



Measuring Mass via Coordinate Cubes

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Abstract: Inspired by a formula of Stern that relates scalar curvature to harmonic functions, we evaluate the mass of an asymptotically flat 3-manifold along faces and edges of a large coordinate cube. In terms of the mean curvature and dihedral angle, the resulting mass formula relates to Gromov’s scalar curvature comparison theory for cubic Riemannian polyhedra. In terms of the geodesic curvature and turning angle of slicing curves, the formula realizes the mass as integration of the angle defect detected by the boundary term in the Gauss–Bonnet theorem.

1. Motivation and Mass Formulae

In [12], Stern gave an intriguing formula relating the scalar curvature of a manifold to the level set of its harmonic functions. In its simplest form, Stern’s formula [12, equation (14)] shows

$$\Delta|\nabla u| = \frac{1}{2|\nabla u|} \left[|\nabla^2 u|^2 + |\nabla u|^2 (R - 2K_\Sigma) \right] \quad (1)$$

near points where $\nabla u \neq 0$, here u is a harmonic function on a Riemannian 3-manifold (M^3, g) , R and K_Σ denote the scalar curvature of g and the Gauss curvature of Σ , the level set of u , respectively. Applications of the formula to closed manifolds and to compact manifolds with boundary were given by Stern [12], and Bray and Stern [4].

If the manifold (M^3, g) is asymptotically flat, by applying Stern’s formula, Bray et al. [3] gave a new elegant proof of the 3-dimensional positive mass theorem, which was originally proved by Schoen and Yau [11], and Witten [13]. Moreover, the result in [3] provides an explicit lower bound of the mass of (M, g) via a single harmonic function.

In the context of asymptotically flat manifolds, an observation of Bartnik [2] was

$$\sum_{i=1}^3 \int_{S_\infty} \frac{1}{2} \frac{\partial}{\partial \nu} |\nabla y^i|^2 d\sigma = 16\pi m(g), \quad (2)$$

where $m(g)$ is the mass of (M, g) , $\{y^i\}$ are harmonic coordinates near infinity, and $\int_{S_\infty}$ denotes the limit of integration along a sequence of suitable surfaces tending to infinity. As $|\nabla y^i|$ approaches 1 sufficiently fast, it can be checked (2) is equivalent to

$$\sum_{i=1}^3 \int_{S_\infty} \frac{\partial}{\partial \nu} |\nabla y^i| d\sigma = 16\pi m(g). \quad (3)$$

In view of (1) and (3), it is desirable to seek a formula that computes the ADM mass (see [1]) solely in terms of geometric data of the level sets of y^i near infinity. In this paper, we derive formulae of this nature. As the level sets of y^i are simply coordinate planes, we are thus prompted to compute $m(g)$ on the boundary of large coordinate cubes.

A Riemannian 3-manifold (M, g) is called asymptotically flat with a metric falloff rate τ if there exists a coordinate chart $\{x^i\}$, outside a compact set, in which the metric coefficients satisfies

$$g_{ij} = \delta_{ij} + O(|x|^{-\tau}), \quad \partial g_{ij} = O(|x|^{-\tau-1}), \quad \partial \partial g_{ij} = O(|x|^{-\tau-2}). \quad (4)$$

The scalar curvature R of g is assumed to be integrable so that $m(g)$ is defined.

Geometric mass formula. Let (M^3, g) be an asymptotically flat 3-manifold with metric falloff rate $\tau > \frac{1}{2}$. Given any large constant $L > 0$, let ∂Cube_L denote the boundary of a coordinate cube with side length $2L$ centered at the coordinate origin. Let H be the mean curvature of the face of ∂Cube_L with respect to the outward unit normal ν in (M, g) . Let $\mathcal{E}_L$ be the set of all edges of ∂Cube_L . Along each edge in $\mathcal{E}_L$, let θ be the angle between ν on the two adjacent faces. Then, as $L \rightarrow \infty$,

$$m(g) = -\frac{1}{8\pi} \int_{\partial \text{Cube}_L} H d\sigma + \frac{1}{8\pi} \int_{\mathcal{E}_L} \left(\frac{\pi}{2} - \theta \right) ds + O(L^{1-2\tau}). \quad (5)$$

Here $d\sigma$ and ds are the area and the length measure with respect to g , respectively. Moreover, in terms of the curve $C_t^{(k)}$ which is the intersection of ∂Cube_L and the coordinate plane $\{x^k = t\}$,

$$m(g) = \frac{1}{8\pi} \sum_{k=1}^3 \int_{-L}^L \left(2\pi - \beta_t^{(k)} - \int_{C_t^{(k)}} \kappa^{(k)} ds \right) dt + O(L^{1-2\tau}). \quad (6)$$

Here $\kappa^{(k)}$ is the geodesic curvature of $C_t^{(k)}$ in $\{x^k = t\}$ and $\beta_t^{(k)}$ denotes the sum of the turning angle of $C_t^{(k)}$ at its four vertices.

We give a few remarks regarding these formulae.

Remark 1. Though our discussion is motivated by (1) and (3), formulae (5) and (6) do not assume $\{x^i\}$ to be harmonic.

Remark 2. In terms of the dihedral angle α between the two adjacent faces at an edge, (5) is equivalent to

$$m(g) = -\frac{1}{8\pi} \int_{\partial \text{Cube}_L} H \, d\sigma + \frac{1}{8\pi} \int_{\mathcal{E}_L} \left(\alpha - \frac{\pi}{2} \right) ds + O(L^{1-2\tau}). \quad (7)$$

In [8], Gromov proposed and outlined the proof of a scalar curvature comparison theorem for polyhedra – let (D^3, g) be a cube-type Riemannian polyhedron with faces F_j , let α_{ij} be the dihedral angle between two adjacent faces F_i and F_j , then the following can not simultaneously hold:

- the scalar curvature R of (D, g) is nonnegative;
- the mean curvature H of all faces F_j is nonnegative; and
- the dihedral angle function $\alpha_{ij} < \frac{\pi}{2}$ for all i and j .

Li [9] established the corresponding rigidity case under the assumption $\alpha_{ij} \leq \frac{\pi}{2}$. (Further investigation of Gromov’s scalar curvature polyhedral comparison theory and edge metrics was given by Li and Mantoulidis [10].) Now suppose (M^3, g) is a complete, asymptotically flat manifold with nonnegative scalar curvature. It follows from the positive mass theorem and formula (7) that

$$-\frac{1}{8\pi} \int_{\partial \text{Cube}_L} H \, d\sigma + \frac{1}{8\pi} \int_{\mathcal{E}_L} \left(\alpha - \frac{\pi}{2} \right) ds \geq 0 \quad (8)$$

for large L . These large cubes in (M, g) provide examples for which Gromov’s above pointwise assumptions on H and α_{ij} may be promoted to an integral inequality.

Remark 3. Heuristically, if ∂Cube_L could be isometrically embedded in $\mathbb{R}^3$ as the boundary of a standard cube, the right side of (5) would represent the corresponding Brown-York mass of ∂Cube_L . In this context, formula (5) resembles the convergence of Brown-York mass to $m(g)$ (see [7]). This resemblance indeed suggests an advantage of (5) as it does not invoke any use of isometric embeddings.

Remark 4. In (6), the quantity $2\pi - \beta_t^{(k)} - \int_{C_t^{(k)}} \kappa^{(k)} ds$ measures the angle defect of the large portion of the coordinate plane $\{x^k = t\}$ inside the cube. Formula (6) shows the mass of (M^3, g) is computable from integrating such angle defects associated to all coordinate planes. (In the setting of asymptotically conical surfaces, the angle defect can be defined as the 2-d “mass” of those surfaces, see [5] for instance.)

Remark 5. Formulae (6) is different from the mass formula of Bray–Kazaras–Khuri–Stern [3, equation(6.27)]. We will examine this difference in Sect. 3.

If $\{x^i\}$ are taken to be harmonic coordinates, upon integration and applying the Gauss–Bonnet theorem, formulae (1), (3) and (6) then imply a lower bound of $m(g)$ in the same manner as in [3]. For instance, suppose M has no boundary, consider

$$U = (u^1, u^2, u^3) : (M^3, g) \rightarrow (\mathbb{R}^3, g_0)$$

to be a harmonic map, which is a diffeomorphism near infinity such that g satisfies condition (4). (By the construction of harmonic coordinates in [2, 6], this map U always exists.) Suppose the regular level set $\Sigma_t^{(i)}$ of all the u^i is connected so that $\chi(\Sigma_t^{(i)}) \leq 1$, for instance this is always satisfied if M is $\mathbb{R}^3$, then it follows from (1), (3) and (6) that

$$\begin{aligned}
& 24\pi \mathfrak{m}(g) \\
&= \lim_{L \rightarrow \infty} \sum_{k=1}^3 \int_{\partial \text{Cube}_L} \frac{\partial}{\partial \nu} |\nabla u^k| d\sigma + \lim_{L \rightarrow \infty} \sum_{k=1}^3 \int_{-L}^L \left[(2\pi - \beta_t^{(k)}) - \int_{C_t^{(k)}} \kappa^{(k)} ds \right] dt \\
&\geq \sum_{i=1}^3 \int_M \frac{1}{2} \left[\frac{1}{|\nabla u^i|} |\nabla^2 u^i|^2 + R |\nabla u^i| \right] dV. \tag{9}
\end{aligned}$$

We emphasize that (9) is weaker than the theorem of Bray et al. [3], because the bound of $\mathfrak{m}(g)$ in [3] uses only a single harmonic function.

2. Calculation on the Cubic Boundary

We will verify (5) and (6) by elementary calculation. Let $\{x^i\}$ be a coordinate chart of (M^3, g) , outside a compact set, in which (4) holds. Given a large constant $L > 0$, let ∂Cube_L be the boundary of the coordinate cube with side length $2L$ centered at the coordinate origin. More precisely, for each $i \in \{1, 2, 3\}$ and $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$, define the faces

$$\begin{aligned}
F_{+,L}^{(i)} &= \{(x^1, x^2, x^3) \mid x^i = L, |x^j| \leq L, |x^k| \leq L\}, \\
F_{-,L}^{(i)} &= \{(x^1, x^2, x^3) \mid x^i = -L, |x^j| \leq L, |x^k| \leq L\}.
\end{aligned}$$

Then

$$\partial \text{Cube}_L = \cup_{i=1}^3 \left(F_{+,L}^{(i)} \cup F_{-,L}^{(i)} \right).$$

For any $i \neq j$, define the edges

$$E_{+,+,L}^{(ij)} = F_{+,L}^{(i)} \cap F_{+,L}^{(j)}, \quad E_{+,-,L}^{(ij)} = F_{+,L}^{(i)} \cap F_{-,L}^{(j)},$$

and

$$E_{-+,L}^{(ij)} = F_{-,L}^{(i)} \cap F_{+,L}^{(j)}, \quad E_{-,-,L}^{(ij)} = F_{-,L}^{(i)} \cap F_{-,L}^{(j)}.$$

Let ν denote the outward unit g -normal to ∂Cube_L . Then

$$\nu = \begin{cases} \frac{\nabla x^i}{|\nabla x^i|} & \text{on } F_{+,L}^{(i)} \\ -\frac{\nabla x^i}{|\nabla x^i|} & \text{on } F_{-,L}^{(i)}. \end{cases} \tag{10}$$

Along the edge $E_{+,+,L}^{(ij)}$, let $\theta_{+,+,L}^{(ij)}$ be the angle between ν on the two adjacent faces. Then

$$\begin{aligned}
\cos \theta_{+,+,L}^{(ij)} &= \left\langle \frac{\nabla x^i}{|\nabla x^i|}, \frac{\nabla x^j}{|\nabla x^j|} \right\rangle \\
&= |\nabla x^i|^{-1} |\nabla x^j|^{-1} g^{ij} \\
&= (1 + O(L^{-\tau})) (-g_{ij} + O(L^{-2\tau})) \\
&= -g_{ij} + O(L^{-2\tau}), \tag{11}
\end{aligned}$$

where we used the fact $g^{ij} = -g_{ij} + O(L^{-2\tau})$, if $i \neq j$. Similarly, define the angle $\theta_{+, -, L}^{(ij)}$, $\theta_{-, +, L}^{(ij)}$, $\theta_{-, -, L}^{(ij)}$ along the edges $E_{+, -, L}^{(ij)}$, $E_{-, +, L}^{(ij)}$, $E_{-, -, L}^{(ij)}$, respectively, and we have

$$\begin{aligned}\cos \theta_{-, +, L}^{(ij)} &= g_{ij} + O(L^{-2\tau}) \\ \cos \theta_{+, -, L}^{(ij)} &= g_{ij} + O(L^{-2\tau}) \\ \cos \theta_{-, -, L}^{(ij)} &= -g_{ij} + O(L^{-2\tau}).\end{aligned}\tag{12}$$

We are also interested in the intersection between ∂Cube_L and coordinate planes. Given any $t \in [-L, L]$, let $P_t^{(k)}$ denote the coordinate 2-plane $\{x^k = t\}$. Let

$$C_t^{(k)} = \partial \text{Cube}_L \cap P_t^{(k)}$$

be the “square” like curve, consisting of four coordinate curves on the faces $F_{\pm, L}^{(i)}$, $i \neq k$. Along $C_t^{(k)}$, let $\kappa^{(k)}$ denote the g -geodesic curvature of $C_t^{(k)}$ in $P_t^{(k)}$ with respect to the outward g -unit normal $\bar{v}$.

Along $C_t^{(k)} \cap F_{+, L}^{(i)}$, $\bar{v} = \partial_{x^i} + O(L^{-\tau})$. Let $j \in \{1, 2, 3\} \setminus \{k, i\}$, then

$$\begin{aligned}\kappa^{(k)} &= -\frac{1}{g_{jj}} \langle \nabla_{\partial_{x^j}} \partial_{x^j}, \bar{v} \rangle \\ &= -\langle \nabla_{\partial_{x^j}} \partial_{x^j}, \partial_{x^i} \rangle + O(L^{-2\tau-1}) \\ &= -\Gamma_{jj}^i + O(L^{-2\tau-1}) \\ &= \frac{1}{2} g_{jj, i} - g_{ij, j} + O(L^{-2\tau-1}).\end{aligned}\tag{13}$$

Similarly, along $C_t^{(k)} \cap F_{-, L}^{(i)}$, $\bar{v} = -\partial_{x^i} + O(L^{-\tau})$ and

$$\begin{aligned}\kappa^{(k)} &= -\frac{1}{g_{jj}} \langle \nabla_{\partial_{x^j}} \partial_{x^j}, -\partial_{x^i} \rangle + O(L^{-2\tau-1}) \\ &= -\left(\frac{1}{2} g_{jj, i} - g_{ij, j} \right) + O(L^{-2\tau-1}).\end{aligned}\tag{14}$$

On ∂Cube_L , let H be the mean curvature of its faces in (M, g) with respect to v . Then, on $F_{+, L}^{(i)}$,

$$\begin{aligned}H &= -\sum_{j \neq i, k \neq i} g^{jk} \langle \nabla_{\partial_{x^j}} \partial_{x^k}, v \rangle \\ &= -\sum_{j \neq i} \langle \nabla_{\partial_{x^j}} \partial_{x^j}, \partial_{x^i} \rangle + O(L^{-2\tau-1}) \\ &= \sum_{k \neq i} \kappa^{(k)} + O(L^{-2\tau-1}).\end{aligned}\tag{15}$$

Similarly, (15) holds on $F_{-, L}^{(i)}$ too.

Finally, we measure the turning angle of $C_t^{(k)}$ at each of its vertices. At the vertex $C_t^{(k)} \cap E_{+,+,L}^{(ij)}$, let $\beta_{+,+,L}^{(ij)}$ denote the turning angle of $C_t^{(k)}$, i.e. the angle between ∂_{x^j} and $-\partial_{x^i}$, then

$$\cos \beta_{+,+,L}^{(ij)} = -g_{ji}. \quad (16)$$

Similarly, if $\beta_{+,-,L}^{(ji)}$, $\beta_{-,-,L}^{(ij)}$, $\beta_{-,+,L}^{(ji)}$ denote the turning angle of $C_t^{(k)}$ at vertices in $E_{+,-,L}^{(ji)}$, $E_{-,-,L}^{(ij)}$, $E_{-,+,L}^{(ji)}$, respectively, then

$$\cos \beta_{+,-,L}^{(ji)} = g_{ij}, \quad \cos \beta_{-,-,L}^{(ij)} = -g_{ji}, \quad \cos \beta_{-,+,L}^{(ji)} = g_{ij}. \quad (17)$$

We define $\beta_t^{(k)}$ to be the sum of the four turning angles of $C_t^{(k)}$ at its vertices. Then

$$\beta_t^{(k)} = \frac{1}{2} \sum_{\{i,j\}=\{1,2,3\} \setminus \{k\}} \left(\beta_{+,+,L}^{(ij)} + \beta_{+,-,L}^{(ji)} + \beta_{-,-,L}^{(ij)} + \beta_{-,+,L}^{(ji)} \right). \quad (18)$$

The factor $\frac{1}{2}$ here is because of the symmetry $\beta_{\mu,\lambda,L}^{(ij)} = \beta_{\lambda,\mu,L}^{(ji)}$ for any indices $i \neq j$ and any sign symbols $\mu, \lambda \in \{+, -\}$. (Similarly, $\theta_{\mu,\lambda,L}^{(ij)} = \theta_{\lambda,\mu,L}^{(ji)}$.)

We now turn to the mass $m(g)$ of (M^3, g) . By [2, Proposition 4.1] (and the fact Stokes' theorem holds on cubic domains), $m(g)$ can be computed by

$$m(g) = \lim_{L \rightarrow \infty} \frac{1}{16\pi} \int_{\partial \text{Cube}_L} \sum_{j,k} (g_{jk,j} - g_{jj,k}) v^k d\sigma. \quad (19)$$

Since $v = \partial_{x^i} + O(L^{-\tau})$ on $F_{+,L}^i$ and $v = -\partial_{x^i} + O(L^{-\tau})$ on $F_{-,L}^i$, (19) simplifies to

$$\begin{aligned} & 16\pi m(g) \\ &= \lim_{L \rightarrow \infty} \sum_i \left(\int_{F_{+,L}^i} \sum_{j \neq i} (g_{ji,j} - g_{jj,i}) d\sigma - \int_{F_{-,L}^i} \sum_{j \neq i} (g_{ji,j} - g_{jj,i}) d\sigma \right). \end{aligned} \quad (20)$$

On $F_{+,L}^{(i)}$, by (13) and (15),

$$\begin{aligned} & \int_{F_{+,L}^{(i)}} \sum_{j \neq i} (g_{ji,j} - g_{jj,i}) d\sigma \\ &= \int_{F_{+,L}^{(i)}} \sum_{j \neq i} (-g_{ji,j}) d\sigma - 2 \int_{F_{+,L}^{(i)}} \sum_{k \neq i} \kappa^{(k)} d\sigma + O(L^{1-2\tau}) \\ &= \int_{F_{+,L}^{(i)}} \sum_{j \neq i} (-g_{ji,j}) d\sigma - 2 \int_{F_{+,L}^{(i)}} H d\sigma + O(L^{1-2\tau}). \end{aligned} \quad (21)$$

On each face and edge, let $d\sigma_0, ds_0$ denote the area and length measure with respect to the background Euclidean metric g_0 . Then

$$\begin{aligned}
 \int_{F_{+,L}^{(i)}} \sum_{j \neq i} g_{ji,j} d\sigma &= \int_{F_{+,L}^{(i)}} \sum_{j \neq i} g_{ji,j} d\sigma_0 + O(L^{1-2\tau}) \\
 &= \sum_{j \neq i} \left[\int_{E_{+,+,L}^{(ij)}} g_{ji} ds_0 + \int_{E_{+,-,L}^{(ij)}} (-g_{ji}) ds_0 \right] + O(L^{1-2\tau}) \\
 &= \sum_{j \neq i} \left[\int_{E_{+,+,L}^{(ij)}} g_{ji} ds + \int_{E_{+,-,L}^{(ij)}} (-g_{ji}) ds \right] + O(L^{1-2\tau}) \quad (22) \\
 &= \sum_{j \neq i} \left[- \int_{E_{+,+,L}^{(ij)}} \cos \theta_{+,+,L}^{(ij)} ds - \int_{E_{+,-,L}^{(ij)}} \cos \theta_{+,-,L}^{(ij)} ds \right] + O(L^{1-2\tau}), \\
 &= (-1) \sum_{j \neq i} \left[\int_{E_{+,+,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{+,+,L}^{(ij)} \right) ds + \int_{E_{+,-,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{+,-,L}^{(ij)} \right) ds \right] + O(L^{1-2\tau}),
 \end{aligned}$$

where we have used (11) and (12). It follows from (21) and (22) that

$$\begin{aligned}
 &\int_{F_{+,L}^{(i)}} \sum_{j \neq i} (g_{ji,j} - g_{jj,i}) d\sigma \\
 &= \sum_{j \neq i} \left[\int_{E_{+,+,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{+,+,L}^{(ij)} \right) ds + \int_{E_{+,-,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{+,-,L}^{(ij)} \right) ds \right] \quad (23) \\
 &\quad - 2 \int_{F_{+,L}^{(i)}} H d\sigma + O(L^{1-2\tau}).
 \end{aligned}$$

Similarly, on $F_{-,L}^{(i)}$,

$$\begin{aligned}
 &\int_{F_{-,L}^{(i)}} \sum_{j \neq i} (g_{ji,j} - g_{jj,i})(-1) d\sigma \\
 &= \sum_{j \neq i} \left[\int_{E_{-+,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{-+,L}^{(ij)} \right) ds + \int_{E_{--,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{--,L}^{(ij)} \right) ds \right] \quad (24) \\
 &\quad - 2 \int_{F_{-,L}^{(i)}} H d\sigma + O(L^{1-2\tau}).
 \end{aligned}$$

By (20), (23) and (24), we have

$$\begin{aligned}
 &16\pi m(g) \\
 &= \sum_{j \neq i} \left[\int_{E_{+,+,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{+,+,L}^{(ij)} \right) ds + \int_{E_{+,-,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{+,-,L}^{(ij)} \right) ds \right. \\
 &\quad \left. + \int_{E_{-+,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{-+,L}^{(ij)} \right) ds + \int_{E_{--,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{--,L}^{(ij)} \right) ds \right]
 \end{aligned}$$

$$-2 \int_{\partial \text{Cube}_L} H d\sigma + O(L^{1-2\tau}), \quad (25)$$

which verifies (5). Note that each edge of ∂Cube_L is counted twice in (25).

Next, we write $m(g)$ in terms of the geodesic curvature and turning angles of $C_t^{(k)}$. By (11), (12) and (16), (17), we have

$$\theta_{\mu,\lambda,L}^{(ij)} = \beta_{\mu,\lambda,L}^{(ij)} + O(L^{-2\tau}), \quad (26)$$

for any indices $i \neq j$ and any sign symbols $\mu, \lambda \in \{+, -\}$. Thus,

$$\begin{aligned} & \sum_{j \neq i} \left[\int_{E_{+,+,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{+,+,L}^{(ij)} \right) ds + \int_{E_{+,-,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{+,-,L}^{(ij)} \right) ds \right. \\ & \quad \left. + \int_{E_{-,+,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{-,+,L}^{(ij)} \right) ds + \int_{E_{-,-,L}^{(ij)}} \left(\frac{\pi}{2} - \theta_{-,-,L}^{(ij)} \right) ds \right] \\ &= \sum_{j \neq i} \left[\int_{E_{+,+,L}^{(ij)}} \left(\frac{\pi}{2} - \beta_{+,+,L}^{(ij)} \right) ds_0 + \int_{E_{+,-,L}^{(ij)}} \left(\frac{\pi}{2} - \beta_{+,-,L}^{(ij)} \right) ds_0 \right. \\ & \quad \left. + \int_{E_{-,+,L}^{(ij)}} \left(\frac{\pi}{2} - \beta_{-,+,L}^{(ij)} \right) ds_0 + \int_{E_{-,-,L}^{(ij)}} \left(\frac{\pi}{2} - \beta_{-,-,L}^{(ij)} \right) ds_0 \right] + O(L^{1-2\tau}) \\ &= \sum_k \int_{-L}^L 2 \left(2\pi - \beta_t^{(k)} \right) dt + O(L^{1-2\tau}). \end{aligned} \quad (27)$$

By (15), we also have

$$\begin{aligned} & \int_{\partial \text{Cube}_L} H d\sigma = \sum_i \int_{F_{+,L}^{(i)} \cup F_{-,L}^{(i)}} H d\sigma \\ &= \sum_i \int_{F_{+,L}^{(i)} \cup F_{-,L}^{(i)}} \sum_{k \neq i} \kappa^{(k)} ds_0 + O(L^{1-2\tau}) \\ &= \sum_k \sum_{i \neq k} \int_{F_{+,L}^{(i)} \cup F_{-,L}^{(i)}} \kappa^{(k)} ds_0 + O(L^{1-2\tau}) \\ &= \sum_k \sum_{i \neq k} \int_{-L}^L \left(\int_{C_t^{(k)} \cap (F_{+,L}^{(i)} \cup F_{-,L}^{(i)})} \kappa^{(k)} ds_0 \right) dt + O(L^{1-2\tau}) \\ &= \sum_k \int_{-L}^L \left(\int_{C_t^{(k)}} \kappa^{(k)} ds_0 \right) dt + O(L^{1-2\tau}) \\ &= \sum_k \int_{-L}^L \left(\int_{C_t^{(k)}} \kappa^{(k)} ds \right) dt + O(L^{1-2\tau}). \end{aligned} \quad (28)$$

Therefore, by (25), (27) and (28), we have

$$\begin{aligned} & 16\pi m(g) \\ &= 2 \sum_k \int_{-L}^L \left(2\pi - \beta_t^{(k)} - \int_{C_t^{(k)}} \kappa^{(k)} ds \right) dt + O(L^{1-2\tau}), \end{aligned} \quad (29)$$

which verifies (6).

3. Relation to the Mass Formula in [3]

In formulae (5) and (6), the coordinates $\{x^i\}$ used in defining ∂Cube_L and $C_t^{(k)}$ do not need to be harmonic. If $\{x^i\}$ are harmonic, (6) and (3) then imply

$$\sum_{k=1}^3 \int_{\partial\text{Cube}_L} \frac{\partial}{\partial \nu} |\nabla x^k| d\sigma + \int_{-L}^L \left[(2\pi - \beta_t^{(k)}) - \int_{C_t^{(k)}} \kappa^{(k)} ds \right] dt = 24\pi m(g) + o(1). \quad (30)$$

This formula is weaker than that of Bray et al. [3], which indicates, without summing over k , each summand above tends to $8\pi m(g)$, provided $\Delta x^k = 0$.

We now examine the summand in (30). Let k, i, j be fixed indices so that they are distinct from each other. Similar to how (22) is derived, by (16) and (17),

$$\begin{aligned} 2 \int_{-L}^L (2\pi - \beta_t^{(k)}) dt &= - \int_{F_{+,L}^i} g_{ij,j} d\sigma_0 + \int_{F_{-,L}^i} g_{ij,j} d\sigma_0 \\ &\quad - \int_{F_{+,L}^j} g_{ij,i} d\sigma_0 + \int_{F_{-,L}^j} g_{ij,i} d\sigma_0 + O(L^{1-2\tau}). \end{aligned}$$

By (13) and (14),

$$\begin{aligned} &2 \int_{-L}^L \left[- \int_{C_t^{(k)}} \kappa^{(k)} ds \right] dt \\ &= \int_{-L}^L \left[\int_{C_t^{(k)} \cap F_{+,L}^i} (2g_{ij,j} - g_{jj,i}) ds + \int_{C_t^{(k)} \cap F_{-,L}^i} (-2g_{ij,j} + g_{jj,i}) ds \right. \\ &\quad \left. + \int_{C_t^{(k)} \cap F_{+,L}^j} (2g_{ji,i} - g_{ii,j}) ds + \int_{C_t^{(k)} \cap F_{-,L}^j} (-2g_{ji,i} + g_{ii,j}) ds \right] dt \\ &= \int_{F_{+,L}^i} (2g_{ij,j} - g_{jj,i}) d\sigma_0 + \int_{F_{-,L}^i} (-2g_{ij,j} + g_{jj,i}) d\sigma_0 \\ &\quad + \int_{F_{+,L}^j} (2g_{ji,i} - g_{ii,j}) d\sigma_0 + \int_{F_{-,L}^j} (-2g_{ji,i} + g_{ii,j}) d\sigma_0 + O(L^{1-2\tau}). \end{aligned}$$

Thus, the boundary term in the Gauss–Bonnet theorem satisfies

$$\begin{aligned} &2 \int_{-L}^L \left[(2\pi - \beta_t^{(k)}) - \int_{C_t^{(k)}} \kappa^{(k)} ds \right] dt \\ &= \int_{F_{+,L}^i} (g_{ij,j} - g_{jj,i}) d\sigma_0 - \int_{F_{-,L}^i} (g_{ij,j} - g_{jj,i}) d\sigma_0 \\ &\quad + \int_{F_{+,L}^j} (g_{ji,i} - g_{ii,j}) d\sigma_0 - \int_{F_{-,L}^j} (g_{ji,i} - g_{ii,j}) d\sigma_0 + O(L^{1-2\tau}). \end{aligned} \quad (31)$$

(If summing over $k \in \{1, 2, 3\}$, this again gives (29).)

We next compute $|\nabla x^k|$ and Δx^k . Since $|\nabla x^k|^2 = g^{kk}$,

$$2 \frac{\partial}{\partial v} |\nabla x^k| = -g_{kk,m} v^m + O(L^{-2\tau-1}).$$

Hence,

$$\begin{aligned} 2 \int_{\partial \text{Cube}_L} \frac{\partial}{\partial v} |\nabla x^k| d\sigma &= - \int_{F_{+,L}^k} g_{kk,k} d\sigma_0 + \int_{F_{-,L}^k} g_{kk,k} d\sigma_0 \\ &\quad - \int_{F_{+,L}^i} g_{kk,i} d\sigma_0 + \int_{F_{-,L}^i} g_{kk,i} d\sigma_0 \\ &\quad - \int_{F_{+,L}^j} g_{kk,j} d\sigma_0 + \int_{F_{-,L}^j} g_{kk,j} d\sigma_0 + O(L^{1-2\tau}). \end{aligned} \quad (32)$$

The term $g_{kk,k}$ appears in

$$\Delta x^k = \sum_{m \neq k} \left(\frac{1}{2} g_{mm,k} - g_{km,m} \right) - \frac{1}{2} g_{kk,k} + O(L^{-2\tau-1}) \quad (33)$$

Thus,

$$\begin{aligned} &2 \int_{\partial \text{Cube}_L} \frac{\partial}{\partial v} |\nabla x^k| d\sigma - \int_{F_{+,L}^k} 2\Delta x^k d\sigma + \int_{F_{-,L}^k} 2\Delta x^k d\sigma \\ &= \int_{F_{+,L}^k} \sum_{m \neq k} (g_{km,m} - g_{mm,k}) + \sum_{m \neq k} g_{km,m} d\sigma_0 \\ &\quad - \int_{F_{-,L}^k} \sum_{m \neq k} (g_{km,m} - g_{mm,k}) + \sum_{m \neq k} g_{km,m} d\sigma_0 \\ &\quad - \int_{F_{+,L}^i} g_{kk,i} d\sigma_0 + \int_{F_{-,L}^i} g_{kk,i} d\sigma_0 \\ &\quad - \int_{F_{+,L}^j} g_{kk,j} d\sigma_0 + \int_{F_{-,L}^j} g_{kk,j} d\sigma_0 + O(L^{1-2\tau}). \end{aligned} \quad (34)$$

Therefore, adding (31) and (34), and using (20), we have

$$\begin{aligned} &2 \int_{\partial \text{Cube}_L} \frac{\partial}{\partial v} |\nabla x^k| d\sigma + 2 \int_{-L}^L \left[(2\pi - \beta_t^{(k)}) - \int_{C_t^{(k)}} \kappa^{(k)} ds_g \right] dt \\ &= 16\pi \mathfrak{m}(g) + \int_{F_{+,L}^k} 2\Delta x^k d\sigma_0 - \int_{F_{-,L}^k} 2\Delta x^k d\sigma_0 + O(L^{1-2\tau}) \\ &\quad + \int_{F_{+,L}^k} (g_{ki,i} + g_{kj,j}) d\sigma_0 - \int_{F_{-,L}^k} (g_{ki,i} + g_{kj,j}) d\sigma_0 \\ &\quad - \int_{F_{+,L}^i} g_{ik,k} d\sigma_0 + \int_{F_{-,L}^i} g_{ik,k} d\sigma_0 - \int_{F_{+,L}^j} g_{jk,k} d\sigma_0 + \int_{F_{-,L}^j} g_{jk,k} d\sigma_0. \end{aligned} \quad (35)$$

The last two lines in (35) cancel upon integration by parts. Thus,

$$\begin{aligned} & \int_{\partial \text{Cube}_L} \frac{\partial}{\partial v} |\nabla x^k| d\sigma + \int_{-L}^L \left[(2\pi - \beta_t^{(k)}) - \int_{C_t^{(k)}} \kappa^{(k)} ds \right] dt \\ &= 8\pi m(g) + \int_{F_{+,L}^k} \Delta x^k d\sigma_0 - \int_{F_{-,L}^k} \Delta x^k d\sigma_0 + O(L^{1-2\tau}), \end{aligned} \quad (36)$$

which is the formula in [3, equation (6.27)] if $\Delta x^k = 0$.

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