VARIETIES OF PLANES ON INTERSECTIONS OF THREE QUADRICS

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ABSTRACT. We study the geometry of spaces of planes on smooth complete intersections of three quadrics, with a view toward rationality questions.

1. INTRODUCTION

This note studies the geometry of smooth complete intersections of three quadrics $X \subset \mathbb{P}^n$, with a view toward rationality questions over nonclosed fields k. We review what is known over $\mathbb{C}$:

- X is irrational for $n \leq 6$ [Bea77];
- X may be either rational or irrational for $n = 7$, and the rational ones are dense in moduli [HPT18];
- X is always rational for $n \geq 8$ [Tju75, Cor. 5.1].

The analysis in higher dimensions relies on the geometry of planes in X. Indeed, when X contains a plane P defined over k then projection from P gives a birational map

$$\pi_P : X \overset{\sim}{\dashrightarrow} \mathbb{P}^{n-3}.$$

This leads us to study the variety of planes $F_2(X) \subset \mathrm{Gr}(3, n+1)$. When $n \geq 12$, the geometry of these varieties gives a quick and uniform proof of rationality over finite fields and function fields of complex curves (see Theorem 1).

We are therefore interested in the intermediate cases $n = 8, 9, 10, 11$, and especially in $n = 8$ and 9. For generic $X \subset \mathbb{P}^8$, the variety $F_2(X)$ is finite of degree 1024 (see Proposition 5); we explore the geometry of the associated configurations of planes in $\mathbb{P}^8$. We then focus most on the case $n = 9$. Here the variety $F_2(X)$ is a threefold of general type with complicated geometry – we analyze its numerical invariants.

Our original motivation was to understand certain *singular* complete intersections of three quadrics in $\mathbb{P}^9$ associated with universal torsors over degree 4 del Pezzo surfaces fibered in conics over $\mathbb{P}^1$ [CTS87]. Conjectures of Colliot-Thélène and Sansuc predict that such torsors

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are *rational* when they admit a point, over number fields. Rationality of torsors has significant arithmetic applications, e.g., to proving the uniqueness of the Brauer–Manin obstruction to the Hasse principle and weak approximation. It has geometric consequences as well, e.g., the construction of new examples of nonrational but stably rational threefolds over $\mathbb{C}$. The geometry of *smooth* intersections, presented here, turned out to be quite rich and interesting on its own.

Here is a road map of the paper: Section 2 presents uniform proofs of rationality for high-dimensional cases. Section 3 is devoted to determinantal presentations of plane curves arising as degeneracy loci of the net of quadrics. In Sections 4 and 5, we turn to numerical invariants of the variety of planes. Our main results concern the computation of degrees and cohomology of $F_2(X)$, for X a smooth intersection of three quadrics, in $\mathbb{P}^8$ and $\mathbb{P}^9$.

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2. Uniform rationality in high dimensions

Theorem 1. *Let k be a finite field or the function field of a complex curve. Suppose that $X \subset \mathbb{P}^n, n \geq 12$, is a smooth complete intersection of three quadrics. Then X is rational.*

Proof. The results of [DM98] show that for X generic over $\bar{k}$, the variety $F_2(X) \subset \mathrm{Gr}(3, n+1)$ is smooth and connected of the expected dimension $3n - 24$. The adjunction formula implies that $F_2(X)$ has canonical class

$$K_{F_2(X)} = (12 - n - 1)\sigma_1;$$

here σ_1 is the hyperplane class from the Plücker embedding – see Section 4 for details. In particular, $F_2(X)$ is Fano for $n \geq 12$.

Suppose k is a function field. When $F_2(X)$ is smooth of the expected dimension, we have $F_2(X)(k) \neq \emptyset$, by the Graber-Harris-Starr theorem [GHS03]. What if $F_2(X)$ is singular or fails to have the expected dimension? However, we still have rational points, by an argument of Starr: Suppose that $\mathcal{X}_\circ \to B$ is a family of varieties over a complex projective curve B corresponding to a morphism $B \to \mathrm{Hilb}$ to the appropriate Hilbert scheme. There exists a one-parameter family of curves $\mathcal{C}_t$, meeting the locus over which F_2 is smooth of the expected dimension, such that $\mathcal{C}_0$ contains B as an irreducible component. (Take the $\mathcal{C}_t$ to be complete intersection curves in Hilb.) For the induced families $\mathcal{X}_t \to \mathcal{C}_t$, the fibrations $F_2(\mathcal{X}_t) \to \mathcal{C}_t$ admit sections for most t. This remains true as $t \to 0$; restricting to B gives a section of $F_2(\mathcal{X}_\circ) \to B$.

A similar argument holds over finite fields, using Esnault's Theorem [Esn03] and the specialization version due to Fakhruddin and Rajan [FR05]. □

3. Determinantal representations

3.1. **Recollection of invariants.** We recall formulas that may be obtained from Appendix I of Hirzebruch's *Topological Methods*, specifically, [Hir66, Th. 22.1.1].

Proposition 2. *Suppose that $n = 2m$. Then we have*

$$\chi(X) = -4m(m-1)$$

and

$$h^{m-1,m-2}(X) = h^{m-2,m-1}(X) = \binom{2m}{2} - 1 = 2m^2 - m - 1.$$

The other Hodge numbers of weight $2m-3$ vanish.

Proposition 3. *Suppose that $n = 2m-1$. Then we have*

$$\chi(X) = 4m(m-1)$$

and

$$h^{m-1,m-3}(X) = h^{m-3,m-1}(X) = \binom{m-1}{2}, h^{m-2,m-2}(X) = 3m^2 - 3m + 2.$$

The odd case follows from [O'G86] and [Las89]; the even case from [Bea77]. For reference, we give a tabulation of nontrivial middle Hodge numbers

n	Hodge numbers
4	5 5
5	1 20 1
6	14 14
7	3 38 3
8	27 27
9	6 62 6
10	44 44

3.2. **Interpretation of cohomology.** Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of three quadrics containing a plane P. It follows that $n \geq 7$ and X is of special moduli when $n = 7$, as the expected dimension of the Fano variety of planes is

$$\operatorname{expdim} F_2(X) = 3(n-2) - 3 \cdot 6 = 3n - 24.$$

This case is studied in depth in [HPT18].

Projection from P induces a birational map $X \overset{\sim}{\dashrightarrow} \mathbb{P}^{n-3}$. The center Y of the inverse has dimension $n-5$ and admits a fibration in quadric hypersurfaces of relative dimension $n-7$ over $\mathbb{P}^2$. This is instrumental in computing and interpreting the cohomology of X.

Suppose n is odd. Then Y is fibered in even-dimensional quadrics and its cohomology is governed by the associated double cover

$$S \to \mathbb{P}^2,$$

branched over the degeneracy locus D, and the associated Brauer class $\eta \in \operatorname{Br}(S)[2]$, when $n \geq 9$. It is possible to interpret $F_2(X)$ via moduli spaces of vector bundles over S [Bho86].

When n is even, Y is fibered in odd-dimensional quadrics, degenerate over a plane curve D of degree $n+1$. The cohomology is obtained via a Prym construction arising from the associated double cover of D.

3.3. **Equations of the center.** Given a vector space or bundle V, $\mathbb{P}(V)$ denotes the projective space of lines in V.

In general, write

$$\mathcal{E} := \mathcal{O}^3_{\mathbb{P}^{n-3}} \oplus \mathcal{O}_{\mathbb{P}^{n-3}}(-1) \hookrightarrow \mathcal{O}^{n+1}_{\mathbb{P}^{n-3}}$$

so that $\mathbb{P}(\mathcal{E}) = \operatorname{Bl}_P(\mathbb{P}^n)$ with bundle morphism $\varpi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^{n-3}$. Quadrics containing P correspond to elements of

$$\Gamma(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \varpi^*\mathcal{O}_{\mathbb{P}^{n-3}}(1)) = \Gamma(\mathcal{E}^*(1)),$$

and complete intersections of three such forms correspond to linear transformations

$$\mathcal{O}^3_{\mathbb{P}^{n-3}} \to \mathcal{E}^*(1) \simeq \mathcal{O}_{\mathbb{P}^{n-3}}(1)^3 \oplus \mathcal{O}_{\mathbb{P}^{n-3}}(2).$$

We are interested in their degeneracy loci, whose equations are given by minors of

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \\ Q_1 & Q_2 & Q_3 \end{pmatrix} \tag{1}$$

where the L_{ij} are linear and the Q_i are quadratic; maximal minors yield one cubic equation and three quartic equations.

Let $Y \subset \mathbb{P}^{n-3}$ denote the subscheme given by the maximal minors, which generically has codimension two. The $(n+1)$-dimensional linear series $\Gamma(\mathcal{I}_Y(4))$ induces a birational map

$$\mathbb{P}^{n-3} \dashrightarrow X \subset \mathbb{P}^n$$

inverse to π_P.

The determinantal equations (1) for Y yield a rational map

$$\phi : Y \dashrightarrow \mathbb{P}^2,$$

assigning to each rank-two matrix its one-dimensional kernel. This fails to be defined along the locus $Z \subset Y$ where the matrix (1) has rank one. See Section 3.5 for more information about its geometry.

We shall analyze the cohomology of X using the cohomology of natural resolutions of Y, expressed via ϕ as quadric bundles over $\mathbb{P}^2$.

3.4. **Linear algebra.** We start with some linear algebra on generic $m \times 3$ ($m \geq 3$) matrices $L = (L_{ij})$. Consider the stratification by rank

$$R_1 := \mathbb{P}^{m-1} \times \mathbb{P}^2 \subset R_2 \subset \mathbb{P}^{3m-1}$$

where $\dim(R_2) = 2m + 1$. Over R_1, we may write

$$L_{ij} = u_i v_j, \quad i = 1, \ldots, m,\ j = 1, 2, 3.$$

We obtain two small resolutions of R_2 by keeping track of just the kernel or the image of L, respectively. For example, we may consider

$$\widehat{R_2} \subset \mathbb{P}^{3m-1}_{L_{ij}} \times \mathbb{P}^2_{x_j}$$

given by the equations

$$L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The exceptional locus $\widehat{E} \subset \widehat{R_2}$ is a $\mathbb{P}^1$-bundle over R_1 – for each rank one matrix we extract the one-dimensional subspaces of its kernel. It has codimension $m - 1$ in $\widehat{R_2}$.

3.5. **Generic behavior.** Assume that $n = 11$ and the L_{ij} are linearly independent.

Consider the locus where the upper 3×3 matrix has rank at most one; here we may write $L_{ij} = u_i v_j, i, j = 1, 2, 3$. The remaining equations involving the Q_i take the form

$$u_1 v_2 Q_1 - u_1 v_1 Q_2 = u_2 v_2 Q_1 - u_2 v_1 Q_2 = \cdots = 0.$$

Writing $q_i = Q_i(u_1 v_1, \ldots, u_3 v_3)$ and dividing by the u_i, we obtain

$$v_2 q_1 - v_1 q_2 = v_3 q_1 - v_1 q_3 = v_2 q_3 - v_3 q_2 = 0.$$

The indeterminacy locus for the kernel map ϕ is a surface $Z \subset \mathbb{P}^2 \times \mathbb{P}^2$.

Resolving the kernel map $\phi : Y \dashrightarrow \mathbb{P}^2$ yields a small resolution $\widehat{Y} \to Y$ with center Z. Write the kernel of (1) as $[x_1, x_2, x_3]$ so that

$$[L_{12}L_{23} - L_{13}L_{23}, -L_{11}L_{23} + L_{13}L_{23}, L_{11}L_{22} - L_{12}L_{21}] \sim [x_1, x_2, x_3]$$

etc. The rank two matrices with this kernel correspond to a codimension-three linear subspace

$$\Lambda_{[x_1,x_2,x_3]} \simeq \mathbb{P}^5 \subset \mathbb{P}^8.$$

The 3×3 minors involving the Q_i are all proportional to

$$Q_1x_1 + Q_2x_2 + Q_3x_3.$$

Hence

$$\phi : \widehat{Y} \to \mathbb{P}^2$$

is a quadric bundle of relative dimension four.

3.6. **Degenerate cases in small dimension.**
$\mathbf{n = 5}$: Suppose we have a complete intersection of three quadrics in $\mathbb{P}^5$ containing a plane

$$X = P \cup_B U.$$

Here $\pi_P : U \to \mathbb{P}^2$ is birational, realizing $U = \mathrm{Bl}_Y(\mathbb{P}^2)$, where

$$\phi : Y = \{y_1, \dots, y_9\} \hookrightarrow \mathbb{P}^2$$

is a generic collection of nine points. The imbedding

$$U \hookrightarrow \mathbb{P}^5$$

is via quartics vanishing at those nine points.

Write g for the hyperplane class on $\mathbb{P}^2$ and its restriction to plane curves. Note there is a canonical cubic curve $W \supset Y$ where the determinant of linear forms vanishes. For each divisor

$$\Sigma \equiv_W 4g - Y$$

there is a unique quartic form – modulo the defining equation of W – cutting out $\Sigma \cup Y$.

$\mathbf{n = 6}$: Now consider a complete intersection of three quadrics in $\mathbb{P}^6$ containing a plane

$$P \subset X \subset \mathbb{P}^6.$$

The threefold X is a nodal Fano threefold, with six singularities along P. Note that X depends on 21 parameters, codimension six in the parameter space of all complete intersections.

In this case, we have

$$Y \subset W \subset \mathbb{P}^3,$$

where W is a smooth cubic surface, Y is a smooth curve with

$$\deg(Y) = 9, \quad \mathrm{genus}(Y) = 9,$$

residual to a twisted cubic $\Sigma \subset W$ in the complete intersection of W and a quartic. Write

$$\mathrm{Pic}(W) = \langle L, E_1, E_2, E_3, E_4, E_5, E_6 \rangle, \quad L = [\Sigma],$$

where the E_i are pairwise disjoint (-1)-classes with $L \cdot E_i = 0$. Then we have

$$[Y] = 11L - 4E_1 - \ldots - 4E_6$$

and the residual twisted cubic to Σ, with divisor class

$$5L - 2E_1 - \ldots - 2E_6,$$

realizes Y is a septic plane curve with six nodes. The corresponding linear series induces $\phi : Y \to \mathbb{P}^2$.

The intermediate Jacobian of a minimal resolution $\widetilde{X} \to X$ is isomorphic to the Jacobian of Y.

$\mathbf{n = 7}$: The simplest smooth case, obtained by taking a codimension-four linear section of the generic case. For generic choices of the linear section, $Z = \emptyset$ and $\phi : Y \to \mathbb{P}^2$ is a double cover branched over a special octic plane curve. Thus we have

$$Y \subset W := \{\det(L_{ij}) = 0\} \subset \mathbb{P}^4,$$

where W is a cubic threefold with six ordinary double points [HT10].

The surface Y has Picard group

	K	h
K	2	7
h	7	9

where K is the canonical class – pulled back from $\mathbb{P}^2$ – and h is associated with the embedding

$$Y \hookrightarrow W \hookrightarrow \mathbb{P}^4.$$

There is a second morphism $Y \to \mathbb{P}^2$ associated with the divisor $2h - K$ because

$$\begin{aligned}\chi(2h-K) &= \frac{(2h-K)^2 - (2h-K)K}{2} + 4 \\ &= \frac{36 - 28 + 2 - 14 + 2}{2} + 4 = 3.\end{aligned}$$

The residual intersection to Y in the complete intersection of W and a quartic hypersurface is a cubic scroll Σ. By [HT10], Y admits two families of such scrolls, each parametrized by $\mathbb{P}^2$. An adjunction computation shows that

$$\Sigma \cap Y \equiv_Y 2h - K.$$

Remark 4. The lattice $\langle K, h\rangle$ has discriminant -31. Consider the lattice

$$\left\langle g^2, P\right\rangle \subset H^4(X, \mathbb{Z})$$

under the intersection pairing, where g is the hyperplane class. A Chern-class computation gives $P^2 = 4$ whence

$$\begin{array}{c|cc} & g^2 & P \\ \hline g^2 & 8 & 1 \\ P & 1 & 4 \end{array},$$

which has discriminant 31. The birational parametrization of X induces an isomorphism of Hodge structures

$$H^2(Y, \mathbb{Z})(-1) \supset \langle K, h\rangle^{\perp} \simeq \left\langle g^2, P\right\rangle^{\perp} \subset H^4(X, \mathbb{Z}).$$

$\mathbf{n = 8}$: Here the L_{ij} are linear forms on $\mathbb{P}^5$ so the determinant cubic is singular along an elliptic normal curve. We again have $Z = \emptyset$ and the center $Y \subset \mathbb{P}^5$ of the mapping to $\mathbb{P}^8$ is a conic bundle over $\mathbb{P}^2$. The equations of $\widehat{Y} \subset \mathbb{P}^5 \times \mathbb{P}^2$ may be written:

$$L_{i1}x_1 + L_{i2}x_2 + L_{i3}x_3 = 0, \quad Q_1x_1 + Q_2x_2 + Q_3x_3 = 0. \tag{2}$$

The linear forms induce

$$0 \to K \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \to 0$$

and the quadratic form induces a symmetric map

$$K \to K^*(1).$$

The degeneracy $C \subset \mathbb{P}^2$ has degree 9 since

$$c_1(K^*(1)) - c_1(K) = 9c_1(\mathcal{O}_{\mathbb{P}^2}(1)).$$

We count parameters using for equations of type (2):

- the linear terms depend on 45 parameters, the dimension of $\mathrm{Gr}(3, \Gamma(\mathcal{O}_{\mathbb{P}^5\times\mathbb{P}^2}(1,1)))$;
- the quadratic term depends on 43 paramaters, the dimension of the projectivization of

$$\Gamma(\mathcal{O}_{\mathbb{P}^5\times\mathbb{P}^2}(2,1))/\langle \text{ linear terms}\rangle\,;$$

- coordinate changes on $\mathbb{P}^5 \times \mathbb{P}^2$ account for 43 parameters.

Thus we are left with a total of 46 parameters – the number of moduli of a plane curve of degree 9. We know that C comes with a distinguished double cover $\widetilde{C} \to C$ but even after fixing this we obtain a total of 1024 determinantal representations (see Proposition 5).

$\mathbf{n = 9}$: Here the L_{ij} are linear forms on $\mathbb{P}^6$. This allows us to use the normal form

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ A_5 & A_1 & A_4 \\ A_6 & A_7 & A_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The rank-one locus is a degree-six del Pezzo surface. The equations of $\widehat{Y} \subset \mathbb{P}^6 \times \mathbb{P}^2$ may be written:

$$\begin{aligned} A_1x_1 + A_2x_2 + A_3x_3 = A_5x_1 + A_1x_2 + A_4x_3 = A_6x_1 + A_7x_2 + A_1x_3 &= 0 \\ Q_1x_1 + Q_2x_2 + Q_3x_3 &= 0. \end{aligned}$$

The linear forms induce

$$0 \to K \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 7} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \to 0$$

and the quadratic form induces a symmetric map

$$K \to K^*(1)$$

with degeneracy $C \subset \mathbb{P}^2$ of degree 10.

Fixing C and the double cover $\widetilde{C} \to C$, the various determinant representations as above are parametrized by the planes in X, i.e., by the threefold $F_2(X)$.

4. Schubert calculus of the variety of planes

Now assume that the variety of planes

$$F_2(X) \subset \mathrm{Gr}(3, n+1)$$

is smooth of the expected dimension $3n - 24$.

Consider the canonical exact sequence over $\mathrm{Gr}(3, n+1)$

$$0 \to S \to V \otimes \mathcal{O} \to Q \to 0.$$

The defining equations

$$X = \{F_1 = F_2 = F_3 = 0\}, \quad F_1, F_2, F_3 \in \mathrm{Sym}^2(V^*),$$

induce sections $f_1, f_2, f_3 \in \mathrm{Sym}^2(S^*)$ so that

$$F_2(X) = \{f_1 = f_2 = f_3 = 0\}.$$

We compute Chern classes. We use the notation of [GH94, pp. 197 ff.], e.g., the Chern classes of S are the Schubert cycles

$$c(S) = 1 - \sigma_1 + \sigma_{1,1} - \sigma_{1,1,1}.$$

Computing via the 'SchurRings' or 'Schubert2' packages of `Macaulay2` [GS], we obtain

$$[F_2(X)] = 512(\sigma_{9,6,3}+2\sigma_{9,5,4}+2\sigma_{8,7,3}+6\sigma_{8,6,4}+4\sigma_{8,5,5}+4\sigma_{7,7,4}+8\sigma_{7,6,5}+2\sigma_{6,6,6}).$$

The paper [DM98] explores these formulas more systematically; our formula extends the tabulation on [DM98, p. 563]. See also [Jia12] for more information on Noether-Lefschetz questions for Fano schemes.

Proposition 5. *When $n = 8$ and X generic, the variety $F_2(X) \subset \mathrm{Gr}(3, 9)$ has dimension zero and*

$$[F_2(X)] = 1024[\text{point}].$$

Question 6. Describe the Galois action arising in this case. To what extent is it governed by the Galois representation on the intermediate Jacobian?

Frank Sottile and his collaborators (see, e.g., [HRS18]) are developing computational approaches to Galois groups of enumerative questions. Computations with Taylor Brysiewicz indicate the group in this case might be smaller than the symmetric group $\mathfrak{S}_{1024}$ but a full computation appears difficult with existing techniques and computational resources.

Recently, Hashimoto and Kadets [HK20] analyzed Galois groups of zero-dimensional Fano varieties of linear subspaces in complete intersections using the classification of multiply transitive groups. Except for a few exceptional cases – like cubic surfaces and even-dimensional complete intersections of two quadrics – the Galois group always contains the alternating group. Thus in our situation the Galois group is either the alternating group $\mathfrak{A}_{1024}$ or $\mathfrak{S}_{1024}$.

Remark 7. We propose to construct these examples synthetically: Fix seven generic planes

$$P_1, \dots, P_7 \subset \mathbb{P}^8$$

which depend up to projectivity on $18 \cdot 7 - 80 = 46$ parameters. Write

$$\Pi = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_7$$

and note that

$$h^0(\mathcal{I}_\Pi(2)) = h^0(\mathcal{O}_{\mathbb{P}^8}(2)) - 7 \cdot 6 = 3.$$

Thus there is a *unique* complete intersection of three quadrics $X \supset \Pi$ which we expect is smooth. How do we construct the other 1017 planes of $F_2(X)$?

We focus next on the case $n = 9$ where $F_2(X) \subset \mathrm{Gr}(3, 10)$ is a threefold with class

$$[F_2(X)] = 512(4\sigma_{7,7,4} + 8\sigma_{7,6,5} + 2\sigma_{6,6,6})$$

so we have

$$\deg(F_2(X)) = 11264 = 11 \cdot 2^{10}.$$

Since $K_{F_2(X)} = 2\sigma_1$ we have

$$K^3_{F_2(X)} = 11 \cdot 2^{13}.$$

Proposition 8. [DM98, Th. 3.4] *When $F_2(X)$ is smooth of the expected dimension we have*

$$h^{0,1}(F_2(X)) = h^{1,0}(F_2(X)) = 0.$$

Consider the rank 18 vector bundle

$$\mathcal{E} = \mathrm{Sym}^2(S^*)^{\oplus 3}$$

so there exists a section $s \in \Gamma(\mathrm{Gr}(3, 10), \mathcal{E})$ with $F_2(X) = \{s = 0\}$. The exact sequence

$$0 \to T_{F_2(X)} \to T_{\mathrm{Gr}(3,10)}|F_2(X) \to N_{F_2(X)/\,\mathrm{Gr}(3,10)} \to 0$$

and the interpretation of the normal bundle as $\mathcal{E}|F_2(X)$ allows us to compute all the Chern classes of $T_{F_2(X)}$:

$$\begin{aligned} c_1(T_{F_2(X)}) &= -2\sigma_1 \\ c_2(T_{F_2(X)}) &= 8\sigma_2 - 3\sigma_{1,1} \\ c_3(T_{F_2(X)}) &= -20\sigma_3 - \sigma_{2,1} + 8\sigma_{1,1,1}. \end{aligned}$$

A computation with the 'SchurRings' or 'Schubert2' packages of `Macaulay2` gives

$$\chi(\mathcal{O}_{F_2(X)}) = c_1(T_{F_2(X)})c_2(T_{F_2(X)})/24 = -2816 = -2^8 \cdot 11$$

and

$$\chi(F_2(X)) = c_3(T_{F_2(X)}) = -36,864 = -2^{12} \cdot 3^2.$$

The Hirzebruch-Riemann-Roch formula gives

$$\chi(\mathcal{O}_{F_2(X)}(1)) = 0,$$

which is to be expected, as Serre duality gives

$$h^i(\mathcal{O}_{F_2(X)}(1)) = h^{3-i}(\omega_{F_2(X)}(-1)) = h^{3-i}(\mathcal{O}_{F_2(X)}(1)).$$

Similarly, we obtain $\chi(\mathcal{O}_{F_2(X)}(2)) = 2816$. We also get

$$\chi(\mathcal{O}_{F_2(X)}(3)) = h^0(\mathcal{O}_{F_2(X)}(3)) = 16,896 = 2^9 \cdot 3 \cdot 11,$$

where the first equality is Kodaira vanishing. Thus we have enough data to extract the Hilbert polynomial of $F_2(X)$

$$\chi(\mathcal{O}_{F_2(X)}(m)) = \frac{2^7 \cdot 11}{3}(m-1)(5(m-1)^2-2).$$

Hirzebruch-Riemann-Roch allows us to compute $\chi(\Omega^1_{F_2(X)})$ as well

$$\chi(\Omega^1_{F_2(X)}) = 15,616 = 2^8 \cdot 61.$$

5. Koszul computations

5.1. **General set-up.** Let $\mathcal{E}$ denote a vector bundle and $s \in \Gamma(\mathcal{E})$ a section such that the degeneracy locus $Z = \{s = 0\}$ has codimension equal to the rank R of $\mathcal{E}$. Then we have a resolution

$$0 \to \bigwedge^R \mathcal{E}^* \to \bigwedge^{R-1} \mathcal{E}^* \to \cdots \to \mathcal{E}^* \to \mathcal{O} \to \mathcal{O}_Z \to 0$$

and the Koszul complex

$$0 \to \bigwedge^R \mathcal{E}^* \to \bigwedge^{R-1} \mathcal{E}^* \to \cdots \to \mathcal{E}^* \to \mathcal{O} \to 0.$$

The arrow

$$\iota_s : \bigwedge^n \mathcal{E}^* \to \bigwedge^{n-1} \mathcal{E}^*$$

is contraction by s. The hypercohomology spectral sequence gives

$$E_1^{p,q} = H^q(\bigwedge^{-p} \mathcal{E}^*) \Rightarrow H^{p+q}(\mathcal{O}_Z),$$

where $p = -R, \dots, 0$.

Following [Man91], we may use this to compute the cohomology of degeneracy loci over Grassmannians $\mathbb{G}(n, d-1) = \mathrm{Gr}(n+1, V)$. Given a d-dimensional vector space V and integers $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_k = 0$ for $k > d$, let $\Gamma^\lambda V$ denote the associated Schur functor. We observe the convention $\Gamma^\lambda V = 0$ for sequences of integers λ failing the decreasing or vanishing conditions.

We keep track of the decompositions using the dictionary between Schubert calculus and tensor products

$$\Gamma^\lambda V \otimes \Gamma^\mu V = \sum_\nu c^\nu_{\lambda,\mu} \Gamma^\nu V,$$

where the multiplicies are defined by

$$\sigma_\lambda \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu} \sigma_\nu.$$

5.2. **Planes in intersections of three quadrics in $\mathbb{P}^9$.** Consider the bundle

$$\mathcal{E} = \mathrm{Sym}^2(S^*)^{\oplus 3}$$

and $s \in \Gamma(\mathcal{E})$ with $F_2(X) = \{s = 0\}$. We have the associated Koszul resolution

$$0 \to \bigwedge^{18} \mathcal{E}^* \to \cdots \to \bigwedge^{r} \mathcal{E}^* \to \cdots \to \mathcal{E}^* \to \mathcal{O} \to \mathcal{O}_{F_2(X)} \to 0.$$

The terms of the Koszul complex decompose into direct sums of products

$$\bigwedge^{e_1} \mathrm{Sym}^2(S) \otimes \bigwedge^{e_2} \mathrm{Sym}^2(S) \otimes \bigwedge^{e_3} \mathrm{Sym}^2(S), \quad e_1 + e_2 + e_3 = r.$$

The hypercohomology spectral sequence gives

$$E_1^{p,q} = H^q(\bigwedge^{-p} \mathcal{E}^*) \Rightarrow H^{p+q}(\mathcal{O}_{F_2(X)}),$$

where $p = -18, \ldots, 0$.

Recall the Weyl character formula for $\mathrm{GL}(V)$ representations:

$$\dim \Gamma^\lambda(V) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Assume S has rank three then

$$\mathrm{rank}\, \Gamma^{(a_1,a_2,a_3)} S = (a_1 - a_2 + 1)(a_1 - a_3 + 2)(a_2 - a_3 + 1)/2.$$

We observe the dictionary

$$\begin{array}{rcl}
\mathrm{Sym}^2(S) & \sim & \sigma_2 \\
\bigwedge^2 \mathrm{Sym}^2(S) & \sim & \sigma_{3,1} \\
\bigwedge^3 \mathrm{Sym}^2(S) & \sim & \sigma_{3,3} + \sigma_{4,1,1} \\
\bigwedge^4 \mathrm{Sym}^2(S) & \sim & \sigma_{4,3,1} \\
\bigwedge^5 \mathrm{Sym}^2(S) & \sim & \sigma_{4,4,2} \\
\bigwedge^6 \mathrm{Sym}^2(S) & \sim & \sigma_{4,4,4}
\end{array}$$

whence

$$\begin{aligned}
\mathrm{Sym}^2(S) &\simeq \Gamma^{(2)}(S)\\
\bigwedge^2 \mathrm{Sym}^2(S) &\simeq \Gamma^{(3,1)}(S)\\
\bigwedge^3 \mathrm{Sym}^2(S) &\simeq \Gamma^{(3,3)}(S) \oplus \Gamma^{(4,1,1)}(S)\\
\bigwedge^4 \mathrm{Sym}^2(S) &\simeq \Gamma^{(4,3,1)}(S)\\
\bigwedge^5 \mathrm{Sym}^2(S) &\simeq \Gamma^{(4,4,2)}(S)\\
\bigwedge^6 \mathrm{Sym}^2(S) &\simeq \Gamma^{(4,4,4)}(S)
\end{aligned}$$

Note also [Jia12, Cor. 1.3] that

$$H^j(\mathrm{Gr}(3,10), \Gamma^\lambda S) = 0$$

whenever j is not divisible by 7 and for all but one value of j.

We tabulate the representations $\Gamma^a S, a = (a_1, a_2, a_3)$ appearing in each $\bigwedge^r \mathcal{E}^*$:

r	decomposition
0	$\Gamma^{(0,0,0)}$
1	$3\Gamma^{(2)}$
2	$3\Gamma^{(4)} + 6\Gamma^{(3,1)} + 3\Gamma^{(2,2)}$
3	$\Gamma^{(6)} + 8\Gamma^{(5,1)} + 9\Gamma^{(4,2)} + 10\Gamma^{(4,1,1)} + 10\Gamma^{(3,3)} + 8\Gamma^{(3,2,1)} + \Gamma^{(2,2,2)}$
4	$3\Gamma^{(7,1)} + 9\Gamma^{(6,2)} + 15\Gamma^{(6,1,1)} + 18\Gamma^{(5,3)} + 24\Gamma^{(5,2,1)} + 6\Gamma^{(4,4)}$ $+33\Gamma^{(4,3,1)} + 9\Gamma^{(4,2,2)} + 15\Gamma^{(3,3,2)}$
5	$3\Gamma^{(8,2)} + 6\Gamma^{(8,1,1)} + 9\Gamma^{(7,3)} + 24\Gamma^{(7,2,1)} + 18\Gamma^{(6,4)} + 54\Gamma^{(6,3,1)}$ $+21\Gamma^{(6,2,2)} + 6\Gamma^{(5,5)} + 48\Gamma^{(5,4,1)} + 54\Gamma^{(5,3,2)} + 39\Gamma^{(4,4,2)} + 30\Gamma^{(4,3,3)}$
6	$\Gamma^{(9,3)} + 8\Gamma^{(9,2,1)} + 8\Gamma^{(8,4)} + 27\Gamma^{(8,3,1)} + 19\Gamma^{(8,2,2)} + 9\Gamma^{(7,5)}$ $+64\Gamma^{(7,4,1)} + 62\Gamma^{(7,3,2)} + 10\Gamma^{(6,6)} + 53\Gamma^{(6,5,1)} + 117\Gamma^{(6,4,2)} + 56\Gamma^{(6,3,3)}$ $+46\Gamma^{(5,5,2)} + 88\Gamma^{(5,4,3)} + 38\Gamma^{(4,4,4)}$
7	$3\Gamma^{(10,3,1)} + 6\Gamma^{(10,2,2)} + 3\Gamma^{(9,5)} + 24\Gamma^{(9,4,1)} + 27\Gamma^{(9,3,2)} + 6\Gamma^{(8,6)}$ $+42\Gamma^{(8,5,1)} + 93\Gamma^{(8,4,2)} + 36\Gamma^{(8,3,3)} + 3\Gamma^{(7,7)} + 48\Gamma^{(7,6,1)} + 132\Gamma^{(7,5,2)}$ $+144\Gamma^{(7,4,3)} + 66\Gamma^{(6,6,2)} + 138\Gamma^{(6,5,3)} + 114\Gamma^{(6,4,4)} + 60\Gamma^{(5,5,4)}$
8	$3\Gamma^{(11,3,2)} + 9\Gamma^{(10,5,1)} + 24\Gamma^{(10,4,2)} + 9\Gamma^{(10,3,3)} + 3\Gamma^{(9,7)} + 24\Gamma^{(9,6,1)}$ $+75\Gamma^{(9,5,2)} + 72\Gamma^{(9,4,3)} + 24\Gamma^{(8,7,1)} + 102\Gamma^{(8,6,2)} + 168\Gamma^{(8,5,3)} + 99\Gamma^{(8,4,4)}$ $+69\Gamma^{(7,7,2)} + 168\Gamma^{(7,6,3)} + 213\Gamma^{(7,5,4)} + 96\Gamma^{(6,6,4)} + 75\Gamma^{(6,5,5)}$
9	$\Gamma^{(12,3,3)} + 9\Gamma^{(11,5,2)} + 8\Gamma^{(11,4,3)} + 9\Gamma^{(10,7,1)} + 36\Gamma^{(10,6,2)} + 63\Gamma^{(10,5,3)}$ $+28\Gamma^{(10,4,4)} + \Gamma^{(9,9)} + 8\Gamma^{(9,8,1)} + 63\Gamma^{(9,7,2)} + 128\Gamma^{(9,6,3)} + 142\Gamma^{(9,5,4)}$ $+28\Gamma^{(8,8,2)} + 142\Gamma^{(8,7,3)} + 216\Gamma^{(8,6,4)} + 146\Gamma^{(8,5,5)} + 146\Gamma^{(7,7,4)}$ $+160\Gamma^{(7,6,5)} + 20\Gamma^{(6,6,6)}$

The terms in red contribute to the cohomology [Jia12, Th. 1.2]. The representations for $r > 9$ may be read off via duality

$$\bigwedge^r \mathcal{E}^* = \det(\mathcal{E}^*) \otimes \bigwedge^{18-r} \mathcal{E}, \quad \Gamma^{(a_1,a_2,a_3)} \times \Gamma^{(12-a_3,12-a_2,12-a_1)} \to \Gamma^{(12,12,12)}.$$

r	decomposition
18	$\Gamma^{(12,12,12)}$
17	$3\Gamma^{(12,12,10)}$
16	$3\Gamma^{(12,12,8)} + 6\Gamma^{(12,11,9)} + 3\Gamma^{(12,10,10)}$
15	$\Gamma^{(12,12,6)} + 8\Gamma^{(12,11,7)} + 9\Gamma^{(12,10,8)} + 10\Gamma^{(11,11,8)} + 10\Gamma^{(12,9,9)} + 8\Gamma^{(11,10,9)}$
	$+\Gamma^{(10,10,10)}$
14	$3\Gamma^{(12,11,5)} + 9\Gamma^{(12,10,6)} + 15\Gamma^{(11,11,6)} + 18\Gamma^{(12,9,7)} + 24\Gamma^{(11,10,7)} + 6\Gamma^{(12,8,8)}$
	$+33\Gamma^{(11,9,8)} + 9\Gamma^{(10,10,8)} + 15\Gamma^{(10,9,9)}$
13	$3\Gamma^{(12,10,4)} + 6\Gamma^{(11,11,4)} + 9\Gamma^{(12,9,5)} + 24\Gamma^{(11,10,5)} + 18\Gamma^{(12,8,6)} + 54\Gamma^{(11,9,6)}$
	$+21\Gamma^{(10,10,6)} + 6\Gamma^{(12,7,7)} + 48\Gamma^{(11,8,7)} + 54\Gamma^{(10,9,7)} + 39\Gamma^{(10,8,8)} + 30\Gamma^{(9,9,8)}$
12	$\Gamma^{(12,9,3)} + 8\Gamma^{(11,10,3)} + 8\Gamma^{(12,8,4)} + 27\Gamma^{(11,9,4)} + 19\Gamma^{(10,10,4)} + 9\Gamma^{(12,7,5)}$
	$+64\Gamma^{(11,8,5)} + 62\Gamma^{(10,9,5)} + 10\Gamma^{(12,6,6)} + 53\Gamma^{(11,7,6)} + 117\Gamma^{(10,8,6)}$
	$+56\Gamma^{(9,9,6)} + 46\Gamma^{(10,7,7)} + 88\Gamma^{(9,8,7)} + 38\Gamma^{(8,8,8)}$
11	$3\Gamma^{(11,9,2)} + 6\Gamma^{(10,10,2)} + 3\Gamma^{(12,7,3)} + 24\Gamma^{(11,8,3)} + 27\Gamma^{(10,9,3)} + 6\Gamma^{(12,6,4)}$
	$+42\Gamma^{(11,7,4)} + 93\Gamma^{(10,8,4)} + 36\Gamma^{(9,9,4)} + 3\Gamma^{(12,5,5)} + 48\Gamma^{(11,6,5)} + 132\Gamma^{(10,7,5)}$
	$+144\Gamma^{(9,8,5)} + 66\Gamma^{(10,6,6)} + 138\Gamma^{(9,7,6)} + 114\Gamma^{(8,8,6)} + 60\Gamma^{(8,7,7)}$
10	$3\Gamma^{(10,9,1)} + 9\Gamma^{(11,7,2)} + 24\Gamma^{(10,8,2)} + 9\Gamma^{(9,9,2)} + 3\Gamma^{(12,5,3)} + 24\Gamma^{(11,6,3)}$
	$+75\Gamma^{(10,7,3)} + 72\Gamma^{(9,8,3)} + 24\Gamma^{(11,5,4)} + 102\Gamma^{(10,6,4)} + 168\Gamma^{(9,7,4)}$
	$+99\Gamma^{(8,8,4)} + 69\Gamma^{(10,5,5)} + 168\Gamma^{(9,6,5)} + 213\Gamma^{(8,7,5)} + 96\Gamma^{(8,6,6)} + 75\Gamma^{(7,7,6)}$

And then we record those contributing cohomology in degree j, using [Jia12, Th. 1.2]. Note that the sequence

$$(-1,-2,-3,-4,-5,-6,-7,a_1-8,a_2-9,a_3-10)$$

must have no repeating integers as these yield 'singular' weights. Thus we must have $a_1 \geq 8$; if $a_1 = 8$ then $a_2 = 1, 0$, etc.

j	weights a contributing
0	$(0,0,0)$
7	no contributions in $\bigwedge^4 \mathcal{E}^*$
	$(8,1,1)$
	no contributions in $\bigwedge^6 \mathcal{E}^*$
14	$(9,9,0)$
	$(10,9,1),(9,9,2)$
	$(10,10,2),(11,9,2)$
	no contributions in $\bigwedge^{12} \mathcal{E}^*$
21	$(10,10,10)$
	$(12,10,10)$
	$(12,12,10)$
	$(12,12,12)$

The degree 7 cohomology contributes through

$$H^7(\bigwedge^5 \mathcal{E}^*) = H^7((\Gamma^{(8,1,1)}S)^{\oplus 6}) \simeq (\Gamma^{(1,1,1,1,1,1,1,1,1,1)}V)^{\oplus 6} \simeq (\det(V))^{\oplus 6} \simeq \mathbb{C}^6.$$

Here V is the standard 10-dimensional representation.

Proposition 9. *We have*

$$H^0(\mathcal{O}_{F_2(X)}) = \mathbb{C}, \quad H^1(\mathcal{O}_{F_2(X)}) = 0.$$

For higher degrees, we have

$$\mathbb{C}^6 \simeq E_1^{-5,7} = E_2^{-5,7} = \cdots = E_\infty^{-5,7} = H^2(\mathcal{O}_{F_2(X)}).$$

Consequently, we deduce that

$$h^3(\mathcal{O}_{F_2(X)}) = 2816 + 1 + 6 - 0 = 2823.$$

Proof. The only term in degree zero is $E_1^{(0,0)}$ and there are no terms in degree one, which gives the degree 0 and 1 cohomology; see also Prop. 8 and [DM98]. The spectral sequence is supported at the following values of (p,q):

$$(0,0), (-5,7), (-9,14), (-10,14), (-11,14),$$
$$(-15,21), (-16,21), (-17,21), (-18,21).$$

All the values after $(-5,7)$ have degree $p+q \geq 3$ and $p \leq -5$, thus do not receive arrows from $E_r^{-5,7}$. And clearly there are no maps from the degree $(0,0)$ term, abutting to $\Gamma(\mathcal{O}_{F_2(X)})$. This yields the equalities asserted above. □

The degree 14 cohomology contributes through

$$\begin{aligned}
H^{14}(\bigwedge^{11} \mathcal{E}^*) &= H^{14}((\Gamma^{(10,10,2)}S)^{\oplus 6}) \oplus (\Gamma^{(11,9,2)}S)^{\oplus 3})) \\
&= (\Gamma^{(3,3,2,2,2,2,2,2,2,2)}V)^{\oplus 6} \oplus (\Gamma^{(4,2,2,2,2,2,2,2,2,2)}V)^{\oplus 3} \\
&\simeq (\mathbb{C}^{45})^6 \oplus (\mathbb{C}^{55})^3 \simeq \mathbb{C}^{435} \\
H^{14}(\bigwedge^{10} \mathcal{E}^*) &= H^{14}((\Gamma^{(9,9,2)}S)^{\oplus 9}) \oplus (\Gamma^{(10,9,1)}S)^{\oplus 3})) \\
&= (\Gamma^{(2,2,2,2,2,2,2,2,2,2)}V)^{\oplus 9} \oplus (\Gamma^{(3,2,2,2,2,2,2,2,2,1)}V)^{\oplus 3} \\
&\simeq (\mathbb{C})^9 \oplus (\mathbb{C}^{99})^3 \simeq \mathbb{C}^{306} \\
H^{14}(\bigwedge^{9} \mathcal{E}^*) &= H^{14}(\Gamma^{(9,9)}S) \\
&= \Gamma^{(2,2,2,2,2,2,2,2,2,0)}V \simeq \mathbb{C}^{55}
\end{aligned}$$

The degree 21 cohomology contributes through:

$$\begin{aligned}
H^{21}(\bigwedge^{18}\mathcal{E}^*) &= H^{21}(\Gamma^{12,12,12}S)\\
&= \Gamma^{(5,5,5,3,3,3,3,3,3,3)}V \simeq \mathbb{C}^{4950}\\
H^{21}(\bigwedge^{17}\mathcal{E}^*) &= H^{21}((\Gamma^{12,12,10}S)^{\oplus 3})\\
&= (\Gamma^{(5,5,3,3,3,3,3,3,3,3)}V)^{\oplus 3} \simeq (\mathbb{C}^{825})^3 \simeq \mathbb{C}^{2475}\\
H^{21}(\bigwedge^{16}\mathcal{E}^*) &= H^{21}((\Gamma^{12,10,10}S)^{\oplus 3})\\
&= (\Gamma^{(5,3,3,3,3,3,3,3,3,3)}V)^{\oplus 3} \simeq (\mathbb{C}^{55})^3 \simeq \mathbb{C}^{165}\\
H^{21}(\bigwedge^{15}\mathcal{E}^*) &= H^{21}(\Gamma^{10,10,10}S)\\
&= \Gamma^{(3,3,3,3,3,3,3,3,3,3)}V \simeq \mathbb{C}
\end{aligned}$$

Remark 10. Standard properties of spectral sequences imply

$$-2816 = \chi(\mathcal{O}_{F_2(X)}) = \sum(-1)^{p+q}H^q(\bigwedge^{-p}\mathcal{E}^*) = \sum(-1)^{a+p}H^{7a}(\bigwedge^{-p}\mathcal{E}^*).$$

We compute

$$-2816 = 1 + 6 + (-55 + 306 - 435) + (-4950 + 2475 - 165 + 1),$$

thus everything checks.

Question 11. The simplest structure on the spectral sequence would be to have exactness in all degrees where there is no contribution to cohomology, i.e., degeneration at E_2. Thus the only nonzero terms in the second page would be

$$E_2^{-18,21} \simeq \mathbb{C}^{4950-2475+165-1} = \mathbb{C}^{2639}, \quad E_2^{-11,14} \simeq \mathbb{C}^{435-306+55} = \mathbb{C}^{184},$$

as well as

$$E_2^{-5,7} \simeq \mathbb{C}^6, \quad E_2^{0,0} \simeq \mathbb{C}.$$

Does this degeneration occur? It would imply a nontrivial filtration on the holomorphic three-forms of $F_2(X)$.

Proposition 12. *The Hodge diamond of $F_2(X)$ is*

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 6 & & 62 & & 6 & \\
2823 & & 15684 & & 15684 & & 2823
\end{array}$$

Proof. We recall that

- The Picard rank $\rho(F_2(X)) = 1$ when X is very general [Jia12, Th. 0.3].
- Let $\Sigma \to \mathbb{P}^2$ denote the double cover branched along degeneracy curve; there exists an isometry on primitive cohomology

$$H^6(X,\mathbb{Z})_{\mathrm{prim}} \hookrightarrow H^2(\Sigma,\mathbb{Z})(-2)_{\mathrm{prim}},$$

realizing the former as an index-two sublattice of the latter [O'G86, Th. 0.1].

The latter observation gives the Hodge numbers of X displayed in Section 3.1.

The incidence correspondence

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & F_2(X) \end{tikzcd}$$

induces Abel-Jacobi maps

$$\alpha_1 : g_* f^* : H^6(X,\mathbb{Z}) \to H^2(F_2(X),\mathbb{Z})(-2),$$

$$\alpha_2 : f_* g^* : H^4(F_2(X),\mathbb{Z}) \to H^6(X,\mathbb{Z})(1).$$

Letting

$$L : H^2(F_2(X),\mathbb{Z}) \to H^4(F_2(X),\mathbb{Z})$$

denote intersection by the hyperplane class σ_1, the composition

$$\alpha_2 \circ L \circ \alpha_1 : H^6(X,\mathbb{Z}) \to H^6(X,\mathbb{Z})$$

forces the cohomology of X and $F_2(X)$ to be tightly intertwined. Such constructions are used to establish Grothendieck's version of the Hodge conjecture for varieties with sparse Hodge diamond; see for example [Voi10].

For our immediate purpose, we can apply the Main Theorem of [Shi04] to conclude that the cylinder map α_1 injects the primitive cohomology $H^6(X,\mathbb{Z})_{\mathrm{prim}}$ into $H^2(F_2(X),\mathbb{Z})$. Since we computed

$$h^{0,2}(F_2(X)) = 6$$

in Proposition 9, we conclude that

$$H^2(F_2(X),\mathbb{Q})_{\mathrm{prim}} \simeq H^6(X,\mathbb{Q})(2)_{\mathrm{prim}} \oplus \mathbb{Q}(-1)^N$$

for some N. The last summand is of Hodge-Tate type. However, $N = 0$ by Jiang's result on the Picard group. $\square$

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