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Bergman–Einstein metrics, a generalization of Kerner's theorem and Stein spaces with spherical boundaries

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Abstract. We give an affirmative solution to a conjecture of Cheng proposed in 1979 which asserts that the Bergman metric of a smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^n, n \geq 2$, is Kähler–Einstein if and only if the domain is biholomorphic to the ball. We establish a version of the classical Kerner theorem for Stein spaces with isolated singularities which has an immediate application to construct a hyperbolic metric over a Stein space with a spherical boundary.

1. Introduction

Canonical metrics are important objects under study in Complex Analysis of Several Variables. Since Cheng and Yau proved in [6] the existence of a complete Kähler–Einstein metric over a bounded pseudoconvex domain in $\mathbb{C}^n$ with reasonably smooth boundary, it has become a natural question to understand when the Cheng–Yau metric of a bounded pseudoconvex domain is precisely its Bergman metric. S. Y. Cheng conjectured in 1979 [5] that if the Bergman metric of a smoothly bounded strictly pseudoconvex domain is Kähler–Einstein, then the domain is biholomorphic to the ball. Cheng's conjecture was previously obtained by Fu and Wong [15] and Nemirovski and Shafikov [26] in the case of complex dimension two. There are closely related studies on versions of the Cheng conjecture in terms of metrics defined by other canonical potential functions. The reader is referred to work of Li [19–21] and the references therein on this matter. There are also many other characterizations of the unit ball in terms of various geometric properties of the domains. See, for instance, [27] and [8].

This paper is twofold. One is to present a solution of the Cheng conjecture in any dimensions. The other is to use this opportunity to generalize the classical Kerner theorem [18]

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and the Chern–Ji theorem [8] to Stein spaces with singularities, whose original version is a fundamental tool to obtain the Cheng conjecture. The generalization of the Kerner theorem to singular space might be of independent interest in its own right and may find other applications.

In Section 2, the first part of this paper, we answer affirmatively the Cheng conjecture [5], based on deep works of many mathematicians in the past 40 years:

Theorem 1.1. *The Bergman metric of a smoothly bounded strongly pseudoconvex domain in* $\mathbb{C}^n$ $(n \geq 2)$ *is Kähler–Einstein if and only if the domain is biholomorphic to the ball.*

To verify the Cheng conjecture, we first show that the Einstein property of the Bergman metric over a bounded strongly pseudoconvex domain Ω forces the boundary $\partial\Omega$ to be spherical. Namely, at each point of $\partial\Omega$ there is a small open piece of $\partial\Omega$ that is CR-diffeomorphic to an open piece of the sphere of the same dimension. To prove that, we will fundamentally make use of the work done by Chern and Moser [9], Fefferman [11, 12], Christoffers [10], Fu and Wong [15], etc. Once this is known, as in the work of Nemirovski and Shafikov [25], one can use the classical Kerner theorem [18] or the Chern–Ji extension theorem [8] to prove that Ω is a ball quotient. Then the proof of Theorem 1.1 follows from the Cheng–Yau uniqueness theorem of complete Kähler–Einstein metrics [6] and the classical Qi-Keng Lu theorem [22].

In Section 3, the second part of the paper, we establish a Kerner-type theorem for Stein spaces even with isolated complex singularities. Before stating our next main theorem, we explain needed notations and terminologies:

Let Ω be a Stein space of complex dimension at least two with possibly isolated singularities and connected compact strongly pseudoconvex boundary $M = \partial\Omega$. Write $\mathrm{Reg}(\Omega)$ for the set of smooth points in Ω and $\mathrm{Reg}(\overline{\Omega}) = \mathrm{Reg}(\Omega) \cup M$. We say (f, D) is a continuous CR map element over M into $\mathbb{C}^N$ if D is a simply connected open piece of M and $f : D \to \mathbb{C}^N$ is a continuous CR map for a certain N. Similarly, we say (g, U) is a holomorphic map element over $\mathrm{Reg}(\overline{\Omega})$ into $\mathbb{C}^N$ if U is a simply connected open subset of $\mathrm{Reg}(\overline{\Omega})$ and $g : U \to \mathbb{C}^N$ is a continuous map that is holomorphic in $U \cap \Omega$. We say (f, D) admits a holomorphic continuation along $\sigma : [0, 1] \to \mathrm{Reg}(\overline{\Omega})$ with $\sigma(0) \in D$ if there exists a collection of holomorphic map elements $\{(f_j, U_j)\}_{j=0}^{k}$ on $\mathrm{Reg}(\overline{\Omega})$ such that $f_0 = f$ in a neighborhood of $\sigma(0)$ in $U_0 \cap D$ and there is a partition $0 = t_0 < t_1 < \cdots < t_{k+1} = 1$ such that $\sigma([t_j, t_{j+1}]) \subset U_j$ for all $0 \leq j \leq k$ with $f_j = f_{j+1}$ on $U_j \cap U_{j+1}$ for $0 \leq j \leq k - 1$. Here (f_k, U_k) is called a (holomorphic) branch of (f, D) obtained by holomorphic continuation of (f, D) along σ. Let (f, D) be a CR map element over M as above and fix a plurisubharmonic function $\psi : \mathbb{C}^N \to \mathbb{R}$ such that $\psi(f) \leq 0$ on D. Let $\widehat{\Omega}$ be an open connected subset of $\mathrm{Reg}(\overline{\Omega})$ containing D. We say (f, D) admits holomorphic continuation with ψ-estimate in $\widehat{\Omega}$ along curves if (f, D) can be continued holomorphically along any curve γ in $\widehat{\Omega}$ with $\gamma(0) \in D$ and for each branch (g, U) with $U \subset \widehat{\Omega}$ obtained through holomorphic continuation of (f, D) along a curve γ in $\widehat{\Omega}$ with $\gamma(0) \in D$, we have $\psi(g) \leq 0$ in U. Similarly, for a positive constant C^*, we say (f, D) admits C^*-uniformly bounded holomorphic map continuation along curves in $\widehat{\Omega}$ if for any branch (g, U) of (f, D) with $U \subset \widehat{\Omega}$, we have $|g| \leq C^*$ over U. One similarly defines the notion of continuous CR map continuation of (f, D) along a curve in M, continuous CR map continuation of (f, D) with ψ-estimate along curves in M and the notion of C^*-uniformly bounded CR map continuation along curves in M.

Theorem 1.2. *Let Ω be a Stein space of complex dimension at least two with possibly isolated singularities and connected compact strongly pseudoconvex boundary $M = \partial\Omega$. Let (f, D) be a continuous CR map element into $\mathbb{C}^N$ over M and $\psi : \mathbb{C}^N \to \mathbb{R}$ a plurisubharmonic function such that $\psi(f) \leq 0$ on D. Then the following conclusions hold:*

(1) *Suppose that $\dim_{\mathbb{C}}(\Omega) \geq 3$ and (f, D) admits continuous CR map continuation with ψ-estimate along each curve inside M starting from a point in D. Then (f, D) admits holomorphic continuation with ψ-estimate along any curve in $\mathrm{Reg}(\overline{\Omega})$ starting from a point in D.*

(2) *Suppose that $\dim_{\mathbb{C}}(\Omega) = 2$ and (f, D) admits C^*-uniformly bounded CR map continuation with ψ-estimate along each curve inside M starting from a point in D. Then (f, D) admits C^*-uniformly bounded holomorphic map continuation with ψ-estimate along any curve in $\mathrm{Reg}(\overline{\Omega})$ starting from a point in D.*

Moreover, in both cases, assume there is a holomorphic branch (h, U) of (f, D) in $\mathrm{Reg}(\overline{\Omega})$ such that $\psi(h(p)) = 0$ at some point p in $U \setminus M$. Then $\psi(g) \equiv 0$ for any holomorphic branch (g, V) of (f, D). In particular, $\psi(f) \equiv 0$ on D.

We mention that in Theorem 1.2, one does not have in general the extension of (f, D) along a curve through the singular points, as Example 3.1 shows, even if the singularities are normal. This is very different from the classical Hartogs extension theorem. This also partially demonstrates that the method of the proof in Kerner's paper does not apply to the singular Stein space case. Indeed, Kerner [18] proved that the envelope of holomorphy $\widehat{Y}$ of the universal cover Y of a domain D_0 over a Stein manifold X is the universal cover of the envelope of holomorphy $\widehat{D_0}$ of D_0. (Here, $\widehat{Y}, Y, \widehat{D_0}$ are domains over X.) Once this is established, the multiple-valued Hartogs extension theorem follows as an immediate consequence, for the multiple-valued map becomes single-valued map in the universal covering space. Example 3.1 shows that when a complex manifold is a domain over a singular Stein space, the envelope of holomorphy of its universal cover is in general no longer the universal cover of its envelope of holomorphy.

For the proof of Theorem 1.2, we will employ a different but in fact more elementary and self-contained argument than those used in [18] and [8]. Ours is based on the Lewy and Baouendi–Treves extension theorem [2], Morse function theory and the Phragmén–Lindelöff maximum value principle.

An important scenario where Theorem 1.2 is applied is when M is spherical and f is a CR diffeomorphism from a simply connected open piece $D \subset M$ to an open piece in $\partial\mathbb{B}^n$ with $n = \dim_{\mathbb{C}} \Omega$. Here and in what follows, we write $\mathbb{B}^n$ for the standard unit ball in $\mathbb{C}^n$. In this case, by the Alexander theorem [1,4], (f, D) extends as local smooth CR diffeomorphism elements into $\partial\mathbb{B}^n$ (and thus with a uniform bound $C^* = 1$) along each curve inside M that starts from a point in D. Such a map element (f, D) is called a development map element and the multiple-valued CR extension of (f, D) in M along curves is called a multiple-valued development map. Hence, the following is an immediate application of Theorem 1.2:

Corollary 1.3. *Let Ω be a Stein space with $\dim_{\mathbb{C}} \Omega = n \geq 2$ with a connected smooth compact spherical boundary $M = \partial\Omega$. Let (f, D) be a smooth CR development map element. Then (f, D) admits a holomorphic continuation along any curve γ in $\mathrm{Reg}(\overline{\Omega})$ with $\gamma(0) \in D$. Moreover, there is a subgroup Γ of $\mathrm{Aut}(\mathbb{B}^n)$ such that if (g, U) (with U a simply connected*

open subset of $\mathrm{Reg}(\Omega)$*) is a holomorphic branch of* (f, D) *obtained through continuation along a curve, then* g *is a local biholomorphic map from* U *into* $\mathbb{B}^n$ *and all other branches of* (f, D) *defined over* U *are precisely the holomorphic map elements of the form* $(\sigma \circ g, U)$ *with* $\sigma \in \Gamma$.

The following is a very useful consequence of Corollary 1.3:

Corollary 1.4. *Let* $\overline{\Omega}$ *be a Stein space with* $\dim_{\mathbb{C}} \Omega = n \geq 2$ *that has a connected compact smooth boundary. Assume the boundary* $\partial\Omega$ *is spherical. Then there is a unique Kähler metric* ω_0 *over* $\mathrm{Reg}(\Omega)$ *such that the following hold:*

(A) ω_0 *has a constant negative holomorphic sectional curvature.*

(B) ω_0 *is complete at infinity. Namely, for each number* $R > 0$ *and* $p_0 \in \mathrm{Reg}(\Omega)$*, the ball centered at* $p_0 \in \mathrm{Reg}(\Omega)$ *with radius* R *(with respect to* ω_0*) has a compact closure in* Ω.

(C) *For a certain* $p \in M$ *there are a small neighborhood* U_p *of* p *in* $\overline{\Omega}$ *and a diffeomorphic map* F *from* U_p *to* V_q *that is holomorphic over* $U_p \cap \mathrm{Reg}(\Omega)$*, where* V_q *is a certain neighborhood of* q *in* $\overline{\mathbb{B}}^n$ *with* $F(p) = q$ *and* $F(M \cap U_p) = V_q \cap \partial\mathbb{B}^n$ *such that* $F^*(\omega_{\mathbb{B}^n}) = \omega_0$ *on* $U_p \setminus M$*. Here* $\omega_{\mathbb{B}^n}$ *is the Bergman metric of the unit ball* $\mathbb{B}^n \subset \mathbb{C}^n$.

When $\overline{\Omega}$ is a smoothly bounded spherical domain in a complex Euclidean space, by studying the projective extension (see [7]) of the Cartan–Chern–Moser structure bundle over M into the interior of $\overline{\Omega}$, Chern and Ji in [8] showed that, under the hypothesis in Corollary 1.3, the development map element (f, D) extends along any curve in $\overline{\Omega}$ as bimeromorphic maps. The paper of Burns and Ryu [3] also mentioned a preprint of Burns in the 1990s ([3, Reference [5]]) and indicated that certain results similar to those in Corollaries 1.4 and 1.3 were obtained in that preprint of Burns. Since Burns' preprint does not seem to be available to a reader, Corollaries 1.3 and 1.4 thus serve as accessible complete proofs of these very useful results for the study of spherical Stein spaces.

When the Ω in Corollary 1.4 is not smooth, the hyperbolic (i.e., having a constant negative holomorphic sectional curvature) metric may not be complete at the singular point. For instance, as proved in Huang [16], if $\overline{\Omega}$ is embedded in a complex Euclidean space with $\partial\Omega$ spherical and algebraic, then Ω has exactly one singular point which is a finite quotient singularity of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$. The naturally inherited hyperbolic metric satisfies all properties stated Corollary 1.4 and is not complete at the singular point (i.e., there is a Cauchy sequence in $\mathrm{Reg}(\Omega)$ with respect to the metric ω_0 that converges to the singular point). The following example shows that the hyperbolic metric in Corollary 1.4 may not be unique (even up to scaling) in general in the one-dimensional case:

Let Δ be the unit disk and let X be the singular Riemann surface in $\mathbb{C}^2$ given by

$$\{(z, w) \in \mathbb{C}^2 : w^2 = z^3\} \cap \Delta^2.$$

It is the image of the map $t \to (t^2, t^3)$, $t \in \Delta$ and has an isolated singularity at $(0, 0)$. Note $X^* := X \setminus \{(0, 0)\}$ is biholomorphic to punctured disk $\Delta^* := \Delta \setminus \{0\}$. The canonical metric on Δ^* induces a metric ω_1 on X^* and the Bergman metric on Δ^* induces a metric ω_2 on X^*. Notice that both metrics are complete at infinity. But ω_1 is complete near the singularity, while ω_2 is not. We claim that both ω_1 and ω_2 satisfy properties in (C) of Corollary 1.4. Indeed, ω_2 is identical with the Bergman metric of Δ near boundary and thus has this property. To understand ω_1, we look at the covering map $\pi(\xi) = e^{i\xi}$ from the upper half plane

$\mathcal{H} = \{\xi \in \mathbb{C} : \operatorname{Im}\xi > 0\}$ to Δ^*. Note that π maps biholomorphically $\{\xi \in \mathcal{H} : 0 < \operatorname{Re}\xi < 2\pi\}$ to an open subset of Δ^* whose boundary contains an open piece of the circle. From this, one sees that the induced canonical hyperbolic metric on Δ^* and thus ω_1 on X^* satisfies (C) of Corollary 1.4. Hence we have two very distinct metrics on X^* that both satisfy properties in (A), (B) and (C) of Corollary 1.4.

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2. Proof of Theorem 1.1

Let $\Omega = \{z \in \mathbb{C}^n : \rho(z) > 0\}$ be a strictly pseudoconvex domain with a smooth defining function ρ. In [11], Fefferman showed that the Bergman kernel function $K(z) = K(z, \overline{z})$ of Ω has the asymptotic expansion

$$K(z) = \frac{\phi(z)}{\rho^{n+1}(z)} + \psi(z)\log\rho(z),$$

where $\phi, \psi \in C^\infty(\overline{\Omega})$ and $\phi|_{\partial\Omega} \neq 0$. In particular, if the boundary $\partial\Omega$ of Ω is spherical, then ψ vanishes to infinite order at the boundary $\partial\Omega$.

We first recall the notion of Fefferman defining functions or Fefferman approximate solutions. Consider the following Monge–Ampère-type equation introduced in [12]:

$$J(u) := (-1)^n \det\begin{pmatrix} u & u_{\overline{\beta}} \\ u_\alpha & u_{\alpha\overline{\beta}} \end{pmatrix} = u^{n+1} \det\left(\left(\log\frac{1}{u}\right)_{\alpha\overline{\beta}}\right) = 1 \quad \text{in } \Omega,$$

with $u = 0$ on $b\Omega$. Fefferman proved that for any bounded strictly pseudoconvex domain Ω with smooth boundary, there is a smooth positive defining function r of Ω such that

$$J(r) = 1 + O(r^{n+1}),$$

which is called a Fefferman approximate solution or a Fefferman defining function of Ω. Moreover, if r_1, r_2 are two Fefferman approximate solutions, then $r_1 - r_2 = O(\rho^{n+2})$, where ρ is a given defining function of Ω.

We next recall the Moser normal form theory [9] and the notion of Fefferman scalar boundary invariants ([13]): Let $M \subset \mathbb{C}^n$ be a real analytic strictly pseudoconvex hypersurface containing $p \in \mathbb{C}^n$. Then there exists a coordinates system $z = (\xi, w) := (\xi_1, \ldots, \xi_{n-1}, w)$ such that in the new coordinates, $p = 0$ and M is defined near p by an equation of the form

$$u = |\xi|^2 + \sum_{|\alpha|,|\beta|\geq 2,\, l\geq 0} A^l_{\alpha\overline{\beta}}\xi^\alpha\overline{\xi}^\beta v^l, \tag{2.1}$$

where $w = u + iv$, and α, β are lists of indices between 1 and $n-1$ (here $|\alpha|$ and $|\beta|$ denote their lengths). Moreover, the coefficients $A^l_{\alpha\overline{\beta}} \in \mathbb{C}$ satisfy the following:

- $A^l_{\alpha\overline{\beta}}$ is symmetric with respect to permutation of the indices in α.
- It holds $\overline{A^l_{\alpha\overline{\beta}}} = A^l_{\beta\overline{\alpha}}$.

- We have $\operatorname{tr} A^l_{22\bar{}} = 0$, $\operatorname{tr}^2 A^l_{23\bar{}} = 0$ and $\operatorname{tr}^3 A^l_{33\bar{}} = 0$, where $A^l_{p\bar{q}}$ is the symmetric tensor $[A^l_{\alpha\bar{\beta}}]_{|\alpha|=p,\,|\beta|=q}$ on $\mathbb{C}^{n-1}$ and the traces are the usual tensorial traces with respect to $\delta_{i\bar{j}}$. Namely, if we write
$$\mathcal{A}^l_{p\bar{q}}(\xi,\bar{\xi}) = \sum_{|\alpha|=p,\,|\beta|=q} A^l_{\alpha\bar{\beta}}\xi^\alpha\bar{\xi}^\beta,$$
then we have, for each l,
$$\Delta\big(\mathcal{A}^l_{22\bar{}}(\xi,\bar{\xi})\big) = \Delta^2\big(\mathcal{A}^l_{23\bar{}}(\xi,\bar{\xi})\big) = \Delta^3\big(\mathcal{A}^l_{33\bar{}}(\xi,\bar{\xi})\big) \equiv 0.$$
Here Δ is the standard Laplacian operator in ξ.

Here (2.1) is called a normal form of M at p. When M is merely smooth, the expansion is in the formal sense. We call $[A^l_{\alpha\bar{\beta}}]$ the normal form coefficients. Recall that a boundary scalar invariant at $p \leftrightarrow 0$, or briefly an invariant of weight $w \geq 0$, is a polynomial P in the normal form coefficients $[A^l_{\alpha\bar{\beta}}]$ of $\partial\Omega$ satisfying certain transformation laws. (See [13] for more details on this transformation law.) Using a Fefferman defining function in the asymptotic expansion of the Bergman kernel function

$$K(z) = \frac{\phi(z)}{r^{n+1}(z)} + \psi(z)\log r(z), \tag{2.2}$$

with $\phi, \psi \in C^\infty(\overline{\Omega})$, $\phi|_{\partial\Omega} \neq 0$, then $\phi \bmod r^{n+1}$, $\psi \bmod r^\infty$ are locally determined. Moreover, if $\partial\Omega$ is in its normal form at $p = 0 \in b\Omega$, then any Taylor coefficient at 0 of ϕ of order $\leq n$, and that of ψ of any order is a universal polynomial in the normal form coefficients $[A^l_{\alpha\bar{\beta}}]$. (See Boutet and Sjöstrand [4] and a related argument in [13].) In particular, we state the following result from [10]:

Proposition 2.1 ([10]). *Let Ω be as above and suppose that $\partial\Omega$ is in the Moser normal form up to sufficiently high order. Let r be a Fefferman defining function, and let ϕ, ψ be as in* (2.2)*. Then*
$$\phi = \frac{n!}{\pi^n} + O(r^2)$$
and $P_2 = \frac{\phi - \frac{n!}{\pi^n}}{r^2}|_{\partial\Omega}$ defines an invariant of weight 2 *at* 0*. Furthermore, if $n = 2$, then $P_2 = 0$. If $n \geq 3$, then $P_2 = c_n\|A^0_{22\bar{}}\|^2$ for some universal constant $c_n \neq 0$.*

As mentioned earlier, Theorem 1.1 is known in the case of $n = 2$ in [15] and [26]. We next assume that $n \geq 3$.

Proof of Theorem 1.1. It is well known that the Bergman metric of the unit ball is complete and hyperbolic, and in particular Kähler–Einstein. Moreover, the Bergman metric is invariant under biholomorphic transformations. Thus if a domain Ω is biholomorphic to the unit ball, then its Bergman metric is Kähler–Einstein. It remains to prove the converse statement. Assume Ω is a smoothly bounded strongly pseudoconvex domain and its Bergman metric is Kähler–Einstein. Recall the Fefferman asymptotic expansion

$$K(z) = \frac{\phi(z)}{\rho^{n+1}(z)} + \psi(z)\log\rho(z) = \frac{\phi + \rho^{n+1}\psi\log\rho}{\rho^{n+1}} \quad \text{for } z \in \Omega \tag{2.3}$$

with $\phi, \psi \in C^\infty(\overline{\Omega})$ and $\phi|_{\partial\Omega} \neq 0$, where $\rho \in C^\infty(\overline{\Omega})$ is a smooth defining function of Ω

with $\Omega = \{z \in \mathbb{C}^n : \rho(z) > 0\}$. Since $K(z) > 0$ for $z \in \Omega$, we have

$$\phi + \rho^{n+1}\psi \log \rho > 0 \quad \text{for } z \in \Omega.$$

Thus

$$(K)^{-\frac{1}{n+1}}(z) = \frac{\rho}{(\phi + \rho^{n+1}\psi \log \rho)^{\frac{1}{n+1}}}$$

is well-defined in Ω.

We notice that the Kähler–Einstein condition of the Bergman metric is equivalent to the fact that $\log K(z)$ is a Kähler–Einstein potential function of Ω. More precisely, we have

$$J\left[\left(\frac{\pi^n}{n!}K(z)\right)^{-\frac{1}{n+1}}\right] = 1$$

for $z \in \Omega$. (See [15]). Let $r_0(z) := (\frac{\pi^n}{n!}K)^{-\frac{1}{n+1}}$. We hence have that $r_0(z) > 0$ and $J(r_0) = 1$ in Ω. We next recall the following result of Fu and Wong [15]:

Proposition 2.2. *Let $\Omega = \{z \in \mathbb{C}^n : \rho > 0\}$ be a bounded strongly pseudoconvex domain with a smooth defining function ρ. If the Bergman metric of Ω is Kähler–Einstein, then the coefficient of the logarithmic term in Fefferman's expansion* (2.3) *vanishes to infinite order at $b\Omega$, i.e., $\psi = O(\rho^k)$ for any $k > 0$.*

As a consequence, $\phi + \rho^{n+1}\psi \log \rho$ extends smoothly to a neighborhood of $\overline{\Omega}$. Since $\phi|_{\partial\Omega} \neq 0$, we have

$$\phi + \rho^{n+1}\psi \log \rho > 0 \quad \text{for all } z \in \overline{\Omega}.$$

Hence r_0 extends smoothly to a neighborhood of $\overline{\Omega}$ and it is then easy to conclude that r_0 is a Fefferman defining function of Ω. Then from the way r_0 was constructed, it follows that

$$K(z) = \frac{n!}{\pi^n} r_0^{-(n+1)}. \tag{2.4}$$

Comparing (2.4) with (2.2), we arrive at the conclusion that if we let $r = r_0$ in (2.2), then $\phi \equiv \frac{n!}{\pi^n}$. Then it follows from Proposition 2.1 that $P_2 = c_n \|A^0_{2\bar{2}}\|^2 = 0$ at $p \in \partial\Omega$ if $\partial\Omega$ is in the Moser normal form up to sufficiently high order at p with $A^0_{2\bar{2}}$ being the Chern–Moser–Weyl tensor at p. Consequently, $A^0_{2\bar{2}} = 0$ in each Moser normal coordinates at each point in $\partial\Omega$, for $c_n \neq 0$. That is, every boundary point of $\partial\Omega$ is a CR umbilical point. We now apply a similar argument of Nemirovski and Shafikov in [26] to show Ω is holomorphically equivalent to the unit ball by applying Corollary 1.4 (or the Chern–Ji [8] or Kerner extension theorem [18]) and the Qi-Keng Lu uniformization theorem [22] as follows:

Since $\partial\Omega$ is now spherical, we fix a point $p_0 \in \partial\Omega$ and an open piece U of p_0 in $\partial\Omega$ such that there is a smooth CR diffeomorphism F from U to an open piece of the unit sphere in $\mathbb{C}^n$. Now, by Corollary 1.4, we obtain a well-defined complete Kähler metric ω_0 on Ω, which is of constant negative holomorphic sectional curvature. (Note that ω_0 is complete for Ω is assumed to be smooth.) Now, by the uniqueness of the complete Kähler–Einstein metric over Ω_0 (see [6]), since the Bergman metric on Ω is assumed to be Kähler–Einstein, we conclude that the Bergman metric of Ω is proportional to ω_0. Finally, by the classical Qi-Keng Lu uniformization theorem [22], Ω is biholomorphic to the ball. □

3. Proof of Theorem 1.2

In this section, we give a proof for Theorem 1.2. We assume that Ω is a Stein space with a smooth compact strongly pseudoconvex boundary $M = \partial\Omega$. Let (f, D) be a CR map element over M into $\mathbb{C}^N$ and ψ a plurisubharmonic function over $\mathbb{C}^N$, as in Theorem 1.2. Write the singular set of Ω to be $\mathrm{sing}(\Omega)$, which has at most finitely many points. Write $\overline{\Omega} = \Omega \cup M$, $\mathrm{Reg}(\Omega) = \Omega \setminus \mathrm{sing}(\Omega)$ and $\mathrm{Reg}(\overline{\Omega}) = \overline{\Omega} \setminus \mathrm{sing}(\Omega)$ as before.

As mentioned in the introduction, (f, D), in general, does not admit holomorphic continuation across a singular point (even a normal singular point) of Ω, as demonstrated by the following example.

Example 3.1. Let $\overline{\Omega}$ be the Stein space with boundary defined by

$$\overline{\Omega} = \left\{ W = (w_1, w_2, w_3) \in \mathbb{C}^3 : \sum_{j=1}^{3} |w_j|^2 \le 1,\ w_2^2 = 2w_1 w_3 \right\}.$$

Let $\pi : \overline{\mathbb{B}^2} \to \overline{\Omega}$ be given by $\pi(z_1, z_2) = (z_1^2, \sqrt{2} z_1 z_2, z_2^2)$. Note that π is 2 to 1 covering from $\mathbb{B}^2 \setminus \{0\}$ to $\Omega \setminus \{0\}$ and $\pi(0) = 0$. Fix a point $p_0 = (\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}) \in M := \partial\Omega$. Let D be a small simply connected open piece of M containing p_0 and (f, D) a CR mapping element given by $f(W) = (\sqrt{w_1}, \sqrt{w_3})$. Here

$$\sqrt{w} = \sqrt{|w|} e^{i\frac{\theta}{2}}$$

for $w = |w| e^{i\theta}$ with $-\pi < \theta < \pi$. Notice that f maps D into $\partial\mathbb{B}^2$ and thus (f, D) is a development map element. Notice that M is spherical. By Corollary 1.3, (f, D) admits uniformly bounded holomorphic map continuation along curves inside $\mathrm{Reg}(\overline{\Omega})$. It does not admit a holomorphic map continuation along a certain curve γ in $\overline{\Omega}$ with $\gamma(0) = p_0$ and $\gamma(1) = 0$.

Indeed, to see the claim made in Example 3.1, set γ_1 be a curve in $\overline{\Omega}$ such that $\gamma_1(0) = p_0$ and $\gamma_1(1) = (\epsilon, 0, 0)$ for some small $\epsilon > 0$ such that γ_1 never intersects $\{w_1 = 0\}$ and let the curve γ_2 in $\overline{\Omega}$ be given by $\gamma_2(t) = (\epsilon t, 0, 0), 0 \le t \le 1$. Write $\gamma = \gamma_1 + (-\gamma_2)$. Note that $\gamma([0, 1))$ does not pass through $\{w_1 = 0\}$. We know for every $0 \le t_0 < 1$, if we write (h, V) for the branch we obtain at $\gamma(t_0)$ with $h = (h_1, h_2)$ on V, then h_1 equals either $[-\sqrt{w_1}]_{\gamma(t_0)}$ or $[\sqrt{w_1}]_{\gamma(t_0)}$. Without loss of generality, we assume it is the latter when $t_0 (\neq 1)$ is close to 1. Suppose we can extend (f, D) holomorphically along γ to get a holomorphic branch (g, U) at $\gamma(1) = 0$ (in particular, $0 \in U$). Write $g = (g_1, g_2)$ on U. Then $g_1 = \sqrt{w_1}$ near $(\epsilon_0, 0, 0)$ for a sufficiently small ϵ_0. But this is impossible as we can find a loop σ in U given by $\sigma(t) = (\epsilon_0 e^{2\pi i t}, 0, 0) \in \overline{\Omega},\ 0 \le t \le 1$, so that we get a different branch when applying holomorphic continuation to $\sqrt{w_1}$ along σ.

The proof of Theorem 1.2 will be split into several steps to be established in the following two Section 3.1 and 3.2. Before proceeding to the proof, we first fix a Morse plurisubharmonic defining function ρ of $\overline{\Omega}$. More precisely, we choose a bounded plurisubharmonic exhaustion function $\rho : \overline{\Omega} \to [-\infty, 0]$ of Ω such that $\rho \equiv 0$ on M, $\rho < 0$ in Ω and $\rho(z) = -\infty$ if and only if z is a singular point of Ω. In addition, $d\rho|_M \neq 0$ and ρ is smooth and strongly plurisubharmonic on $\mathrm{Reg}(\Omega)$. Moreover, ρ has only finitely many critical points in $\mathrm{Reg}(\Omega)$ and they are all non-degenerate. The existence of such a ρ is guaranteed by the assumption on $\overline{\Omega}$ and Morse function theory. (The local existence of such a function near a singular point can be

found in Milnor [24]. Away from singular points, we refer to the book of Forstnerič [14] for such a construction. Then one applies the Morse approximation to get our ρ.)

For clarity, we fix a Riemannian metric ds^2 over $\mathrm{Reg}(\overline{\Omega})$ which induces a distance function $\widehat{d}(x, y)$ for $x, y \in \mathrm{Reg}(\overline{\Omega})$. Write X_ρ for the dual vector field of $-d\rho$ with respect to ds^2 over $\mathrm{Reg}(\overline{\Omega})$ away from the critical points of ρ.

3.1. Proof of Theorem 1.2: Part I. This step is the same for the two cases (1)and (2) in Theorem 1.2 and it aims to prove (f, D) admits holomorphic map continuation in a tube neighborhood of M in $\overline{\Omega}$. We emphasize that the boundedness assumption in case (2) is not needed in this subsection to derive the extension. We choose three finite open convex cover $\{W_j^{(k)}\}_{j=1}^m$, $k = 1, 2, 3$, of $(M, ds^2|_M)$ with

$$W_j^{(1)} \subset\subset W_j^{(2)} \subset\subset W_j^{(3)} \quad \text{for each } j\,.$$

Moreover, we make $W_j^{(3)}$ sufficiently small for each j so that a neighborhood of $\overline{W_j^{(3)}}$ on M is CR diffeomorphic to a strongly pseudoconvex hypersurface in $\mathbb{C}^n$. Write D_j for the union of all smooth holomorphic disks attached to $W_j^{(3)}$ which can be deformed through a continuous family of disks to points in $W_j^{(3)}$. For $0 < \epsilon_1 \ll 1$ and $1 \le k \le 3$, we let $\widehat{W}_{j,\epsilon_1}^{(k)}$ be the open subset of $\overline{\Omega}$ obtained by flowing each point $p \in W_j^{(k)}$ along the orbit of X_ρ (where X_ρ is as defined right before Section 3.1) with time $0 \le t < \epsilon_1$. Note we can find an $\epsilon_1 > 0$ sufficiently small such that

$$\widehat{W}_{j,\epsilon_1}^{(2)} \subset\subset (D_j \cup W_j^{(3)}) \quad \text{for each } j\,,$$

and that $\widehat{W}_{j,\epsilon_1}^{(2)}$ is topological trivial (recall $W_j^{(2)}$ is chosen to be convex). In particular, for each point $q \in \widehat{W}_{j,\epsilon_1}^{(2)}$, there is a small embedded holomorphic disk Δ_q containing q attached to $W_j^{(3)}$ that is contained in D_j and can be continuously deformed to a point on $W_j^{(3)}$. Fix such an ϵ_1. Write for $r_2 < r_1 \le 0$,

$$\overline{\Omega}_{r_2,r_1} = \{p \in \overline{\Omega} : r_2 < \rho \le r_1\}.$$

We emphasize that $\overline{\Omega}_{r_2,r_1}$ only contains its outer boundary but not its inner boundary.

Let $0 < \epsilon_2 \ll \epsilon_1$ be small enough such that

$$\overline{\Omega}_{-\epsilon_2,0} \subset \bigcup_{j=1}^m \widehat{W}_{j,\epsilon_1}^{(2)}.$$

Define $J_{\epsilon_2} : \overline{\Omega}_{-\epsilon_2,0} \to M$ for the retract of $\overline{\Omega}_{-\epsilon_2,0}$ to M which maps every point p in $\overline{\Omega}_{-\epsilon_2,0}$ through the orbit of X_ρ to the corresponding point on M. Note that J_{ϵ_2} is a smooth map for a small ϵ_2. By the Lewy–Baouendi–Treves theorem, we see that every continuous CR function h on $W_j^{(3)}$ extends to a holomorphic function in $\widehat{W}_{j,\epsilon_1}^{(2)}$ that is continuous up to $W_j^{(2)}$. Let $\gamma : [0, 1] \to \overline{\Omega}_{-\epsilon_2,0}$ be a curve. There is a corresponding curve $\widehat{\gamma} := J_{\epsilon_2} \circ \gamma$ on M. By making ϵ_1, ϵ_2 sufficiently small, we note from the definition of J_{ϵ_2} that $\gamma(t) \subset \widehat{W}_{j,\epsilon_1}^{(k)}$ for some $1 \le j \le m$, $1 \le k \le 3$ if and only if $\widehat{\gamma}(t) \subset W_j^{(k)}$. We next prove the following lemma.

Lemma 3.2. *Let (f, D) and ψ be as in Theorem* 1.2. *Fix a curve $\gamma : [0, 1] \to \overline{\Omega}_{-\epsilon_2,0}$ with $\gamma(0) = p_0 \in D$ and let $\widehat{\gamma}$ be as above.*

(1) *We can find $\{W_{j_l}^{(2)}\}_{l=0}^k$ and $\{\widehat{W}_{j_l,\epsilon_1}^{(2)}\}_{l=0}^k$ with $0 \le j_l \le m$, together with continuous CR map elements $\{(f_l, W_{j_l}^{(2)})\}_{l=0}^k$ on M and holomorphic map elements $\{(g_l, \widehat{W}_{j_l,\epsilon_1}^{(2)})\}_{l=0}^k$*

in $\mathrm{Reg}(\overline{\Omega})$ *such that the following hold:*

(a) *The point* $p_0 \in W_{j_0}^{(2)}$ *and* $f_0 = f$ *in a neighborhood of* p_0 *on* M.

(b) *There is a partition* $0 = \delta_0 < \delta_1 < \cdots < \delta_{k+1} = 1$ *of* $[0,1]$ *such that*

$$\gamma([\delta_l, \delta_{l+1}]) \subset \widehat{W}_{j_l,\epsilon_1}^{(2)}, \widehat{\gamma}([\delta_l, \delta_{l+1}]) \subset W_{j_l}^{(2)}$$

for $0 \le l \le k$.

(c) $f_l = g_l$ *on* $W_{j_l}^{(2)}$ *for* $0 \le l \le k$.

(d) $f_l = f_{l+1}$ *on* $W_{j_l}^{(2)} \cap W_{j_{l+1}}^{(2)}$ *and* $g_l = g_{l+1}$ *on* $\widehat{W}_{j_l,\epsilon_1}^{(2)} \cap \widehat{W}_{j_{l+1},\epsilon_1}^{(2)}$ *for* $0 \le l \le k-1$.

Consequently, $\{(g_l, \widehat{W}_{j_l,\epsilon_1}^{(2)})\}_{l=0}^{k}$ *(resp.* $\{(f_l, W_{j_l}^{(2)})\}_{l=0}^{k}$*) induces a holomorphic (resp. CR) continuation of* (f, D) *along* γ *(resp.* $\widehat{\gamma}$*).*

(2) *It holds that* $\psi \circ g_l \le 0$ *on* $\widehat{W}_{j_l,\epsilon_1}^{(2)}$ *for all* $0 \le l \le k$. *Moreover, assume that there is some* l_0 *with* $0 \le l_0 \le k$ *and a point* $q \in \widehat{W}_{j_{l_0},\epsilon_1}^{(2)} \setminus M$ *such that* $\psi \circ g_{l_0}(q) = 0$. *Then* $\psi(g) \equiv 0$ *for any holomorphic branch* (g, V) *of* (f, D) *in* $\overline{\Omega}_{-\epsilon_2,0}$.

Proof of Lemma 3.2. By the uniform continuity of $\widehat{\gamma}$ on $[0,1]$ and the Lebesgue lemma, we can find some $\epsilon > 0$ such that for any sub-interval I^* of $[0,1]$ with length bounded by ϵ, there exists some $1 \le j(I^*) \le m$ satisfying $\widehat{\gamma}(I^*) \subset W_{j(I^*)}^{(1)}$. Note $p_0 = \gamma(0) = \widehat{\gamma}(0)$ is contained in $W_{j_0}^{(1)}$ for some $1 \le j_0 \le m$. Set $\delta_0 = 0$ and let $\delta_1 \in (0,1]$ be the (unique) number (if exists) such that $\widehat{\gamma}([0,\delta_1)) \subset W_{j_0}^{(1)}$ but $\widehat{\gamma}(\delta_1) \notin W_{j_0}^{(1)}$. Note here that we choose j_0 such that δ_1 takes the largest value and thus we must have $\delta_1 \ge \epsilon$ if it exists. If such a number δ_1 does not exist, this means that $\widehat{\gamma}([0,1]) \subset W_{j_0}^{(1)}$ and consequently $\gamma([0,1]) \subset \widehat{W}_{j_0,\epsilon_1}^{(1)} \cap \overline{\Omega}_{-\epsilon_2,0}$. Note first by the continuous CR map continuation assumption and the monodromy theorem, the germ of f at p_0 extends to a CR function f_0 on $W_{j_0}^{(3)}$ as $W_{j_0}^{(3)}$ is simply connected. Secondly $(f_0, W_{j_0}^{(2)})$ can be extended to a holomorphic map element $(g_0, \widehat{W}_{j_0,\epsilon_1}^{(2)})$. Then $(g_0, \widehat{W}_{j_0,\epsilon_1}^{(2)})$ induces a holomorphic continuation of (f, D) along γ and the first part of lemma is established.

Now assume that such a δ_1 exists. First, as above, the germ of f at p_0 extends to a CR function f_0 on $W_{j_0}^{(3)}$ and f_0 extends to a holomorphic function element $(g_0, \widehat{W}_{j_0,\epsilon_1}^{(2)})$ (thus (c) holds for $l = 0$). We then look at $\widehat{\gamma}(\delta_1)$. Note $\widehat{\gamma}(\delta_1) \in W_{j_0}^{(2)} \subset W_{j_0}^{(3)}$ and there exists some $1 \le j_1 \le m$ such that $\widehat{\gamma}(\delta_1) \in W_{j_1}^{(1)}$. By the same reason as above, the germ of f_0 at $\widehat{\gamma}(\delta_1)$ extends to a continuous CR map f_1 on $W_{j_1}^{(3)}$ and f_1 extends to a holomorphic map element $(g_1, \widehat{W}_{j_1,\epsilon_1}^{(2)})$ (thus condition (c) holds for $l = 1$). Note that f_1 and f_0 coincide near $\widehat{\gamma}(\delta_1)$. Moreover, since $W_{j_0}^{(3)} \cap W_{j_1}^{(3)}$ and $\widehat{W}_{j_0,\epsilon_1}^{(2)} \cap \widehat{W}_{j_1,\epsilon_1}^{(2)}$ are simply connected by our convexity assumption, we conclude that

$$f_0 = f_1 \quad \text{on } W_{j_0}^{(3)} \cap W_{j_1}^{(3)} \qquad \text{and} \qquad g_1 = g_0 \quad \text{in } \widehat{W}_{j_0,\epsilon_1}^{(2)} \cap \widehat{W}_{j_1,\epsilon_1}^{(2)}$$

(thus (d) holds for $l = 0$).

Then we pick the (unique) number δ_2 (if exists) such that

$$\widehat{\gamma}([\delta_1, \delta_2)) \subset W_{j_1}^{(1)} \quad \text{and} \quad \widehat{\gamma}(\delta_2) \notin W_{j_1}^{(1)}.$$

Note again we choose j_1 such that δ_2 takes the largest possible value if it exists and thus we must have $\delta_2 - \delta_1 \ge \epsilon$. And we run the same procedure as above to obtain $1 \le j_2 \le m$ such that $\widehat{\gamma}(\delta_2) \in W_{j_2}^{(1)}$, along with a CR map element $(f_2, W_{j_2}^{(3)})$ and a holomorphic map element $(g_2, \widehat{W}_{j_2,\epsilon_1}^{(2)})$ such that $f_1 = f_2$ on $W_{j_1}^{(3)} \cap W_{j_2}^{(3)}$, and $g_1 = g_2$ in $\widehat{W}_{j_1,\epsilon_1}^{(2)} \cap \widehat{W}_{j_2,\epsilon_1}^{(2)}$.

By repeating the above procedure for at most $[\frac{1}{\epsilon}]+1$ times, we arrive at some positive number δ_k such that $\delta_k+\epsilon\geq 1$. More precisely, we obtain a partition

$$0=\delta_0<\delta_1<\cdots<\delta_k<\delta_{k+1}=1$$

of $[0,1]$ and a collection of integers $1\leq j_1,\ldots,j_k\leq m$ such that $\widehat{\gamma}([\delta_l,\delta_{l+1}))\in W^{(1)}_{j_l}$ and $\widehat{\gamma}(\delta_{l+1})\notin W^{(1)}_{j_l}$ for $1\leq l\leq k$ (in particular, one has $\widehat{\gamma}([\delta_l,\delta_{l+1}])\in W^{(2)}_{j_l}$ for all k). Moreover, there are a collection of CR map elements $\{(f_l,W^{(3)}_{j_l})\}_{l=0}^k$ and holomorphic map elements $\{(g_l,\widehat{W}^{(2)}_{j_l,\epsilon_1})\}_{l=1}^k$ satisfying condition (c) and (d). This proves part (1) of the lemma.

To prove part (2), we note for every $p\in\widehat{W}^{(2)}_{j_l,\epsilon_1}\setminus M$, there exists a small holomorphic disk $\Delta_p\subset\widehat{W}^{(3)}_{j_l,\epsilon_1}$ attached to $W^{(3)}_{j_l}$ such that $p\in\Delta_p$. Moreover, $\psi\circ g_l$ is subharmonic in Δ_p, continuous up to $\partial\Delta_p$, and agrees with $\psi\circ f_l$ on $\partial\Delta_p$. By the assumption that (f,D) admits a continuous CR map continuation with ψ-estimate in M and thus in particular $\psi\circ f_l\leq 0$ on $\partial\Delta_p\subset W^{(3)}_{j_l}$, we conclude by the maximum principle that $\psi\circ g_l(p)\leq 0$. As p is arbitrary, we have $\psi\circ g_l\leq 0$ on $\widehat{W}^{(2)}_{j_l,\epsilon_1}$ for all $0\leq l\leq k$. Now suppose there is some $0\leq l_0\leq k$ and a point $q\in\widehat{W}^{(2)}_{j_{l_0},\epsilon_1}\setminus M$ such that $\psi\circ g_{l_0}(q)=0$. This means $\psi\circ g_{l_0}$ achieves its maximum at an interior point q. Since $\psi\circ g_{l_0}$ is subharmonic in $\widehat{W}^{(2)}_{j_{l_0},\epsilon_1}$, we conclude that

$$\psi\circ g_{l_0}\equiv 0\quad\text{in }\widehat{W}^{(2)}_{j_{l_0},\epsilon_1}.$$

As $g_l=g_{l+1}$ for each $0\leq l\leq k$ on $\widehat{W}^{(2)}_{j_l,\epsilon_1}\cap\widehat{W}^{(2)}_{j_{l+1},\epsilon_1}$, we have, for each l, $\psi\circ g_l$ attains its maximum at an interior point and thus is constant. In particular, $\psi\circ g_0\equiv 0$ in $\widehat{W}^{(2)}_{j_0,\epsilon_1}$ and $\psi\circ f_0=0$ in $W^{(2)}_{j_0}$. Now let (g,U) be a holomorphic branch of (f,D) in $\overline{\Omega}_{-\epsilon_2,0}$. Then there is a path σ in $\overline{\Omega}_{-\epsilon_2,0}$ connecting p_0 to some point $p_1\in U$ such that (g,U) is obtained by holomorphic continuation of (f,D) along σ. By part (1) of Lemma 3.2, writing $\widehat{\sigma}:=J_{\epsilon_1}\circ\sigma$, there exist holomorphic map elements $\{(\widetilde{g}_l,\widehat{W}^{(2)}_{i_l,\epsilon_1})\}_{l=0}^{\nu}$ and CR map elements $\{(\widetilde{f}_l,W^{(2)}_{i_l,\epsilon_1})\}_{l=0}^{\nu}$ that satisfy the conditions (a), (b), (c), and (d), and induce a holomorphic continuation and a CR continuation of (f,D) along σ and $\widehat{\sigma}$. By the monodromy theorem, we have $g=\widehat{g}_\nu$ near q in $U\cap\widehat{W}^{(2)}_{i_\nu,\epsilon_1}$. Note we have $\widetilde{f}_0=f_0$ near p_0 and $\widetilde{g}_0=g_0$ near p_0. Then $\psi\circ\widetilde{g}_0\equiv 0$ on $\widehat{W}^{(2)}_{i_0,\epsilon_1}$. Applying the maximum principle for subharmonic functions as above, we obtain that $\psi\circ\widetilde{g}_l\equiv 0$ in $\widehat{W}^{(2)}_{i_l,\epsilon_1}$ for every $0\leq l\leq\nu$. In particular, we have $\psi\circ g\equiv 0$ on U. This proves Lemma 3.2. □

Summarizing the arguments above, we have the following:

Lemma 3.3. *Let (f,D) be as in Theorem* 1.2. *The continuous CR map element (f,D) admits holomorphic continuation in $\overline{\Omega}_{-\epsilon_2,0}$ with ψ-estimate. In the dimension-two case, if (f,D) admits C^*-uniformly bounded CR map extension along curves in M, then it also admits C^*-uniformly bounded holomorphic continuation with ψ-estimate in $\overline{\Omega}_{-\epsilon_2,0}$. Moreover, if there is a holomorphic branch (h,U) of (f,D) in $\overline{\Omega}_{-\epsilon_2,0}$ such that $\psi(h(p))=0$ at some point p in $U\setminus M$. Then $\psi(g)\equiv 0$ for every holomorphic branch (g,V) of (f,D) in $\overline{\Omega}_{-\epsilon_2,0}$.*

3.2. Proof of Theorem 1.2: Part II. In this subsection, we finish the proof of Theorem 1.2. The treatment for the two cases are different. We will apply the method of continuous families of holomorphic curves which is a typical machinery in the study of holomorphic continuation problem. The use of the Morse function theory to study the holomorphic continuation of multiple-valued holomorphic maps near the boundary to the interior of the pseudoconvex

domain in $\mathbb{C}^n$ with $n \geq 2$ appeared in [17, Section 5]. In the paper by Merkel and Porten [23], they employed the Morse function theory to re-investigate the Hartogs extension theorem of single-valued holomorphic functions. In our argument here, besides the Morse function theory, the Phragmén–Lindelöf principle will play a fundamental role in the case of complex dimension two.

To start with, we make the following definition. Let (f, D) be as in Theorem 1.2 and assume the hypotheses in Theorem 1.2. Let $\widetilde{\Omega}$ be a connected open subset of $\mathrm{Reg}(\overline{\Omega})$ with $D \subset \widetilde{\Omega}$. We say $\widetilde{\Omega}$ has the extendability property if, in the three or higher-dimensional case, (f, D) admits holomorphic continuation along cures in $\widetilde{\Omega}$ with ψ-estimate. And in the dimension-two case, (f, D) is assumed to admit uniformly bounded holomorphic continuation with a bound $C^* > 0$ along curves in $\widetilde{\Omega}$, that also has the ψ-estimate. Here C^* is the least upper bound for the super-norms of the CR map branches of (f, D) obtained along curves inside M. To establish Theorem 1.2, we will need to prove $\mathrm{Reg}(\overline{\Omega})$ has the extendability property. Now set

$$A = \{a < 0 : \ \overline{\Omega}_{a,0} \text{ has the extendability property}\}.$$

We first note that A is not empty as $[-\epsilon_2, 0) \subset A$, where ϵ_2 is chosen as in Section 3.1. Set $b = \inf(A) < 0$. If $b = -\infty$, Theorem 1.2 holds trivially. We will therefore assume $b > -\infty$ in what follows. Note it follows from the definition of b that $\overline{\Omega}_{b,0}$ has the extendability property. Write $\inf(\rho) := \inf\{\rho(z) : z \in \overline{\Omega} \setminus \mathrm{sing}(\Omega)\}$. Before proceeding to the three different scenarios, we recall the following result of Huang-Ji that will be used several times in our later discussions:

Proposition 3.4 ([17, Lemma 5.2]). *The following statements hold:*

(A) *Let $r > 0$ and let γ be a curve in $\overline{\Omega}_{-r,0}$ with $\gamma(0) \in D$ and $\gamma(1) \in M$. Then γ can be continuously deformed inside $\overline{\Omega}_{-r,0}$ with both endpoints fixed to a new curve inside M. Together with the monodromy theorem, one further has:*

(B) *Let $0 < r_1 < r_2$ and suppose that $\overline{\Omega}_{-r_2,0}$ has the extendability property. Then for each $q \in \overline{\Omega}_{-r_1,0}$, the holomorphic map extension of (f, D) along a curve γ inside $\overline{\Omega}_{-r_2,0}$ with $\gamma(0) \in D$, $\gamma(1) = q$ produces the same branch as that by continuing (f, D) along a certain other curve inside $\overline{\Omega}_{-r_1,0}$ with the same endpoints.*

Case I. We first consider the easiest case when $b = \inf(\rho)$. In this case, since $b > -\infty$, we must have that $\mathrm{sing}(\Omega) = \emptyset$. Write $\rho^{-1}(c)$ for the level set $\{z \in \overline{\Omega} : \rho(z) = c\}$. Note in this case we have $\overline{\Omega}_{b,0} = \mathrm{Reg}(\overline{\Omega}) \setminus \rho^{-1}(b)$ and $\rho^{-1}(b)$ is a finite set. Fix any $\widehat{q} \in \rho^{-1}(b)$. By assumption, $\widehat{q}$ is an isolated critical point of ρ and the real Hessian of ρ is strictly positive definite at $\widehat{q}$. We find a small strongly pseudoconvex domain $U_{\widehat{q}}$ containing $\widehat{q}$ with $\overline{U_{\widehat{q}}}$ diffeomorphic to the closed ball and $\overline{U_{\widehat{q}}} \cap \rho^{-1}(b) = \widehat{q}$. To prove Theorem 1.2, we fix an arbitrary curve γ in $\mathrm{Reg}(\overline{\Omega})$ with $\gamma(0) \in D$. Suppose $\gamma([0, 1]) \cap \rho^{-1}(b) = \widehat{q}$ and let $t_0 \in [0, 1]$ be the first one with $\gamma(t_0) = \widehat{q}$ and let $t_1 < t_0$ be the largest one with $\gamma(t_1) \in \partial U_{\widehat{q}}$. We can then continue (f, D) along $\gamma|_{[0,t_1]}$ to get a holomorphic branch (g, U) with $\gamma(t_1) \in U$. Notice that $\partial U_{\widehat{q}}$ is simply connected, we can further continue (g, U) along curves inside the boundary of $U_{\widehat{q}}$ to get a single valued holomorphic map in a neighborhood of $\partial U_{\widehat{q}}$. Now, applying the classical Hartogs theorem, we get a holomorphic function h over $\overline{U_{\widehat{q}}}$ which together with a certain holomorphic map continuation of (f, D) along $\gamma|_{[0,t_1]}$ to get a holomorphic continuation of (f, D) across $\widehat{q}$ along $\gamma|_{[0,t^*]}$ with $t^* - t_0 > \eta$ for a certain fixed positive number η

(here η can be taken to be a positive number such that, if $\gamma(t') = \widehat{q}$ for some $0 \le t' \le 1$, then $\gamma(t) \in U_{\widehat{q}}$ for all $|t - t'| \le \eta$). Continuing this process, we obtain a holomorphic map continuation of (f, D) along γ. The same argument can be applied even if γ goes through several points in $\rho^{-1}(b)$. This proves the existence of the holomorphic continuation in Theorem 1.2. The desired ψ-estimate and the C^*-boundedness in the theorem follow from the maximum principle in an obvious way.

Case II. Suppose that $b > \inf(\rho)$. We assume that $b > \inf(\rho)$ and $M_b := \rho^{-1}(b)$ has no critical points of ρ. In this case, we note $M_{b'}$ is a compact smooth strongly pseudoconvex hypersurface in Ω for $b' \approx b$. Recall by the definition of b, $\overline{\Omega}_{b,0}$ has the extendability property. Then we apply the same argument in Section 3.1 with b' being sufficiently close but greater than b to obtain a small $\epsilon > 0$ such that (f, D) admits holomorphic map extension along curves in $\overline{\Omega}_{b-\epsilon,0}$. Applying maximum principle, we thus conclude $\overline{\Omega}_{b-\epsilon,0}$ has the extendability property. This, however, contradicts with the definition of b. Thus Case II cannot occur.

Case III. We assume that $b > \inf(\rho)$ and ρ has critical points on M_b. Let $p \in \rho^{-1}(b)$ be a critical point of ρ. Then choose a neighborhood U_p of p such that p is the only critical point of ρ in U_p. And choose certain holomorphic coordinates z on U_p such that $z(p) = 0$ and ρ takes the following normal form near $p \leftrightarrow 0$:

$$\rho = |z|^2 + 2\,\mathrm{Re}\sum_{j=1}^{n} \lambda_j z_j^2 + O(|z|^3) + b.$$

Here we have $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n < \infty$ and $\lambda_j \ne \frac{1}{2}$ by the non-degeneracy assumption. Recall that the z_j-direction is called elliptic if $0 \le \lambda_j < \frac{1}{2}$, and hyperbolic if $\lambda_j > \frac{1}{2}$. Also, in some smooth coordinates x on U_p with $x(p) = 0$, we have

$$\rho(x) = \sum_{j=1}^{m} x_j^2 - \sum_{m+1}^{2n} x_j^2 + b.$$

By the plurisubharmonicity, we have $m \ge n$. For a small number $\epsilon > 0$, set

$$V_\epsilon := \{q \in U_p : |x(q)| < \epsilon\} \subset\subset U_p.$$

One directly verifies that $V_\epsilon \cap \overline{\Omega}_{b,0}$ is connected for any small $\epsilon > 0$. We will need the following crucial lemma:

Lemma 3.5. *For a sufficiently small $\epsilon_3 > 0$, it holds, for every $q \in V_{\epsilon_3} \cap \overline{\Omega}_{b,0}$, that if $[g]_q$ is the germ of a holomorphic branch of (f, D) obtained by holomorphic continuation along a curve in $\overline{\Omega}_{b,0}$, then $[g]_q$ extends to a single-valued holomorphic function in V_{ϵ_3} with ψ-estimate (and with the C^*-uniform boundedness in the dimension-two case).*

Proof. We choose holomorphic coordinates z in a small neighborhood $U = U_p$ of p mentioned above. It holds that for a sufficiently small δ and a small $|z|$ that

$$\rho \ge (1 - \delta)|z|^2 + 2\,\mathrm{Re}\sum_{j=1}^{n} \lambda_j z_j^2 + b. \tag{3.1}$$

(a) We first assume ρ has an elliptic direction at $z(p) = 0$, say the z_1-direction (i.e., $\lambda_1 < \frac{1}{2}$). Write Δ for the unit disk in $\mathbb{C}$. Fix small numbers $0 < \widetilde{\epsilon} \ll 1$ and $0 < \eta \ll 1$. We define for $0 < t < \widetilde{\epsilon}$, a continuous family (parametrized by t) of holomorphic disks with boundary $\phi_t : \overline{\Delta} \to U_0 \subset\subset U_p$ given by $\phi_t(\xi) = (\eta\xi, \eta t, 0, \ldots, 0)$. By (3.1), we have for $\xi \in \overline{\Delta}$,

$$\rho \circ \phi_t(\xi) \geq (1-\delta)\eta^2|\xi|^2 + \lambda_1\eta^2(\xi^2 + \overline{\xi}^2) + (1 - \delta + 2\lambda_2)\eta^2 t^2 + b. \tag{3.2}$$

Choosing δ small enough such that $\frac{\lambda_1}{1-\delta} < \frac{1}{2}$, we have

$$\rho \circ \phi_t(\xi) \geq b + (1 - \delta + 2\lambda_2)\eta^2 t^2 > b.$$

Thus we have $\phi_t(\overline{\Delta}) \subset\subset \overline{\Omega}_{b,0}$. Now for $\xi \in \partial\Delta$, it follows from (3.2) again that

$$\rho(\phi_t(\xi)) \geq (1 - \delta - 2\lambda_1)\eta^2 + b. \tag{3.3}$$

In the following context, we set $d(z, w) = \max\{|z_j - w_j| : 1 \leq j \leq n\}$ for $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$. Set $d(X, Y) = \inf\{d(z, w) : z \in X, w \in Y\}$ for two subsets X, Y of $\mathbb{C}^n$. Write $\mathbb{P}(z_0, r) \subset \mathbb{C}^n$, $r > 0$, for the polydisk $\{z \in \mathbb{C}^n : d(z, z_0) < r\}$.

Note (3.3) implies there exists a positive number A_1 independent of t such that

$$d(\phi_t(\partial\Delta), \partial(\overline{\Omega}_{b,0} \cap U)) \geq A_1 \quad \text{for all } t. \tag{3.4}$$

Note there also exists a positive number A_2 independent of t such that

$$d(\phi_t(\overline{\Delta}), \partial U) \geq A_2 \quad \text{for all } t. \tag{3.5}$$

Set $A = \min\{A_1, A_2\}$. Now pick $\epsilon > 0$ sufficiently small such that $V_\epsilon \subset\subset U$ and $d(z, 0) < \frac{A}{2}$ whenever $z \in V_\epsilon$. Fix any branch $[g]_q$ with $q \in V_\epsilon \cap \overline{\Omega}_{b,0}$ as in the assumption of Lemma 3.5. As $V_\epsilon \cap \overline{\Omega}_{b,0}$ is connected, we can first extend $[g]_q$ holomorphically along certain curve in $V_\epsilon \cap \overline{\Omega}_{b,0}$ to obtain a new branch $[h]_{\widehat{q}}$ with $\widehat{q} = \phi_t(0) = (0, \eta t, 0, \ldots, 0) \in V_\epsilon \cap \phi_t(\overline{\Delta})$ for sufficiently small $0 < t < \widetilde{\epsilon}$.

We also note that

$$\mathrm{d}(\mathbf{0}, \phi_t(0)) = \eta t \to 0 \quad \text{as } t \to 0.$$

As for each fixed small t, $\phi_t(\overline{\Delta}) \subset\subset \overline{\Omega}_{b,0}$, we can extend $[h]_{\widehat{q}}$ holomorphically along any curve inside a small neighborhood of $\phi_t(\overline{\Delta})$. Since $\phi_t(\overline{\Delta})$ is simply connected, $[h]_{\widehat{q}}$ extends to a well-defined holomorphic function, which we still denote by h, in this small neighborhood of $\phi_t(\overline{\Delta})$. Moreover, if $q_0 \in \phi_t(\partial\Delta)$, then by (3.4) and (3.5), we can extends $[h]_{q_0}$ to a holomorphic function in the polydisk $\mathbb{P}(q_0, A)$. By the continuity principle, we conclude that for any $w \in \phi_t(\overline{\Delta})$, the Taylor expansion (in z-coordinates) of $[h]_w$ about w converges in $\mathbb{P}(w, A)$. In particular, $[h]_{\widehat{q}}$ with $\widehat{q} = \phi_t(0)$ extends to a holomorphic function, still called h, in $\mathbb{P}(\widehat{q}, A)$ and note for sufficiently small t, $\mathbb{P}(\widehat{q}, A)$ contains V_ϵ. By the uniqueness of holomorphic maps, we have $[g]_q = [h]_q$. In this way, we extends $[g]_q$ to a holomorphic map h in V_ϵ.

To obtain the ψ-estimate on h, we need to shrink V_ϵ. For small $\lambda, \mu > 0$, define a family (parametrized by $\tau \in \mathbb{C}^{n-1}$) of holomorphic disks $\varphi_\tau(\xi) : \Delta \to U$ given by $\varphi_\tau(\xi) = (\lambda\xi, \tau)$ with $|\tau| < \mu$. Note there exists $c > 0$ that only depends on ϵ such that if we choose $\mu, \lambda < c$, then $\varphi_\tau(\overline{\Delta}) \subset V_\epsilon$ for any $|\tau| < \mu$. Now we fix $0 < \mu \ll \lambda < c$ such that $\varphi_\tau(\partial\Delta) \subset \overline{\Omega}_{b,0}$ for any $|\tau| < \mu$. (The existence of such μ, λ is due to (3.1) and the fact that $\lambda_1 < \frac{1}{2}$.) Then we note there is a small $0 < \epsilon_3 < \epsilon$ which only depends on λ and μ such that $V_{\epsilon_3} \subset \bigcup_{\|\tau\| < \mu} \varphi_\tau(\overline{\Delta})$.

We claim V_{ϵ_3} is the desired region in Lemma 3.5. Indeed, for any $q \in V_{\epsilon_3} \cap \overline{\Omega}_{b,0}$, if $[g]_q$ is a holomorphic branch of (f, D) in $\overline{\Omega}_{b,0}$, then by the above argument, $[g]_q$ extends to a holomorphic map h in V_ϵ. Furthermore, as $V_\epsilon \cap \overline{\Omega}_{b,0}$ is connected, for each τ and any $q_0 \in \varphi_\tau(\partial\Delta)$, we can find a path in $V_\epsilon \cap \overline{\Omega}_{b,0}$ connecting q to q_0. This shows that $[h]_{q_0}$ is a branch of (f, D) obtained by holomorphic continuation along a certain curve in $\overline{\Omega}_{b,0}$. By the assumption on $\overline{\Omega}_{b,0}$, we conclude $[h]_{q_0}$ satisfies the ψ-estimate for every $q_0 \in \varphi_\tau(\partial\Delta)$. Now by the maximum principle for subharmonic functions, h also satisfies the ψ-estimate on $\varphi_\tau(\overline{\Delta})$ for all τ. In particular, h satisfies the ψ-estimate in V_{ϵ_3}. The C^*-uniform boundedness in the dimension-two case also follows from the maximum principle.

(b) Next we consider the case where there are no elliptic directions (i.e., all $\lambda_j > \frac{1}{2}$). When $n \geq 3$, the argument is similar to that in case (a). We replace the holomorphic disks $\phi_t(\xi)$ by

$$\phi_t(\xi) = \eta(\sqrt{\lambda_2}\xi, i\sqrt{\lambda_1}\xi, t, 0, \ldots, 0), \quad 0 < t < \widetilde{\epsilon},$$

where η and $\widetilde{\epsilon}$ are fixed small positive numbers. Then by (3.1) we have for $\xi \in \overline{\Delta}$,

$$\rho \circ \phi_t(\xi) \geq (1-\delta)(\lambda_1+\lambda_2)\eta^2|\xi|^2 + (1-\delta)\eta^2 t^2 + 2\lambda_3\eta^2 t^2 + b > b.$$

Thus we have $\phi_t(\overline{\Delta}) \subset\subset \overline{\Omega}_{b,0}$ for all t. Furthermore, if $\xi \in \partial\Delta$, it yields that

$$\rho \circ \phi_t(\xi) \geq (1-\delta)(\lambda_1+\lambda_2)\eta^2 + b. \tag{3.6}$$

Replacing (3.3) by (3.6), the same argument in (a) yields that for sufficiently small ϵ, and for every $q \in V_\epsilon \cap \overline{\Omega}_{b,0}$, if $[g]_q$ is a holomorphic branch of (f, D) in $\overline{\Omega}_{b,0}$, then $[g]_q$ extends to a single-valued holomorphic function h in V_ϵ. It remains to establish the desired ψ-estimate. We will also need to shrink V_ϵ. For fixed small $\lambda, \mu > 0$, we define a family (parametrized by $\tau \in \mathbb{C}$, $\chi \in \mathbb{C}^{n-2}$ with $|\tau|, |\chi| < \mu$) of Riemann surfaces with boundaries $\overline{E}_{\tau,\chi}$ as follows:

$$\overline{E}_{\tau,\chi} := \{z = (z_1, z_2, \chi) \in U : \lambda_1 z_1^2 + \lambda_2 z_2^2 = \tau,\ 2\lambda_1|z_1|^2 + 2\lambda_2|z_2|^2 \leq \lambda^2\}.$$

Note that there exists $\widehat{\lambda} > 0$ such that if we choose $0 < \lambda, \mu < \widehat{\lambda}$, then $\overline{E}_{\tau,\chi} \subset V_\epsilon$ for all $|\tau|, |\chi| < \mu$. Furthermore, note for $z \in \partial\overline{E}_{\tau,\chi}$, we have

$$\begin{aligned}\rho(z) &\geq (1-\delta)|z|^2 + 2\,\mathrm{Re}\,\tau + 2\,\mathrm{Re}\sum_{j=3}^{n}\lambda_j z_j^2 + b \\ &\geq (1-\delta)\frac{\lambda^2}{2\lambda_1+2\lambda_2} - 2\mu - 2\mu^2\left(\sum_{j=3}^{n}\lambda_j\right) + b.\end{aligned}$$

Thus we can choose $0 < \mu \ll \lambda < \widehat{\lambda}$ such that $\partial\overline{E}_{\tau,\chi} \subset \overline{\Omega}_{b,0}$ for all $|\tau|, |\chi| < \mu$. Fix such a pair λ and μ. Then there is a small $0 < \epsilon_3 < \epsilon$ which only depends on λ and μ such that $V_{\epsilon_3} \subset \bigcup_{|\tau|,|\chi|<\mu} \overline{E}_{\tau,\chi}$. Now a quite similar argument as that in (a), which we skip, shows that V_{ϵ_3} is the desired region in Lemma 3.5.

(c) We now consider the subtler case when $n = 2$ and both directions are non-elliptic: $\lambda_1, \lambda_2 > \frac{1}{2}$. Recall in this case, we have additionally assumed the holomorphic continuation of (f, D) in $\overline{\Omega}_{b,0}$ along curves is C^*-uniformly bounded. Fix $0 < \widetilde{\epsilon} < \eta \ll 1$. Consider a continuous family (parametrized by t) of Riemann surfaces $\overline{E}_t$, for $0 < t < \widetilde{\epsilon}$, in U defined by

$$\overline{E}_t := \{z = (z_1, z_2) \in U : \lambda_1 z_1^2 + \lambda_2 z_2^2 = t,\ 2\lambda_1|z_1|^2 + 2\lambda_2|z_2|^2 \leq \eta^2\}.$$

We apply a holomorphic change of coordinates

$$w_1 = \sqrt{\lambda_1} z_1 + i\sqrt{\lambda_2} z_2, \quad w_2 = \sqrt{\lambda_1} z_1 - i\sqrt{\lambda_2} z_2,$$

i.e., $z_1 = \frac{w_1+w_2}{2\sqrt{\lambda_1}}, z_2 = \frac{w_1-w_2}{2\sqrt{\lambda_2} i}$. Then $\overline{E}_t$ is defined as follows in the new coordinates:

$$w_1 w_2 = t, \quad |w_1|^2 + |w_2|^2 \le \eta^2, \text{ or } w_2 = \frac{t}{w_1}, \ |w_1|^2 + \left|\frac{t}{w_1}\right|^2 \le \eta^2.$$

This shows that $\overline{E}_t$ is the graph of $w_2 = \frac{t}{w_1}$ over a certain annulus $e^{a_1} \le |w_1| \le e^{a_2}$ in the w_1-plane for $a_2 > a_1$ depending smoothly on η and t. Furthermore, by using the w-coordinates, we note that $\overline{E}_t$ is covered by a closed strip $\mathcal{S}_t := \{\xi \in \mathbb{C} : a_1 \le \mathrm{Re}(\xi) \le a_2\}$. Indeed, the map $\pi(\xi) = (e^{\xi}, te^{-\xi})$ gives a covering map from $\mathcal{S}_t$ to $\overline{E}_t$. In particular, writing ∂E_t for the boundary of $\overline{E}_t$, $\pi(\{\mathrm{Re}(\xi) = a_1\})$ corresponds to one component of ∂E_t, and $\pi(\{\mathrm{Re}(\xi) = a_2\})$ corresponds to the other.

Note by (3.1), for any point $z = (z_1, z_2) \in \overline{E}_t$, we have

$$\rho(z) \ge (1-\delta)\|z\|^2 + 2t + b > b.$$

Hence we have $\overline{E}_t \subset \overline{\Omega}_{b,0}$. On the other hand, if $z = (z_1, z_2)$ is on the boundary ∂E_t of $\overline{E}_t$, it follows from (3.1) that

$$\rho(z) \ge (1-\delta)\|z\|^2 + 2t + b > (1-\delta)\frac{\eta^2}{2\lambda_1 + 2\lambda_2} + b.$$

As before, this implies there exists a positive numbers A_1, A_2 independent of t such that

$$d(\partial E_t, \partial(\overline{\Omega}_{b,0} \cap U)) \ge A_1, \quad d(E_t, \partial U) \ge A_2 \quad \text{for all } t. \tag{3.7}$$

Set $A = \min\{A_1, A_2\}$. Let ϵ be a small positive number such that $V_\epsilon \subset U$ and $d(z, 0) < A$ whenever $z \in V_\epsilon$. Write for $0 < t < \widetilde{\epsilon}$,

$$z_t = \left(\sqrt{\frac{t}{\lambda_1}}, 0\right).$$

Notice that $z_t \in \overline{E}_t$ and $z_t \to 0$ as $t \to 0$. Now fix a branch $[g]_q$ with $q \in V_\epsilon \cap \overline{\Omega}_{b,0}$ as in the assumption of Lemma 3.5. As $V_\epsilon \cap \overline{\Omega}_{b,0}$ is connected, we can first extend $[g]_q$ holomorphically along certain curve in $V_\epsilon \cap \overline{\Omega}_{b,0}$ to obtain a new branch $[g_0]_{z_t}$ for sufficiently small $0 < t < \widetilde{\epsilon}$. Then note $\overline{E}_t$ and thus a small neighborhood of it are connected(but not simply connected). We can thus extend $[g_0]_{z_t}$ to a multiple-valued map, still denoted by g, in a small neighborhood V of $\overline{E}_t$. By the assumption on the holomorphic continuation in $\overline{\Omega}_{b,0}$, the norm of any branch of g is bounded by C^*. Since $\overline{E}_t \subset \overline{\Omega}_{b,0} \cap U$, there exists a constant $r_t > 0$ depending on t such that the polydisk $\mathbb{P}(z, r_t) \subset \overline{\Omega}_{b,0} \cap U$ for any point $z \in \overline{E}_t$. Then any branch $[g]_z$ of g at z extends to a single-valued holomorphic map, still denoted by g, in $\mathbb{P}(z, r_t)$. We apply the Cauchy estimate to g on $\mathbb{P}(z, r_t)$ to obtain for any multiindex $\alpha = (\alpha_1, \alpha_2)$,

$$|D^\alpha g(z)| \le \frac{C^* \alpha!}{(r_t)^{|\alpha|}}. \tag{3.8}$$

Let p be a point on $\partial\overline{E}_t$. By (3.7), we see that $\mathbb{P}(p, A) \subset \overline{\Omega}_{b,0} \cap U$. Then any branch $[g]_p$ of g at p extends to a single-valued holomorphic map $\widehat{g}$ in $\mathbb{P}(p, A)$. As $\overline{E}_t \cup \mathbb{P}(p, A) \subset \overline{\Omega}_{b,0}$, by the C^*-boundedness assumption on holomorphic continuation in $\overline{\Omega}_{b,0}$, we have $|\widehat{g}| \le C^*$

in $\mathbb{P}(p, A)$. It then follows from the Cauchy estimate that for any multiindex $\alpha = (\alpha_1, \alpha_2)$,

$$|D^\alpha \widehat{g}(p)| \le \frac{C^*\alpha!}{A^{|\alpha|}}. \tag{3.9}$$

By the monodromy theorem, the multiple-valued g (respectively its derivatives $D^\alpha g$) on $\overline{E}_t$ lifts to a single-valued holomorphic map h (respectively its derivatives denoted by h_α) in the covering $\mathcal{S}_t$ which is holomorphic in the interior of $\mathcal{S}_t$ and continuous to the boundary $\partial\mathcal{S}_t$. Moreover, it follows from (3.8) that

$$|h_\alpha(\xi)| \le \frac{C^*\alpha!}{(r_t)^{|\alpha|}} \quad \text{for all } \xi \in \mathcal{S}_t.$$

Thus h_α is bounded on $\mathcal{S}_t$. It follows from (3.9) that

$$|h_\alpha(\xi_0)| \le \frac{C^*\alpha!}{A^{|\alpha|}} \quad \text{for } \xi_0 \in \partial\mathcal{S}_t.$$

We apply the Phragmén–Lindelöf principle to h_α on $\mathcal{S}_t$ to obtain that for all $\xi \in \mathcal{S}_t$,

$$|h_\alpha(\xi)| \le \frac{C^*\alpha!}{A^{|\alpha|}}.$$

This implies that for every $z \in \overline{E}_t$, we have

$$|D^\alpha g(z)| \le \frac{C^*\alpha!}{A^{|\alpha|}} \quad \text{for any branch of } g \text{ at } z.$$

In particular, it holds at $z = z_t \in \overline{E}_t$ and for the branch $[g_0]_{z_t}$. Hence the Taylor expansion of $[g_0]_{z_t}$ about z_t is convergent in $\mathbb{P}(z_t, A)$, and thus $[g_0]_{z_t}$ extends to a holomorphic map in $\mathbb{P}(z_t, A)$. In particular, $[g_0]_{z_t}$ extends to a holomorphic map h in V_ϵ as $V_\epsilon \subset \mathbb{P}(z_t, A)$ for small $t > 0$.

Finally, we prove the ψ-estimate on h. We will also need to shrink V_ϵ. For fixed small $\lambda, \mu > 0$, we define a family (parametrized by $\tau \in \mathbb{C}$) of Riemann surfaces with boundary $\overline{E}_\tau$, $|\tau| < \mu$, given by

$$\overline{E}_\tau := \{z = (z_1, z_2) \in U : \lambda_1 z_1^2 + \lambda_2 z_2^2 = \tau,\ 2\lambda_1|z_1|^2 + 2\lambda_2|z_2|^2 \le \lambda^2\}.$$

Then we apply the same argument as in (a) or (b) to get the desired region V_{ϵ_3}, C^*-boundedness and the ψ-estimate.

This finally finishes the proof of Lemma 3.5. □

We now continue the proof in Case III. To this end, we let $\{p_1, \dots, p_k\}$ be the critical points of ρ on M_b. By Lemma 3.5, we can choose for each j with $1 \le j \le k$ small simply connected neighborhoods $V_{1,j} \subset\subset V_{2,j} \subset\subset V_{3,j}$ of p_j such that any branch $[g]_q$ of (f, D) with $q \in V_{3,j} \cap \overline{\Omega}_{b,0}$ extends to a holomorphic map in $V_{3,j}$ with ψ-estimate (and also with C^*-boundedness in the two-dimensional case). Moreover, by making $V_{1,j}$ sufficiently small, we can assume $V_{3,j} \setminus V_{1,j}$ is connected for all j. Let δ_1 be such that if $z_1 \in V_{2,j}$ and $\widehat{d}(z_1, z_2) < 2\delta_1$, then $z_2 \in V_{3,j}$. Now choose $0 < \epsilon' \ll 1$ such that there is a continuous retract J of $(\overline{\Omega}_{b-\epsilon',0} \setminus \bigcup_{j=1}^k V_{1,j}) \cup \overline{\Omega}_{b+\epsilon',0}$ into $\overline{\Omega}_{b+\epsilon',0}$ (see Section 3.1). Here J maps a point

in $\overline{\Omega}_{b+\epsilon',0}$ to itself and maps the other points along the orbit of X_ρ to $M_{b+\epsilon'}$. Then by applying a similar argument as in Section 3.1, we obtain a sufficiently small $\epsilon' > 0$ such that (f, D) admits holomorphic extension along curves in $\overline{\Omega}_{b-\epsilon',0} \setminus \bigcup_{j=1}^{k} V_{1,j}$ with ψ-estimate (and also with C^*-boundedness in the two-dimensional case). More precisely, write $\mathcal{D}$ for the union of all small holomorphic disks attached to $M_{b+\epsilon'}$ that can be continuously deformed to a point in $M_{b+\epsilon'}$. Choosing ϵ' small, we make $\overline{\Omega}_{b-\epsilon',b+\epsilon'} \setminus \bigcup_{j=1}^{k} V_{1,j}$ be contained in $\mathcal{D} \cup M_{b+\epsilon'}$. We can further make ϵ' small such that for any curve σ in $\overline{\Omega}_{b-\epsilon',0} \setminus \bigcup_{j=1}^{k} V_{1,j}$ with $\sigma(0) \in D$, we can deform σ to a $\widetilde{\sigma}$ in $\overline{\Omega}_b$ through J such that $\sigma(0) = \widetilde{\sigma}(0)$ and $\sigma(t), \widetilde{\sigma}(t)$ are sufficiently close for each $t \in [0, 1]$. Moreover, the holomorphic continuation of (f, D) along the two curves are induced by the same branch at each t.

The above argument proved that $\overline{\Omega}_{b-\epsilon',0} \setminus \bigcup_{j=1}^{k} V_{1,j}$ has the extendability property.

Claim. *The set $\overline{\Omega}_{b-\epsilon',0}$ has the extendability property.*

Proof of Claim. Let γ be any curve in $\overline{\Omega}_{b-\epsilon',0}$ with $\gamma(0) \in D$. We first find a δ^* such that whenever $\gamma(t) \in \overline{V_{1,j}}$ for some $1 \le j \le k$ and $|t - t'| < \delta^*$ with $t, t' \in [0, 1]$, we have $\gamma(t') \in V_{2,j}$. Note if $\gamma([0, 1]) \subset \overline{\Omega}_{b-\epsilon',0} \setminus \bigcup_{j=1}^{k} V_{1,j}$, then the proof is done. Now assume that there is some t_1^* such that $\gamma([0, t_1^*)) \subset \overline{\Omega}_{b-\epsilon',0} \setminus \bigcup_{j=1}^{k} V_{1,j}$ and $\gamma(t_1^*) \in \overline{V_{1,j_0}}$ for some $1 \le j_0 \le k$. Fix a $\widehat{t_1}$ with $0 < \widehat{t_1} < t_1^*$ but sufficiently close to t_1^* such that $\gamma(t) \in V_{2,j_0}$ when $\widehat{t_1} \le t \le t_1^*$. By the above argument, deforming $\gamma([0, \widehat{t_1}])$ if necessary, we can assume there is some $0 < t_0 < \widehat{t_1}$ such that $\gamma([0, t_0]) \subset \overline{\Omega}_{b,0}$ and $\widehat{d}(\gamma(t), \gamma(\widehat{t_1})) < 2\delta_1$ if $t_0 \le t \le \widehat{t_1}$. But since $\gamma(\widehat{t_1}) \in V_{2,j_0}$, we have $\gamma(t) \in V_{3,j_0}$ when $t_0 \le t \le \widehat{t_1}$ (and thus $\gamma(t) \in V_{3,j_0}$ for $t_0 \le t \le t_1^*$). Writing $[f]_{\gamma(t)}$ for the germ of a branch of (f, D) at $\gamma(t)$ obtained by holomorphic continuation along $\gamma|_{[0,t_1^*)}$, we conclude $[f]_{\gamma(t_0)}$ is the germ of a branch of (f, D) obtained by holomorphic continuation along a curve in $\overline{\Omega}_{b,0}$ to a point in $\overline{\Omega}_{b,0} \cap V_{3,j_0}$. It follows from our assumption on $V_{3,j}$ that $[f]_{\gamma(t_0)}$ extends to a holomorphic map h in V_{3,j_0} with ψ-estimate and thus $[f]_{\gamma(t_0)}$ extends as germs of h along $\gamma(t)$ for $t_0 \le t \le t_1^*$. If $\gamma([t_1^*, 1]) \subset V_{3,j_0}$, then clearly (f, D) admits holomorphic continuation along γ with ψ-estimate (and also with C^*-boundedness in the two -dimensional case). If $\gamma([t_1^*, 1]) \not\subset V_{3,j_0}$, then there exists some t_2 with $t_1^* < t_2 < 1$ such that $\gamma([t_1^*, t_2)) \subset V_{2,j_0}$ and $\gamma(t_2) \in V_{3,j_0} \setminus V_{2,j_0}$. Since $\gamma(t_1^*) \in \overline{V_{1,j_0}}$, we must have $|t_2 - t_1^*| \ge \delta^*$. Moreover, by the proceeding argument, we can deform $\gamma([t_0, t_2])$ in V_{3,j_0} with endpoints fixed such that γ avoids V_{1,j_0} and we still get the same branch at $\gamma(t_2)$. In summary, we can extend (f, D) along $\gamma|_{[0,t_2]}$ and we do obtain the same branch at $\gamma(t_2)$ by continuing (f, D) along a certain other curve in $\overline{\Omega}_{b-\epsilon',0} \setminus \bigcup_{j=1}^{k} V_{1,j}$. Hence this branch is also a branch of (f, D) through continuation along curves in $\overline{\Omega}_{b-\epsilon',0} \setminus \bigcup_{j=1}^{k} V_{1,j}$. Next we consider $\gamma([t_2, 1])$. If $\gamma([t_2, 1]) \subset \overline{\Omega}_{b-\epsilon',0} \setminus \bigcup_{j=1}^{k} V_{1,j}$, then the proof is done again. Otherwise we repeat the above argument for at most $[\frac{1}{\delta^*}] + 1$ times to arrive at $\gamma(1)$. This completes the proof of the claim. □

The above claim gives a contradiction to the definition of b and thus Case III cannot occur. This finishes the proof of part (1) and (2) in Theorem 1.2.

To see the last part of Theorem 1.2, we assume there is a holomorphic branch (h, U) of (f, D) in $\mathrm{Reg}(\overline{\Omega})$ such that $\psi(h(p)) = 0$ for some $p \in U \setminus M$. But by part (1) and (2) of Theorem 1.2, $\psi(h) \le 0$ on U. Since $\psi \circ h$ is subharmonic, it follows from the maximum principle that $\psi(h) \equiv 0$ in U. Let σ be the curve in $\mathrm{Reg}(\overline{\Omega})$ along which we obtain (h, U) by applying holomorphic continuation to (f, D). Applying the maximum principle finitely

many times backward of σ, we have $\psi \circ f_0 \equiv 0$ over U_0. Here (f_0, U_0) is a holomorphic branch of (f, D) with $U_0 \cap D \neq \emptyset$ and $f_0|_{U_0 \cap D} = f|_{U_0 \cap D}$. The statements in the last part of Theorem 1.2 then follows easily. This finally completes the proof of Theorem 1.2. □

4. Proofs of Corollaries 1.3, 1.4 and 4.1

Let Ω be as in Corollary 1.4 with $n = \dim_{\mathbb{C}} \Omega$ and let (f, D) be a CR diffeomorphism from a certain simply connected open subset D to an open piece of the boundary of $\mathbb{B}^n$. Take $\psi(z) = |z|^2 - 1$ for $z \in \mathbb{C}^n$. As explained in the introduction, (f, D) extends to a multiple-valued development map over M. Applying Theorem 1.2, we conclude that (f, D) admits holomorphic map continuation along curves in $\mathrm{Reg}(\overline{\Omega})$ with ψ-estimate. Since f is not constant, it follows that for any branch (g, U) of (f, D) with $U \subset \mathrm{Reg}(\Omega)$, we have $\psi(g) = |g|^2 - 1 < 0$ in U, or $g(U) \subset \mathbb{B}^n$.

By the Alexander theorem, for any $p \in M$ and for two germs of CR branches $[f_1]_p$ and $[f_2]_p$ of (f, D), we must have $f_2 = G \circ f_1$ near p for some automorphism G of $\mathbb{B}^n$. By Proposition 3.4, even if we extends along curves inside $\mathrm{Reg}(\overline{\Omega})$ with endpoints in M, we will not get more branches than extending through curves just insider M. Let Γ_p be the collection of all such $G's$. Apparently, from the uniqueness of holomorphic functions, Γ_p is a subgroup of the automorphism group of the unit ball. We also have $\Gamma_q = \Gamma_p$ for $q \in M$ sufficiently close to p. Since M is connected, $\Gamma := \Gamma_p$ is independent of $p \in M$. Let $\overline{\Omega}_{-\epsilon_2,0}$ be as in Section 3.1. It follows readily from the construction there of the holomorphic continuation in $\overline{\Omega}_{-\epsilon_2,0}$ that the germs of two holomorphic branches $[g_1]_q$ and $[g_2]_q$ of (f, D) at some $q \in \overline{\Omega}_{-\epsilon_2,0}$ obtained through curves inside $\overline{\Omega}_{-\epsilon_2,0}$ must satisfy $g_2 = G \circ g_1$ for some automorphism $G \in \Gamma$. Conversely, if $[g_1]_q$ is the germ of a branch of (f, D) in $\overline{\Omega}_{-\epsilon_2,0}$, then so is $[G \circ g_1]_q$ for any $G \in \Gamma$. By Proposition 3.4, this is also the case when extending along curves inside $\mathrm{Reg}(\Omega)$. Since for any curve γ inside $\mathrm{Reg}(\overline{\Omega})$, the holomorphic continuation along γ induces a one-to-one correspondence between the set of germs of branches at the endpoints of γ, we easily see the following statement:

Let $[h_1]_q$ be the germ of a branch of (f, D) at $q \in \mathrm{Reg}(\Omega)$. Any other germ $[h_2]_q$ of a holomorphic map at q is the germ of a certain branch of (f, D) at q if and only if $[h_2]_q = [G \circ h_1]_q$ for some $G \in \Gamma$.

Now we define the complex analytic hyper-variety $E \subset \mathrm{Reg}(\Omega)$ to be such that, for any branch (f^*, U^*) of (f, D), $E \cap U^*$ is the zero of the Jacobian of f^*. Then we see from the above claim that E is well-defined and is independent of the choice of the chosen branch. Since $E \cap \Omega_{-\epsilon_2,0} = \emptyset$, we see that $E = \emptyset$. Namely, f^* is always a local biholomorphism. This completes the proof of Corollary 1.3. □

We now define the hyperbolic metric ω_0 on $\mathrm{Reg}(\Omega)$ in the following way. Writing $\omega_{\mathbb{B}^n}$ for the Bergman metric on $\mathbb{B}^n$, for any holomorphic branch (g, V) of (f, D) in $\mathrm{Reg}(\Omega)$, we define $\omega_0 = g^*(\omega_{\mathbb{B}^n})$ on V. Then ω_0 is a Kähler metric which is independent of the choice of (g, V) as the Bergman metric on $\mathbb{B}^n$ is invariant under automorphisms. Thus the metric ω_0 is well-defined on $\mathrm{Reg}(\Omega)$. Finally, we notice that for any $p \in M$, there are a neighborhood W_p of p in $\mathrm{Reg}(\overline{\Omega})$ and a smooth diffeomorphism F from W_p to a certain open subset W'_q of $\overline{\mathbb{B}^n}$ such that (i) $\partial_0 W_p := W_p \cap M$ is a connected open subset of M containing p and $\partial_0 W'_q := W'_q \cap \partial\mathbb{B}^n$ is a connected open subset of the unit sphere $\partial\mathbb{B}^n$ containing q, (ii) F is

CR diffeomorphism from $\partial_0 W_p$ to $\partial_0 W_p'$ that extends to a biholomorphism from $W_p \cap \Omega$ to $W_p' \cap \mathbb{B}^n$ with $F(p) = q$, and (iii) $\omega_0 = F^*(\omega_{\mathbb{B}^n})$ on $W_p \setminus M$. This shows that ω_0 satisfies the properties in (A), (B) and (C) of Corollary 1.4.

Next, let ω be a Kähler metric over $\mathrm{Reg}(\Omega)$ with properties in (A), (B) and (C) of Corollary 1.4. Then ω is real analytic. And near a certain $p \in M$, we have the system $\{U_p, V_q, F\}$ with $F(p) = q$ as in (C) of Corollary 1.4 corresponding to ω. As above, we have a similar system $\{U_p^0, V_q^0, F^0\}$ with $F^0(p) = q$ for ω_0. By the Alexander theorem, $F = \sigma \circ F^0$ for a certain $\sigma \in \mathrm{Aut}(\mathbb{B}^n)$. Since $F^*(\omega_{\mathbb{B}^n}) = \omega$, $(F^0)^*(\omega_{\mathbb{B}^n}) = \omega_0$ and σ is an isometry for $\omega_{\mathbb{B}^n}$, we see that $\omega = \omega_0$ near p. Hence we see $\omega = \omega_0$ over $\mathrm{Reg}(\Omega)$.

This completes the proof of Corollary 1.4. □

We finish off the paper by presenting one more application of Theorem 1.2.

A Stein space Ω with isolated complex singularities can be exhausted by Stein spaces with smooth compact strongly pseudoconvex boundaries. Indeed, let F be a proper holomorphic map from Ω into $\mathbb{C}^N$ that is injective and regular at smooth points [14, Theorem 2.4.1]. By Sard's theorem, most level sets of $\rho = |F|^2$ are smooth and strongly pseudoconvex, from which we can pick a sequence exhausting Ω. Then Theorem 1.2 gives the following:

Corollary 4.1. *Let Ω be a Stein space of complex dimension $n \geq 2$ with possibly isolated singularities. Let K be a compact subset of Ω such that $\Omega \setminus K$ is connected. Assume that (f, D) is a holomorphic map element in $\mathrm{Reg}(\Omega) \setminus K$ that extends holomorphically along any curve γ in $\mathrm{Reg}(\Omega) \setminus K$ with $\gamma(0) \in D$. Then the following conclusions hold:*

(A) *Suppose that $n = \dim_{\mathbb{C}}(\Omega) \geq 3$. Then (f, D) admits holomorphic continuation along any curve in $\mathrm{Reg}(\Omega)$ that starts from a point in D.*

(B) *Suppose that $n = \dim_{\mathbb{C}}(\Omega) = 2$ and (f, D) admits uniformly bounded holomorphic map continuation along any curve γ inside $\mathrm{Reg}(\Omega) \setminus K$ with $\gamma(0) \in D$. Then (f, D) admits uniform bounded holomorphic map continuation along any curve in $\mathrm{Reg}(\Omega)$ that starts from a point in D.*

Proof. Let γ be a curve in $\mathrm{Reg}(\Omega)$ with $p_0 = \gamma(0) \in D$. We first connect p_0 by a curve σ to a certain point $q \in M \subset \Omega \setminus K$. Here M is the compact smooth strongly pseudoconvex boundary of a certain Stein space Ω_M containing γ. Let $[h]_q$ be the germ at q of the holomorphic map obtained by continuing (f, D) through σ. Then by the assumption, $[h]_q$ can be continued along any curves inside M starting from q (with a uniform bound when the dimension is two). By Theorem 1.2, we see that $[h]_q$ can be continued holomorphically along the curve $(-\sigma) + \gamma$, from which the assertion in the corollary follows easily. □

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