NONEXISTENCE OF NNSC FILL-INS WITH LARGE MEAN CURVATURE

PENGZI MIAO

Abstract. In this note we show that a closed Riemannian manifold does not admit a fill-in with nonnegative scalar curvature if the mean curvature is point-wise large. Similar result also holds for fill-ins with a negative scalar curvature lower bound.

Consider a 2-sphere S^2 with the standard round metric γ_o of area 4π. If H is a function on S^2 with $H > 2$, there does not exist any compact Riemannian 3-manifold (Ω, g) with nonnegative scalar curvature, with boundary such that $\partial\Omega$ is isometric to (S^2, γ_o) and has mean curvature H. This can be derived as a consequence of the Riemannian positive mass theorem [14, 22], formulated on manifolds with corner along hypersurfaces [12, 19].

For an arbitrary closed orientable surface Σ, a pair (γ, H) is called Bartnik data on Σ if γ and H denote a metric and a function on Σ, respectively. The question whether (Σ, γ, H) bounds a compact 3-manifold with nonnegative scalar curvature, with boundary isometric to (Σ, γ) and having mean curvature H is closely tied to the quasi-local mass problem of (Σ, γ, H) (see [1, 7] for instance).

In general, let Σ^{n-1} be a closed $(n-1)$-dimensional manifold. Given a metric γ and a function H on Σ, we say a compact orientable Riemannian 3-manifold (Ω, g) is a nonnegative scalar curvature (NNSC) fill-in of (Σ, γ, H) if $\partial\Omega$ is isometric to (Σ, γ) and the mean curvature of $\partial\Omega$ equals H. In [5], Gromov showed, if Ω is a spin manifold and if (Ω^n, g) is a NNSC fill-in of (Σ, γ, H), then

$$\min_{\Sigma} H \leq \frac{n-1}{\mathrm{Rad}(\Sigma, \gamma)}, \tag{1}$$

where $\mathrm{Rad}(\Sigma, \gamma)$ is a constant only depending on (Σ, γ), known as the (hyper)spherical radius of (Σ, γ). As a result, spin NNSC fill-ins of (Σ, γ, H) do not exist for large mean curvature function H.

One can drop the requirement on H when studying the geometry of fill-ins. We say (Ω, g) is a fill-in of (Σ, γ) if $\partial\Omega$ is isometric to (Σ, γ). Interaction among the scalar curvature, the total boundary mean curvature, and the volume of fill-ins were studied in [19, 8, 11, 13, 18, 17].

In [6], Gromov raised the following existence question of fill-ins with positive scalar curvature.

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Question 1 ([6]) *If $\Sigma = \partial\Omega$ for a compact manifold Ω and γ is a Riemannian metric on Σ, does γ extend to a Riemannian metric g on Ω with positive scalar curvature?*

This question recently has been answered affirmatively by Shi, Wang and Wei [17].

Theorem 1 (Shi-Wang-Wei [17])**.** *Let Ω^n be a compact n-dimensional manifold with boundary Σ. Then any metric γ on Σ can be extended to a Riemannian metric g on Ω with positive scalar curvature.*

To construct such an extension, the authors made an ingenious use of the parabolic method employed in [2, 19]. More precisely, they started with a metric $\bar{g}$ on Ω with positive scalar curvature (whose existence is guaranteed by [4, 9] for instance), and construct a suitable transition metric on $\Sigma \times [0,1]$, which connects γ on $\Sigma \times \{0\}$ to a (large) constant scaling of the induced metric from $\bar{g}$ on $\Sigma = \Sigma \times \{1\}$, via the parabolic method.

Applying their proof of Theorem 1 and the positive mass theorem, Shi, Wang and Wei [17] further proved the following nonexistence theorem on NNSC fill-ins. Hereinafter, the dimension n denotes a dimension for which the Riemannian positive mass theorem holds. (See the recent work of Schoen-Yau [16] and references therein.)

Theorem 2 (Shi-Wang-Wei [17])**.** *Suppose a closed manifold Σ^{n-1} can be topologically embedded in $\mathbb{R}^n$. Given any Riemannian metric γ on Σ, there exists a constant h_0, depending on Σ, γ, and the embedding of Σ in $\mathbb{R}^n$, such that, if $\min_\Sigma H \geq h_0$, there do not exist NNSC fill-ins of (Σ, γ, H).*

It is the purpose of this note to show that no NNSC fill-ins exist for any Σ, if H is large. The proof makes use of Shi-Wang-Wei's Theorem 1 above and Schoen-Yau's result on closed manifolds with positive scalar curvature [15].

Theorem 3. *Let Σ^{n-1} be the boundary of some compact n-dimensional manifold Ω. Given any Riemannian metric γ on Σ, there exists a constant H_0, depending on γ and Ω, such that, if $\min_\Sigma H \geq H_0$, there do not exist NNSC fill-ins of (Σ, γ, H).*

Proof. Let p be an interior point in Ω. Near p, form a connected sum of Ω with T^n, where T^n is the n-dimensional torus. Denote the resulting manifold by $\widetilde{\Omega} = \Omega \# T^n$, then $\partial\widetilde{\Omega} = \partial\Omega$.

Given the metric γ on Σ, apply Theorem 1 to $\Sigma = \partial\widetilde{\Omega}$, one obtains a metric g with positive scalar curvature on $\widetilde{\Omega}$ such that g induces the metric γ on Σ. Let $H_{\tilde{\Omega}}$ be the mean curvature of Σ in $(\widetilde{\Omega}, g)$ with respect to the inward unit normal.

Now suppose $(M, g_{_M})$ is a compact manifold with nonnegative scalar curvature so that ∂M is isometric to (Σ, γ). Let $H_{_M}$ be the mean curvature of $\Sigma = \partial M$ in $(M, g_{_M})$ with respect to the outward unit normal. Suppose

$$\min_\Sigma H_{_M} \geq \max_\Sigma H_{\tilde{\Omega}}. \tag{2}$$

Consider the manifold $(\widetilde{M}, \tilde{g})$ obtained by gluing $(M, g_{_M})$ and $(\widetilde{\Omega}, g)$ along their common boundary (Σ, γ). The metric $\tilde{g}$ has nonnegative scalar curvature in M, has positive scalar curvature in $\tilde{\Omega}$, and satisfies (2) across Σ in $\widetilde{M}$. By the interpretation of (2) as the metric having nonnegative distributional scalar curvature across Σ [12], one expects $\tilde{g}$ can be mollified to produce a smooth positive scalar curvature metric on $\widetilde{M}$.

This expectation can be verified via results in [20] (also see [10]) for instance. By Corollary 4.8 in [20], there exists a sequence of smooth metrics $\{g_i\}$ on $\widetilde{M}$ such that g_i has nonnegative scalar curvature and g_i converges to $\tilde{g}$ in C^∞ norm on compact sets away from Σ. Since $\tilde{g} = g$ has positive scalar curvature on $\tilde{\Omega}$, this implies g_i has positive scalar curvature somewhere in $\tilde{\Omega}$. Consequently, $\widetilde{M}$ supports a metric with positive scalar curvature.

However, by construction, $\widetilde{M}$ has topology

$$\widetilde{M} = K \# T^n, \tag{3}$$

where K is an n-dimensional closed orientable manifold obtained by gluing M and Ω along their common boundary Σ. By Schoen-Yau's result on closed manifolds [15] (also see [16]), $\widetilde{M}$ does not admit a metric with positive scalar curvature.

This gives a contradiction to (2). The claim follows by taking $H_0 = \max_\Sigma H_{\tilde{\Omega}}$. □

Combined with a trick of Gromov [5], Theorem 3 implies a similar result for fill-ins with a negative scalar curvature lower bound.

Theorem 4. *Let Σ^{n-1} be the boundary of some compact n-dimensional manifold Ω. Let $\sigma > 0$ be a constant. Given a Riemannian metric γ on Σ, there exists a constant H_σ, depending on γ, Ω and σ, such that, if $\min_\Sigma H \geq H_\sigma$, (Σ, γ, H) does not have fill-ins with scalar curvature bounded below by $-\sigma$.*

Proof. Let $(S^m(r), g_{_S})$ denote a standard m-dimensional round sphere with radius r. Here $m \geq 2$ and r is chosen so that $m(m-1) = r^2\sigma$.

Suppose $(M, g_{_M})$ is a fill-in of (Σ, γ) with $R(g_{_M}) \geq -\sigma$. Following [5], consider the Riemannian product $(N^{n+m}, g_{_N}) = (M, g_{_M}) \times (S^m(r), g_{_S})$. Then $R(g_{_N}) \geq 0$, and $\partial N = \Sigma \times S^m(r)$ has mean curvature $H_{_M}$ in $(N, g_{_N})$. The claim follows by taking $H_\sigma = H_0$, where H_0 is the constant given in Theorem 3 when it is applied to $\Sigma \times S^m(r) = \partial(\Omega^n \times S^m(r))$ with the metric $\gamma + g_{_S}$. □

We conclude this note with some questions open to the author's knowledge. Given a pair (Σ, γ) with Σ being the boundary of some compact manifold, let $\mathcal{F}_{(\Sigma,\gamma)}$ denote the set of NNSC fill-ins of (Σ, γ). Shi-Wang-Wei's extension theorem shows $\mathcal{F}_{(\Sigma,\gamma)} \neq \emptyset$. However, the fill-in produced in [17] has a feature that its boundary Σ has negative mean curvature, i.e. the mean curvature vector of Σ points outward. This leads to a question similar with but different to Question 1:

Question 2. *If Σ is the boundary of some compact manifold and γ is a Riemannian metric on Σ, is $\mathcal{F}^+_{(\Sigma,\gamma)} \neq \emptyset$? Here $\mathcal{F}^+_{(\Sigma,\gamma)}$ denotes the set of NNSC fill-ins of (Σ, γ) so that Σ has positive mean curvature.*

From the point of view of quasi-local mass [19, 21], the topology of fill-ins are allowed to vary. If the topology of fill-ins is fixed to be some Ω, a recent result of

Carlotto and Li [3] completely determines the topology of these Ω in 3-dimension, assuming Ω admits a positive scalar curvature metric with mean convex boundary.

In terms of the set $\mathcal{F}_{(\Sigma,\gamma)}$, Theorem 3 shows

$$\sup_{(M,g_M)\in\mathcal{F}_{(\Sigma,\gamma)}} \min_{\Sigma} H_M < \infty. \tag{4}$$

Considering Gromov's result (1), it would be interesting to know if the left side of (4) can be estimated explicitly by a metric quantity of (Σ,γ).

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(Pengzi Miao) Department of Mathematics, University of Miami, Coral Gables, FL 33146, USA

E-mail address: pengzim@math.miami.edu