

Pointwise Inequalities for Sobolev Functions on Outward Cuspidal Domains

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We show that the 1st-order Sobolev spaces $W^{1,p}(\Omega_\psi)$, $1 < p \leq \infty$, on cuspidal symmetric domains Ω_ψ can be characterized via pointwise inequalities. In particular, they coincide with the Hajlasz–Sobolev spaces $M^{1,p}(\Omega_\psi)$.

1 Introduction

Optimal definitions for Sobolev spaces are crucial in analysis. It was a remarkable discovery of Hajlasz [3] that distributionally defined Sobolev functions can be characterized using pointwise estimates in the context of Sobolev extension domains. This, in part, has played a crucial role in defining Sobolev spaces for general metric measure spaces. Here, we show that for certain cuspidal domains the pointwise characterization holds without any additional assumptions. These domains do not admit extensions for Sobolev functions. Given a domain $\Omega \subset \mathbb{R}^n$, we denote by $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, the usual 1st-order Sobolev space consisting of all functions $u \in L^p(\Omega)$ whose 1st-order

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distributional partial derivatives also belong to $L^p(\Omega)$. If $\Omega = \mathbb{R}^n$, then any Sobolev function u satisfies the pointwise inequality

$$|u(x) - u(y)| \leq C|x - y| (M[|\nabla u|](x) + M[|\nabla u|](y)) \quad (1.1)$$

at Lebesgue points of u , where $M[|\nabla u|]$ is the Hardy–Littlewood maximal function of $|\nabla u|$, see [1, 2, 3, 6]. Motivated by this, Hajlasz introduced [3] the space $M^{1,p}(\Omega)$ consisting of all those $u \in L^p(\Omega)$ for which there exists a set $E \subset \Omega$ of n -measure zero and a function $0 \leq g \in L^p(\Omega)$ so that

$$|u(x) - u(y)| \leq |x - y| (g(x) + g(y)), \quad (1.2)$$

whenever $x, y \in \Omega \setminus E$.

One has $M^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$ as sets for $1 < p \leq \infty$, and the norms are comparable once $M^{1,p}(\mathbb{R}^n)$ is equipped with the natural norm. Also, for $1 \leq p \leq \infty$, one always has $M^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ and the inclusion is strict for $p = 1$ for any domain Ω , see [5].

A natural question to ask is:

For which domains $\Omega \subset \mathbb{R}^n$ do we have $M^{1,p}(\Omega) = W^{1,p}(\Omega)$?

Indeed, these two spaces coincide if there is a bounded extension operator from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R}^n)$, for a given $1 < p \leq \infty$. When $p = \infty$ and Ω is bounded, this is the case if Ω is quasiconvex and actually the equality is equivalent to quasiconvexity under these assumptions. This follows from [4, Theorem 7]. Moreover, for $1 < p < \infty$, under the assumption that

$$|B(x, r)| \leq C|B(x, r) \cap \Omega| \quad (1.3)$$

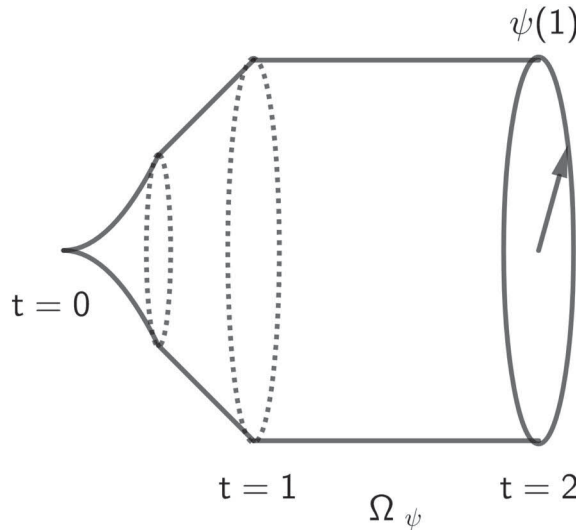
for every $x \in \Omega$ and every $0 < r < 1$, where $|\cdot|$ refers to n -measure, $M^{1,p}(\Omega) = W^{1,p}(\Omega)$ implies the existence of such an extension operator. Indeed, in this case the spaces coincide precisely when such an extension operator exists. For this, see [4]. Using this fact, it is easy to exhibit domains Ω for which $M^{1,p}(\Omega) = W^{1,p}(\Omega)$ fails for all p ; for example, take $\Omega \subset \mathbb{R}^2$ to be the unit disk minus the interval $[0, 1)$ on the real axis.

In this paper, we consider this question for cuspidal domains of the form

$$\Omega_\psi := \left\{ (t, x) \in (0, 1) \times \mathbb{R}^{n-1}; |x| < \psi(t) \right\} \cup \left\{ (t, x) \in [1, 2) \times \mathbb{R}^{n-1}; |x| < \psi(1) \right\}, \quad (1.4)$$

where $\psi: (0, 1] \rightarrow (0, \infty)$ is a left-continuous increasing function. (Left continuity is required just to get Ω_ψ open. The term “increasing” is used in the non-strict sense.) The seemingly strange cylindrical annexes are included only to exclude other singularities than the cuspidal one. It is crucial to note that these domains will not, except for limited special cases, be Sobolev extension domains, and thus the methods from [4] do not apply.

It is easy to check that $\Omega_\psi \subset \mathbb{R}^n$ is a domain. If $\lim_{t \rightarrow 0} \frac{\psi(t)}{t} = 0$, then the measure density condition (3) fails, and hence, by [4], there can not exist any bounded extension operator from $W^{1,p}(\Omega_\psi)$ to $W^{1,p}(\mathbb{R}^n)$. However, according to a somewhat surprising result by Romanov [7], one still has $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$ if $\psi(t) = t^s$ with $s > 1$ and $p > \frac{1+(n-1)s}{n}$. Actually, Romanov proved this statement for a domain, which is bi-Lipschitz equivalent to Ω_ψ when $\psi(t) = t^s$, but bi-Lipschitz transforms preserve both Sobolev and Hajlasz–Sobolev spaces.



We show that the above restriction on p is superfluous and that ψ being of the form $\psi(t) = t^s$ can be relaxed to being any left-continuous increasing function.

Theorem 1.1. Let $\psi: (0, 1] \rightarrow (0, \infty)$ be a left-continuous increasing function. Define the corresponding cuspidal domain Ω_ψ as in (4). Then $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$ for all $1 < p \leq \infty$ with equivalence of norms.

As a consequence of the bi-Lipschitz invariance stated above, the conclusion $M^{1,p}(\Omega) = W^{1,p}(\Omega)$ then holds for all bi-Lipschitz images of Ω_ψ . Thus, our result covers the result obtained by Romanov.

2 Definitions and Preliminaries

In what follows, $\Omega \subset \mathbb{R}^n$ is always a domain. We write

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} := \{z := (t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}\}.$$

Throughout the paper, we consider a left-continuous increasing function $\psi: (0, 1] \rightarrow (0, \infty)$, extend the definition of ψ to the interval $(0, 2)$ by setting

$$\psi(t) = \psi(1), \text{ for every } t \in (1, 2),$$

and write

$$\Omega_\psi = \{(t, x) \in (0, 2) \times \mathbb{R}^{n-1}; |x| < \psi(t)\}.$$

Typically, c or C will be constants that depend on various parameters and may differ even on the same line of inequalities. The Euclidean distance between points x, y in the Euclidean space $\mathbb{R}^n$ is denoted by $|x - y|$. The open m -dimensional ball of radius r centered at the point x is denoted by $B^m(x, r)$.

The space of locally integrable functions is denoted by $L^1_{\text{loc}}(\Omega)$. For every measurable set $Q \subset \mathbb{R}^n$ with $0 < |Q| < \infty$ and every non-negative measurable or integrable function f on Q we define the integral average of f over Q by

$$\oint_Q f(w) \, dw := \frac{1}{|Q|} \int_Q f(w) \, dw.$$

Let us give the definitions of Sobolev space $W^{1,p}(\Omega)$ and Hajlasz–Sobolev space $M^{1,p}(\Omega)$.

Definition 1. We define the 1st-order Sobolev space $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, as the set

$$\{u \in L^p(\Omega); \nabla u \in L^p(\Omega; \mathbb{R}^n)\}.$$

Here $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right)$ is the weak (or distributional) gradient of a locally integrable function u .

We equip $W^{1,p}(\Omega)$ with the non-homogeneous norm:

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$$

for $1 \leq p \leq \infty$, where $\|f\|_{L^p(\Omega)}$ denotes the usual L^p -norm for $p \in [1, \infty]$.

For $u \in L^p(\Omega)$, we denote by $\mathcal{D}_p(u)$ the class of functions $0 \leq g \in L^p(\Omega)$ for which there exists $E \subset \Omega$ with $|E| = 0$, so that

$$|u(z_1) - u(z_2)| \leq |z_1 - z_2| (g(z_1) + g(z_2)), \text{ for } z_1, z_2 \in \Omega \setminus E.$$

Definition 2. We define the Hajlasz–Sobolev space $M^{1,p}(\Omega)$, $1 \leq p \leq \infty$, as the set

$$\left\{ u \in L^p(\Omega), \mathcal{D}_p(u) \neq \emptyset \right\}.$$

We equip $M^{1,p}(\Omega)$ with the non-homogeneous norm:

$$\|u\|_{M^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \inf_{g \in \mathcal{D}_p(u)} \|g\|_{L^p(\Omega)},$$

for $1 \leq p \leq \infty$.

3 Maximal Functions

We will define two maximal functions. The 1st one, $M^t[f]$, will vary only the 1st component t , and the 2nd $M^x[f]$ will vary the x -component. For every $x \in B^{n-1}(0, \psi(1))$ set

$$S_x := \{t \in \mathbb{R}; (t, x) \in \Omega_\psi\}.$$

Let $f: \Omega_\psi \rightarrow \mathbb{R}$ be measurable and let $(t, x) \in \Omega_\psi$. We define the one-dimensional maximal function in the direction of the 1st variable by setting

$$M^t[f](t, x) := \sup_{[a,b] \ni t} \int_{[a,b] \cap S_x} |f(s, x)| \, ds. \quad (3.1)$$

The supremum is taken over all intervals $[a, b]$ containing t .

On the other hand, the 2nd maximal function will be defined for functions $f: (0, 2) \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}$. For every point $(t, x) \in (0, 2) \times \mathbb{R}^{n-1}$, we define the $(n-1)$ -

dimensional maximal function $M^\chi[f]$ by setting

$$M^\chi[f](t, x) := \sup_{B^{n-1}(x', r) \ni x} \int_{B^{n-1}(x', r)} |f(t, y)| \, dy, \quad (3.2)$$

where we take the supremum over the $(n-1)$ -dimensional balls for which $x \in B^{n-1}(x', r)$. The next lemmas tell us that both M^τ and M^χ enjoy the usual L^p -boundedness property. See [8] for the statements and proofs of the classical boundedness results.

Lemma 3.1. Let $1 < p < \infty$. Then for every $f \in L^p(\Omega_\psi)$, $M^\tau[f]$ is measurable and we have

$$\int_{\Omega_\psi} |M^\tau[f](z)|^p \, dz \leq C \int_{\Omega_\psi} |f(z)|^p \, dz, \quad (3.3)$$

where the constant C is independent of f .

Proof. Since the maximal function comes out the same if we consider only segments with rational endpoints, it preserves measurability. Fubini's theorem implies that $f(\cdot, x) \in L^p(S_x)$ for almost every $x \in B^{n-1}(0, \psi(1))$. By the L^p -boundedness of the classical Hardy–Littlewood maximal function on the interval S_x , for such x we have

$$\int_{S_x} |M^\tau[f](t, x)|^p \, dt \leq C \int_{S_x} |f(t, x)|^p \, dt, \quad (3.4)$$

where the constant C is independent of f and x . By combining the inequality (4) and Fubini's theorem together, we obtain

$$\begin{aligned} \int_{\Omega_\psi} |M^\tau[f](t, x)|^p \, dx \, dt &= \int_{B^{n-1}(0, \psi(1))} \int_{S_x} |M^\tau[f](t, x)|^p \, dt \, dx \\ &\leq C \int_{B^{n-1}(0, \psi(1))} \int_{S_x} |f(t, x)|^p \, dt \, dx \\ &= C \int_{\Omega_\psi} |f(t, x)|^p \, dx \, dt. \quad \blacksquare \end{aligned}$$

Lemma 3.2. Let $1 < p < \infty$. Then for every $f \in L^p((0, 2) \times \mathbb{R}^{n-1})$, $M^\chi[f]$ is measurable and we have

$$\int_{(0, 2) \times \mathbb{R}^{n-1}} |M^\chi[f](z)|^p \, dz \leq C \int_{(0, 2) \times \mathbb{R}^{n-1}} |f(z)|^p \, dz, \quad (3.5)$$

where the constant C is independent of f .

Proof. Again, the maximal function preserves measurability, as it comes out the same if we consider only balls with rational centers and radii (a point is rational if all its coordinates are rational). By Fubini's theorem, $f(t, \cdot) \in L^p(\mathbb{R}^{n-1})$ for almost every $t \in (0, 2)$. By the L^p -boundedness of the Hardy–Littlewood maximal operator we have

$$\int_{\mathbb{R}^{n-1}} |M^\chi[f](t, x)|^p dx \leq C \int_{\mathbb{R}^{n-1}} |f(t, x)|^p dx,$$

where the positive constant C is independent of f and t . Then Fubini's theorem gives

$$\begin{aligned} \int_{(0,2) \times \mathbb{R}^{n-1}} |M^\chi[f](z)|^p dz &= \int_0^2 \int_{\mathbb{R}^{n-1}} |M^\chi[f](t, x)|^p dx dt \\ &\leq C \int_0^2 \int_{\mathbb{R}^{n-1}} |f(t, x)|^p dx dt \\ &\leq C \int_{(0,2) \times \mathbb{R}^{n-1}} |f(z)|^p dz. \end{aligned}$$

■

4 Proof of the Main Theorem

Let us begin by sketching a simple proof for Theorem 1.1 in the Euclidean plane $\mathbb{R}^2$, for $1 < p < \infty$. In this case, the maximal function $M^\chi[f]$, with respect to the x -coordinate, can be replaced by

$$\tilde{M}^\chi[f](t, x) := \sup_{[z,w] \ni x} \int_{\{y \in [z,w]; (t,y) \in \Omega_\psi\}} |f(t, y)| dy, \quad (4.1)$$

for every $(t, x) \in \Omega_\psi$. As in Lemma 3.1 we obtain

$$\int_{\Omega_\psi} |\tilde{M}^\chi[f](z)|^p dz \leq C \int_{\Omega_\psi} |f(z)|^p dz. \quad (4.2)$$

By [3], there is a bounded inclusion $\iota: M^{1,p}(\Omega_\psi) \hookrightarrow W^{1,p}(\Omega_\psi)$. To show that ι is an isomorphism, it suffices to show that its inverse ι^{-1} is both densely defined and bounded on $W^{1,p}(\Omega_\psi)$. Let $C^1(\Omega_\psi)$ be the set of continuously differentiable functions. Since $C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$ is dense in $W^{1,p}(\Omega_\psi)$, it suffices to show that $C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi) \subset M^{1,p}(\Omega_\psi)$ and that for each $u \in C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$ we have $\|u\|_{M^{1,p}(\Omega_\psi)} \lesssim \|u\|_{W^{1,p}(\Omega_\psi)}$.

Fix $u \in C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$. Let $z_1 := (t_1, x_1), z_2 := (t_2, x_2) \in \Omega_\psi$ be arbitrary. Without loss of generality, we assume $0 < t_1 \leq t_2 < 2$. From the definition of Ω_ψ , the

point $z' := (t_2, x_1)$ is also in Ω_ψ . Using the triangle inequality, we have

$$|u(z_1) - u(z_2)| \leq |u(z_1) - u(z')| + |u(z') - u(z_2)|. \quad (4.3)$$

Since $u \in C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$, the fundamental theorem of calculus implies

$$|u(z_1) - u(z')| \leq \int_{t_1}^{t_2} |\nabla u(s, x_1)| \, ds \leq |z_1 - z_2| M^\tau[|\nabla u|](z_1) \quad (4.4)$$

and

$$|u(z') - u(z_2)| \leq \left| \int_{x_1}^{x_2} |\nabla u(t_2, y)| \, dy \right| \leq |z_1 - z_2| \tilde{M}^\chi[|\nabla u|](z_2). \quad (4.5)$$

Combining inequalities (3), (4), and (5) together, we have

$$|u(z_1) - u(z_2)| \leq |z_1 - z_2| \left(M^\tau[|\nabla u|](z_1) + \tilde{M}^\chi[|\nabla u|](z_2) \right) \leq |z_1 - z_2| (g(z_1) + g(z_2)),$$

where

$$g(z) := M^\tau[|\nabla u|](z) + \tilde{M}^\chi[|\nabla u|](z).$$

By inequalities (3) and (2), we have

$$\int_{\Omega_\psi} |g(z)|^p \, dz \leq C \int_{\Omega_\psi} |\nabla u(z)|^p \, dz,$$

which immediately gives that $g \in \mathcal{D}_p(u)$, and $\|u\|_{M^{1,p}(\Omega_\psi)} \leq C \|u\|_{W^{1,p}(\Omega_\psi)}$.

In higher dimensions, we have to work harder. Let us fix some notation.

Let $\eta: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ be a smooth cut-off function such that $\eta = 1$ on $B^{n-1}(0, 1)$ and $\eta = 0$ on the complement of $B^{n-1}(0, 2)$. Consider the standard extension operator $E^R: W^{1,p}(B^{n-1}(0, R)) \rightarrow W^{1,p}(\mathbb{R}^{n-1})$ given by

$$E^R u(x) = \begin{cases} u(x), & |x| < R, \\ 0, & |x| = R, \\ u\left(\frac{R^2}{|x|^2} x\right) \eta\left(\frac{x}{R}\right), & |x| > R. \end{cases}$$

Then,

$$\|\nabla E^R u\|_{L^p(\mathbb{R}^{n-1})} \leq C \|\nabla u\|_{L^p(B^{n-1}(0,R))} \quad (4.6)$$

with C independent of u and R .

Let $u \in W^{1,p}(\Omega_\psi)$ be arbitrary, $1 < p < \infty$. Extend the function u to $(0, 2) \times \mathbb{R}^{n-1}$ by setting

$$\tilde{u}(t, \cdot) = E^{\psi(t)}(u(t, \cdot)), \quad t \in (0, 2). \quad (4.7)$$

Denoting the gradient with respect to the x -variable by ∇^x , from (1) we immediately obtain

$$|\tilde{u}(z_1) - \tilde{u}(z_2)| \leq C|z_1 - z_2|(M^x[|\nabla^x \tilde{u}|](z_1) + M^x[|\nabla^x \tilde{u}|](z_2)) \quad (4.8)$$

for a.e. $t \in (0, 2)$ and a.e. $z_1, z_2 \in \{t\} \times \mathbb{R}^{n-1}$. It is easily seen, when $u \in C^1(\Omega_\psi)$, that the function $\tilde{u}$ and $\nabla^x \tilde{u}$ are measurable on $(0, 2) \times \mathbb{R}^{n-1}$. In fact, it could be shown that both of these would be measurable even if u were just in $W^{1,p}(\Omega_\psi)$.

Next, we prove the main estimate.

Lemma 4.1. Let $z_1 = (t_1, x_1), z_2 := (t_2, x_2) \in \Omega_\psi$ be two points with $t_1 < t_2$. Suppose that $u \in W^{1,p}(\Omega_\psi) \cap C^1(\Omega_\psi)$ and that $\tilde{u}$ is its extension given by (7). Then we have

$$\begin{aligned} |u(z_1) - u(z_2)| \leq & C|z_1 - z_2|(M^\tau[|\nabla u|](z_1) + M^\tau[M^x[|\nabla^x \tilde{u}|]](z_1) + \\ & M^\tau[|\nabla u|](z_2) + M^\tau[M^x[|\nabla^x \tilde{u}|]](z_2)). \end{aligned} \quad (4.9)$$

Proof. Similarly to the 2D argument, we will compare the change in the function via additional values $\tilde{u}(s, x_i)$ for some $s \in (0, 2)$. Without knowing exactly which s yields an optimal estimate, we will instead average over a range of possible s with the hope that, on average, the differences are better controlled. Indeed, let

$$\begin{aligned} T_2 &= \min \left\{ 2, t_2 + \frac{t_2 - t_1}{2} \right\}, \\ T_1 &= T_2 - \frac{t_2 - t_1}{2}. \end{aligned}$$

Notice that $t_2 \in [T_1, T_2]$ and $[T_1, T_2] \times \{x_1, x_2\} \subset \Omega_\psi$. When we average over different possible $s \in [T_1, T_2]$ and use the triangle inequality we obtain that

$$\begin{aligned}
 |u(z_2) - u(z_1)| &\leq \underbrace{\left| \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} |u(t_2, x_2) - u(s, x_2)| \, ds \right|}_I \\
 &+ \underbrace{\left| \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} |u(s, x_2) - u(s, x_1)| \, ds \right|}_II \\
 &+ \underbrace{\left| \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} |u(s, x_1) - u(t_1, x_1)| \, ds \right|}_III. \tag{4.10}
 \end{aligned}$$

First, we estimate the terms I and III . Let $i \in \{1, 2\}$. If $t_i < s$, by the fundamental theorem of calculus, we have the following:

$$|u(t_i, x_i) - u(s, x_i)| \leq \int_{t_i}^s |\nabla u(r, x_i)| \, dr \leq |t_i - s| M^\tau[|\nabla u|](z_i) \leq 3(T_2 - T_1) M^\tau[|\nabla u|](z_i). \tag{4.11}$$

Similarly, (11) holds also if $t_i \geq s$. Integrating with respect to s we obtain

$$I \leq 3(T_2 - T_1) M^\tau[|\nabla u|](z_2) \leq 2|z_2 - z_1| M^\tau[|\nabla u|](z_2) \tag{4.12}$$

and

$$III \leq 3(T_2 - T_1) M^\tau[|\nabla u|](z_1) \leq 2|z_2 - z_1| M^\tau[|\nabla u|](z_1) \tag{4.13}$$

Next, we apply (8) to the 2nd term:

$$\begin{aligned}
 II &\leq \frac{C|x_1 - x_2|}{T_2 - T_1} \int_{T_1}^{T_2} (M^\chi[|\nabla^\chi \tilde{u}|](s, x_1) + M^\chi[|\nabla^\chi \tilde{u}|](s, x_2)) \, ds \\
 &\leq C|x_1 - x_2| \left(\frac{1}{T_2 - t_1} \int_{t_1}^{T_2} M^\chi[|\nabla^\chi \tilde{u}|](s, x_1) \, ds + \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} M^\chi[|\nabla^\chi \tilde{u}|](s, x_2) \, ds \right) \\
 &\leq C|z_1 - z_2| (M^\tau[M^\chi[|\nabla^\chi \tilde{u}|]](z_1) + M^\tau[M^\chi[|\nabla^\chi \tilde{u}|]](z_2)). \tag{4.14}
 \end{aligned}$$

Finally, by combining inequalities (12), (13), (14), and (10), we obtain the desired inequality (9). ■

Recall that a domain Ω is quasiconvex if there exists a $C \geq 1$ such that, for every pair of points $x, y \in \Omega$, there is a rectifiable curve $\gamma \subset \Omega$ joining x to y so that $\text{len}(\gamma) \leq C|x - y|$.

Proof. Proof of Theorem 1.1 Because Ω_ψ is quasiconvex for every ψ , the case of $p = \infty$ is a consequence of [4, Theorem 7]. Thus, fix $1 < p < \infty$. By [3], we know that there is a bounded inclusion $\iota: M^{1,p}(\Omega_\psi) \hookrightarrow W^{1,p}(\Omega_\psi)$. To show that ι is an isomorphism, it suffices to show that the dense subspace $C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$ of $W^{1,p}(\Omega_\psi)$ is contained in $M^{1,p}(\Omega_\psi)$ and that the restricted inverse $\iota^{-1}|_{C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)}$ is defined and bounded.

Let $u \in C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$ be arbitrary, and define $\tilde{u}$ as in (7). Set

$$\hat{g}(z) = M^\tau[|\nabla u|](z) + M^\chi[|\nabla^\chi \tilde{u}|](z) + M^\tau[M^\chi[|\nabla^\chi \tilde{u}|]](z). \quad (4.15)$$

By (8) and Lemma 4.1, for every $z_1, z_2 \in \Omega_\psi$, we get the estimate

$$|u(z_1) - u(z_2)| \leq C|z_1 - z_2|(\hat{g}(z_1) + \hat{g}(z_2)).$$

Hence, (2) holds for $g := C\hat{g}$ for a suitable constant $C > 1$. The triangle inequality gives

$$\int_{\Omega_\psi} |g(z)|^p dz \leq C \left(\int_{\Omega_\psi} M^\tau[|\nabla u|](z)^p dz + \int_{\Omega_\psi} M^\chi[|\nabla^\chi \tilde{u}|](z)^p dz + \int_{\Omega_\psi} M^\tau[M^\chi[|\nabla^\chi \tilde{u}|]](z)^p dz \right).$$

Lemmata 3.1 and 3.2 and (6) lead to the estimates

$$\int_{\Omega_\psi} |M^\tau[|\nabla u|](z)|^p dz \leq C \int_{\Omega_\psi} |\nabla u(z)|^p dz$$

and

$$\begin{aligned} \int_{\Omega_\psi} |M^\tau[M^\chi[|\nabla^\chi \tilde{u}|]](z)|^p dz &\leq C \int_{\Omega_\psi} M^\chi[|\nabla^\chi \tilde{u}|](z)^p dz \leq C \int_{(0,2) \times \mathbb{R}^{n-1}} |\nabla^\chi \tilde{u}(z)|^p dz \\ &\leq C \int_0^2 \int_{\mathbb{R}^{n-1}} |\nabla^\chi \tilde{u}(t, x)|^p dx dt \leq C \int_0^2 \int_{B(0, \psi(t))} |\nabla^\chi u(t, x)|^p dx dt \\ &\leq C \int_{\Omega_\psi} |\nabla u(z)|^p dz, \end{aligned}$$

which imply that $g \in \mathcal{D}_p(u)$ and that $\|u\|_{M^{1,p}(\Omega_\psi)} \leq C\|u\|_{W^{1,p}(\Omega_\psi)}$. That is, $\iota^{-1}|_{C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)}$ is both well-defined and bounded. ■

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