$  is in fact a single-valued holomorphic section of  $\mathcal{H}^{m-n}$ . We then extend  $\mathcal{H}^{m-n}$  across  $\tilde{U}$  by decreeing these sections to form a holomorphic frame for the vector bundle. Call this extension  $\mathcal{H}_{\tilde{U}}^{m-n}$ .

It is then standard (see again [25], Chapter IV) that the Hodge bundle  $F_{\tilde{U}^o}^{m-n} := (f_*\Omega_{\tilde{M}/\tilde{U}}^{m-n})|_{\tilde{U}^o} \subset \mathcal{H}^{m-n}$  has a natural extension  $F_{\tilde{U}}^{m-n} \subset \mathcal{H}_{\tilde{U}}^{m-n}$  to  $\tilde{U}$ .

Next note that the form

$$\Omega^{rel} := \iota(\partial/\partial y_1, \dots, \partial/\partial y_n)\Omega$$

is a well-defined section of  $\tilde{f}_*\Omega_{\tilde{f}^{-1}(U)/U}^{m-n}$  and thus pulls back to a well-defined section  $\Omega_{\tilde{U}^o}^{rel}$  of  $F_{\tilde{U}^o}^{m-n}$ . Furthermore, the function  $\varphi_U$  given in the statement of the theorem satisfies at a point  $y \in U$

$$\varphi_U(y) = (-1)^{\frac{(m-n)^2}{2}} \int_{\tilde{f}^{-1}(y)} \Omega^{rel} \wedge \bar{\Omega}^{rel}.$$

We will show that the section  $\Omega_{\tilde{U}^o}^{rel}$  of  $F_{\tilde{U}^o}^{m-n}$  extends to a meromorphic section of  $F_{\tilde{U}}^{m-n}$  and investigate the order of the poles of this section along the discriminant locus. For this it is sufficient to restrict to one-parameter families. Let  $S_i \subset U$  be the disc on which all the coordinates  $y_j$  are constant and nonzero except for  $y_i \in \Delta$ . Let  $\tilde{S}_i = \mu^{-1}(S_i)$ . Note that  $\tilde{S}_i$  is a disjoint union of discs. Denote by  $\tilde{M}_i \rightarrow S_i$  and  $\tilde{M}_i \rightarrow \tilde{S}_i$  the basechanges of  $\tilde{M} \rightarrow \tilde{N}$  and  $\tilde{M} \rightarrow \tilde{U}$  respectively. By choosing  $S_i$  generally, we can assume that  $\tilde{M}_i \rightarrow S_i$  is a normal crossings morphism, possibly with a nonreduced singular fiber over  $0 \in S_i$ . Call this fiber  $\tilde{Y}$ . In particular, restricting to  $S_i$ , we can consider  $\Omega^{rel}$  as a meromorphic section of  $\tilde{f}_*\Omega_{\tilde{M}_i/S_i}^{m-n}(\log \tilde{Y})$ .

Going back to our problem, we first determine the order of pole of  $\Omega^{rel}$  at 0 as a section of this bundle. Locally on  $\tilde{M}_i$ , near a general point of an irreducible component  $Y_j$  of  $\tilde{Y}$ , the map  $\tilde{M}_i \rightarrow S_i$  is given by  $y_i = z_1^\beta$ , with  $\beta \geq 1$  and  $z_1, \dots, z_{m-n+1}$  coordinates on  $\tilde{M}_i$ . We can write

$$\Omega_i := \iota(\partial_{y_1}, \dots, \widehat{\partial_{y_i}}, \dots, \partial_{y_n})\Omega,$$

as a form on  $\tilde{M}_i$  locally as

$$\Omega_i = \psi_i dz_1 \wedge \dots \wedge dz_{m-n+1},$$

for some holomorphic function  $\psi_i$ . We have  $\psi_i = z_1^k \phi$  where  $|\phi| > 0$  at the generic point of  $Y_j$ .

In our local coordinate description, the vector field on  $\tilde{M}_i$  given by  $\beta^{-1}z_1^{-\beta+1}\partial_{z_1}$  is a lift of  $\partial_{y_i}$ . Thus  $\Omega^{rel}$  as a section of  $\Omega_{\tilde{M}_i/S_i}^{m-n}(\log \tilde{Y})$  is locally given by

$$\pm \frac{\psi_i}{\beta z_1^{\beta-1}} dz_2 \wedge \dots \wedge dz_{m-n+1} = \phi z_1^{k+1-\beta} dz_2 \wedge \dots \wedge dz_{m-n+1}.$$

We now need to pull-back  $\Omega^{rel}$  to  $\Omega_{\tilde{U}^o}^{rel}$  and study this section as a section of  $F_{\tilde{U}}^{m-n}$ . The stable reduction theorem [39] gives a resolution of singularities  $\bar{M}'_i \rightarrow \bar{M}_i$  such that the composed map  $\bar{M}'_i \rightarrow \bar{S}_i$  is normal crossings. So we have a diagram

$$\begin{array}{ccccc} \bar{M}'_i & \xrightarrow{\pi'} & \bar{M}_i & \xrightarrow{\pi} & \tilde{M}_i \\ & \searrow \tilde{f}' & \downarrow \tilde{f} & & \downarrow \tilde{f} \\ & & \bar{S}_i & \xrightarrow{\mu} & S_i \end{array}$$

Furthermore, the map  $\pi'$  is a toric resolution of singularities by the construction of [39], Chapter II. In particular, locally  $\pi'$  and  $\pi$  can be described as dominant morphisms of toric varieties of the same dimension. On such toric charts, by [56], Prop. 3.1, the sheaves of logarithmic differentials  $\Omega_{\bar{M}_i/\bar{S}_i}^{m-n}(\log \tilde{Y})$ ,  $\Omega_{\tilde{M}_i/\bar{S}_i}^{m-n}(\log \tilde{Y})$ , and  $\Omega_{\bar{M}'_i/\bar{S}_i}^{m-n}(\log \tilde{Y}')$  are trivial vector bundles generated by exterior products of logarithmic differentials of toric monomials (here  $\tilde{Y}$  and  $\tilde{Y}'$  denote the fibers over zero), and thus  $\pi^* \Omega_{\bar{M}_i/\bar{S}_i}^{m-n}(\log \tilde{Y}) \cong \Omega_{\bar{M}_i/\bar{S}_i}^{m-n}(\log \tilde{Y})$  and  $(\pi')^* \Omega_{\bar{M}_i/\bar{S}_i}^{m-n}(\log \tilde{Y}) \cong \Omega_{\bar{M}'_i/\bar{S}_i}^{m-n}(\log \tilde{Y}')$ . Furthermore,

$$\begin{aligned} \tilde{f}'_* \Omega_{\bar{M}'_i/\bar{S}_i}^{m-n}(\log \tilde{Y}') &\cong \tilde{f}_* \pi'^* \Omega_{\bar{M}_i/\bar{S}_i}^{m-n}(\log \tilde{Y}') \\ &\cong \tilde{f}_* \pi'^* (\pi')^* \Omega_{\bar{M}_i/\bar{S}_i}^{m-n}(\log \tilde{Y}) \\ &\cong \tilde{f}_* ((\pi'_* \mathcal{O}_{\bar{M}'_i}) \otimes \Omega_{\bar{M}_i/\bar{S}_i}^{m-n}(\log \tilde{Y})) \\ &\cong \tilde{f}_* \Omega_{\bar{M}_i/\bar{S}_i}^{m-n}(\log \tilde{Y}). \end{aligned}$$

It also follows from [68] (see also [25], Chapter VII) that

$$F_{\tilde{U}}^{m-n}|_{S_i} \cong \tilde{f}'_* \Omega_{\bar{M}'_i/\bar{S}_i}^{m-n}(\log \tilde{Y}').$$

Thus, in order to understand the behaviour of  $\Omega_{\tilde{U}^o}^{rel}$  as a section of  $F_{\tilde{U}}^{m-n}$ , it is sufficient to pull back  $\Omega^{rel}$  to  $\Omega_{\bar{M}_i/\bar{S}_i}^{m-n}(\log \tilde{Y})$  and understand the behaviour of this form as a section of  $\tilde{f}_* \Omega_{\bar{M}_i/\bar{S}_i}^{m-n}(\log \tilde{Y})$ .

Again, we do this locally near the inverse image of a general point of an irreducible component of  $\tilde{Y}$ . Using the same notation as before, we know that  $\bar{M}_i$  is locally given by the normalization of the equation  $w_i^{m_i} = z_1^\beta$ . Note that  $\beta|m_i$ , so a local description of the normalization is given by an equation  $w_i^{m_i/\beta} = \xi z_1$  for  $\xi$  a  $\beta$ -th root of unity. Thus  $\Omega^{rel}$  pulls back to

$$\psi_i w_i^{\frac{-m_i(\beta-1)}{\beta}} dz_2 \wedge \cdots \wedge dz_{m-n+1} = \phi w_i^{\frac{-m_i(\beta-1-k)}{\beta}} dz_2 \wedge \cdots \wedge dz_{m-n+1}.$$

Note that

$$w_i^{\frac{-m_i(\beta-1-k)}{\beta}} = y_i^{\frac{1+k}{\beta}-1}.$$

Thus letting  $\beta_i$  and  $k_i$  be the numbers such that  $\frac{1+k_i}{\beta_i} = \min\{\frac{1+k}{\beta}\}$  among irreducible components of  $\tilde{Y}$  (and let  $Y_j$  be one of the components where this minimum is achieved),

we find  $w_i^{m_i(\beta_i-1-k_i)/\beta_i} \Omega_{\bar{U}^o}^{rel}$  extends to a holomorphic section of  $\Omega_{\bar{M}_i/\bar{S}_i}^{m-n}(\log \bar{Y})$ , hence yields a holomorphic section of  $F_{\bar{U}}^{m-n}|_{\bar{S}_i} = \bar{f}'_* \Omega_{\bar{M}'_i/\bar{S}_i}^{m-n}(\log \bar{Y}')$ .

Now set

$$\Omega^{norm} := \prod_{i=1}^{\ell} w_i^{m_i(\beta_i-1-k_i)/\beta_i} \Omega_{\bar{U}^o}^{rel}.$$

This now extends to a holomorphic section of  $F_{\bar{U}}^{m-n}$ . By construction, since  $|\phi| > 0$  at the generic point of  $Y_j$ , we see that  $\Omega^{norm} \neq 0$  at the generic point of  $Y_j$ . Thus we can write  $\Omega^{norm}$ , as a section of  $\mathcal{H}_{\bar{U}}^{m-n}$ , as

$$\Omega^{norm} = \sum_{i=1}^s h_i(w) \sigma_i(w),$$

for  $h_i$  holomorphic functions on  $\bar{U}$ . We then compute, with  $\langle \cdot, \cdot \rangle$  denoting the cup product followed by evaluation on the fundamental class

$$H^{m-n}(f^{-1}(t_0), \mathbb{C}) \times H^{m-n}(f^{-1}(t_0), \mathbb{C}) \rightarrow \mathbb{C},$$

that

$$\begin{aligned} & \int_{\bar{f}^{-1}(w)} \Omega^{norm} \wedge \bar{\Omega}^{norm} \\ &= \left\langle \sum_{i=1}^s h_i(w) \sigma_i(w), \sum_{j=1}^s \bar{h}_j(w) \bar{\sigma}_j(w) \right\rangle \\ &= \left\langle \sum_{i=1}^s e^{-\sum_{k=1}^{\ell} N_k \log w_k / 2\pi \sqrt{-1}} h_i e_i, \sum_{j=1}^s e^{\sum_{p=1}^{\ell} N_p \log \bar{w}_p / 2\pi \sqrt{-1}} \bar{h}_j e_j \right\rangle. \end{aligned}$$

Note the exponentials can be expanded in a finite power series because the  $N_i$  are nilpotent, and hence a term in the above expression is

$$C \cdot h_i \bar{h}_j \prod_{k=1}^{\ell} (\log w_k)^{d_k} (\log \bar{w}_k)^{d'_k} \left\langle \prod_k N_k^{d_k} e_i, \prod_k N_k^{d'_k} e_j \right\rangle.$$

Here the constant  $C$  only depends on the powers  $d_k, d'_k$  occurring. We can assume that we have chosen the imaginary part of  $\log w_k$  to lie between 0 and  $2\pi$  (this is equivalent to choosing the branch of  $e_i(y)$ ). Keeping in mind that the  $h_j$  are holomorphic on  $\bar{U}$ , after shrinking  $\bar{U}$  we can assume that  $|h_j|$  are bounded by some constant, and so we see that the above term is bounded by a sum of a finite number of expressions of the form

$$C' \prod_{k=1}^{\ell} (-\log |w_k|)^{d''_k}.$$

Thus the entire integral is bounded by an expression of the form

$$C \left( 1 - \sum_{k=1}^{\ell} \log |w_k| \right)^d,$$

for suitable choice of constant  $C$  and exponent  $d$ .

Returning to  $\Omega_{\tilde{U}^o}^{rel}$ , we see that

$$(-1)^{\frac{(m-n)^2}{2}} \int_{\tilde{f}^{-1}(w)} \Omega_{\tilde{U}^o}^{rel} \wedge \bar{\Omega}_{\tilde{U}^o}^{rel} \leq C \prod_{i=1}^{\ell} |w_i|^{-2m_i(\beta_i-1-k_i)/\beta_i} \left(1 - \sum_{k=1}^{\ell} \log |w_k|\right)^d.$$

Using  $y_i = w_i^{m_i}$  then gives one of the inequalities in (2.1), with  $\alpha_i = \frac{1+k_i}{\beta_i}$ .

To prove the other inequality, we want to show that

$$(-1)^{\frac{(m-n)^2}{2}} \int_{f^{-1}(y)} \Omega^{rel} \wedge \bar{\Omega}^{rel} \geq C^{-1} \prod_{i=1}^{\ell} |y_i|^{-2(1-\frac{k_i+1}{\beta_i})}$$

for  $y \in U \setminus \tilde{D} = (\Delta^*)^{\ell} \times \Delta^{n-\ell} \subset \tilde{N} \setminus E$ . Let  $y_1 = w_1^{m_1}, \dots, y_{\ell} = w_{\ell}^{m_{\ell}}, y_{\ell+1} = w_{\ell+1}, \dots, y_n = w_n$  be the covering  $\mu : \tilde{U}^o \rightarrow U^o$  such that the monodromy operators are unipotent, and let

$$\Omega^{norm} = \prod_{i=1}^{\ell} w_i^{m_i(\beta_i-1-k_i)/\beta_i} \Omega_{\tilde{U}^o}^{rel}.$$

Then our goal is equivalent to proving

$$(-1)^{\frac{(m-n)^2}{2}} \int_{\tilde{f}^{-1}(w)} \Omega^{norm} \wedge \bar{\Omega}^{norm} \geq C^{-1} > 0. \quad (2.2)$$

Let  $Z$  be the zero divisor of  $\Omega^{norm}$ , which is regarded as a section of the line bundle  $F_{\tilde{U}}^{m-n}$ . Then  $Z \subset \bar{D} = \{w_1 \dots w_{\ell} = 0\}$  the discriminant locus. To prove (2.2) it is enough to show that  $Z = \emptyset$ .

Assume that  $Z \neq \emptyset$ . Note that  $Z$  is a divisor, and thus  $Z$  contains an irreducible component of  $\bar{D}$ , say  $\{w_i = 0\}$ . Let  $S_i$  be a disc on which all  $y_j$  are constant nonzero except for  $y_i$ , i.e.  $S_i = \{y_1 = c_1, \dots, y_i = \bar{c}_i, \dots, y_n = c_n\}$ , and  $\tilde{S}_i = \mu^{-1}(S_i)$ . Hence  $w_i$  is the coordinate on  $\tilde{S}_i$ , and  $\tilde{S}_i \cap Z = \{w_i = 0\} \neq \emptyset$ . Let  $\tilde{M}_i \rightarrow S_i$  and  $\bar{M}_i \rightarrow \tilde{S}_i$  be the base changes. From above,  $\bar{M}_i \rightarrow S_i$  is normal crossing morphism with a possible nonreduced central fiber  $\tilde{Y}$ .

As before, let  $Y_j$  be one of the components of  $\tilde{Y}$  where the  $\min\{\frac{1+k}{\beta}\}$  is achieved (notation as before). Then in local coordinates on  $\bar{M}_i$  as before, near the preimage of  $Y_j$  in  $\bar{M}_i$  under the map  $q : \bar{M}_i \rightarrow M_i$ , we have that  $\Omega_{\tilde{U}^o}^{rel}$  pulls back to

$$\phi w_i^{\frac{-m_i(\beta-1-k)}{\beta}} dz_2 \wedge \dots \wedge dz_{m-n+1},$$

where  $\phi$  is not identically zero, and so  $\Omega^{norm}$  pulls back to

$$\phi dz_2 \wedge \dots \wedge dz_{m-n+1},$$

with  $\phi \not\equiv 0$ , which is a contradiction because the preimage  $q^{-1}(\tilde{Y})$  under the map  $q : \bar{M}_i \rightarrow M_i$  is mapped into  $Z = \{\Omega^{norm} = 0\}$  by  $\bar{M}_i \rightarrow \tilde{S}_i \hookrightarrow \tilde{U}$ . Hence  $Z = \emptyset$ .  $\square$

Next, we translate the local estimate in Theorem 2.1 into a global statement. To do this, we need to recall the notion of Kähler metrics with conical singularities. We write  $E = \bigcup_{j=1}^{\mu} E_j$  for the decomposition of  $E$  in irreducible components. Given real numbers  $0 < \alpha_i \leq 1$ ,  $1 \leq i \leq \mu$ , there is a well-defined notion of *conical Kähler metrics*  $\omega_{\text{cone}}$  on  $\tilde{N}$  with singularities along  $E$  with cone angle  $2\pi\alpha_i$  along each component  $E_i$ . Any such metric is a smooth Kähler metric on  $\tilde{N} \setminus E$  such that in any local chart  $U$  (a unit polydisc with coordinates  $(w_1, \dots, w_n)$ ) centered at a point of  $E$  adapted to the normal crossings structure (so  $E \cap U$  is given by  $w_1 \cdots w_k = 0$  for some  $1 \leq k \leq n$ , and say that  $\{w_i = 0\} = E_{j_i} \cap U$  for some  $1 \leq j_i \leq \mu$  and all  $1 \leq i \leq k$ ) we have that on  $U \setminus \{w_1 \cdots w_k = 0\}$  the metric  $\omega_{\text{cone}}$  is uniformly equivalent to the model

$$\sum_{i=1}^k \frac{\sqrt{-1} dw_i \wedge d\bar{w}_i}{|w_i|^{2(1-\alpha_{j_i})}} + \sum_{i=k+1}^n \sqrt{-1} dw_i \wedge d\bar{w}_i.$$

We also fix a defining section  $s_i$  of the divisor  $E_i$  and a smooth Hermitian metric  $h_i$  on  $\mathcal{O}(E_i)$ , for all  $1 \leq i \leq \mu$ . An explicit conical Kähler metric  $\omega_{\text{cone}}$  on  $N_0$  with cone angle  $2\pi\alpha_i$  along each component  $E_i$  is given by

$$\omega_{\text{cone}} = \omega_{\tilde{N}} + \delta \sqrt{-1} \partial \bar{\partial} \left( \sum_{i=1}^{\mu} |s_i|_{h_i}^{2\alpha_i} \right),$$

for some small  $\delta > 0$ , where  $\omega_{\tilde{N}}$  is any given Kähler metric on  $\tilde{N}$ .

The following is then the global version of Theorem 2.1, which was stated as (1.5) in the Introduction:

**Theorem 2.3.** *There is a constant  $C > 0$  and natural numbers  $d \in \mathbb{N}$ ,  $0 \leq p \leq \mu$  and rational numbers  $\beta_i > 0$ ,  $1 \leq i \leq p$  and  $0 < \alpha_i \leq 1$ ,  $p+1 \leq i \leq \mu$ , such that on  $\pi^{-1}(N_0)$  we have*

$$C^{-1} \prod_{j=1}^p |s_j|_{h_j}^{2\beta_j} \omega_{\text{cone}}^n \leq (\pi^* \omega)^n \leq C \prod_{j=1}^p |s_j|_{h_j}^{2\beta_j} \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^d \omega_{\text{cone}}^n, \quad (2.3)$$

where  $\omega_{\text{cone}}$  is a conical metric with cone angle  $2\pi\alpha_i$  along the components  $E_i$  with  $p+1 \leq i \leq \mu$ .

*Proof.* This makes use of the above results from Hodge theory. First of all, as explained for example in [78, Section 4], on  $N_0$  the metric  $\omega$  satisfies

$$\omega^n = c f_*((-1)^{\frac{m^2}{2}} \Omega \wedge \bar{\Omega}) \quad (2.4)$$

for some explicit constant  $c > 0$ . Recall that at the beginning of this Section we have constructed a sequence of blowups  $\pi : \tilde{N} \rightarrow N$  such that  $E = \pi^{-1}(D)$  is an snc divisor, and then we have the base-change  $M \times_N \tilde{N} \rightarrow M$  and its resolution  $\tilde{M} \rightarrow M \times_N \tilde{N}$  which, when composed, gives a holomorphic map  $p : \tilde{M} \rightarrow M$ . We also have the new fibration  $\tilde{f} : \tilde{M} \rightarrow \tilde{N}$  so that we have the commutative diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{p} & M \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{N} & \xrightarrow{\pi} & N \end{array}$$

For clarity, call  $\tilde{\Omega} = p^*\Omega$ , which is a holomorphic  $n$ -form on  $\tilde{M}$  which in general has zeros. Then on  $\tilde{N} \setminus E$  we actually have equality

$$\pi^* f_*(\Omega \wedge \overline{\Omega}) = \tilde{f}_*(\tilde{\Omega} \wedge \overline{\tilde{\Omega}}), \quad (2.5)$$

since  $p, \pi$  are biholomorphisms when we are away from the singular fibers. Combining (2.4) and (2.5) we get

$$(\pi^* \omega)^n = c \tilde{f}_*((-1)^{\frac{m^2}{2}} \tilde{\Omega} \wedge \overline{\tilde{\Omega}})$$

on  $\tilde{N} \setminus E$ . Now Theorem 2.1 gives us an estimate for the behavior of the RHS locally, when compared with a Euclidean volume form, near any given point of  $E$ , of the form

$$\begin{aligned} C^{-1} \prod_{i=1}^{\ell} |y_i|^{-2(1-\alpha_i)} &\leq \frac{c \tilde{f}_*((-1)^{\frac{m^2}{2}} \tilde{\Omega} \wedge \overline{\tilde{\Omega}})}{i dy_1 \wedge d\bar{y}_1 \wedge \cdots \wedge i dy_n \wedge d\bar{y}_n} \\ &\leq C \prod_{i=1}^{\ell} |y_i|^{-2(1-\alpha_i)} \left(1 - \sum_{i=1}^{\ell} \log |y_i|\right)^d, \end{aligned}$$

where  $\alpha_i \in \mathbb{Q}_{>0}$ . We keep those  $\alpha_i$  which satisfy  $\alpha_i \leq 1$ , and for the others we define  $\beta_j = -1 + \alpha_j \in \mathbb{Q}_{>0}$  so that we can write (with abuse of notation)

$$\prod_{k=1}^{\ell} |y_k|^{-2(1-\alpha_k)} = \prod_i |y_i|^{-2(1-\alpha_i)} \prod_j |y_j|^{2\beta_j},$$

and so that this shows that (2.3) holds locally, in the sense that given any point  $x \in E$  we can find a neighborhood  $U$  of  $x$  in  $\tilde{N}$  together with  $\ell, C, d, p, \alpha_i, \beta_i$  (which all depend on  $U$ ), such that

$$\frac{C^{-1} \prod_j |y_j|^{2\beta_j} \omega_{\tilde{N}}^n}{\prod_i |y_i|^{2(1-\alpha_i)}} \leq (\pi^* \omega)^n \leq \left(1 - \sum_{i=1}^{\ell} \log |y_i|\right)^d \frac{C \prod_j |y_j|^{2\beta_j} \omega_{\tilde{N}}^n}{\prod_i |y_i|^{2(1-\alpha_i)}}, \quad (2.6)$$

holds on  $U$ . Fix attention on one irreducible component  $E_1$  of  $E$ , and cover  $E_1$  by such open sets  $U_j$ , and say that on each  $U_j$  we have  $E_1 \cap U_j = \{y_1 = 0\}$ . We wish to show that the number  $\alpha_1$  (which give the cone angle along  $E_1$  if  $\alpha_1 \leq 1$ ) or  $\beta_1$  (which give the order of zero along  $E_1$  if  $\alpha_1 > 1$ ) that we get is the same on all these  $U_j$ 's. On a nonempty intersection of two such  $U_j$ 's, we can look at points approaching  $E_1$  in this intersection, which do not come close to any of the other components of  $E$ . Then the blowup rate (or order of vanishing) of the function in (2.6) along  $\{y_1 = 0\}$  must be the same in both sets, which thanks to (2.6) implies that the corresponding numbers  $\alpha_1$  (or  $\beta_1$ ) are equal. Repeating this for all components of  $E$ , this proves (2.3).  $\square$

### 3. The Codimension 2 Part of the Discriminant

Still working in the Calabi–Yau setting, our next goal is to show that the codimension 2 (or more) subset  $D^{(2)} \cap N_0$  does not contribute to the singularities of  $\omega$  on  $N_0$ . For this, we fix a Kähler metric  $\omega_N$  on  $N$  (in the sense of analytic spaces [52] if  $N$  is not smooth), and we define a smooth positive function  $\mathcal{F}$  defined on  $N_0$  by

$$\frac{(-1)^{\frac{m^2}{2}} f_*(\Omega \wedge \overline{\Omega})}{\omega_N^n} = \mathcal{F}, \quad (3.1)$$

which satisfies

$$\sqrt{-1} \partial \bar{\partial} \log \mathcal{F} = -\omega_{\text{WP}} + \text{Ric}(\omega_N),$$

while the metric  $\omega$  on  $N_0$  satisfies

$$\omega^n = c \mathcal{F} \omega_N^n, \quad (3.2)$$

for some constant  $c > 0$ .

Our next observation is that the components of  $D$  of codimension 2 or higher do not contribute to the singularities of  $\mathcal{F}$ . More precisely, write  $N' = N^{\text{reg}} \setminus D^{(1)}$  and  $D' = D^{(2)} \cap N'$ , so  $D'$  is codimension 2 or more inside the complex manifold  $N'$ , and  $N' \setminus D' = N_0$ .

**Proposition 3.1.** *The function  $\mathcal{F}$  on  $N_0$  extends to a smooth positive function on all of  $N'$ , and the current  $\omega$  on  $N$  is a smooth Kähler metric on  $N'$ .*

*Proof.* For this we use the well-known relation between the Weil–Petersson form and the Hodge metric on the Hodge bundle, that we now recall. The reader is referred to e.g. [25] for the relevant background. For  $y \in N_0$  denote by  $M_y = f^{-1}(y)$  the fiber over  $y$  and let  $P \subset H^{n-m}(M_y, \mathbb{C}) =: H$  be the primitive cohomology given by the Kähler class  $[\omega_M|_{M_y}]$ ,  $H_{\mathbb{Z}} := P \cap H^{n-m}(M_y, \mathbb{Z})$ ,  $k^{p,q} := \dim_{\mathbb{C}} P \cap H^q(M_y, \Omega_{M_y}^p)$  ( $p+q = n-m$ ), and let  $Q$  be the polarization form on  $H$ . Let  $\mathcal{D}$  be the classifying space for integral polarized Hodge structures of type  $\{H_{\mathbb{Z}}, k^{p,q}, Q\}$ , and let

$$\mathcal{P} : N_0 \rightarrow \mathcal{D}/\Gamma,$$

be the corresponding period map, where  $\Gamma$  is the monodromy group. This map is holomorphic, locally liftable and horizontal. Since  $\mathcal{D}$  has a Hermitian metric with negative holomorphic sectional curvature in the horizontal directions [26], a result of Griffiths–Schmid [26, Corollary 9.8] shows that  $\mathcal{P}$  admits a holomorphic extension  $\overline{\mathcal{P}} : N' \rightarrow \mathcal{D}/\Gamma$  across the analytic set  $D'$  of codimension 2 or more.

In our situation, the form  $\omega_{\text{WP}}$  on  $N_0$  is in fact equal to the pullback  $\mathcal{P}^* \omega_H$  of the Hodge Kähler metric on  $\mathcal{D}/\Gamma$  by [74]. It follows therefore that  $\omega_{\text{WP}}$  admits a smooth extension  $\overline{\omega}_{\text{WP}}$  to  $N'$ , which is equal to  $\overline{\mathcal{P}}^* \omega_H \geq 0$ .

Given any  $x \in D'$ , there is an open subset  $x \in U \subset N'$ ,  $U$  biholomorphic to a ball, where we can write

$$-\overline{\omega}_{\text{WP}} + \text{Ric}(\omega_N) = \sqrt{-1} \partial \bar{\partial} u,$$

for some smooth function  $u$ . Note that  $U \cap N_0$  is simply connected. On the other hand, if on  $N_0$  we write  $v = \log \mathcal{F}$ , then the difference  $w = u - v$  is pluriharmonic on  $U \cap N_0$ .



In particular,  $\partial w$  is  $d$ -closed on  $U \cap N_0$ , and since  $H^1(U \cap N_0, \mathbb{C}) = 0$ , it is  $d$ -exact, i.e.  $\partial w = dw'$  for some complex function  $w'$  on  $U \cap N_0$ . Then  $\bar{\partial} w' = 0$  so  $w'$  is holomorphic, and also  $d(w' + \overline{w'} - w) = 0$  so  $w' + \overline{w'} - w = c$  on  $U \cap N_0$ . Therefore  $w$  equals the real part of the holomorphic function  $h = 2w' - c$  on  $U \cap N_0$ . By Hartogs,  $h$  extends to a holomorphic function on  $U$ , and so  $w$  (and hence also  $v$ ) extends smoothly on  $U$ . This shows the desired extension property of  $\mathcal{F}$ .

Lastly, the fact that  $\omega$  is a smooth Kähler metric on  $N'$  follows from the way in which  $\omega$  (solving (3.2)) is constructed in [65, §3.2] via an approximation procedure, after we know that  $\mathcal{F}$  is smooth and positive on  $N'$ .  $\square$

It is interesting to point out the following extension of the results in [84]:

**Proposition 3.2.** *Let  $M$  be a projective manifold with  $K_M \cong \mathcal{O}_M$ ,  $N$  a compact Kähler manifold and  $f : M \rightarrow N$  a surjective holomorphic map with connected fibers. If the locus  $D \subset N$  of critical values of  $f$  has codimension 2 or more, then  $f|_{M \setminus f^{-1}(D)} : M \setminus f^{-1}(D) \rightarrow N \setminus D$  must be a holomorphic fiber bundle (i.e.  $f$  is isotrivial).*

The result in [84] (see also [86, Theorem 3.3]) is the same theorem under the stronger assumption that  $D = \emptyset$ . On the other hand, this result is of course false if  $D$  is allowed to have components of codimension 1.

*Proof.* Since  $f$  has connected fibers and  $N$  is smooth we have  $f_*\mathcal{O}_M \cong \mathcal{O}_N$ . Combining this with  $K_M \cong \mathcal{O}_M$  and with the projection formula we get that

$$\mathcal{O}_N \cong f_*(K_{M/N}) + K_N,$$

i.e.

$$-K_N \cong f_*(K_{M/N}).$$

In particular the pushforward  $f_*(K_{M/N})$  is indeed a line bundle on  $N$ . On  $N \setminus D$  we have a smooth Hermitian metric on this line bundle given locally by

$$(-1)^{\frac{(m-n)^2}{2}} \int_{M_y} \Omega_y \wedge \overline{\Omega}_y,$$

(where  $M_y = f^{-1}(y)$ ) whose curvature form on  $N \setminus D$  is exactly the Weil-Petersson form  $\omega_{\text{WP}} \geq 0$ . From the proof of Proposition 3.1 we know that  $\omega_{\text{WP}}$  admits a smooth extension  $\overline{\omega}_{\text{WP}}$  to  $N$ . On the other hand, there is another smooth metric on  $-K_N$  whose curvature form on  $N$  is  $\text{Ric}(\omega_N)$ , and so there exists a smooth function  $u$  on  $N \setminus D$  such that

$$\text{Ric}(\omega_N) + \sqrt{-1} \partial \bar{\partial} u = \omega_{\text{WP}}.$$

Since  $D \subset N$  has codimension 2 or more, the Grauert-Remmert extension theorem [24] shows that  $u$  extends to a quasi-psh function  $\underline{u}$  on  $N$  which satisfies  $\text{Ric}(\omega_N) + \sqrt{-1} \partial \bar{\partial} \underline{u} \geq 0$  weakly as currents. The difference  $\overline{\omega}_{\text{WP}} - (\text{Ric}(\omega_N) + \sqrt{-1} \partial \bar{\partial} \underline{u})$  is a closed real  $(1, 1)$  current which is the difference of two positive currents, and which is supported on the analytic set  $D$  of codimension at least 2, hence by the Federer support theorem this current vanishes. It follows that  $\sqrt{-1} \partial \bar{\partial} \underline{u}$  is smooth on all of  $N$ , and hence so is  $\underline{u}$  by regularity of the Laplacian. This means that  $\text{Ric}(\omega_N) + \sqrt{-1} \partial \bar{\partial} \underline{u} \geq 0$  is a smooth semipositive form on  $N$  cohomologous to  $c_1(-K_N)$ , and so by Yau's Theorem [91] the manifold  $N$  admits a Kähler metric with nonnegative Ricci curvature.

As in [84], we can apply the Schwarz Lemma [84, Lemma 3.3] to the map  $\overline{\mathcal{P}} : N \rightarrow \mathcal{D}/\Gamma$ , to conclude that  $\overline{\mathcal{P}}$  is constant on  $N$ . Therefore  $\mathcal{P}$  is constant on  $N \setminus D$ , and as in [84, Proof of Theorem 3.1] this implies that the map  $f|_{M \setminus f^{-1}(D)}$  is a holomorphic fiber bundle.  $\square$

#### 4. The Case when the Discriminant Locus is an SNC Divisor

In this section we prove the estimate (1.3), under the assumption that  $N$  is smooth and  $D^{(1)}$  is a simple normal crossings divisor.

We start with a preliminary observation related to the volume estimate in Theorem 2.3, in the general setting of Theorem 1.1. Recall that we have constructed a resolution  $\pi : \tilde{N} \rightarrow N$  of  $N$  such that  $\pi^{-1}(D) = E$  is a simple normal crossings divisor,  $E = \cup_{i=1}^{\mu} E_i$ , and we have fixed defining sections  $s_i$  and metrics  $h_i$  for  $\mathcal{O}(E_i)$ . Let  $\tilde{D} \subset \tilde{N}$  be the proper transform of  $D^{(1)}$ , and write

$$E = F \cup \tilde{D},$$

where  $F$  is a  $\pi$ -exceptional divisor.

We fix a Kähler metric  $\omega_{\tilde{N}}$  on  $\tilde{N}$  and a Kähler metric  $\omega_N$  on  $N$  (in the sense of analytic spaces [52] if  $N$  is not smooth), and define a smooth nonnegative function  $J_{\pi}$  on  $\tilde{N}$  by

$$\pi^* \omega_N^n = J_{\pi} \omega_{\tilde{N}}^n, \quad (4.1)$$

which is clearly positive on  $\pi^{-1}(N_0)$ .

Going back to the volume form estimate (2.3), we wish to understand which of the components of  $F$  or  $\tilde{D}$  actually have zeros (i.e.  $\beta_j > 0$ ) or conical singularities (i.e.  $\alpha_i < 1$ ). The first observation is the following:

**Lemma 4.1.** *The function  $\mathcal{F}$  on  $N_0$  defined in (3.1) satisfies*

$$\mathcal{F} \geq C^{-1}. \quad (4.2)$$

*Proof.* This is a consequence of results of Song–Tian [65], as follows. First, as shown in [65, Lemma 3.3], on  $M \setminus S$  we can write

$$f^* \mathcal{F} = \frac{(-1)^{\frac{m^2}{2}} \Omega \wedge \overline{\Omega}}{f^* \omega_N^n \wedge \omega_{\text{SRF}}^{m-n}}, \quad (4.3)$$

where  $\omega_{\text{SRF}}$  is a *semi-Ricci-flat form* on  $M \setminus S$  defined by  $\omega_{\text{SRF}} = \omega_M + \sqrt{-1} \partial \bar{\partial} \rho$ , where  $\omega_M$  is any given Kähler metric on  $M$  with fiber volume  $\int_{M_y} \omega_M^{m-n} = 1$  for all  $y \in N_0$ , and  $\rho$  is the unique smooth function on  $M \setminus S$  such that

$$\omega_{\text{SRF}}|_{M_y} = \omega_M|_{M_y} + \sqrt{-1} \partial \bar{\partial} \rho|_{M_y},$$

is the unique Ricci-flat Kähler metric on  $M_y$  cohomologous to  $\omega_M|_{M_y}$  (for all  $y \in N_0$ ), with the normalization that  $\int_{M_y} \rho|_{M_y} \omega_M|_{M_y}^{m-n} = 0$ , which is obtained by a fiberwise application of Yau’s Theorem [91].

We can then argue in a similar way as the proof of [65, Proposition 3.2]. Explicitly, given any  $y \in N_0$ , on the fiber  $M_y$  the volume forms  $(\omega_M|_{M_y})^{m-n}$  and  $(\omega_{\text{SRF}}|_{M_y})^{m-n}$  have the same total volume, and hence there is a point  $x_y \in M_y$  such that

$$\frac{(\omega_M|_{M_y})^{m-n}}{(\omega_{\text{SRF}}|_{M_y})^{m-n}}(x_y) = 1.$$

Then by (4.3) we have

$$\mathcal{F}(y) = \frac{(-1)^{\frac{m^2}{2}} \Omega \wedge \overline{\Omega}}{f^* \omega_N^n \wedge \omega_{\text{SRF}}^{m-n}}(x_y) = \frac{(-1)^{\frac{m^2}{2}} \Omega \wedge \overline{\Omega}}{f^* \omega_N^n \wedge \omega_M^{m-n}}(x_y) \geq C^{-1},$$

since on  $M$  we have  $f^* \omega_N^n \wedge \omega_M^{m-n} \leq C \omega_M^n$  and  $(-1)^{\frac{m^2}{2}} \Omega \wedge \overline{\Omega} \geq C^{-1} \omega_M^n$ .  $\square$

Pulling back (3.2) via  $\pi$ , we can use Lemma 4.1 to get on  $\pi^{-1}(N_0)$

$$(\pi^* \omega)^n = c(\mathcal{F} \circ \pi) \pi^* \omega_N^n = c(\mathcal{F} \circ \pi) J_\pi \omega_N^n \geq C^{-1} J_\pi \omega_N^n,$$

while by (2.3)

$$(\pi^* \omega)^n \leq C \prod_{j=1}^p |s_j|_{h_j}^{2\beta_j} \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^d \frac{\omega_N^n}{\prod_{i=p+1}^{\mu} |s_i|_{h_i}^{2(1-\alpha_i)}},$$

and so

$$C^{-1} J_\pi \leq \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^d \frac{\prod_{j=1}^p |s_j|_{h_j}^{2\beta_j}}{\prod_{i=p+1}^{\mu} |s_i|_{h_i}^{2(1-\alpha_i)}}. \quad (4.4)$$

Given a component of  $\tilde{D}$ , given by  $\{s_i = 0\}$  for some  $1 \leq i \leq \mu$ , near its generic point we have that  $J_\pi > 0$  so by (4.4) we must have  $i > p$  (i.e. there is no zero for  $(\pi^* \omega)^n$  near this point).

On the other hand, for components of  $F$ ,  $(\pi^* \omega)^n$  may have either a zero or a conical type singularity, but if it has a zero, it cannot vanish more than  $J_\pi$  (up to much smaller logarithmic terms), thanks again to (4.4).

In particular, if  $N$  is smooth and  $D^{(1)}$  has simple normal crossings, then by definition we can take  $\tilde{N} = N$ ,  $\pi = \text{Id}$  and  $E = \tilde{D} = D$ , so this discussion shows that in this case all the  $\alpha_i$  that appear in Theorem 2.1 must be  $\leq 1$ , or equivalently that there are no  $\beta_j$ 's in Theorem 2.3 (i.e.  $p = 0$ ). In other words, this shows that:

**Corollary 4.2.** *Assume that  $N$  is smooth and  $D^{(1)} = \bigcup_{j=1}^{\mu} D_j$  has simple normal crossings. Then there are constants  $C, d > 0$  and rational numbers  $0 < \alpha_j \leq 1, 1 \leq j \leq \mu$ , such that on  $N_0$  we have*

$$C^{-1} \omega_{\text{cone}}^n \leq \omega^n \leq C \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^d \omega_{\text{cone}}^n, \quad (4.5)$$

where  $\omega_{\text{cone}}$  is a conical metric with cone angle  $2\pi\alpha_j$  along the component  $D_j, 1 \leq j \leq \mu$ .

We can now give the proof of (1.3).

*Proof of (1.3).* On  $N$  we have the continuous function  $\varphi$  (normalized with  $\sup_N \varphi = 0$ ), which satisfies

$$\omega = \omega_N + \sqrt{-1} \partial \bar{\partial} \varphi,$$

and solves the Monge–Ampère equation

$$\omega^n = (\omega_N + \sqrt{-1} \partial \bar{\partial} \varphi)^n = c(-1)^{\frac{n^2}{2}} f_*(\Omega \wedge \bar{\Omega}) = \frac{\psi}{\prod_{j=1}^{\mu} |s_j|_{h_j}^{2(1-\alpha_j)}} \omega_N^n, \quad (4.6)$$

on  $N_0$  (and also globally on  $N$  in the weak sense), where the smooth function  $\psi$  on  $N_0$  is defined by this last equality. In particular,

$$\begin{aligned} \sqrt{-1} \partial \bar{\partial} \log(1/\psi) &= \text{Ric}(\omega) - \text{Ric}(\omega_N) - \sum_{j=1}^{\mu} (1 - \alpha_j) \sqrt{-1} \partial \bar{\partial} \log |s_j|_{h_j}^2 \\ &\geq -\text{Ric}(\omega_N) + \sum_{j=1}^{\mu} (1 - \alpha_j) R_{h_j} \\ &\geq -C \omega_N, \end{aligned} \quad (4.7)$$

holds on  $N_0$ . Thanks to Corollary 4.2, on  $N_0$  we have that

$$C^{-1} \leq \psi \leq C \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^d. \quad (4.8)$$

As shown in (4.7) function  $\log(1/\psi)$  is quasi-psh, and thanks to (4.8) it is bounded above near  $N \setminus N_0$ , and so it extends uniquely to a quasi-psh function on  $N$  (still denoted by  $\log(1/\psi)$ ) which satisfies the same inequality (4.7) by a classical result of Grauert–Remmert [24].

Thanks to Demailly’s regularization theorem [14], we can find smooth functions  $u_j$  on  $N$  which decrease pointwise to  $\log(1/\psi)$  as  $j \rightarrow \infty$  and which satisfy  $\sqrt{-1} \partial \bar{\partial} u_j \geq -C \omega_N$  and  $u_j \leq C$  on  $N$ .

We now employ a trick from Datar–Song [13]. For each  $i = 1, \dots, \mu$ , let  $\omega_{\text{cone},i}$  be a conical Kähler metric with cone angle  $2\pi\alpha_i$  along the component  $D_i$ , and smooth everywhere else. For example, we can take

$$\omega_{\text{cone},i} = \omega_N + \delta \sqrt{-1} \partial \bar{\partial} |s_i|_{h_i}^{2\alpha_i},$$

and let also  $\eta_i = \delta |s_i|_{h_i}^{2\alpha_i}$ . Then for all  $j \geq 1$  let

$$G_{j,i} = \prod_{k \neq i} \left( |s_k|_{h_k}^2 + \frac{1}{j} \right)^{1-\alpha_k},$$

which are smooth functions on  $N$  which satisfy

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\log G_{j,i} &= \sum_{k\neq i}(1-\alpha_k)\sqrt{-1}\partial\bar{\partial}\log\left(|s_k|_{h_k}^2+\frac{1}{j}\right) \\ &\geq \sum_{k\neq i}(1-\alpha_k)\frac{|s_k|_{h_k}^2}{|s_k|_{h_k}^2+\frac{1}{j}}\sqrt{-1}\partial\bar{\partial}\log|s_k|_{h_k}^2 \\ &\geq -C\omega_N,\end{aligned}$$

everywhere on  $N$ , for a constant independent of  $j$ .

We can now consider the complex Monge–Ampère equation

$$(\omega_N + \sqrt{-1}\partial\bar{\partial}\varphi_{j,i})^n = c_{j,i}e^{-u_j}\frac{\omega_N^n}{G_{j,i}|s_i|_{h_i}^{2(1-\alpha_i)}}, \quad (4.9)$$

where  $c_{j,i}$  are constants that are obtained by integrating the equation, so  $c_{j,i} \rightarrow 1$  as  $j \rightarrow \infty$ , the functions  $\varphi_{j,i}$  are normalized by  $\sup_N \varphi_{j,i} = 0$ , and  $\omega_{j,i} := \omega_N + \sqrt{-1}\partial\bar{\partial}\varphi_{j,i}$  are now conical Kähler metrics with cone angle  $2\pi\alpha_i$  along  $D_i$  (only), and smooth away from  $D_i$ . This last fact follows from the main result of [37] or [30] (see also [5, 6] for earlier weaker results and [1] for more recent work). It is quickly verified that the ratio  $\omega_{j,i}^n/\omega_N^n$  has  $L^p$  norm uniformly bounded independent of  $j$ , for some  $p > 1$ , and so by Kołodziej [42] we have

$$\sup_N |\varphi_{j,i}| \leq C,$$

and since the right hand side of (4.9) converges to the right hand side of (4.6) (say in  $L^1(N)$ ), Kołodziej’s stability theorem [43] gives us that  $\varphi_{j,i} \rightarrow \varphi$  in  $C^0(N)$  as  $j \rightarrow \infty$ .

Our goal is to prove that

$$\omega_{j,i} \geq C^{-1}\omega_{\text{cone},i}, \quad (4.10)$$

holds on  $N_0$ , for a constant  $C$  independent of  $j$ . If this is proved, then it follows from this and (4.9) (by standard arguments) that on any compact subset  $K \subset N_0$  we have uniform  $C^k$  bounds for  $\omega_{j,i}$  independent of  $j$  (but depending on  $K$ ), and since  $\varphi_{j,i} \rightarrow \varphi$  uniformly, we conclude that  $\omega_{j,i} \rightarrow \omega$  in  $C_{\text{loc}}^\infty(N_0)$  as  $j \rightarrow \infty$  (for all  $i$ ). Therefore

$$\omega \geq C^{-1}\omega_{\text{cone},i},$$

holds on  $N_0$ , and summing these up for  $i = 1, \dots, \mu$  we obtain

$$\omega \geq C^{-1}\omega_{\text{cone}},$$

on  $N_0$ , which is half of (1.2). To prove the other half of (1.2) it suffices to use (4.6) and (4.8) to obtain

$$\text{tr}_{\omega_{\text{cone}}}\omega \leq (\text{tr}_{\omega}\omega_{\text{cone}})^{n-1}\frac{\omega^n}{\omega_{\text{cone}}^n} \leq C\frac{\omega^n}{\omega_{\text{cone}}^n} \leq C\psi \leq C\left(1 - \sum_{i=1}^{\mu}\log|s_i|_{h_i}\right)^d,$$

on  $N_0$ , which thus proves (1.2).

So it remains to prove (4.10). The argument is similar to the one in [13]. First of all, by a calculation in the appendix of [37] (or the simpler proof in [61, Lemma 3.14] due to J. Sturm), the bisectional curvature of  $\omega_{\text{cone},i}$  on  $N \setminus D_i$  has a uniform upper bound. Second of all, the Ricci curvature of  $\omega_{j,i}$  on  $N \setminus D_i$  satisfies

$$\begin{aligned} \text{Ric}(\omega_{j,i}) &= \sqrt{-1} \partial \bar{\partial} u_j + \text{Ric}(\omega_N) + \sqrt{-1} \partial \bar{\partial} \log G_{j,i} \\ &\quad + (1 - \alpha_i) \sqrt{-1} \partial \bar{\partial} \log |s_i|_{h_i}^2 \geq -C \omega_N, \end{aligned}$$

independent of  $j$ . Combining these two into Yau's Schwarz Lemma calculation [92] gives us

$$\Delta_{\omega_{j,i}} \log \text{tr}_{\omega_{j,i}} \omega_{\text{cone},i} \geq -C \text{tr}_{\omega_{j,i}} \omega_{\text{cone},i} - C,$$

on  $N \setminus D_i$ . To ensure that the maximum is achieved away from  $D_i$ , we compute

$$\Delta_{\omega_{j,i}} \log(|s_i|_{h_i}^{2\gamma} \text{tr}_{\omega_{j,i}} \omega_{\text{cone},i}) \geq -C \text{tr}_{\omega_{j,i}} \omega_{\text{cone},i} - C,$$

where we used that  $\omega_N \leq C \omega_{\text{cone},i}$  on  $N \setminus D_i$ , and the constant  $C$  does not depend on  $\gamma > 0$  small. On the other hand

$$\Delta_{\omega_{j,i}} (\varphi_{j,i} - \eta_i) = n - \text{tr}_{\omega_{j,i}} \omega_{\text{cone},i},$$

so we get the differential inequality

$$\Delta_{\omega_{j,i}} \log(|s_i|_{h_i}^{2\gamma} \text{tr}_{\omega_{j,i}} \omega_{\text{cone},i} - A(\varphi_{j,i} - \eta_i)) \geq \text{tr}_{\omega_{j,i}} \omega_{\text{cone},i} - C,$$

for  $A$  large and uniform (independent of  $j$ ). The maximum principle can then be applied because  $\omega_{j,i}$  and  $\omega_{\text{cone},i}$  are conical metrics with the same cone angles along  $D_i$  hence the function  $\text{tr}_{\omega_{j,i}} \omega_{\text{cone},i}$  is uniformly bounded above on  $N \setminus D_i$ , so the maximum is achieved away from  $D_i$ . Using the uniform  $C^0$  bound for  $\varphi_{j,i}$ , we conclude that

$$\text{tr}_{\omega_{j,i}} \omega_{\text{cone},i} \leq \frac{C}{|s_i|_{h_i}^{2\gamma}},$$

holds on  $N \setminus D_i$  with constant independent of  $\gamma$  and  $j$ , so we can let  $\gamma \rightarrow 0$  and get (4.10).  $\square$

## 5. The Main Theorem

In this section we prove the main Theorem 1.1, in the Calabi–Yau setting.

Let  $\pi : \tilde{N} \rightarrow N$  be the composition of blowups constructed in Sect. 2, with  $\tilde{N}$  smooth,  $\pi^{-1}(D) = E$  a simple normal crossings divisor, and  $\pi$  an isomorphism on the locus where  $D^{(1)}$  is snc. We write  $E = \cup_{i=1}^{\mu} E_i$ , and we have fixed defining sections  $s_i$  and metrics  $h_i$  for  $\mathcal{O}(E_i)$ .

If we denote by  $F = \cup_i F_i \subset E$  the exceptional components of  $E$ , then it is well-known (see e.g. [58, Lemma 7]) that for all  $0 < \delta \ll 1$  the form

$$\pi^* \omega_N - \delta \sum_i R_{F_i}, \tag{5.1}$$

is a Kähler metric on  $\tilde{N}$ , where  $R_{F_i}$  is the curvature of a certain Hermitian metric on  $\mathcal{O}(F_i)$ .

Next, from Sect. 2 we obtain a conical structure  $\{E_i, 2\pi\alpha_i\}$  on  $\tilde{N}$ , and we fix a conical Kähler metric  $\omega_{\text{cone}}$  with these cone angles around the  $E_i$ 's. We will choose it of the form  $\omega_{\text{cone}} = \omega_{\tilde{N}} + \sqrt{-1}\partial\bar{\partial}\eta$ , where

$$\eta = C^{-1} \sum_i |s_i|_{h_i}^{2\alpha_i},$$

for some  $C > 0$  sufficiently large. Note that of course we have  $\omega_{\tilde{N}} \leq C\omega_{\text{cone}}$ . We also obtain the order of zeros  $\beta_j$  (along some components of  $F$ , different from the components that have conical singularities), and we define

$$H = \prod_j |s_j|_{h_j}^{2\beta_j},$$

where the product is only over those components  $F_j$  of  $F$  which have a nontrivial order of zero  $\beta_j > 0$ . On  $\tilde{N} \setminus F$  we have

$$\sqrt{-1}\partial\bar{\partial} \log H = - \sum_j \beta_j R_{F_j}. \quad (5.2)$$

We can then define a smooth function  $\psi$  on  $\tilde{N} \setminus E$  by

$$\psi = \frac{\pi^* \omega^n \prod_i |s_i|^{2(1-\alpha_i)}}{H \omega_{\tilde{N}}^n}, \quad (5.3)$$

which satisfies

$$C^{-1} \leq \psi \leq C \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^d, \quad (5.4)$$

by Theorem 2.3. We thus have the Monge–Ampère equation

$$(\pi^* \omega_N + \sqrt{-1}\partial\bar{\partial}\pi^* \varphi)^n = \psi H \frac{\omega_{\tilde{N}}^n}{\prod_i |s_i|^{2(1-\alpha_i)}}. \quad (5.5)$$

The first step is to regularize equation (5.5) as follows. On  $\tilde{N} \setminus E$  we have

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial} \log(1/\psi) &= \text{Ric}(\pi^* \omega) - \text{Ric}(\omega_{\tilde{N}}) + \sum_i (1 - \alpha_i) R_i + \sqrt{-1}\partial\bar{\partial} \log H \\ &\geq -\text{Ric}(\omega_{\tilde{N}}) + \sum_i (1 - \alpha_i) R_i - \sum_j \beta_j R_{F_j}, \end{aligned} \quad (5.6)$$

and note that the RHS is a smooth form on all of  $\tilde{N}$ . Also  $\log(1/\psi)$  is bounded above near  $E$  by (5.4), so by Grauert–Remmert [24] it extends to a global quasi-psh function on  $\tilde{N}$  which satisfies (5.6) in the weak sense. Thanks to (5.4), the extension has vanishing

Lelong numbers, so we can approximate it using Demailly's regularization theorem [14] by a decreasing sequence of smooth functions  $u_j$  which satisfy

$$\sqrt{-1}\partial\bar{\partial}u_j \geq -\text{Ric}(\omega_{\tilde{N}}) + \sum_i (1 - \alpha_i)R_i - \sum_j \beta_j R_{F_j} - \frac{1}{j}\omega_{\tilde{N}}, \quad (5.7)$$

on all of  $\tilde{N}$ . We also have that  $\sqrt{-1}\partial\bar{\partial}\log H \geq -C\omega_{\tilde{N}}$  weakly on  $\tilde{N}$  and so  $\log H$  can also be regularized by smooth functions  $v_k$  with  $\sqrt{-1}\partial\bar{\partial}v_k \geq -C\omega_{\tilde{N}}$  and  $v_k \leq C$  on  $\tilde{N}$ .

Let  $\omega_{j,k} = \pi^*\omega_N + \frac{1}{j}\omega_{\tilde{N}} + \sqrt{-1}\partial\bar{\partial}\varphi_{j,k}$  be a solution on  $\tilde{N}$  of

$$\omega_{j,k}^n = c_{j,k}e^{v_k - u_j} \frac{\omega_{\tilde{N}}^n}{\prod_i |s_i|^{2(1-\alpha_i)}}, \quad (5.8)$$

with  $\omega_{j,k}$  a cone metric with the same cone angles as  $\omega_{\text{cone}}$ . This exists thanks to the main result of [37] or [30] (see also [5, 6] for earlier weaker results and [1] for more recent work). In particular,  $\omega_{j,k}$  is smooth on  $\tilde{N} \setminus E$ .

It is quickly verified that the ratio  $\omega_{j,k}^n/\omega_{\tilde{N}}^n$  has  $L^p$  norm uniformly bounded independent of  $j$ , for some  $p > 1$ , and so by [17] we have

$$\sup_{\tilde{N}} |\varphi_{j,k}| \leq C,$$

and since  $e^{v_k - u_j} \rightarrow H\psi$ , we conclude using the stability theorem in [17] that as  $j, k \rightarrow \infty$  we have  $c_{j,k} \rightarrow 1$  and  $\varphi_{j,k} \rightarrow \pi^*\varphi$  in  $C^0(\tilde{N})$ , where  $\pi^*\varphi$  solves (5.5).

We also introduce a partial regularization, which we denote by  $\omega_j = \pi^*\omega_N + \frac{1}{j}\omega_{\tilde{N}} + \sqrt{-1}\partial\bar{\partial}\varphi_j$ , which are Kähler metrics on  $\tilde{N} \setminus E$  which solve

$$\omega_j^n = c_j H e^{-u_j} \frac{\omega_{\tilde{N}}^n}{\prod_i |s_i|^{2(1-\alpha_i)}}. \quad (5.9)$$

Again we have  $c_j \rightarrow 1$ , and  $c_{j,k} \rightarrow c_j$  for each  $j$  fixed as  $k \rightarrow \infty$ . The solution  $\varphi_j \in C^0(\tilde{N})$  exists by [11, 17, 65, 78], and are smooth away from  $E$  by Yau [91]. Again we have that  $\varphi_{j,k} \rightarrow \varphi_j$  in  $C^0(\tilde{N})$  for each  $j$  fixed as  $k \rightarrow \infty$ .

**Proposition 5.1.** *For each  $j$  there is a constant  $C_j$  which depends on  $j$  such that on  $\tilde{N} \setminus E$  we have*

$$\text{tr}_{\omega_{\text{cone}}} \omega_{j,k} \leq C_j, \quad (5.10)$$

for all  $k$ .

This, together with (5.8) and standard arguments, implies that  $\omega_{j,k} \rightarrow \omega_j$  locally smoothly away from  $E$ , and so on  $\tilde{N} \setminus E$  we also have

$$\text{tr}_{\omega_{\text{cone}}} \omega_j \leq C_j. \quad (5.11)$$



*Proof.* This follows from the estimates of Guenancia-Păun [30]. Specifically, as in [30, proof of Theorem A], we make one further approximation of (5.8) by a smooth PDE

$$\omega_{j,k,\varepsilon}^n = c_{j,k,\varepsilon} e^{v_k - u_j} \frac{\omega_{\tilde{N}}^n}{\prod_i (|s_i|^2 + \varepsilon^2)^{(1-\alpha_i)}}, \quad (5.12)$$

where  $\omega_{j,k,\varepsilon} = \pi^* \omega_N + \frac{1}{j} \omega_{\tilde{N}} + \sqrt{-1} \partial \bar{\partial} \varphi_{j,k,\varepsilon}$  is now a smooth Kähler metric on  $\tilde{N}$ . Again we have  $\varphi_{j,k,\varepsilon} \rightarrow \varphi_{j,k}$  uniformly as  $\varepsilon \rightarrow 0$ , and the claim is then that

$$\mathrm{tr}_{\omega_\varepsilon} \omega_{j,k,\varepsilon} \leq C_j, \quad (5.13)$$

where  $C_j$  does not depend on  $k$  or  $\varepsilon$ , and  $\omega_\varepsilon = \pi^* \omega_N + \frac{1}{j} \omega_{\tilde{N}} + \sqrt{-1} \partial \bar{\partial} \chi_{j,\varepsilon}$  is a family of Kähler metrics (which also depend on  $j$ ) that as  $\varepsilon \rightarrow 0$  approximate a ( $j$ -dependent) conical metric, as constructed in [30, Section 3]. This, together with (5.12) and standard arguments, implies that  $\omega_{j,k,\varepsilon} \rightarrow \omega_{j,k}$  locally smoothly away from  $E$  as  $\varepsilon \rightarrow 0$ , and then (5.10) follows because we allow all the constants to depend on  $j$ .

The proof of (5.13) follows from the arguments for their Laplacian estimate in [30, Proposition 1], which however cannot be applied directly because they make one extra hypothesis which for us is not satisfied. The key points for us, which make the estimate independent of  $k$ , is that  $v_k \leq C$ , and  $\sqrt{-1} \partial \bar{\partial} v_k \geq -C \omega_{\tilde{N}}$ .

Here are the details: using the notation of [30], the functions  $\tilde{\varphi} = \varphi_{j,k,\varepsilon} - \chi_{j,\varepsilon}$ , are uniformly bounded (independent of  $j, k, \varepsilon$ ) and satisfy  $\omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi} = \omega_{j,k,\varepsilon}$ . We therefore rewrite (5.12) as

$$(\omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi})^n = e^{\psi^+} \omega_\varepsilon^n,$$

where

$$e^{\psi^+} = c_{j,k,\varepsilon} e^{v_k - u_j} \frac{\omega_{\tilde{N}}^n}{\prod_i (|s_i|^2 + \varepsilon^2)^{(1-\alpha_i)} \omega_\varepsilon^n} = c_{j,k,\varepsilon} e^{v_k - u_j} e^{F_\varepsilon},$$

where  $F_\varepsilon$  is exactly defined as in [30]. Defining  $\Psi_\varepsilon$  as in [30, Section 4], they show that  $\sqrt{-1} \partial \bar{\partial} F_\varepsilon \geq -(C_j \omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \Psi_\varepsilon)$ , and so we get that

$$\begin{aligned} \sqrt{-1} \partial \bar{\partial} \psi^+ &\geq -(C_j \omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \Psi_\varepsilon) + \sqrt{-1} \partial \bar{\partial} (v_k - u_j) \\ &\geq -(C_j \omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \Psi_\varepsilon) - C_j \omega_{\tilde{N}} \\ &\geq -(C_j \omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \Psi_\varepsilon), \end{aligned}$$

using that  $\omega_\varepsilon \geq C_j^{-1} \omega_{\tilde{N}}$ . This is half of inequality (iii) in [30, Proposition 1]. The second half requires that  $|\psi^+| \leq C_j$ , but this is not true since  $v_k$  decreases to  $\log H$  which is  $-\infty$  along some components of  $F$ , so we only have that  $\psi^+ \leq C_j$ . However, this is enough for their arguments, as the lower bound for  $\psi^+$  is not used. Therefore, the main calculation of [30, Proposition 1] gives

$$\Delta_{\omega_{j,k,\varepsilon}} (\log \mathrm{tr}_{\omega_\varepsilon} \omega_{j,k,\varepsilon} - A_j \tilde{\varphi} + 3\Psi_\varepsilon) \geq \mathrm{tr}_{\omega_{j,k,\varepsilon}} \omega_\varepsilon - C_j,$$

and at a maximum point we get  $\mathrm{tr}_{\omega_{j,k,\varepsilon}} \omega_\varepsilon \leq C_j$  and so

$$\mathrm{tr}_{\omega_\varepsilon} \omega_{j,k,\varepsilon} \leq C_j \frac{\omega_{j,k,\varepsilon}^n}{\omega_\varepsilon^n} = C_j e^{\psi^+} \leq C_j,$$

as required.  $\square$

Thanks to Proposition 5.1, we can apply the maximum principle to quantities involving  $\text{tr}_{\omega_{\text{cone}}} \omega_j$ , provided we add a term which goes to  $-\infty$  along  $E$  (arbitrarily slowly). Using this, we first prove the main result, which uses Tsuji's trick [87], and from which we will quickly derive Theorem 1.1:

**Theorem 5.2.** *On  $\tilde{N} \setminus E$  we have*

$$\text{tr}_{\omega_{\text{cone}}} \pi^* \omega \leq \frac{C\psi}{|s_F|^{2A}}, \quad (5.14)$$

where  $F$  is the union of the  $\pi$ -exceptional divisors, and  $|s_F|^2$  is a shorthand for  $\prod_i |s_{F_i}|_{h_i}^2$ .

*Proof.* It is enough to show that on  $\tilde{N} \setminus E$  we have

$$\text{tr}_{\omega_{\text{cone}}} \omega_j \leq \frac{C e^{-u_j}}{|s_F|^{2A}}, \quad (5.15)$$

with  $C$  now independent of  $j$ , since again this together with (5.9) and standard arguments implies that  $\omega_j \rightarrow \pi^* \omega$  locally smoothly away from  $E$ .

Following [30] we define  $\Psi = C \sum_i |s_i|_{h_i}^{2\rho}$ , for some small  $\rho > 0$  and large  $C > 0$ , which can be chosen so that on  $\tilde{N} \setminus E$  the curvature of  $\omega_{\text{cone}}$  satisfies

$$\text{Rm}(\omega_{\text{cone}}) \geq -(C\omega_{\text{cone}} + \sqrt{-1} \partial \bar{\partial} \Psi) \otimes \text{Id}, \quad (5.16)$$

see [30, (4.3)].

To prove (5.15) we shall apply the maximum principle to the quantity

$$Q = \log \text{tr}_{\omega_{\text{cone}}} \omega_j + n\Psi + u_j - A^2 \varphi_j + Ab\eta + \varepsilon \log |s_E|^2 + A \log |s_F|^2,$$

where  $A$  large,  $b > 0$  is small and  $0 < \varepsilon \leq \frac{1}{j}$ . The terms  $n\Psi + u_j - A^2 \varphi_j + Ab\eta$  are all bounded on  $\tilde{N}$  (with bounds independent of  $j$  except for  $u_j$ ), while the term  $\log \text{tr}_{\omega_{\text{cone}}} \omega_j$  is bounded above on  $\tilde{N} \setminus E$  (depending on  $j$ ) by Proposition 5.1. Since the term  $\varepsilon \log |s_E|^2$  goes to  $-\infty$  on  $E$ , the quantity  $Q$  achieves a global maximum on  $\tilde{N} \setminus E$ . All the following computations are at an arbitrary point of  $\tilde{N} \setminus E$ .

The first claim is that on  $\tilde{N} \setminus E$  we have

$$\Delta_{\omega_j} (\log \text{tr}_{\omega_{\text{cone}}} \omega_j + n\Psi) \geq -C \text{tr}_{\omega_j} \omega_{\text{cone}} - \frac{\text{tr}_{\omega_{\text{cone}}} \text{Ric}(\omega_j)}{\text{tr}_{\omega_{\text{cone}}} \omega_j}. \quad (5.17)$$

Indeed by the Aubin–Yau's second order calculation we have

$$\begin{aligned} & \Delta_{\omega_j} (\log \text{tr}_{\omega_{\text{cone}}} \omega_j + n\Psi) \\ & \geq \frac{1}{\text{tr}_{\omega_{\text{cone}}} \omega_j} \left( -\text{tr}_{\omega_{\text{cone}}} \text{Ric}(\omega_j) + \text{Rm}(\omega_{\text{cone}})_{p\bar{q}}^{k\bar{\ell}} g_j^{p\bar{q}} g_{j,k\bar{\ell}} \right) + n \Delta_{\omega_j} \Psi, \end{aligned}$$

and as in [30] we choose coordinates so that at our given point  $\omega_{\text{cone}}$  is the identity while  $\omega_j$  is diagonal with eigenvalues  $\lambda_k > 0$ , so we can write

$$\begin{aligned} \text{Rm}(\omega_{\text{cone}})_{p\bar{q}}^{k\bar{\ell}} g_j^{p\bar{q}} g_{j,k\bar{\ell}} &= \sum_{i,k} \frac{\lambda_i}{\lambda_k} R_{i\bar{i}k\bar{k}} \\ &\geq - \sum_{i,k} \frac{\lambda_i}{\lambda_k} (C + \Psi_{k\bar{k}}), \end{aligned}$$

using (5.16), while we also have  $\sqrt{-1}\partial\bar{\partial}\Psi \geq -C\omega_{\text{cone}}$  by [30, (4.2)], so

$$\begin{aligned}\Delta_{\omega_j}\Psi &= \sum_k \frac{1}{\lambda_k} (C + \Psi_{k\bar{k}}) - C\text{tr}_{\omega_j}\omega_{\text{cone}} \\ &\geq \frac{1}{n\text{tr}_{\omega_{\text{cone}}}\omega_j} \sum_{i,k} \frac{\lambda_i}{\lambda_k} (C + \Psi_{k\bar{k}}) - C\text{tr}_{\omega_j}\omega_{\text{cone}},\end{aligned}$$

and (5.17) follows.

The second claim is that on  $\tilde{N} \setminus E$  we have

$$\Delta_{\omega_j}(\log \text{tr}_{\omega_{\text{cone}}}\omega_j + n\Psi + u_j) \geq -C\text{tr}_{\omega_j}\omega_{\text{cone}}. \quad (5.18)$$

For this, recall that from (5.9), on  $\tilde{N} \setminus E$  we have

$$\begin{aligned}\text{Ric}(\omega_j) &= \sqrt{-1}\partial\bar{\partial}(u_j - \log H) + \text{Ric}(\omega_{\tilde{N}}) - \sum_i (1 - \alpha_i)R_i \\ &\geq -\frac{1}{j}\omega_{\tilde{N}} \geq -\frac{C}{j}\omega_{\text{cone}},\end{aligned}$$

using (5.2) and (5.7), and also

$$\begin{aligned}\Delta_{\omega_j}u_j &= \text{tr}_{\omega_j}\text{Ric}(\omega_j) + \text{tr}_{\omega_j}\left(-\sum_j \beta_j R_{F_j}\right) - \text{tr}_{\omega_j}\text{Ric}(\omega_{\tilde{N}}) + \text{tr}_{\omega_j}\left(\sum_i (1 - \alpha_i)R_i\right) \\ &\geq \text{tr}_{\omega_j}\text{Ric}(\omega_j) - C\text{tr}_{\omega_j}\omega_{\text{cone}}\end{aligned}$$

so

$$\Delta_{\omega_j}(\log \text{tr}_{\omega_{\text{cone}}}\omega_j + n\Psi + u_j) \geq -C\text{tr}_{\omega_j}\omega_{\text{cone}} - \frac{\text{tr}_{\omega_{\text{cone}}}\text{Ric}(\omega_j)}{\text{tr}_{\omega_{\text{cone}}}\omega_j} + \text{tr}_{\omega_j}\text{Ric}(\omega_j) \quad (5.19)$$

The subclaim is now that

$$-\frac{\text{tr}_{\omega_{\text{cone}}}\text{Ric}(\omega_j)}{\text{tr}_{\omega_{\text{cone}}}\omega_j} + \text{tr}_{\omega_j}\text{Ric}(\omega_j) \geq -\frac{C}{j}\text{tr}_{\omega_j}\omega_{\text{cone}}. \quad (5.20)$$

Indeed,

$$\begin{aligned}-\frac{\text{tr}_{\omega_{\text{cone}}}\text{Ric}(\omega_j)}{\text{tr}_{\omega_{\text{cone}}}\omega_j} + \text{tr}_{\omega_j}\text{Ric}(\omega_j) &= -\frac{\text{tr}_{\omega_{\text{cone}}}(\text{Ric}(\omega_j) + \frac{C}{j}\omega_{\text{cone}})}{\text{tr}_{\omega_{\text{cone}}}\omega_j} \\ &\quad + \text{tr}_{\omega_j}\left(\text{Ric}(\omega_j) + \frac{C}{j}\omega_{\text{cone}}\right) + \frac{Cn}{j\text{tr}_{\omega_{\text{cone}}}\omega_j} - \frac{C}{j}\text{tr}_{\omega_j}\omega_{\text{cone}} \\ &\geq -\frac{\text{tr}_{\omega_{\text{cone}}}(\text{Ric}(\omega_j) + \frac{C}{j}\omega_{\text{cone}})}{\text{tr}_{\omega_{\text{cone}}}\omega_j} + \text{tr}_{\omega_j}\left(\text{Ric}(\omega_j) + \frac{C}{j}\omega_{\text{cone}}\right) - \frac{C}{j}\text{tr}_{\omega_j}\omega_{\text{cone}},\end{aligned}$$

and if at any given point we choose coordinates so that  $\omega_{\text{cone}}$  is the identity and  $\omega_j$  is diagonal with eigenvalues  $\lambda_k > 0$ , while  $(\text{Ric}(\omega_j) + \frac{C}{j}\omega_{\text{cone}}) \geq 0$  has entries  $A_{i\bar{j}}$ , then we note that

$$-\frac{1}{\sum_k \lambda_k} \sum_{\ell} A_{\ell\bar{\ell}} + \sum_{\ell} \lambda_{\ell}^{-1} A_{\ell\bar{\ell}} \geq 0,$$

which proves (5.20). Combining (5.20) with (5.19) we obtain (5.18).

Continuing our calculation, and setting for simplicity  $R_F = \sum_i R_{F_i}$ , we have

$$\begin{aligned} \Delta_{\omega_j} Q &\geq -C \operatorname{tr}_{\omega_j} \omega_{\text{cone}} + A^2 \operatorname{tr}_{\omega_j} \pi^* \omega_N + A b \operatorname{tr}_{\omega_j} \sqrt{-1} \partial \bar{\partial} \eta - A^2 n \\ &\quad - \varepsilon \operatorname{tr}_{\omega_j} R_E - A \operatorname{tr}_{\omega_j} R_F \\ &\geq -C \operatorname{tr}_{\omega_j} \omega_{\text{cone}} + A \operatorname{tr}_{\omega_j} \left( A_0 \pi^* \omega_N - R_F + b \sqrt{-1} \partial \bar{\partial} \eta \right) - A^2 n, \end{aligned}$$

for all  $A_0 \leq A$ . We choose first  $A_0$  such that  $A_0 \pi^* \omega_N - R_F$  is a Kähler metric  $\hat{\omega}_{\tilde{N}}$  (cf. (5.1)), and then we choose  $b$  sufficiently small so that  $\hat{\omega}_{\tilde{N}} + b \sqrt{-1} \partial \bar{\partial} \eta = \hat{\omega}_{\text{cone}}$  is a conical Kähler metric with same cone angles as  $\omega_{\text{cone}}$ . This implies that  $\hat{\omega}_{\text{cone}} \geq c \omega_{\text{cone}}$  for some  $c > 0$ , and finally we can choose  $A \geq A_0$  large so that

$$\Delta_{\omega_j} Q \geq \operatorname{tr}_{\omega_j} \omega_{\text{cone}} - C.$$

Therefore at a maximum of  $Q$  (which is not on  $E$ ) we have

$$\operatorname{tr}_{\omega_j} \omega_{\text{cone}} \leq C,$$

and so using (5.9) we get

$$\operatorname{tr}_{\omega_{\text{cone}}} \omega_j \leq (\operatorname{tr}_{\omega_j} \omega_{\text{cone}})^{n-1} \frac{\omega_j^n}{\omega_{\text{cone}}^n} \leq C H e^{-u_j},$$

hence

$$\log \operatorname{tr}_{\omega_{\text{cone}}} \omega_j + u_j \leq C \log H \leq C,$$

and so also  $Q \leq C$ , hence this last one holds everywhere on  $\tilde{N} \setminus E$ . The constants do not depend on  $\varepsilon$ , so we can let  $\varepsilon \rightarrow 0$  and this gives

$$\operatorname{tr}_{\omega_{\text{cone}}} \omega_j \leq \frac{C e^{-u_j}}{|s_F|^{2A}},$$

which is (5.15).  $\square$

*Proof of Theorem 1.1.* We have already proved in Proposition 3.1 that  $\omega$  extends to a smooth Kähler metric across  $D^{(2)}$  (on  $N^{\text{reg}}$ ). First we work on the blown-up manifold  $\tilde{N}$  as above. Theorem 5.2 gives us the estimate (5.14), which combined with (5.3) gives on  $\pi^{-1}(N_0)$

$$\begin{aligned} \operatorname{tr}_{\pi^* \omega} \omega_{\text{cone}} &\leq (\operatorname{tr}_{\omega_{\text{cone}}} \pi^* \omega)^{n-1} \frac{\omega_{\text{cone}}^n}{\pi^* \omega^n} \leq C \frac{\psi^{n-1}}{|s_F|^{2A'}} \frac{1}{H \psi} \\ &\leq \frac{C}{H |s_F|^{2A'}} \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^{\max(d(n-2), 0)} \end{aligned}$$

and so together with (5.14) we obtain on  $\pi^{-1}(N_0)$

$$\begin{aligned} C^{-1} H |s_F|^{2A'} \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^{-\max(d(n-2), 0)} \omega_{\text{cone}} \\ \leq \pi^* \omega \leq \frac{C}{|s_F|^{2A}} \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^d \omega_{\text{cone}}. \end{aligned} \tag{5.21}$$

Now, if  $x \in \tilde{N}$  is any point near which  $\pi$  is an isomorphism (which by construction includes the preimages of all points where  $D^{(1)}$  is snc) then near  $x$  we have  $J_\pi > 0$  (where  $J_\pi$  is defined in (4.1)) and  $|s_F| > 0$ , and from (4.4) we see that we also have  $H > 0$ , and so (1.2) follows directly from (5.21).  $\square$

## 6. Gromov–Hausdorff Collapsed Limits of Ricci-flat Metrics

**6.1. Proof of Corollary 1.3.** In this section we show how Corollary 1.3 follows from the estimate (1.3). The argument is essentially contained in the previous work of the second and third-named authors [86, Theorem 2.1 and §3.4], based on our work [28], where we proved the following:

**Theorem 6.1.** *Suppose that there is a constant  $C > 0$ , rational numbers  $0 < \alpha_i \leq 1$ ,  $1 \leq i \leq \mu$ , and  $A > 0$  such that on  $\pi^{-1}(N_0)$  we have*

$$\pi^* \omega \leq C \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^A \omega_{\text{cone}}, \quad (6.1)$$

where  $\omega_{\text{cone}}$  is a conical metric with cone angle  $2\pi\alpha_i$  along each component  $E_i$ . Then parts (a) and (b) of Conjecture 1.2 hold. If furthermore we have that  $N$  is smooth and  $D^{(1)}$  is snc (so that  $\pi = \text{Id}$ ), then part (c) of Conjecture 1.2 also holds.

*Proof.* The first statement is proved in [86, Theorem 2.1], assuming that  $\omega_{\text{cone}}$  is an orbifold Kähler metric (i.e.  $\alpha_i$  of the form  $\frac{1}{N_i}$  for some  $N_i \in \mathbb{N}$ , for all  $i$ ) and that  $N$  is smooth. The smoothness of  $N$  was actually never used in the proof there. As for the orbifold metric, since given any rational number  $0 < \alpha \leq 1$  we can find an integer  $p$  such that  $1 - \alpha < 1 - \frac{1}{p} < 1$ , we can bound  $\omega_{\text{cone}} \leq \omega_{\text{orb}}$  (where  $\omega_{\text{orb}}$  has orbifold order  $p$  along all  $E_i$ ), so parts (a) and (b) of Conjecture 1.2 follow immediately from [86, Theorem 2.1].

Next, assume that  $N$  is smooth and  $D^{(1)}$  is an snc divisor in  $N$ , and let  $(X, d_X)$  the metric completion of  $(N_0, \omega)$ , which we want to show is homeomorphic to  $N$ . The argument here is contained in [86, §3.4], and we briefly recall it. Thanks to (6.1) and the arguments in [86, §3.4], we know that there are continuous maps

$$\begin{aligned} h : (X, d_X) &\rightarrow (N, \omega_N), \\ p : N &\rightarrow X, \end{aligned}$$

such that  $h$  is Lipschitz and surjective, and  $p$  is surjective and satisfies  $\text{Id} = h \circ p$ . Therefore  $p$  is also injective and hence a homeomorphism.  $\square$

To conclude the proof of Corollary 1.3 it suffices to note that (6.1) follows immediately from (1.3) in the case when  $N$  is smooth and  $D^{(1)}$  is snc (so  $\pi = \text{Id}$ ), and therefore Theorem 6.1 applies.

Of course, when  $N$  is singular or  $D^{(1)}$  fails to be snc, the map  $\pi$  will be nontrivial, and the estimate (5.14) that we proved in the previous section is weaker than (6.1) and not sufficient by itself to carry out the arguments in [86, Theorem 2.1] to prove part (a) of Conjecture 1.2.

**6.2. Geometry of the Kähler Cone.** We conclude this section with a remark that connects our setting where the Ricci-flat Kähler metrics  $\tilde{\omega}_t$  which are cohomologous to  $f^*\omega_N + e^{-t}\omega_M$  and collapse as  $t \rightarrow \infty$ , to the geometry of the Kähler cone  $\mathcal{K} \subset H^{1,1}(M, \mathbb{R})$  of  $M$  as investigated in [36, 50, 90]. This remark is certainly known to the experts, see e.g., the discussion in the introduction of [90]. However, this calculation does not appear to have appeared explicitly.

More precisely, let  $\mathcal{K}_1 = \{\alpha \in \mathcal{K} | \alpha^m = 1\}$  be the space of unit-volume Kähler classes on a Calabi–Yau manifold  $M$ , where we use here the shorthand notation  $\alpha^m = \int_M \alpha^m$ . This is a smooth manifold of dimension  $h^{1,1}(M) - 1$ , and the tangent space at  $\alpha \in \mathcal{K}_1$  is the primitive  $(1, 1)$ -cohomology group  $P^{1,1}(\alpha) = \{\beta \in H^{1,1}(M, \mathbb{R}) | \alpha^{m-1} \cdot \beta = 0\}$ . There is a natural Riemannian metric  $G$  on  $\mathcal{K}_1$  introduced by Huybrechts [36] which is given by

$$G(v_1, v_2) = -\alpha^{m-2} \cdot v_1 \cdot v_2,$$

for  $v_1, v_2 \in P^{1,1}(\alpha)$ .

Wilson pointed out in [90] that degenerations of Kähler classes at finite distance with respect to  $G$  correspond to Calabi–Yau varieties, and other cases are degenerations at infinite distance. Therefore our current case is a degeneration of Kähler classes at infinite distance, which is known to experts. The completeness of the metric  $G$  was also studied in [50, Proposition 4.4].

The following elementary result is formally analogous to the characterization of degenerations of complex structures on Calabi–Yau manifolds with finite Weil–Petersson distance [70, 81, 88, 94] by characterizing our collapsing cohomology classes  $[f^*\omega_N + e^{-t}\omega_M] \in \mathcal{K}$  as those which (after rescaling to volume 1) give paths with infinite  $G$ -length. For readers' convenience, we present the detailed calculations here, and change the parametrization of our path to  $s = e^{-t}$ .

**Proposition 6.2.** *Let  $M^n$  be a compact Calabi–Yau manifold, with classes  $\alpha \in \mathcal{K}$  and  $\alpha_0 \in \partial\mathcal{K}$ . For  $s \in (0, 1]$  let  $\alpha_s = \alpha_0 + s\alpha \in \mathcal{K}$ , let  $V_s = \alpha_s^m$ , and let  $\tilde{\omega}_s \in \alpha_s$  be the Ricci-flat Kähler metric given by [91]. Then the following statements are equivalent:*

- (a)  $\alpha_0^m = 0$ .
- (b)  $(M, \tilde{\omega}_s)$  collapses as  $s \rightarrow 0$ , i.e. for any metric 1-ball  $B_{\tilde{\omega}_s}(p, 1)$ ,

$$\text{Vol}_{\tilde{\omega}_s}(B_{\tilde{\omega}_s}(p, 1)) \rightarrow 0.$$

- (c) If we denote  $\tilde{\alpha}_s = V_s^{-\frac{1}{m}}\alpha_s \in \mathcal{K}_1$ ,  $s \in (0, 1]$ , then the length of the path  $\{\tilde{\alpha}_s\}_{s \in (0, 1]}$  with respect to  $G$  is infinite.
- (d) When  $s \rightarrow 0$ ,  $\tilde{\alpha}_s$  diverges in  $H^{1,1}(M, \mathbb{R})$ .

*Proof.* (a)  $\Rightarrow$  (b) since

$$0 = \alpha_0^m = \lim_{s \rightarrow 0} \alpha_s^m = \lim_{s \rightarrow 0} \text{Vol}_{\tilde{\omega}_s}(M).$$

(b)  $\Rightarrow$  (a): By [60, 77], we have

$$\text{diam}(M, \tilde{\omega}_s) \leq C,$$

for some constant  $C > 0$ . The Bishop–Gromov volume comparison theorem shows

$$\alpha_0^m = \lim_{s \rightarrow 0} \text{Vol}_{\tilde{\omega}_s}(M) \leq C^{2m} \lim_{s \rightarrow 0} \text{Vol}_{\tilde{\omega}_s}(B_{\tilde{\omega}_s}(p, 1)) = 0.$$

(a)  $\Rightarrow$  (c): Let  $0 \leq d < m$  be the numerical dimension of  $\alpha_0$ , which is characterized as the maximum  $\ell \geq 1$  such that  $\alpha_0^\ell \cdot \alpha^{m-\ell} > 0$ , and denote by  $k = m - d \geq 1$ . We have

$$V_s = (\alpha_0 + s\alpha)^m = s^k c_k^m \alpha^k \cdot \alpha_0^{m-k} + O(s^{k+1})$$

where  $c_k^m = \frac{m!}{k!(m-k)!}$ , and the velocity

$$v_s = \dot{\alpha}_s = V_s^{-\frac{1}{m}} \alpha - \frac{1}{m} V_s^{-1} \tilde{\alpha}_s \dot{V}_s.$$

Since  $v_s \cdot \tilde{\alpha}_s^{m-1} = 0$ , we have

$$\begin{aligned} G(v_s, v_s) &= -V_s^{-\frac{1}{m}} \alpha \cdot v_s \cdot \tilde{\alpha}_s^{m-2} \\ &= -V_s^{-\frac{2}{m}} \alpha^2 \cdot \tilde{\alpha}_s^{m-2} + \frac{\dot{V}_s}{m V_s^{1+\frac{1}{m}}} \alpha \cdot \tilde{\alpha}_s^{m-1} \\ &= -V_s^{-1} \alpha^2 \cdot (\alpha_0 + s\alpha)^{m-2} + \frac{\dot{V}_s}{m V_s^2} \alpha \cdot (\alpha_0 + s\alpha)^{m-1}. \end{aligned}$$

We then calculate  $\dot{V}_s = k s^{k-1} c_k^m \alpha^k \cdot \alpha_0^{m-k} + O(s^k)$ ,

$$\begin{aligned} \alpha^2 \cdot (\alpha_0 + s\alpha)^{m-2} &= \begin{cases} s^{k-2} c_{k-2}^{m-2} \alpha^k \cdot \alpha_0^{m-k} + O(s^{k-1}), & \text{if } k \geq 2, \\ O(1) & \text{if } k = 1. \end{cases} \\ \alpha \cdot (\alpha_0 + s\alpha)^{m-1} &= s^{k-1} c_{k-1}^{m-1} \alpha^k \cdot \alpha_0^{m-k} + O(s^k). \end{aligned}$$

Thus

$$G(v_s, v_s) = \frac{1}{s^2} \left( \frac{k(m-k)}{m^2(m-1)} + O(s) \right).$$

We obtain that

$$\int_\varepsilon^1 \sqrt{G(v_s, v_s)} ds \geq -c' \log \varepsilon \rightarrow \infty,$$

as  $\varepsilon \rightarrow 0$ , and so the length of the path  $\{\tilde{\alpha}_s\}_{s \in (0,1]}$  with respect to  $G$  is infinite.

(c)  $\Rightarrow$  (a): Assume that  $\alpha_0^m > 0$ . A similar calculation as above shows that

$$G(v_s, v_s) \leq C,$$

and so the length of the path  $\{\tilde{\alpha}_s\}_{s \in (0,1]}$  with respect to  $G$  is finite, a contradiction.

Finally, (a)  $\Leftrightarrow$  (d) since  $\tilde{\alpha}_s$  diverges if and only if  $V_s \rightarrow 0$ .  $\square$

## 7. The Kähler–Ricci Flow Setting

The setup in this section will be the Kähler–Ricci flow setting described in the Introduction, namely we let  $M$  be an  $m$ -dimensional compact Kähler manifold with  $K_M$  semiample and  $0 < \kappa(M) =: n < m$ , and  $f : M \rightarrow \mathbb{P}^r$  is the semiample fiber space given by the linear system  $|\ell K_M|$  for some  $\ell \geq 1$  sufficiently divisible, with image  $N \subset \mathbb{P}^r$  a normal projective variety of dimension  $n$  (the canonical model of  $M$ ). The map  $f$  has connected fibers, and if as before we let  $D \subset N$  be the singular locus of  $N$  together with the critical values of  $f$  on  $N^{\text{reg}}$ , then every fiber of  $f$  over  $N_0 := N \setminus D$  is a Calabi–Yau  $(m - n)$ -fold. We write  $\omega_N = \frac{1}{\ell} \omega_{FS}|_N$ , so that  $f^* \omega_N$  belongs to  $c_1(K_M)$ , and we fix also a basis  $\{s_i\}$  of  $H^0(M, \ell K_M)$ , which defines the map  $f$ , and let

$$\mathcal{M} = \left( (-1)^{\frac{\ell m^2}{2}} \sum_i s_i \wedge \bar{s}_i \right)^{\frac{1}{\ell}},$$

which is a smooth positive volume form on  $M$  which satisfies

$$\text{Ric}(\mathcal{M}) = -\sqrt{-1} \partial \bar{\partial} \log \mathcal{M} = -f^* \omega_N.$$

On  $N_0$  we then have a twisted Kähler–Einstein metric  $\omega = \omega_N + \sqrt{-1} \partial \bar{\partial} \varphi$  which satisfies (1.1) with  $\lambda = -1$  and is constructed in [11, 17, 65, 78] by solving the Monge–Ampère equation

$$(\omega_N + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^\varphi f_* \mathcal{M},$$

with  $\varphi \in C^0(N)$ ,  $\varphi$  is  $\omega_N$ -psh and smooth on  $N_0$ .

By [64, 65, 83] we know that if  $\omega_0$  is any Kähler metric on  $M$  and  $\omega(t)$  is its evolution by the Kähler–Ricci flow

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}(\omega(t)) - \omega(t), \quad \omega(0) = \omega_0,$$

then  $\omega(t)$  exists for all  $t \geq 0$  and as  $t \rightarrow \infty$  we have that  $\omega(t) \rightarrow f^* \omega$  in  $C_{\text{loc}}^0(M \setminus f^{-1}(D))$ , so  $\omega$  is the collapsed limit of the Kähler–Ricci flow, away from the singular fibers of  $f$ .

The main results of this section are Theorems 1.4 and 1.5. The first ingredient is the following generalization of the estimates in Sect. 2 in Theorem 2.3. Let  $\pi : \tilde{N} \rightarrow N$  be a birational morphism with  $\tilde{N}$  smooth and  $E = \pi^{-1}(D)$  is a simple normal crossings divisor  $E = \bigcup_{j=1}^{\mu} E_j$ , and fix defining sections  $\tau_j$  and smooth metrics  $h_j$  for  $\mathcal{O}(E_j)$ .

**Theorem 7.1.** *There is a constant  $C > 0$  and natural numbers  $d \in \mathbb{N}$ ,  $0 \leq p \leq \mu$  and rational numbers  $\beta_i > 0$ ,  $1 \leq i \leq p$ , and  $0 < \alpha_i \leq 1$ ,  $p + 1 \leq i \leq \mu$ , such that on  $\pi^{-1}(N_0)$  we have*

$$C^{-1} \prod_{j=1}^p |\tau_j|_{h_j}^{2\beta_j} \omega_{\text{cone}}^n \leq \pi^* f_* \mathcal{M} \leq C \prod_{j=1}^p |\tau_j|_{h_j}^{2\beta_j} \left( 1 - \sum_{i=1}^{\mu} \log |\tau_i|_{h_i} \right)^d \omega_{\text{cone}}^n, \quad (7.1)$$

where  $\omega_{\text{cone}}$  is a conical metric with cone angle  $2\pi\alpha_i$  along the components  $E_i$  with  $p + 1 \leq i \leq \mu$ .



*Proof.* As in the proof of Theorem 2.3, it suffices to prove the analogous local statement (as in (2.1)) near an arbitrary point  $y_0 \in E$ . We will use a ramified  $\ell$ -cyclic covering trick to reduce ourselves directly to Theorem 2.1. Recall that  $\ell K_M = f^* \mathcal{O}_{\mathbb{P}^r}(1)$  is semiample, and so we may choose the basis of sections  $s_i \in H^0(M, \ell K_M)$  (which are pullbacks of linear forms on  $\mathbb{P}^r$ ) such that their zero loci  $\{s_i = 0\}$  are smooth, reduced and irreducible effective divisors. At least one of these linear forms doesn't vanish at  $\pi(y_0)$ , and we may assume it is the one that pulls back to  $s_1$  on  $M$ . We can then construct  $v : \hat{M} \rightarrow M$  an  $\ell$ -cyclic covering ramified along  $\{s_1 = 0\}$ , see e.g. [47, Proposition 4.1.6]. It satisfies that  $\hat{M}$  is connected and smooth, with a  $\mathbb{Z}_\ell$ -action with quotient  $M$ , and there is  $\rho \in H^0(\hat{M}, K_{\hat{M}})$  such that  $v^*s_1 = \rho^{\otimes \ell}$ . Note that by construction  $\rho$  doesn't vanish on an open set of the form  $\hat{U} = v^{-1}(f^{-1}(U))$  for some open neighborhood  $U$  of  $\pi(y_0)$  in  $N$ . On  $\hat{U}$  we may then write  $v^*s_j = f_j \rho^{\otimes \ell}$ ,  $j \geq 2$ , for some holomorphic functions  $f_j$ . Then

$$v^* \mathcal{M} = \left( (-1)^{\frac{\ell m^2}{2}} \sum_i v^* s_i \wedge \overline{v^* s_i} \right)^{\frac{1}{\ell}},$$

which is uniformly equivalent to

$$\sum_i \left( (-1)^{\frac{\ell m^2}{2}} v^* s_i \wedge \overline{v^* s_i} \right)^{\frac{1}{\ell}} = (-1)^{\frac{m^2}{2}} \rho \wedge \bar{\rho} \left( 1 + \sum_{i \geq 2} |f_i|^2 \right)^{\frac{1}{\ell}},$$

and so also uniformly equivalent to  $(-1)^{\frac{m^2}{2}} \rho \wedge \bar{\rho}$ . Thus, the asymptotic behavior of  $\pi^* f_* \mathcal{M}$  is, up to constants, the same as the behavior of

$$(-1)^{\frac{m^2}{2}} \pi^* f_* v_*(\rho \wedge \bar{\rho}).$$

This is now almost the same setting as in Theorem 2.1, the only difference is that the fibers of  $f \circ v$  need not be connected anymore. For any  $y \in U$ , write  $(f \circ v)^{-1}(y) = F_1 \cup \dots \cup F_d$ , where the  $F_j$  are the connected components of the fiber. Then  $d|\ell$  and every  $F_j$  is invariant under the induced  $\mathbb{Z}_{\ell/d}$ -action, with  $F_j/\mathbb{Z}_{\ell/d} = f^{-1}(y)$ , while the quotient  $\mathbb{Z}_d$ -action on  $(f \circ v)^{-1}(y)$  interchanges the components. The number  $d$  is locally constant for  $y \in U \setminus D$ , and so we conclude by repeating the same argument as in Theorem 2.1 to any of the components  $F_j$ .  $\square$

*Proof of Theorem 1.4.* As recalled earlier, the twisted Kähler–Einstein metric  $\omega = \omega_N + \sqrt{-1} \partial \bar{\partial} \varphi$  on  $N_0$  satisfies

$$\omega^n = e^\varphi f_* \mathcal{M} = e^\varphi \mathcal{F} \omega_N^n,$$

where  $\varphi \in C^0(N)$ ,  $\mathcal{F}$  is defined by the last equality, and as explained above it is completely analogous to the function  $\mathcal{F}$  in (3.1). It satisfies

$$\sqrt{-1} \partial \bar{\partial} \log \mathcal{F} = -\omega_{WP} + \text{Ric}(\omega_N) + \omega_N.$$

As in Sects. 3 and 4 we see that  $\mathcal{F}$  extends to a smooth positive function across  $D^{(2)} \cap N^{\text{reg}}$ , and it satisfies  $\mathcal{F} \geq C^{-1}$  since the proof of Lemma 4.1 goes through with minimal changes once we notice ([65, Lemma 3.3]) that on  $M \setminus S$  we have

$$f^* \mathcal{F} = \frac{\left( (-1)^{\frac{\ell m^2}{2}} \sum_i s_i \wedge \bar{s}_i \right)^{\frac{1}{\ell}}}{f^* \omega_N^n \wedge \omega_{\text{SRF}}^{m-n}},$$

and again the numerator is strictly positive on  $M$ . As in Sect. 4, the lower bound for  $\mathcal{F}$  implies that if  $N$  is smooth and  $D^{(1)}$  is snc then we actually have on  $N_0$

$$C^{-1} \omega_{\text{cone}}^n \leq \omega^n \leq C \left( 1 - \sum_{i=1}^{\mu} \log |\tau_i|_{h_i} \right)^d \omega_{\text{cone}}^n,$$

while in general we have on  $\tilde{N} \setminus E$

$$C^{-1} H \omega_{\text{cone}}^n \leq (\pi^* \omega)^n \leq C H \left( 1 - \sum_{i=1}^{\mu} \log |\tau_i|_{h_i} \right)^d \omega_{\text{cone}}^n,$$

where  $H = \prod_{j=1}^p |\tau_j|_{h_j}^{2\beta_j}$ . The proof of (1.3) given in Sect. 4 then goes through with the appropriate small changes, defining now the function  $\psi$  by

$$\omega^n = \frac{e^{\varphi} \psi}{\prod_{j=1}^{\mu} |\tau_j|_{h_j}^{2(1-\alpha_j)}} \omega_N^n,$$

and similarly, the proof of Theorem 1.1 given in Sect. 5 goes through, defining now  $\psi$  by

$$\pi^* \omega^n = \frac{e^{\pi^* \varphi} \psi H}{\prod_{j=1}^{\mu} |\tau_j|_{h_j}^{2(1-\alpha_j)}} \omega_{\tilde{N}}^n.$$

In this way, we obtain the proof of Theorem 1.4.  $\square$

We can now give the proof of Theorem 1.5. The outline of the proof is the same as in [67, 76] which deals with the case when  $m = 1$ , and uses the relative volume comparison from [76]. However, new difficulties arise when  $m > 1$ , and Theorem 1.4 will precisely help us overcome these.

*Proof of Theorem 1.5.* Thanks to [66], we have the estimate

$$C^{-1} e^{-(n-m)t} \omega_0^n \leq \omega(t)^n \leq C e^{-(n-m)t} \omega_0^n, \quad (7.2)$$

on  $M \times [0, \infty)$ . Denote by  $(Z, d_Z)$  the metric completion of the twisted Kähler–Einstein manifold  $(N_0, \omega)$ . It is shown in [67, Proposition 2.2] that  $Z$  is compact, and that for every  $p, q \in N_0$  and  $\delta > 0$  there is a path  $\gamma$  in  $N_0$  joining  $p$  and  $q$  whose  $\omega$ -length is at most  $d_Z(p, q) + \delta$  (we can call this the “almost convexity” of  $(N_0, \omega)$  inside its metric completion). Also, in [67, Proposition 2.3] it is proved that when  $N$  is smooth (or an orbifold) then  $Z$  is homeomorphic to  $N$ .

For any  $\varepsilon > 0$  let  $U_\varepsilon \subset N$  be the  $\varepsilon$ -neighborhood of  $D$  with respect to some fixed metric  $\omega_N$  on  $N$ , and let  $\tilde{U}_\varepsilon = f^{-1}(U_\varepsilon) \subset M$ . From the volume form bound (7.2) we see that for any given  $\delta > 0$  there are  $\varepsilon = \varepsilon(\delta) < \delta$ ,  $T > 0$  such that for all  $t \geq T$  we have

$$\frac{\text{Vol}(\tilde{U}_\varepsilon, \omega(t))}{\text{Vol}(M, \omega(t))} \leq \delta. \quad (7.3)$$

**Claim 1.** We have

$$d_{\text{GH}}((Z, d_Z), (N \setminus U_\varepsilon, \omega)) \leq \delta', \quad \text{where } \delta'(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Here is where Theorem 1.4 comes in, by giving us (1.3). Using this (in fact, just the upper bound for  $\omega$  in (1.3)), the arguments in the proof of [86, Theorem 2.1] show that for every point  $p \in U_\varepsilon$  there is a point  $q \in \partial U_\varepsilon$  with  $d_Z(p, q) \leq \delta'$  with  $\delta'(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . We can therefore define a (discontinuous) map  $F : Z \cong N \rightarrow N \setminus U_\varepsilon$  which is the identity on  $N \setminus U_\varepsilon$  and inside  $U_\varepsilon$  it maps  $p$  to  $q$  above (which is not unique, but we just choose any one of them). If  $G : N \setminus U_\varepsilon \rightarrow Z$  denotes the inclusion, then it is now elementary to check that  $F$  and  $G$  are a  $3\delta'$ -GH approximation, since replacing  $p$  by  $q$  only distorts the  $d_Z$ -distance function by  $\delta'$ , and using also the almost convexity property, see the proof of Claim 3 below for a very similar argument. This proves Claim 1.

**Claim 2.** Up to making  $T$  larger, we have

$$d_{\text{GH}}((N \setminus U_\varepsilon, \omega), (M \setminus \tilde{U}_\varepsilon, \omega(t))) \leq \delta,$$

for all  $t \geq T$ .

This is an immediate consequence of the main theorem of [83], which gives locally uniform convergence of  $\omega(t)$  to  $f^*\omega$  on compact sets away from  $S$ .

**Claim 3.** Up to making  $T$  larger, we have

$$d_{\text{GH}}((M \setminus \tilde{U}_\varepsilon, \omega(t)), (M, \omega(t))) \leq \delta', \quad \text{where } \delta'(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0,$$

for all  $t \geq T$ .

It is clear that once we have these 3 claims, then we conclude that

$$(M, \omega(t)) \rightarrow (Z, d_Z),$$

in the Gromov–Hausdorff topology as  $t \rightarrow \infty$ , which would complete the proof of Theorem 1.5.

The proof of Claim 3 uses the relative volume comparison from [76] and the assumption on the Ricci curvature of  $\omega(t)$ , similarly as what is done in [67, 76] when  $m = 1$ . The starting point is that the relative volume comparison and the assumed Ricci bound together imply (by the argument in [76, p.23] or [75, p.12] or [67, Lemma 5.5] which also uses (7.3)) that, up to enlarging  $T$ , we have that for all  $t \geq T$  and for all  $x \in \tilde{U}_\varepsilon$

$$\text{dist}_{(M, \omega(t))}(x, \partial \tilde{U}_\varepsilon) \leq \delta', \quad \text{where } \delta'(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0. \quad (7.4)$$

Once we have this, we can argue as in Claim 1. Define a discontinuous map  $F : M \rightarrow M \setminus \tilde{U}_\varepsilon$  which is the identity on  $M \setminus \tilde{U}_\varepsilon$  and inside  $U_\varepsilon$  it maps  $p$  to some point  $q \in \partial \tilde{U}_\varepsilon$  with  $\text{dist}_{(M, \omega(t))}(p, q) \leq \delta'$  for all  $t \geq T$ . Define also  $G : M \setminus \tilde{U}_\varepsilon \rightarrow M$  to be the inclusion,

and we claim again that  $F$  and  $G$  give a  $3\delta'$ -GH approximation between  $(M, \omega(t))$  and  $(M \setminus \tilde{U}_\varepsilon, \omega(t))$ . Let  $d_t = \text{dist}_{(M, \omega(t))}$  and  $\hat{d}_t = \text{dist}_{(M \setminus \tilde{U}_\varepsilon, \omega(t))}$ . Then we need to show the following properties

$$d_t(x, G(F(x))) \leq \delta', \quad x \in M, \quad (7.5)$$

$$\hat{d}_t(y, F(G(y))) \leq \delta', \quad y \in M \setminus \tilde{U}_\varepsilon, \quad (7.6)$$

$$|d_t(x, x') - \hat{d}_t(F(x), F(x'))| \leq 3\delta', \quad x, x' \in M, \quad (7.7)$$

$$|\hat{d}_t(y, y') - d_t(G(y), G(y'))| \leq 3\delta', \quad y, y' \in M \setminus \tilde{U}_\varepsilon. \quad (7.8)$$

The estimate in (7.5) is trivial when  $x \in M \setminus \tilde{U}_\varepsilon$ , and follows from (7.4) when  $x \in \tilde{U}_\varepsilon$ . Estimate (7.6) is trivial. Next we observe that for every  $x, x' \in M \setminus \tilde{U}_\varepsilon$  and for every  $t \geq T$  we can find a path  $\gamma$  (that depends on  $t$ ) that connects them inside  $M \setminus \tilde{U}_\varepsilon$  and with  $L_{\omega(t)}(\gamma) \leq d_t(x, x') + \delta'$  (this is because  $S = f^{-1}(D)$  has real codimension at least 2 in  $M$  and  $\tilde{U}_\varepsilon$  is comparable to an  $\varepsilon$ -neighborhood of it in a fixed metric on  $M$ ). This implies that

$$d_t(x, x') \leq \hat{d}_t(x, x') \leq d_t(x, x') + \delta'. \quad (7.9)$$

Now, for (7.7) there are three cases to consider. First, suppose both  $x, x' \in M \setminus \tilde{U}_\varepsilon$ , and then (7.7) follows from (7.9). Second, suppose  $x \in M \setminus \tilde{U}_\varepsilon, x' \in \tilde{U}_\varepsilon$ . Then by (7.4) we have  $d_t(x', F(x')) \leq \delta'$ , and so

$$d_t(x, x') \leq d_t(x, F(x')) + \delta' = d_t(F(x), F(x')) + \delta' \leq \hat{d}_t(F(x), F(x')) + \delta',$$

using (7.9). On the other hand by (7.4), (7.9) again

$$\begin{aligned} \hat{d}_t(F(x), F(x')) &\leq d_t(F(x), F(x')) + \delta' \\ &\leq d_t(x, x') + d_t(x', F(x')) + \delta' \leq d_t(x, x') + 2\delta', \end{aligned}$$

proving (7.7) in this case. Thirdly, we suppose  $x, x' \in \tilde{U}_\varepsilon$ , and then by (7.4), (7.9)

$$d_t(x, x') \leq d_t(F(x), F(x')) + 2\delta' \leq \hat{d}_t(F(x), F(x')) + 2\delta',$$

and

$$\begin{aligned} \hat{d}_t(F(x), F(x')) &\leq d_t(F(x), F(x')) + \delta' \\ &\leq d_t(x, x') + d_t(x, F(x)) + d_t(x', F(x')) + \delta' \\ &\leq d_t(x, x') + 3\delta'. \end{aligned}$$

This proves (7.7), and finally (7.8) follows immediately from (7.9).

This in turns concludes the proof of Claim 3, and hence of Theorem 1.5.  $\square$

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## References

1. Aleyasin, S.A.: Calabi problem for manifolds with edge-cone singularities, preprint, [arXiv:1810.07823](https://arxiv.org/abs/1810.07823)
2. Barlet, D.: Développement asymptotique des fonctions obtenues par intégration sur les fibres. *Invent. Math.* **68**(1), 129–174 (1982)
3. Berman, R.J.: K-polystability of  $\mathbb{Q}$ -Fano varieties admitting Kähler-Einstein metrics. *Invent. Math.* **203**(3), 973–1025 (2016)
4. Boucksom, S., Jonsson, M.: Tropical and non-Archimedean limits of degenerating families of volume forms. *J. Éc. Polytech. Math.* **4**, 87–139 (2017)
5. Brendle, S.: Ricci flat Kähler metrics with edge singularities. *Int. Math. Res. Not. IMRN* **24**, 5727–5766 (2013)
6. Campana, F., Guenancia, H., Păun, M.: Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields. *Ann. Sci. Éc. Norm. Supér. (4)* **46**(6), 879–916 (2013)
7. Cattani, E., Kaplan, A., Schmid, W.: Degeneration of Hodge structures. *Ann. Math. (2)* **123**(3), 457–535 (1986)
8. Cattani, E., Kaplan, A., Schmid, W.: Variations of polarized Hodge structure: asymptotics and monodromy. *Hodge theory* (Sant Cugat, 1985). *Lecture Notes in Mathematics*, vol. 1246, pp. 16–31. Springer, Berlin (1987)
9. Chambert-Loir, A., Tschinkel, Y.: Igusa integrals and volume asymptotics in analytic and adelic geometry. *Conflu. Math.* **2**(3), 351–429 (2010)
10. Chen, G., Viaclovsky, J., Zhang, R.: Collapsing Ricci-flat metrics on elliptic K3 surfaces, preprint, [arXiv:1910.11321](https://arxiv.org/abs/1910.11321)
11. Coman, D., Guedj, V., Zeriahi, A.: Extension of plurisubharmonic functions with growth control. *J. Reine Angew. Math.* **676**, 33–49 (2013)
12. Datar, V., Jacob, A., Zhang, Y.: Adiabatic limits of anti-self-dual connections on collapsed K3 surfaces. *J. Differ. Geom.* (to appear)
13. Datar, V., Song, J.: A remark on Kähler metrics with conical singularities along a simple normal crossing divisor. *Bull. Lond. Math. Soc.* **47**(6), 1010–1013 (2015)
14. Demailly, J.-P.: Estimations  $L^2$  pour l’opérateur  $\bar{\partial}$  d’un fibré vectoriel holomorphe semi-positif au-dessus d’une variété kählérienne complète. *Ann. Sci. École Norm. Sup. (4)* **15**(3), 457–511 (1982)
15. Eriksson, D., Montplet, G., Freixas i, Mourougane, C.: Singularities of metrics on Hodge bundles and their topological invariants. *Algebr. Geom.* **5**(6), 742–775 (2018)
16. Eriksson, D., Montplet, G., Freixas i, Mourougane, C.: BCOV invariants of Calabi-Yau manifolds and degenerations of Hodge structures, preprint, [arXiv:1809.05452](https://arxiv.org/abs/1809.05452)
17. Eyssidieux, P., Guedj, V., Zeriahi, A.: Singular Kähler-Einstein metrics. *J. Am. Math. Soc.* **22**(3), 607–639 (2009)
18. Eyssidieux, P., Guedj, V., Zeriahi, A.: Convergence of weak Kähler-Ricci flows on minimal models of positive Kodaira dimension. *Commun. Math. Phys.* **357**(3), 1179–1214 (2018)
19. Fine, J.: Fibrations with constant scalar curvature Kähler metrics and the CM-line bundle. *Math. Res. Lett.* **14**(2), 239–247 (2007)
20. Fong, F.T.-H., Zhang, Z.: The collapsing rate of the Kähler-Ricci flow with regular infinite time singularity. *J. Reine Angew. Math.* **703**, 95–113 (2015)
21. Freed, D.: Special Kähler Manifolds. *Commun. Math. Phys.* **203**(1), 31–52 (1999)
22. Fu, X., Guo, B., Song, J.: Geometric estimates for complex Monge-Ampère equations. *J. Reine Angew. Math.* **765**, 69–99 (2020)
23. Fujita, T.: On Kähler fiber spaces over curves. *J. Math. Soc. Jpn.* **30**(4), 779–794 (1978)
24. Grauert, H., Remmert, R.: Plurisubharmonische Funktionen in komplexen Räumen. *Math. Z.* **65**, 175–194 (1956)
25. Griffiths, P.: *Topics in Transcendental Algebraic Geometry*. *Annals of Mathematics Studies*, 106. Princeton University Press, Princeton, NJ (1984)
26. Griffiths, P., Schmid, W.: Locally homogeneous complex manifolds. *Acta Math.* **123**, 253–302 (1969)
27. Gross, M., Tosatti, V., Zhang, Y.: Collapsing of abelian fibred Calabi-Yau manifolds. *Duke Math. J.* **162**(3), 517–551 (2013)
28. Gross, M., Tosatti, V., Zhang, Y.: Gromov-Hausdorff collapsing of Calabi-Yau manifolds. *Commun. Anal. Geom.* **24**(1), 93–113 (2016)
29. Gross, M., Wilson, P.M.H.: Large complex structure limits of K3 surfaces. *J. Differ. Geom.* **55**(3), 475–546 (2000)
30. Guenancia, H., Păun, M.: Conic singularities metrics with prescribed Ricci curvature: general cone angles along normal crossing divisors. *J. Differ. Geom.* **103**(1), 15–57 (2016)
31. Hein, H.-J.: Gravitational instantons from rational elliptic surfaces. *J. Am. Math. Soc.* **25**(2), 355–393 (2012)

32. Hein, H.-J., Tosatti, V.: Remarks on the collapsing of torus fibered Calabi-Yau manifolds. *Bull. Lond. Math. Soc.* **47**(6), 1021–1027 (2015)
33. Hein, H.-J., Tosatti, V.: Higher-order estimates for collapsing Calabi-Yau metrics, *Camb. J. Math.* (to appear)
34. Hitchin, N.: The moduli space of special Lagrangian submanifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **25**(3–4), 503–515 (1997)
35. Hitchin, N.: The moduli space of complex Lagrangian submanifolds. *Asian J. Math.* **3**(1), 77–91 (1999)
36. Huybrechts, D.: Products of harmonic forms and rational curves. *Doc. Math.* **6**, 227–239 (2001)
37. Jeffres, T., Mazzeo, R., Rubinstein, Y.A.: Kähler-Einstein metrics with edge singularities. *Ann. Math.* (2) **183**(1), 95–176 (2016)
38. Kashiwara, M.: The asymptotic behavior of a variation of polarized Hodge structure. *Publ. Res. Inst. Math. Sci.* **21**(4), 853–875 (1985)
39. Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B.: *Toroidal Embeddings. I. Lecture Notes in Mathematics*, vol. 339. Springer, Berlin (1973)
40. Kim, D.: Canonical bundle formula and degenerating families of volume forms, preprint, [arXiv:1910.06917](https://arxiv.org/abs/1910.06917)
41. Kollár, J.: Subadditivity of the Kodaira dimension: fibers of general type. In: Bando S, Mabuchi T (ed) *Algebraic geometry, Sendai, 1985*, pp 361–398, *Adv. Stud. Pure Math.*, 10, North-Holland, Amsterdam (1987)
42. Kolodziej, S.: The complex Monge-Ampère equation. *Acta Math.* **180**(1), 69–117 (1998)
43. Kolodziej, S.: The Monge-Ampère equation on compact Kähler manifolds. *Indiana Univ. Math. J.* **52**(3), 667–686 (2003)
44. Kontsevich, M., Soibelman, Y.: Homological mirror symmetry and torus fibrations. In: Cox, D.A., Katz, S. (eds.) *Symplectic Geometry and Mirror Symmetry*, 203–263. World Scientific Publishing, Singapore (2001)
45. Kontsevich, M., Soibelman, Y.: Affine Structures and Non-Archimedean Analytic Spaces. In: Atiyah, M.F. (ed.) *The Unity of Mathematics*. Springer, Berlin (2006)
46. Landman, A.: On the Picard-Lefschetz transformation for algebraic manifolds acquiring general singularities. *Trans. Am. Math. Soc.* **181**, 89–126 (1973)
47. Lazarsfeld, R.: *Positivity in Algebraic Geometry I & II*. Springer, Berlin (2004)
48. Li, Y.: On collapsing Calabi-Yau fibrations. *J. Differ. Geom.* (to appear)
49. Li, Y.: A gluing construction of collapsing Calabi-Yau metrics on  $K3$  fibred 3-folds. *Geom. Funct. Anal.* **29**(4), 1002–1047 (2019)
50. Magnússon, G.: The geometry of Kähler cones, preprint, [arXiv:1211.6934](https://arxiv.org/abs/1211.6934)
51. McLean, R.C.: Deformations of calibrated submanifolds. *Commun. Anal. Geom.* **6**(4), 705–747 (1998)
52. Moishezon, B.G.: *Singular Kählerian spaces. Manifolds-Tokyo 1973 Proceedings International Conference*. Tokyo, 1973, pp. 343–351. University of Tokyo Press, Tokyo (1975)
53. Mourougane, C., Takayama, S.: Hodge metrics and positivity of direct images. *J. Reine Angew. Math.* **606**, 167–178 (2007)
54. Mourougane, C., Takayama, S.: Hodge metrics and the curvature of higher direct images. *Ann. Sci. Éc. Norm. Supér.* (4) **41**(6), 905–924 (2008)
55. Mourougane, C., Takayama, S.: Extension of twisted Hodge metrics for Kähler morphisms. *J. Differ. Geom.* **83**(1), 131–161 (2009)
56. Oda, T.: *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 15. Springer, Berlin (1988)
57. Odaka, Y., Oshima, Y.: Collapsing  $K3$  surfaces, tropical geometry and moduli compactifications of Satake, Morgan-Shalen type, preprint, [arXiv:1810.07685](https://arxiv.org/abs/1810.07685)
58. Phong, D.H., Sturm, J.: On the singularities of the pluricomplex Green’s function. In: Fefferman, C., Ionescu, A.D., Phong, D.H., Wainger, S. (eds.) *Advances in analysis. The legacy of Elias M. Stein*, pp. 419–435. Princeton University Press, Princeton (2014)
59. Rong, X., Zhang, Y.: Continuity of Extremal Transitions and Flops for Calabi-Yau Manifolds. Appendix B by Mark Gross. *J. Differ. Geom.* **89**(2), 233–269 (2011)
60. Ruan, W.-D., Zhang, Y.: Convergence of Calabi-Yau manifolds. *Adv. Math.* **228**(3), 1543–1589 (2011)
61. Rubinstein, Y.A.: Smooth and singular Kähler-Einstein metrics. Geometric and spectral analysis. *Contemporary Mathematics, Centre de Recherches Mathématiques Proceedings*, vol. 630, pp. 45–138. American Mathematics Society, Providence, RI (2014)
62. Schmid, W.: Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.* **22**, 211–319 (1973)
63. Schoen, C.: On fiber products of rational elliptic surfaces with section. *Math. Z.* **197**(2), 177–199 (1988)
64. Song, J., Tian, G.: The Kähler-Ricci flow on surfaces of positive Kodaira dimension. *Invent. Math.* **170**(3), 609–653 (2007)
65. Song, J., Tian, G.: Canonical measures and Kähler-Ricci flow. *J. Amer. Math. Soc.* **25**(2), 303–353 (2012)

66. Song, J., Tian, G.: Bounding scalar curvature for global solutions of the Kähler-Ricci flow. *Amer. J. Math.* **138**(3), 683–695 (2016)
67. Song, J., Tian, G., Zhang, Z.: Collapsing behavior of Ricci-flat Kähler metrics and long time solutions of the Kähler-Ricci flow, preprint, [arXiv:1904.08345](https://arxiv.org/abs/1904.08345)
68. Steenbrink, J.: Limits of Hodge structures. *Invent. Math.* **31**(3), 229–257 (1975/76)
69. Stoppa, J.: Twisted constant scalar curvature Kähler metrics and Kähler slope stability. *J. Differ. Geom.* **83**(3), 663–691 (2009)
70. Takayama, S.: On moderate degenerations of polarized Ricci-flat Kähler manifolds. *J. Math. Sci. Univ. Tokyo* **22**(1), 469–489 (2015)
71. Takayama, S.: A filling-in problem and moderate degenerations of minimal algebraic varieties. *Algebr. Geom.* **6**(1), 26–49 (2019)
72. Takayama, S.: Asymptotic expansions of fiber integrals over higher dimensional bases. *J. Reine Angew. Math.* <https://doi.org/10.1515/crelle-2020-0027>
73. Temkin, M.: Functorial desingularization over  $\mathbf{Q}$ : boundaries and the embedded case. *Israel J. Math.* **224**(1), 455–504 (2018)
74. Tian, G.: Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric. In: *Mathematical aspects of string theory* (San Diego, California, 1986). *Advance Series Mathematics Physics*, 1, World Scientific Publishing, Singapore (1987)
75. Tian, G.: Some progresses on Kähler-Ricci flow. *Boll. Unione Mat. Ital.* **12**(1–2), 251–263 (2019)
76. Tian, G., Zhang, Z.: Relative volume comparison of Ricci Flow and its applications, preprint, [arXiv:1802.09506](https://arxiv.org/abs/1802.09506)
77. Tosatti, V.: Limits of Calabi-Yau metrics when the Kähler class degenerates. *J. Eur. Math. Soc. (JEMS)* **11**(4), 755–776 (2009)
78. Tosatti, V.: Adiabatic limits of Ricci-flat Kähler metrics. *J. Differ. Geom.* **84**(2), 427–453 (2010)
79. Tosatti, V.: Degenerations of Calabi-Yau metrics Geometry and Physics in Cracow. *Acta Phys. Polon. B Proc. Suppl.* **4**(3), 495–505 (2011)
80. Tosatti, V.: Calabi-Yau manifolds and their degenerations. *Ann. N.Y. Acad. Sci.* **1260**, 8–13 (2012)
81. Tosatti, V.: Families of Calabi-Yau manifolds and canonical singularities. *Int. Math. Res. Not. IMRN* **20**, 10586–10594 (2015)
82. Tosatti, V.: KAWA lecture notes on the Kähler-Ricci flow. *Ann. Fac. Sci. Toulouse Math.* **27**(2), 285–376 (2018)
83. Tosatti, V., Weinkove, B., Yang, X.: The Kähler-Ricci flow, Ricci-flat metrics and collapsing limits. *Am. J. Math.* **140**(3), 653–698 (2018)
84. Tosatti, V., Zhang, Y.: Triviality of fibered Calabi-Yau manifolds without singular fibers. *Math. Res. Lett.* **21**(4), 905–918 (2014)
85. Tosatti, V., Zhang, Y.: Infinite time singularities of the Kähler-Ricci flow. *Geom. Topol.* **19**(5), 2925–2948 (2015)
86. Tosatti, V., Zhang, Y.: Collapsing hyperkähler manifolds. *Ann. Sci. Éc. Norm. Supér.* **53**(3), 751–786 (2020)
87. Tsuji, H.: Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type. *Math. Ann.* **281**(1), 123–133 (1988)
88. Wang, C.-L.: On the incompleteness of the Weil-Petersson metric along degenerations of Calabi-Yau manifolds. *Math. Res. Lett.* **4**(1), 157–171 (1997)
89. Wang, C.-L.: Quasi-Hodge metrics and canonical singularities. *Math. Res. Lett.* **10**(1), 57–70 (2003)
90. Wilson, P.M.H.: Sectional curvatures of Kähler moduli. *Math. Ann.* **330**(4), 631–664 (2004)
91. Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. *I. Commun. Pure Appl. Math.* **31**, 339–411 (1978)
92. Yau, S.-T.: A general Schwarz lemma for Kähler manifolds. *Am. J. Math.* **100**(1), 197–203 (1978)
93. Yoshikawa, K.-I.: On the boundary behavior of the curvature of  $L^2$ -metrics, preprint, [arXiv:1007.2836](https://arxiv.org/abs/1007.2836)
94. Zhang, Y.: Note on equivalences for degenerations of Calabi-Yau manifolds. *Surveys in Geometric Analysis 2017*, pp. 186–202. Science Press, Beijing (2018)
95. Zhang, Y.S.: Collapsing limits of the Kähler-Ricci flow and the continuity method. *Math. Ann.* **374**(1–2), 331–360 (2019)
96. Zhang, Y.S.: Infinite-time singularity type of the Kähler-Ricci flow II, preprint, [arXiv:1809.01305](https://arxiv.org/abs/1809.01305)

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