

REMARKS ON LARGE TIME BEHAVIOR OF LEVEL-SET MEAN CURVATURE FLOW EQUATIONS WITH DRIVING AND SOURCE TERMS

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ABSTRACT. We study a level-set mean curvature flow equation with driving and source terms, and establish convergence results on the asymptotic behavior of solutions as time goes to infinity under some additional assumptions. We also study the associated stationary problem in details in a particular case, and establish Alexandrov's theorem in two dimensions in the viscosity sense, which is of independent interest.

1. Introduction. In this paper, we study the large time behavior of the viscosity solution u to a degenerate parabolic PDE of the form

$$(C) \quad \begin{cases} u_t - \left(\operatorname{div} \left(\frac{Du}{|Du|} \right) + 1 \right) |Du| = f(x) & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^n, \end{cases}$$

where $n \geq 2$, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the source term, which is nontrivial and satisfies

$$f \in \operatorname{Lip}(\mathbb{R}^n), \quad f \geq 0, \quad \text{and } \operatorname{supp}(f) \subset B(0, R) \text{ for some } R > 0. \quad (1)$$

In other words, $f \geq 0$ is Lipschitz continuous on $\mathbb{R}^n$ such that the support of f is contained in an open ball $B(0, R)$ of radius $R > 0$ centered at the origin. The initial condition $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ is in $BUC(\mathbb{R}^n)$, the space of all bounded uniformly continuous functions on $\mathbb{R}^n$. Here, $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ is a unknown function, and u_t , Du denote the time derivative and the spatial gradient of u , respectively. Note that if the source term f is identically equal to zero, the equation (C) is nothing

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but the level set flow equation of the motion by mean curvature plus a constant ($= 1$) driving force. Its analytic foundation like well-posedness and comparison principle has been well established by adjusting the theory of viscosity solutions [5, 8]. See also e.g. [10] and references therein for further development. We are always concerned with viscosity solutions in this paper, and the term “viscosity” is omitted henceforth.

Equation (C) was derived as a continuum limit of the birth and spread model in [14] (see also [15]) with a motivation to describe crystal growth in supersaturated environments. In [15], we obtained the existence of the large time average of u which was denoted by the *asymptotic speed*. More precisely, we showed that, locally uniformly for $x \in \mathbb{R}^n$,

$$c = \lim_{t \rightarrow \infty} \frac{u(x, t)}{t} \quad (2)$$

exists under assumption (1). We then studied qualitative properties of asymptotic speed $c = c_f$ in some specific settings.

In this paper, as a next step, we investigate the large time asymptotic of u , that is, behavior of $u(x, t)$ as $t \rightarrow \infty$. We establish the convergence result

$$u(x, t) - ct \rightarrow v_\infty(x) \quad \text{locally uniformly for } x \in \mathbb{R}^n \text{ as } t \rightarrow \infty, \quad (3)$$

where v_∞ is a solution to

$$-\left(\operatorname{div}\left(\frac{Dv}{|Dv|}\right) + 1\right)|Dv| = f(x) - c \quad \text{in } \mathbb{R}^n \quad (4)$$

in two specific cases. If (3) holds, we denote by v_∞ the *asymptotic profile* of u with given initial data u_0 . The question on whether (3) holds or not was addressed in [11], and it seems quite difficult to be resolved in the most general setting at this moment.

If one looks at a different type of scaling, it is easy to derive the following scaled asymptotic profile

$$\lim_{\lambda \rightarrow \infty} \frac{u(\lambda x, \lambda t)}{\lambda} = \max\{c(t - |x|), 0\}$$

as stated in [11]. See also [18] for the first order problem.

1.1. Main results. We now describe two specific cases in which we are able to obtain (3). In Case 1, we consider the radially symmetric setting, that is, we assume the following.

- (A1) $f(x) = \tilde{f}(|x|)$ for all $x \in \mathbb{R}^n$, where $\tilde{f} \in C_c^1([0, \infty))$, i.e., $f \in C^1([0, \infty))$ has a compact support in $[0, \infty)$.
- (A2) $u_0(x) = \tilde{u}_0(|x|)$ for all $x \in \mathbb{R}^n$, where $\tilde{u}_0 \in \operatorname{BUC}([0, \infty))$.

In Case 2, we put a different set of assumptions.

- (A3) $f \in C_c^1(\mathbb{R}^n)$ and the set $U := \{x \in \mathbb{R}^n : f(x) = \max_{\mathbb{R}^n} f\}$ contains $B(x_0, n - 1)$ for some $x_0 \in \mathbb{R}^n$.
- (A4) $U = \bigcup_{i \in I} E_i$, where $I \neq \emptyset$ is an index set, and for each $i \in I$, $E_i \subset \mathbb{R}^n$ is a closed set with nonempty interior and C^2 -boundary satisfying $\kappa + 1 \geq 0$ on ∂E_i . Here, for $x \in \partial E_i$, $\kappa(x)$ denotes the mean curvature of ∂E_i in the direction of the outer normal vector to E_i at x .
- (A5) $u_0 \equiv 0$.

It is worth mentioning that (A3) gives us that the asymptotic speed $c = \max_{\mathbb{R}^n} f$ (see [15]), and (A4) implies that the set U grows under the surface evolution $V = \kappa + 1$, where V denotes the normal velocity of the evolving hypersurface. Here are our main convergence results.

Theorem 1.1. *Assume (A1)–(A2). Let u be the solution to (C). Then (3) holds, where v_∞ is a solution to (4).*

Theorem 1.2. *Assume that (A3)–(A5) hold. Let u be the solution to (C). We have (3), where v_∞ is a solution to (4).*

In Case 1, we have a further characterization of v_∞ which is stated as following.

Theorem 1.3. *Assume that (A1)–(A2) hold. Let u be the solution to (C). Then, the asymptotic speed c , and asymptotic profile v_∞ are described by $c = \max_{|x| \geq 1} f(x)$, and $v_\infty(x) = \psi_\infty(|x|)$ for $x \in \mathbb{R}^n$, where ψ_∞ is given by*

$$\psi_\infty(r) = \max \left\{ d(r, s) + v_0(s) : s \in \tilde{\mathcal{A}} \right\}. \quad (5)$$

Here,

$$d(r, s) := \sup_{\gamma \in \mathcal{C}(t, 0; r, s), t > 0} \left\{ \int_0^t \left(\tilde{f}(\gamma(z)) - c \right) dz : \gamma([0, t]) \subset (0, \infty) \right\} \quad (6)$$

for any $r, s \in [0, \infty)$, where we set

$$\begin{aligned} \mathcal{C}(t, 0; r, s) &:= \{ \gamma \in \text{AC}([0, t]; (0, \infty)) : \gamma(t) = r, \gamma(0) = s, \\ &\quad \left| \gamma'(z) + \frac{n-1}{\gamma(z)} \right| \leq 1 \text{ for a.e. } z \in (0, t) \}, \end{aligned}$$

and

$$v_0(r) := \max \{ d(r, \rho) + \tilde{u}_0(\rho) : \rho \in [0, \infty) \}, \quad \tilde{\mathcal{A}} := \{ r \geq n-1 : \tilde{f}(r) = c \}.$$

It is worth noting that the set $\tilde{\mathcal{A}}$ is the set of all equilibrium points associated with our cost function. Moreover, $\tilde{\mathcal{A}}$ plays a role of a *uniqueness set* for the associated stationary problem, which is stated in Theorem 2.5.

1.2. Literature and discussions. In the last two decades, a lot of works have been devoted to the study of large time behavior of viscosity solutions of both first-order and second-order Hamilton-Jacobi equations. In these papers, the Hamiltonian $H = H(x, p)$ is assumed to be convex and coercive in p . See [24, 9, 2, 7, 21, 20, 3, 1, 22, 13] and the references therein for a complete description of the literature. Nevertheless, to the best of our knowledge, large time behavior of (C) has not been studied. It is worth noting that in general, (C) is not convex/concave in the gradient variable, and the methods in aforementioned results do not apply here. Furthermore, the spatial variable x is in the whole $\mathbb{R}^n$ without any periodicity, which is not compact, and hence, another layer of difficulty appears. See [16, 4, 19] for some results on large time behavior on the mean curvature flow in different contexts.

In the two cases described above, we are able to use specific structure of the PDE to obtain large time behavior. More specifically, in the radially symmetric setting (Case 1), it is intuitively clear that solution u to (C) is also radially symmetric, that is, $u(x, t) = \phi(|x|, t)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$ for some $\phi : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$. It turns out that this is indeed the case, and ϕ solves

$$\phi_t - \frac{n-1}{r} \phi_r - |\phi_r| = f(r) \quad \text{in } (0, \infty) \times (0, \infty), \quad (7)$$

a first-order Hamilton-Jacobi equation that is singular at $r = 0$. Set

$$H(r, p) := -\frac{n-1}{r}p - |p| - f(r) \quad \text{for } (r, p) \in (0, \infty) \times \mathbb{R}.$$

It is clear that H is concave but is not coercive in p . We therefore need to treat the problem with care, and indeed, the behavior of solutions at $r \leq n-1$ is rather different from the behavior at $r > n-1$. Some of our ideas are inspired by [24, 7].

The treatment of Case 2 is quite different. For this, we aim at showing that U is a uniqueness set of (4) (to be described carefully later), and then use geometric properties of each set E_i for $i \in I$ to conclude.

Organization of the paper. In Section 2, we study the radially symmetric setting, and give proofs to Theorem 1.1, and Theorem 1.3. Section 3 is devoted to prove Theorem 1.2. In particular, we show that U is a uniqueness of ergodic problem (4) in Proposition 2. Finally, in Section 4, we discuss further on the uniqueness set U under assumption (A3), and show that the matter is quite complex in general as ergodic problem (4) restricted to U (see (26)) might have infinitely many solutions. Therefore, if we do not assume more ((A4)–(A5)), it is not yet known whether convergence (3) holds or not in the nonradially symmetric setting. We also establish Alexandrov's theorem in two dimensions in the viscosity sense, which is of independent interest.

2. Case 1: radially symmetric case. In this section, we *always* assume that (A1)–(A2) hold. In the radially symmetric case, the unique solution u to (C) is represented by $u(x, t) = \phi(|x|, t)$ where ϕ is defined as

$$\begin{aligned} \phi(r, t) = \sup \left\{ \int_0^t \tilde{f}(\gamma(s)) ds + \tilde{u}_0(\gamma(0)) : \gamma([0, t]) \subset (0, \infty), \right. \\ \left. \gamma(t) = r, \left| \gamma'(s) + \frac{n-1}{\gamma(s)} \right| \leq 1 \text{ for a.e. } s \in [0, t] \right\} \quad (8) \end{aligned}$$

for $(r, t) \in (0, \infty) \times [0, \infty)$. Note that $\phi \in \text{UC}((0, \infty) \times [0, \infty))$. See [14, 15].

By using formula (8), it was obtained in [15] that, locally uniformly for $x \in \mathbb{R}^n$,

$$\frac{u(x, t)}{t} \rightarrow c := \max_{|x| \geq n-1} f(x) = \max_{r \geq n-1} \tilde{f}(r) \quad \text{as } t \rightarrow \infty. \quad (9)$$

We now aim at studying finer behavior of $u(x, t)$ as $t \rightarrow \infty$.

2.1. Ergodic problem. In this subsection, we study

$$-\frac{n-1}{r}\psi_r - |\psi_r| = h(r) \quad \text{in } (0, \infty), \quad (10)$$

where we set $h(r) := \tilde{f}(r) - c$ for convenience. We first establish the existence result of solutions to (10).

Theorem 2.1. *There exists a solution $\psi \in C([0, \infty))$ to (10) which is bounded from above. Moreover, ψ has a linear growth at infinity. More precisely,*

$$\lim_{r \rightarrow \infty} \frac{\psi(r)}{1+r} \text{ exists and is negative.}$$

Proof. We construct ψ explicitly as following. Let

$$r_0 = \min \left\{ r \geq n-1 : \tilde{f}(r) = c \right\} = \min \{ r \geq n-1 : h(r) = 0 \}.$$

We first define $\psi(r)$ for $r \in [n-1, \infty)$. For $r > r_0$, we let $\psi_r(r) \leq 0$ and hence,

$$\psi_r(r) = \frac{rh(r)}{r - (n-1)} \quad \text{for } r > r_0. \quad (11)$$

There are two cases to be considered here. Firstly, if $r_0 > n-1$, then we are able to extend the above definition for $r = r_0$ naturally, and clearly, $\psi_r(r_0) = 0$. For $r \in [n-1, r_0)$, we let $\psi_r(r) \geq 0$, that is,

$$\psi_r(r) = \frac{-rh(r)}{r + (n-1)} \quad \text{for } n-1 \leq r < r_0. \quad (12)$$

Secondly, if $r_0 = n-1$, we could also extend (11) and define that

$$\psi_r(n-1) = \lim_{r \rightarrow n-1} \frac{rh(r)}{r - (n-1)} = \lim_{r \rightarrow n-1} \frac{r(h(r) - h(n-1))}{r - (n-1)} = (n-1)h'(n-1) \leq 0.$$

In both cases, it is clear that $\psi \in C^1([n-1, \infty))$ and ψ solves (10) classically in $(n-1, \infty)$.

Next, we define $\psi(r)$ for $r \in [0, n-1)$. For this, we decompose the interval $I = (0, n-1)$ into

$$\begin{aligned} I &= (0, n-1) \\ &= \{r \in I : h(r) < 0\} \cup \{r \in I : h(r) = 0\} \cup \{r \in I : h(r) > 0\} \\ &=: A \cup B \cup C. \end{aligned} \quad (13)$$

For $r \in A$, we let $\psi_r(r) \geq 0$ and define it as in (12). For $r \in B$, we simply let $\psi_r(r) = 0$. And finally, for $r \in C$, we set $\psi_r(r) \leq 0$ as in (11). In all cases, for $r \in (0, 1/2)$,

$$|\psi_r(r)| \leq 2r\|h\|_{L^\infty},$$

which allows us to extend naturally that $\psi_r(0) = 0$. Here $\|h\|_\infty$ denotes the (essential) supremum norm of h . Finally, set $\psi(0) = 0$.

It is clear from the construction that $\psi \in C^1([0, \infty))$ is bounded from above and satisfies the growth condition, and furthermore ψ solves (10) classically in $(0, \infty)$. $\square$

For ψ constructed above, we define

$$v(x) = \psi(|x|) \quad \text{for all } x \in \mathbb{R}^n.$$

Then, v is a solution to (4) (see [15] for a detailed proof). The next lemma concerns behavior of solutions to (10) in $(0, n-1)$.

Lemma 2.2. *Let $A, C \subset (0, n-1)$ be the sets defined by (13), and ψ be an arbitrary solution to (10) and $(a, b) \subset (0, n-1)$. Then, the following holds.*

- (i) *If $(a, b) \subset A$, then $\psi_r(r) \geq 0$ and $\psi_r(r)$ satisfies (12) for $r \in (a, b)$.*
- (ii) *If $(a, b) \subset C$, then $\psi_r(r) \leq 0$ and $\psi_r(r)$ satisfies (11) for $r \in (a, b)$.*
- (iii) *Let ψ_1, ψ_2 be two solutions to (10). Then, $(\psi_1)_r(r) = (\psi_2)_r(r)$ for a.e. $r \in [0, n-1]$. In particular, $\psi_1 - \psi_2$ is constant on $[0, n-1]$.*

Proof. We proceed to prove (i). Assume by contradiction that ψ is not increasing in (a, b) , then, we can find $a < c < d < b$ such that $\psi(c) > \psi(d)$.

If ψ has a local maximum at $y \in (c, d)$, then by the subsolution test at y , we get $0 \leq h(y) < 0$, which is absurd. Thus, ψ does not have a local maximum in (c, d) , and hence, ψ is decreasing in (c, d) . We then have that $\psi_r(r) \leq 0$ for a.e. $r \in (c, d)$. We take one such r and plug it into (10) to yield

$$0 > h(r) = -\frac{n-1}{r}\psi_r(r) - |\psi_r(r)| = \left(\frac{n-1}{r} - 1\right)|\psi_r(r)| \geq 0,$$

which is also absurd.

Therefore, ψ is increasing in (a, b) and $\psi_r(r) \geq 0$ for a.e. $r \in (a, b)$. It is then straightforward to see that $\psi_r(r)$ satisfies (12) for all $r \in (a, b)$.

Since the proof of (ii) is analogous to the above, and we can get (iii) as a straightforward result of (i)–(ii), we omit them. $\square$

We are now interested in behavior of solutions to (10) in $(n-1, \infty)$.

Lemma 2.3. *Let ψ_1, ψ_2 be two solutions to (10). Pick $a, b \in \tilde{\mathcal{A}}$ such that $(a, b) \cap \tilde{\mathcal{A}} = \emptyset$. Assume that $\psi_1(a) = \psi_2(a)$ and $\psi_1(b) = \psi_2(b)$. Then, $\psi_1 = \psi_2$ in (a, b) .*

Proof. We note that, for $r \in (a, b)$,

$$\tilde{H}(r, p) := -\frac{n-1}{r}p - |p| \leq \left(\frac{n-1}{r} - 1\right)|p| = -\frac{r-(n-1)}{r}|p| \leq 0,$$

which tends to $-\infty$ and $|p| \rightarrow \infty$. Hence, ψ_1, ψ_2 are locally Lipschitz in (a, b) and differentiable a.e. there. Of course, at places of differentiability of ψ_i for $i = 1, 2$, $(\psi_i)_r(r)$ takes value of either form (12) or (11).

Besides, as $p \mapsto \tilde{H}(r, p)$ is concave, and $h(r) < 0$ in (a, b) , we conclude that the graphs of ψ_1, ψ_2 cannot have corners from above, and can only have at most one corner from below in (a, b) . Indeed, if ψ_i has a corner from above at $z \in (a, b)$ for some $i \in \{1, 2\}$, then it means that ψ_i has a local maximum at z . We can then apply the viscosity subsolution test at z to deduce that $0 \leq h(z) < 0$, which is absurd. The fact that ψ_i cannot have corners from above implies immediately that it has at most one corner from below.

Thus, for each $i = 1, 2$, there exists at most one point $c_i \in (a, b)$ such that

$$\begin{cases} (\psi_i)_r \leq 0 \text{ and } (\psi_i)_r \text{ satisfies (11) in } (a, c_i), \\ (\psi_i)_r \geq 0 \text{ and } (\psi_i)_r \text{ satisfies (12) in } (c_i, b). \end{cases}$$

Since $\psi_1(a) = \psi_2(a)$ and $\psi_1(b) = \psi_2(b)$, in case that c_1, c_2 exist, it is clear that $c_1 = c_2$. We thus get the desired conclusion. $\square$

Lemma 2.4. *Let ψ_1, ψ_2 be two bounded from above solutions to (10). Assume that $\psi_1(r_0) = \psi_2(r_0)$ and $\psi_1(M) = \psi_2(M)$, where $r_0 = \min\{r \in [n-1, \infty) : h(r) = 0\}$ and $M = \max\{r \in [n-1, \infty) : h(r) = 0\}$. Then, $\psi_1 = \psi_2$ on $[n-1, r_0] \cup [M, \infty)$.*

Proof. We first show that $\psi_1 = \psi_2$ on $[n-1, r_0]$. If $r_0 = n-1$, then there is nothing to prove. We therefore only need to consider the case that $r_0 > n-1$. As noted in the proof above, ψ_i cannot have corners from above, and thus,

$$(\psi_i)_r(r) = \frac{-rh(r)}{r+(n-1)} > 0 \quad \text{for } n-1 \leq r < r_0, i = 1, 2.$$

We use this and the hypothesis that $\psi_1(r_0) = \psi_2(r_0)$ to yield $\psi_1 = \psi_2$ on $[n-1, r_0]$.

We next prove that $\psi_1 = \psi_2$ on $[M, \infty)$. To get this, we aim at showing

$$(\psi_1)_r = (\psi_2)_r \leq 0 \quad \text{on } [M, \infty). \quad (14)$$

Indeed, as ψ_1, ψ_2 cannot have corners from above, and can only have at most one corner from below, if (14) were false, then there would exist $i \in \{1, 2\}$, and $y \geq M$ such that

$$(\psi_i)_r(r) = \frac{-rh(r)}{r + (n-1)} > 0 \quad \text{for all } r > y.$$

This implies that ψ_i is not bounded from above, which is absurd. The proof is complete. $\square$

Combining the results of Lemmas 2.2–2.4, we obtain our main result in this subsection.

Theorem 2.5. *The following holds.*

- (i) *Let ψ_1, ψ_2 be solutions to (10), which are bounded from above. Assume further that $\psi_1 = \psi_2$ on $\tilde{\mathcal{A}}$. Then $\psi_1 = \psi_2$ on $[0, \infty)$.*
- (ii) *Let ξ_1, ξ_2 be, respectively, a subsolution, and a supersolution to (10), which are bounded from above. Assume further that $\xi_1 \leq \xi_2$ on $\tilde{\mathcal{A}}$. Then $\xi_1 \leq \xi_2$ on $[0, \infty)$.*

2.2. Large time behavior. We now study the behavior of $u(x, t) = \phi(|x|, t)$ as $t \rightarrow \infty$.

Proof of Theorem 1.1. Let $\tilde{\phi}(r, t) = \phi(r, t) - ct$ for all $(r, t) \in [0, \infty) \times [0, \infty)$, where c is given by (9). Then $\tilde{\phi}$ solves

$$\tilde{\phi}_t - \frac{n-1}{r} \tilde{\phi}_r - |\tilde{\phi}_r| = h(r) \quad \text{in } (0, \infty) \times (0, \infty).$$

Noting that

$$-\frac{n-1}{r} \tilde{\phi}_r - |\tilde{\phi}_r| = \begin{cases} -\left(\frac{n-1}{r} + 1\right) \tilde{\phi}_r & \text{if } \tilde{\phi}_r \geq 0, \\ -\left(\frac{n-1}{r} - 1\right) \tilde{\phi}_r & \text{if } \tilde{\phi}_r < 0, \end{cases}$$

we easily see that $-\frac{n-1}{r} \tilde{\phi}_r - |\tilde{\phi}_r| \leq 0$ for $r \geq n-1$. In particular, $\tilde{\phi}$ is a supersolution to

$$\tilde{\phi}_t \geq h(r) \quad \text{in } (n-1, \infty) \times (0, \infty).$$

We use the above and the fact that $h(r) = 0$ for $r \in \tilde{\mathcal{A}}$ to yield

$$t \mapsto \tilde{\phi}(r, t) \quad \text{is nondecreasing for } r \in \tilde{\mathcal{A}}. \quad (15)$$

We now find upper and lower bounds for $\tilde{\phi}$. Let ψ be a solution to (10) constructed in Theorem 2.1, and $v(x) = \psi(|x|)$ for $x \in \mathbb{R}^n$. Since ψ is bounded from above, there exists $M > \|u_0\|_{L^\infty}$ such that $v - M \leq u_0 \leq M$ on $\mathbb{R}^n$. By the usual comparison principle [5, 8], we have

$$v(x) - M + ct \leq u(x, t) \leq M + ct \quad \text{on } \mathbb{R}^n \times [0, \infty).$$

Hence,

$$\psi(r) - M \leq \tilde{\phi}(r, t) \leq M \quad \text{on } [0, \infty) \times [0, \infty).$$

Therefore, we are able to define the functions $\psi^\pm \in C([0, \infty))$ as

$$\begin{aligned}\psi^+(r) &:= \limsup_{t \rightarrow \infty}^* \tilde{\phi}(r, t) = \lim_{t \rightarrow \infty} \sup \left\{ \tilde{\phi}(z, s) : |z - r| \leq \frac{1}{s}, s \geq t \right\}, \\ \psi^-(r) &:= \liminf_{t \rightarrow \infty}^* \tilde{\phi}(r, t) = \lim_{t \rightarrow \infty} \inf \left\{ \tilde{\phi}(z, s) : |z - r| \leq \frac{1}{s}, s \geq t \right\},\end{aligned}$$

which are, respectively, a subsolution and a supersolution to (10). By (15), we have $\psi^+ = \psi^-$ on $\tilde{\mathcal{A}}$. In light of Theorem 2.5, we get the conclusion. $\square$

Proof of Theorem 1.3. In light of (15), for any $r \in \tilde{\mathcal{A}}$, we have

$$\begin{aligned}\phi(r, t) - ct &\rightarrow \sup_{\gamma \in \mathcal{C}(t, 0; r, \rho), \rho > 0, t > 0} \left\{ \int_0^t (\tilde{f}(\gamma(\tau)) - c) d\tau + \tilde{u}_0(\rho) \right\} \\ &= v_0(r) \quad \text{as } t \rightarrow \infty.\end{aligned}$$

Due to the uniqueness set property of $\tilde{\mathcal{A}}$ and the concavity of equation (10) with respect to ψ_r , the function ψ_∞ is represented by (5). We refer to [7, Theorem 3.1], [23, Theorem 1.3] (see also [22, Theorem 5.24]) for more details. $\square$

2.3. A generalization. In this subsection, we remove the radial symmetric assumption of initial data u_0 in a weak sense, and give a result on the convergence which slightly generalizes that of Theorem 1.1.

We consider the following initial data

$$w_0(x) = \tilde{u}_0(|x|) + \phi(x),$$

where $\tilde{u}_0 \in \text{BUC}([0, \infty))$, and $\phi \in \text{BUC}(\mathbb{R}^n)$ are given. For $r \geq 0$, set

$$\bar{\phi}(r) := \max_{|x|=r} \phi(x), \quad \text{and} \quad \underline{\phi}(r) := \min_{|x|=r} \phi(x).$$

Define

$$w_0^+(x) := \tilde{u}_0(|x|) + \bar{\phi}(|x|), \quad w_0^-(x) := \tilde{u}_0(|x|) + \underline{\phi}(|x|), \quad (16)$$

$$v_0^\pm(r) := \max \{ d(r, t) + w_0^\pm(t) : t \in [0, \infty) \}, \quad (17)$$

where d is given by (6). Here is a straightforward consequence of Theorem 1.3.

Proposition 1. *Assume that*

$$v_0^+(s) = v_0^-(s) =: v_0(s) \quad \text{for all } s \in \tilde{\mathcal{A}}. \quad (18)$$

Then,

$$u(x, t) - ct \rightarrow \psi_\infty(|x|) \quad \text{locally uniformly for } x \in \mathbb{R}^n \text{ as } t \rightarrow \infty,$$

where c is given by (9), and

$$\psi_\infty(r) = \max \{ d(r, s) + v_0(s) : s \in \tilde{\mathcal{A}} \}.$$

We give two nontrivial examples satisfying condition (18) in Proposition 1.

Example 1. Let $\phi \in C_c(B(0, n-1))$, and $w_0^\pm, v_0^\pm$ be the functions defined by (16), (17), respectively. Then, we can easily see that (18) holds. Indeed, noting if $\gamma(z) \in (0, n-1)$, then

$$\gamma'(z) \leq 1 - \frac{n-1}{\gamma(z)} < 0,$$

which means that we cannot reach $s \in \tilde{\mathcal{A}}$ if we start from $t \in (0, n-1)$, and thus $d(s, t) = -\infty$ for all $s \in \tilde{\mathcal{A}}$, and $t \in (0, n-1)$. This example says that information

of initial data in $B(0, n-1)$ does not matter for the large time behavior for (C) if f and $\tilde{u}_0$ are radially symmetric.

Example 2. Assume that $c := \max_{r \geq 1} \tilde{f}(r) > 0$ and $\|u_0\|_{L^\infty} < R$ for some $R > 0$, where R is given by (1). Let $\phi \in \text{BUC}(\mathbb{R}^n)$ be a function satisfying

$$\|\phi\|_{L^\infty} < R \quad \text{and} \quad \phi = 0 \text{ in } B\left(0, R + \frac{6R}{c}\right).$$

Let $w_0^\pm, v_0^\pm$ be the functions defined by (16), (17), respectively. Then, assumption (18) holds.

Note first that we have $w_0^+(t) = w_0^-(t)$ for all $t \in [0, R + 6R/c]$. Thus, we just need to prove that the maximum of the left hand side of (17) does not achieve for $t > R + 6R/c$. Fix $t > R + 6R/c$, and let γ be an admissible curve in (8) with $\gamma(0) = t, \gamma(T) = s$. If $\gamma(r) > n-1$, then

$$-2 \leq -1 - \frac{n-1}{\gamma(r)} \leq \gamma'(r) \leq 1 - \frac{n-1}{\gamma(r)}.$$

Hence, it takes γ at least $3R/c$ time spending outside of $B(0, R)$ since $s \in \mathcal{A} \subset [n-1, \infty)$. This gives that

$$\int_0^T (\tilde{f}(\gamma(r)) - c) dr \leq -\frac{3R}{c}c = -3R.$$

Therefore, $d(s, t) \leq -3R$, and thus,

$$d(s, t) + w_0^\pm(t) \leq -3R + R = -2R < w_0^\pm(s) = d(s, s) + w_0^\pm(s),$$

which implies (18).

3. Case 2: Non radially symmetric setting (a toy case). In this section, we *always* assume that (A3)–(A5) hold. Due to (A3), the asymptotic speed is given by $c = \max_{\mathbb{R}^n} f > 0$ (see [15]).

3.1. Ergodic problem (4).

Lemma 3.1. *Assume that (A3) holds. Ergodic problem (4) has a solution which is bounded from above.*

Proof. Set

$$\bar{f}(r) := \max_{|x|=r} f(x), \quad \underline{f}(r) := \min_{|x|=r} f(x) \quad \text{for all } r \geq 0.$$

Without loss of generality, we can assume that $B(0, n-1) \subset U$. Then, there exist $n-1 \leq a \leq b$ such that $\underline{f} \leq \bar{f}$ and

$$\begin{cases} \bar{f}(r) = c \text{ for all } r \in [0, a], \text{ and } \bar{f}(r) < c \text{ for all } r > a, \\ \underline{f}(r) = c \text{ for all } r \in [0, b], \text{ and } \underline{f}(r) < c \text{ for all } r > b. \end{cases}$$

Thanks to the construction in Theorem 2.1, there exist solutions $\bar{v}$ and $\underline{v}$, respectively, to (4) with $f(x) = \bar{f}(|x|), \underline{f}(|x|)$ and $c = \max_{\mathbb{R}^n} f$, which are radially symmetric and bounded from above.

By adding a constant, we have further that

$$\underline{v} \leq \bar{v} \text{ on } \mathbb{R}^n, \quad \text{and} \quad \underline{v}(x) = \bar{v}(x) = 0 \text{ in } B(0, a).$$

We thus are able to use the Perron method to construct a solution v to (4) by

$$v(x) = \sup \{w(x) : w \text{ is a subsolution to (4) and } \underline{v} \leq w \leq \bar{v} \text{ on } \mathbb{R}^n\}$$

for $x \in \mathbb{R}^n$. □

Let $\bar{v}, \underline{v}$ be the functions defined in the proof of Lemma 3.1. It is important noting that, as f is compactly supported, $h(r) = -c$ for $r \gg 1$. Thus, integrating the equation (11) with respect to r yields

$$-c|x| - c(n-1)\log(|x|+1) - C \leq \underline{v}(x) \leq \bar{v}(x) \leq -c|x| + C \quad \text{for } x \in \mathbb{R}^n. \quad (19)$$

Define

$$\Gamma_f = \{v : \mathbb{R}^n \rightarrow \mathbb{R} : v \text{ satisfies } \underline{v} \leq v \leq \bar{v}\}.$$

We have the following uniqueness result of solutions to (4) in Γ_f .

Proposition 2. *Assume that $v, w \in \Gamma_f$ are, respectively, a subsolution and a supersolution to (4). Assume further that $v \leq w$ on U . Then $v \leq w$ in $\mathbb{R}^n$.*

Note that considering solutions in Γ_f is enough for our purpose of studying large time behavior to (C).

Proof. Assume by contradiction that there exists $y \in \mathbb{R}^n \setminus U$ such that $v(y) > w(y)$. We can find $\lambda > 1$, which is close enough to 1, such that

$$0 < \max_{\mathbb{R}^n} (\lambda v - w) = \lambda v(z) - w(z)$$

for some $z \in \mathbb{R}^n \setminus U$. Note that $f(z) < c$.

For $\varepsilon > 0$, thanks to (19) and the fact that $\lambda > 1$, we are able to find $x_\varepsilon, y_\varepsilon \in \mathbb{R}^n$ such that

$$\max_{x, y \in \mathbb{R}^n} \left(\lambda v(x) - w(y) - \frac{|x - y|^4}{\varepsilon} \right) = \lambda v(x_\varepsilon) - w(y_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^4}{\varepsilon},$$

and, up to passing to a subsequence,

$$\lim_{\varepsilon \rightarrow 0} (x_\varepsilon, y_\varepsilon) = (z, z) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{|x_\varepsilon - y_\varepsilon|^4}{\varepsilon} = 0.$$

By the Crandall–Ishii Lemma [6, Lemma 3.2], there exist $X_\varepsilon, Y_\varepsilon \in \mathbb{S}^n$ satisfying

$$\begin{pmatrix} X_\varepsilon & 0 \\ 0 & -Y_\varepsilon \end{pmatrix} \leq J + \varepsilon J^2, \quad (20)$$

where

$$J = \frac{4}{\varepsilon} \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} \quad \text{and} \quad Z = |x_\varepsilon - y_\varepsilon|^2 I_n + 2(x_\varepsilon - y_\varepsilon) \otimes (x_\varepsilon - y_\varepsilon).$$

We claim that for $\varepsilon > 0$ sufficiently small, $x_\varepsilon \neq y_\varepsilon$. Indeed, assume otherwise that $x_\varepsilon = y_\varepsilon$, which is quite close to z . Then $f(x_\varepsilon) < c$. Since $\lambda v(x) - \frac{|x - x_\varepsilon|^4}{\varepsilon}$ has a max at x_ε , we employ the viscosity subsolution test to yield

$$-\text{tr}((I_n - q \otimes q)0_n) \leq \lambda(f(x_\varepsilon) - c) < 0.$$

Here, 0_n is the zero matrix of size n , and $q \in \mathbb{R}^n$ is a vector such that $|q| \leq 1$. We get a contradiction immediately.

Therefore, for $\varepsilon > 0$ sufficiently small, $x_\varepsilon \neq y_\varepsilon$. By viscosity subsolution and viscosity supersolution tests, we have

$$-\text{tr}((I_n - p \otimes p)X_\varepsilon) - \frac{4|x_\varepsilon - y_\varepsilon|^3}{\varepsilon} \leq \lambda(f(x_\varepsilon) - c), \quad (21)$$

and

$$-\text{tr}((I_n - p \otimes p)Y_\varepsilon) - \frac{4|x_\varepsilon - y_\varepsilon|^3}{\varepsilon} \geq f(y_\varepsilon) - c. \quad (22)$$

Here, $p = \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|}$. Combine (20)–(22) to yield

$$0 \leq -\text{tr}((I_n - p \otimes p)(X_\varepsilon - Y_\varepsilon)) \leq \lambda(f(x_\varepsilon) - c) - (f(y_\varepsilon) - c).$$

Let $\varepsilon \rightarrow 0$ to get

$$0 \leq (\lambda - 1)(f(z) - c) < 0,$$

which is absurd. $\square$

3.2. Large time behavior. In this subsection, under (A3)–(A5), we give the proof of Theorem 1.2.

Lemma 3.2. *Assume (A3)–(A5). Let u be the viscosity solution to (C). Then,*

$$u(x, t) = ct \quad \text{for all } (x, t) \in U \times [0, \infty).$$

Proof. It is clear that $u(x, t) \leq ct$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$.

To show the reverse inequality for $(x, t) \in U \times [0, \infty)$, we simply show that, for each $i \in I$, the following function

$$\varphi(x, t) = ct \mathbf{1}_{E_i}(x) \quad \text{for } (x, t) \in \mathbb{R}^n \times [0, \infty)$$

is a subsolution to (C), where $\mathbf{1}_E$ denotes the characteristic function of E , i.e., $\mathbf{1}_E(x) = 1$ for $x \in E$ and $\mathbf{1}_E(x) = 0$ if $x \in E^c$. Indeed, take a smooth test function ψ such that $\varphi - \psi$ has a strict global maximum at $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ and $\varphi(x_0, t_0) = \psi(x_0, t_0)$. We only need to check the case that $x_0 \in \partial E_i$ as other cases are straightforward.

First of all, it is clear that $\psi_t(x_0, t_0) \leq c$. If we have that $D\psi(x_0, t_0) = 0$, then $D^2\psi(x_0, t_0) \geq 0$. Thus, for vector $\eta = 0 \in \mathbb{R}^n$, we have

$$\psi_t(x_0, t_0) - (\delta_{ij} - \eta_i \eta_j) \psi_{x_i x_j}(x_0, t_0) \leq c - \Delta \psi(x_0, t_0) \leq c = f(x_0).$$

Secondly, if $D\psi(x_0, t_0) \neq 0$, then $D\psi(x_0, t_0)$ is a normal vector to the level set $\{x \in \mathbb{R}^n : \psi(x, t_0) = \psi(x_0, t_0)\}$. Furthermore, as $\varphi - \psi$ has a strict global maximum at (x_0, t_0) , we deduce that

$$E_i \subset W = \{x \in \mathbb{R}^n : \psi(x, t_0) \geq \psi(x_0, t_0)\}.$$

Therefore, the mean curvature to W at x_0 is smaller than or equal to the mean curvature to E_i at x_0 . In light of (A4),

$$-\text{div} \left(\frac{D\psi(x_0, t_0)}{|D\psi(x_0, t_0)|} \right) \leq 1,$$

which gives

$$\psi_t(x_0, t_0) - \left(\text{div} \left(\frac{D\psi(x_0, t_0)}{|D\psi(x_0, t_0)|} \right) + 1 \right) |D\psi(x_0, t_0)| \leq \psi_t(x_0, t_0) \leq c = f(x_0). \quad \square$$

Remark 1.

(i) By [14, Lemma 5.1], we see that the set $\Gamma(t) = \{x \in \mathbb{R}^n : u(x, t) = ct\}$ moves according to the obstacle problem of the surface evolution $V = \kappa + 1$ with obstacle U . Lemma 3.2 is consistent with this observation.

(ii) Let us consider a front propagation problem $V = 1$ with a source term f . The corresponding equation is

$$u_t - |Du| = f(x) \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

which is a typical example in [24, 1]. Then, the set $U = \{x \in \mathbb{R}^n : f(x) = \max_{\mathbb{R}^n} f\}$ plays a role of not only a *uniqueness set* for the associated ergodic problem but also a *monotonicity set* of u , that is, $t \mapsto u(x, t) - ct$ is monotone for every $x \in U$.

It is important emphasizing here that, in our problem, U in general is not a monotonicity set of u if we assume only (A3). Additional assumptions (A4)–(A5) are really needed in this current approach.

Proof of Theorem 1.2. For $x \in \mathbb{R}^n$, set

$$v(x) = \limsup_{t \rightarrow \infty}^* (u(x, t) - ct), \quad w(x) = \liminf_{t \rightarrow \infty}^* (u(x, t) - ct).$$

It is clear then that $v, w \in \Gamma_f$ and v, w are a subsolution and a supersolution to (4), respectively. By Lemma 3.2, we get further that $v = w = 0$ on U . We then use Proposition 2 to yield the desired result. $\square$

Remark 2. In fact, in this section, we can relax (A4) by the following weaker assumption.

(A4') $U = \cup_{i \in I} E_i$, where $I \neq \emptyset$ is an index set, and for each $i \in I$, $E_i \subset \mathbb{R}^n$ is a closed set with nonempty interior satisfying $-\kappa - 1 \leq 0$ on ∂E_i in the sense of viscosity solutions to be defined in the following discussion.

This relaxation is pretty clear to see as one only needs to use (A4') in the proof of Lemma 3.2.

Let us now explain clearly assumption (A4'). We first recall the notion of set-theoretic solution $\{D_t\}_{t \geq 0} \subset \mathbb{R}^n$ to the surface evolution equation

$$V = \kappa + 1 \quad \text{on } \partial D_t, \quad (23)$$

where V and κ denote the outward normal velocity and the outward mean curvature of ∂D_t , respectively.

Definition 3.3. We say that an evolving family $\{D_t\}_{t \geq 0}$ of open sets D_t in $\mathbb{R}^n$ is a set-theoretic subsolution (resp., supersolution) of (23) if $\mathbf{1}_{D_t}$ is a viscosity subsolution (resp., supersolution) of

$$u_t - \left(\operatorname{div} \left(\frac{Du}{|Du|} \right) + 1 \right) |Du| = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \quad (24)$$

where $\mathbf{1}_{D_t}$ is a characteristic function of D_t , i.e., $\mathbf{1}_{D_t}(x) = 1$ if $x \in D_t$ and $\mathbf{1}_{D_t}(x) = 0$ if $x \notin D_t$.

We say that $\{D_t\}_{t \geq 0}$ is a set-theoretic solution of (23) if it is both a set-theoretic subsolution and a set-theoretic supersolution.

Next, we give the definition of stationary set-theoretic solution for a closed set. Of course, the definition of stationary set-theoretic solution for an open set follows in the same way.

Definition 3.4. Let $E \subset \mathbb{R}^n$ be a closed and bounded set with nonempty interior such that $\partial E = \partial(E^\circ)$. Here, E° is the interior of E .

We say that E satisfies $-\kappa - 1 \leq 0$ on ∂E (resp., $-\kappa - 1 \geq 0$ on ∂E) in the sense of viscosity solutions if $\{D_t\}_{t \geq 0}$ with $D_t = E^\circ$ for all $t \geq 0$ is a set theoretic subsolution (resp., supersolution) to (24).

We say that E satisfies $-\kappa - 1 = 0$ on ∂E if both $-\kappa - 1 \leq 0$ and $-\kappa - 1 \geq 0$ on ∂E hold.

4. **Some further discussions.** In this section, we consider a particular case of U in two dimensions, which is given by a stadium shape, that is,

$$U = \bigcup_{x \in [-a, a]} \overline{B}((x, 0), 1), \quad (25)$$

for some fixed $a > 0$.

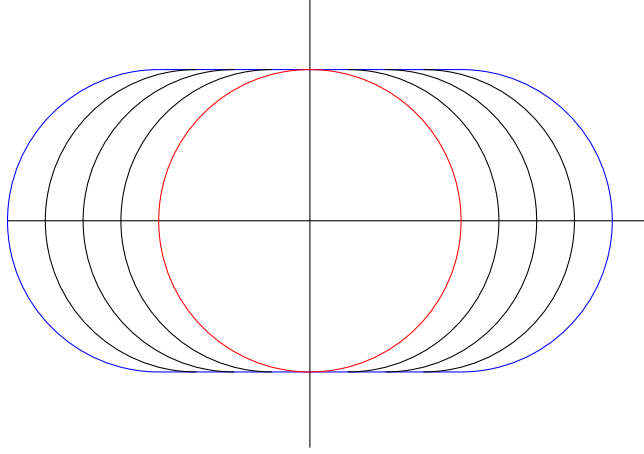


FIGURE 1. U and semicircles of radii 1

Let us now consider (4) in U only. Since U satisfies (A3), we have $c = \max_{\mathbb{R}^2} f$. Thus, the stationary equation (4) in the interior of U reads

$$-\left(\operatorname{div}\left(\frac{Dv}{|Dv|}\right) + 1\right)|Dv| = 0 \quad \text{in } U^\circ. \quad (26)$$

Here, U° is the interior of U .

We first construct particular solutions to (26).

Lemma 4.1. *Assume $n = 2$ and (25). Then, there exist infinitely many solutions $v \in C(U)$ of (26) such that $v = 0$ in $B(0, 1)$ and $v \neq 0$.*

Proof. We provide an explicit construction of a solution v as following.

For $x \in U$ such that $x_1 \leq 0$ or $|x| \leq 1$, we set $v(x) = 0$. For $x \in U$ such that $x_1 > 0$ and $|x| > 1$, we set

$$v(x) = \sqrt{1 - x_2^2} - x_1.$$

Let us explain a bit the definition of v in the later part. For any $0 \leq c \leq a$, setting $\Gamma_c := \{(x_1, x_2) : \sqrt{1 - x_2^2} - x_1 = -c\}$, we can easily see that $\Gamma_c \subset \partial B(c, 1)$. Thus, geometrically, if x stays on a semicircle of radius 1 with center at $(0, c)$ for $c > 0$, then $v(x) = -c$.

One can check analytically that v is a solution to (26) in a straightforward way. Let us instead give a geometrical explanation here. Indeed, for each $b < 0$, the level set $\{x : v(x) = b\}$ is exactly a semicircle of radius 1. Thus, for $x \in \{x : v(x) = b\}$, the mean curvature of the level set at x is exactly 1, which means

$$-\operatorname{div}\left(\frac{Dv(x)}{|Dv(x)|}\right) = 1,$$

which confirms that v is a solution to (26).

Now, to create infinitely many such solutions, we just need to define

$$\tilde{v}(x) = \phi(v(x)) \quad \text{for all } x \in U,$$

where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is any smooth function such that $\phi(0) = 0$. Then $\tilde{v}$ is always a solution to (26). The proof is complete. $\square$

Next, we give a result to characterize a solution obtained in Lemma 4.1.

Lemma 4.2. *Let u be a solution of (26) which is continuous up to the boundary of U . Assume that $u(x) = 0$ for $x \in U$ such that $x_1 \leq 0$ or $|x| \leq 1$. Assume further that u is strictly decreasing in x_1 for $x_1 > 0$ at the boundary, that is, $u(x_1, x_2) > u(\tilde{x}_1, x_2)$ for any $(x_1, x_2), (\tilde{x}_1, x_2) \in \partial U$ with $0 < x_1 < \tilde{x}_1$. Then, the level curves of u for $x_1 > 0$ agree with those of $v(x) = \sqrt{1 - x_2^2} - x_1$.*

To prove Lemma 4.2, we consider a set-theoretic solution of $-\kappa - 1 = 0$ in U defined in Definition 3.4.

Theorem 4.3. *Assume that U is open and convex in $\mathbb{R}^2$. Let $D \subset U$ be an open set and a set-theoretic solution of $-\kappa - 1 = 0$ on $\partial D \cap U$. Let D_i be a connected component of D . Then D_i is convex. Let S_i be a connected component of ∂D_i in U . Then S_i is an open arc of a unit circle.*

For the proof, we first derive convexity of D if D is connected.

Lemma 4.4. *Assume that U is open and convex in $\mathbb{R}^2$. Let $D \subset U$ be an open set and a set-theoretic solution of $-\kappa - 1 \geq 0$ on $\partial D \cap U$. Then each connected component of D must be convex in U .*

Proof. Assume that D is not convex but connected. Then there are points $P_1, P_2 \in D$ such that the line segment $P_1 P_2$ intersects $U \setminus \overline{D}$. We may assume that $P_1 P_2$ is a closed, nontrivial interval of the x_1 -axis. Without loss of generality, we may assume further that there is a connected component K of the closed set $(\overline{U} \setminus D) \cap \{x_2 \leq 0\}$ which contains some part of the segment $P_1 P_2$. The set K is compact since U is convex and D is connected.

Clearly, there is a point $Q = (x_1^*, x_2^*) \in \partial D$ such that

$$x_2^* = \inf \{x_2 \in \mathbb{R} : (x_1, x_2) \in K \text{ for some } x_1 \in \mathbb{R}\}.$$

At the point Q , one is able to use the straight line $x_2 = x_2^*$ as a test function to yield a contradiction since $\mathbf{1}_D$ must be a supersolution of (24). The convexity of each connected component of D follows right away and the proof is now complete. $\square$

Lemma 4.5. *Assume that U is open and convex in $\mathbb{R}^2$. Let $D \subset U$ be an open, convex set and a set-theoretic solution of $-\kappa - 1 = 0$ on $\partial D \cap U$. Then, the boundary of D in U consists of open arcs of unit circle.*

Proof. Since D is convex, ∂D is locally represented as the graphs of concave functions. If ∂D is locally represented as the graph of a concave function $x_2 = f(x_1)$, then f must satisfy the graph equation of $-\kappa - 1 = 0$ in viscosity sense and D is locally represented as $\{(x_1, x_2) : x_2 > f(x_1)\}$ by applying results of [12, Section 5]. Since f is concave, it is twice differentiable a.e. thanks to Alexandrov's theorem for convex functions. Then, f satisfies

$$-\frac{d}{dx} \frac{f'(x)}{(1 + (f'(x))^2)^{1/2}} - 1 = 0$$

almost everywhere. This implies that $x_2 = f(x_1)$ must be a part of the graph of $\sqrt{1 - x_1^2}$ up to translation. $\square$

Theorem 4.3 follows immediately from Lemma 4.4 and Lemma 4.5. If $U = \mathbb{R}^2$, then by Theorem 4.3, D_i is convex so that $S_i = \partial D_i$ and S_i is an open arc of a unit circle. This means that D_i is an open unit disk. It is worth emphasizing that this answers a question raised by [17, Problem C] at least partially.

Corollary 1. *Let $D \subset \mathbb{R}^2$ be an open set and a set-theoretic solution of $-\kappa - 1 = 0$ on ∂D . Then D is an open unit disk if D is connected. In general, each connected component of U is an open unit disk.*

As an application of Theorem 4.3, we consider a kind of Alexandrov's problem in $\mathbb{R}^2$ in viscosity sense. Assume that $\{D_t\}_{t \in I}$ is an open evolution of $V = \kappa + 1$ and $E_t = \overline{D_t}$ is a closed evolution for $V = \kappa + 1$ as in [10]. In other words, there is a uniformly continuous function u in $\mathbb{R}^2 \times \bar{I}$ such that u solves (24) in $\mathbb{R}^2 \times I$ in viscosity sense with the property that

$$\begin{aligned} E_t &= \{x \in \mathbb{R}^2 : u(x, t) \geq 0\} \\ D_t &= \{x \in \mathbb{R}^2 : u(x, t) > 0\}. \end{aligned}$$

for $t \in I$. In general, the set $\{u \geq 0\}$ can be strictly larger than $\overline{\{u > 0\}}$ even if initially $\{x : u(x, 0) \geq 0\}$ is the closure of $\{x : u(x, 0) > 0\}$. Such a phenomenon is called fattening as pointed out by [8] for the mean curvature flow (see also [10]).

Theorem 4.6. *Let D be a stationary set-theoretic solution of $V = \kappa + 1$ in $\mathbb{R}^2$. Assume that $\overline{D}$ is a closed evolution for $V = \kappa + 1$ in $\mathbb{R}^2$. Then D consists of a family of unit disks whose closures are mutually disjoint.*

Proof. Because of Corollary 1 each component of D is an open unit disk. If the closure touches, it is known that a fattening phenomenon occurs as pointed out by [25]. Then $E = \overline{D}$ cannot be a closed evolution. This is a contradiction, and hence, the closure of each connected component does not intersect. The proof is now complete. $\square$

Proof of Lemma 4.2. By Theorem 4.3, we see that a negative level set of u in U consists of open arcs of unit circles. The endpoints of these arcs must be on the boundary of U $((x_1, x_2)$ with $0 < x_1 \leq a$ and $x_2 = \pm 1$, and the right most semicircle of ∂U). Both endpoints of an arc cannot be on $x_2 = 1$ because of the assumption on the monotonicity of u on the boundary. Similarly, both endpoints of an arc cannot be on $x_2 = -1$. Also, it is not possible that, for a given arc, one endpoint is on $x_2 = 1$ and one endpoint is on the right most semicircle of ∂U as then u cannot be continuous because each level set nearby must be an arc of a unit circle that has the same property (one endpoint is on $x_2 = 1$ and one endpoint is on the right most semicircle of ∂U), and two of these level sets must intersect in U .

Thus, we conclude that each negative level curve of u contains a semicircle with one endpoint on $x_2 = 1$ and the other endpoint on $x_2 = -1$ or an arc of a unit circle with two endpoints both on the right most semicircle of ∂U . Since $u(x) = 0$ for $x \in U$ such that $x_1 \leq 0$ or $|x| \leq 1$, and u is strictly decreasing in x_1 for $x_1 > 0$ at the boundary, the second scenario cannot happen. Therefore, the level curves of $\{u(x) = c\}$ for $c < 0$ give the same foliation as those of v .

Note that one can only compare u with v by its level curves because all functions $\theta \circ v$ with nondecreasing continuous θ such that $\theta(0) = 0$ solve (26). $\square$

We now conclude the paper by pointing out that the result of Lemma 4.1 gives us some further intuition that it is quite complicated to get convergence result (3) in the most general setting. It gives us a fact that it is not easy to understand solutions of (26), and more generally, solutions of (4), and we probably need to be able to characterize all solutions to (26) in order to proceed further.

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