

Graph Coloring using Coupled Oscillator-based Dynamical Systems

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Abstract— Graph coloring is a NP-hard problem, and computing the solution on a digital computer entails an exponential increase in the computing resources (time, memory) with increasing problem size. This has motivated the search for alternate and more efficient non-Boolean approaches. Here, we experimentally demonstrate the solution to this problem using the phase dynamics of coupled oscillators. Using a 30-oscillator IC platform with reconfigurable all-to-all coupling and minimal post-processing, our approach achieves 98% accuracy in detecting (near-) optimal solutions within 1 color of the optimal solution in comparison to the 77% accuracy achieved with the heuristic Johnson algorithm. Additionally, we propose a new local search-based post-processing scheme to improve the quality of the coloring solution. Finally, using circuit simulations, we demonstrate the scalability and speed up ($\sim 100\times$) achievable with the above approach in larger graphs.

Keywords— coupled oscillators, graph coloring, analog computing

I. INTRODUCTION

The graph coloring problem is an archetypal combinatorial optimization problem defined as: Given a graph G where each vertex is assigned a ‘color’ (label), compute the minimum number of colors such that no two nodes that share a common edge have the same color. While this graph theoretic problem can be mapped to a broad spectrum of practical applications in areas ranging from image processing to data mining, solving this problem efficiently using conventional digital computers is fundamentally challenging; even approximate high-quality solutions, which are useful in many practical cases, are difficult to compute [1]. This is because the graph coloring belongs to the NP-hard computational complexity class of problems where exponentially increasing resources are required as the problem sizes increase. This limitation has motivated an effort to realize alternate computing paradigms and platforms that can accelerate such problems.

Dynamical systems of synchronized oscillators, wherein discrete (digital) sequential computing is replaced by continuous time evolution of the system, offer unique advantages when solving such problems. Each oscillator evolves in parallel, and the memory is distributed along with the “compute” operations allowing the system to search the high-dimensional solution space efficiently [2]. Prior work by Parihar *et al.* [3] on graph coloring laid the theoretical foundation of this approach and demonstrated the coloring of up to 5 node graphs using insulator-metal phase transition (VO_2)-based oscillators. However, scaling this emerging device technology is challenging and can preclude the

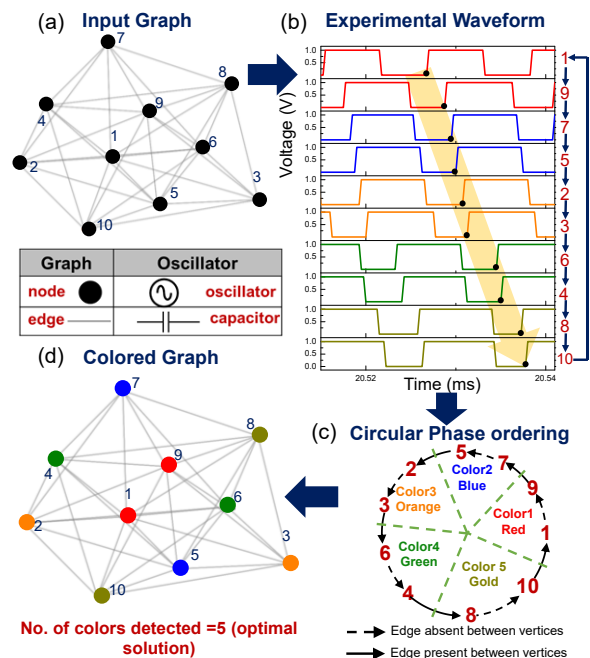


Fig. 1. (a) A representative 10 node graph solved using the coupled relaxation oscillators. (b) Measured outputs of the topologically equivalent coupled oscillator circuit. (c) Corresponding ordering of phases experimentally measured from the time domain waveform; and (d) Resulting coloring solution for the graph (=5, which is optimal).

evaluation of the computational primitive in larger oscillator systems. Other oscillator works [4][5] focused on graph coloring have relied on circuit simulations. In contrast, works on oscillator-based computing using the more mature CMOS technology have not focused on graph coloring [6][7]; Ahmed *et al.* [6] recently showcased the dynamics of coupled oscillators as an Ising machine (Max-Cut problem was addressed) in a planar configuration. Planar graphs are 4 colorable [8] (i.e. any planar graph requires a maximum of 4 colors to color it optimally) and can be solved in polynomial time. Thus, the dynamics of the oscillators when solving the computationally hard non-planar graphs remains to be explored. The ability to compute the Maximum Independent Set (MIS) using CMOS-based coupled oscillators was recently demonstrated by the authors in [9]. Computing the optimal MIS only entails obtaining the cardinality of the largest partition in the phase ordering (as discussed in the following section) while optimal graph coloring requires the correct number of partitions in the phase ordering of the oscillators. In other words, optimal MIS can be obtained even if the phase ordering (required for graph coloring) is sub-optimal (as long

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as the largest partition of nodes contains the optimal number of nodes). This makes graph coloring- the focus of the present work- a significantly harder problem to solve using coupled oscillators.

II. GRAPH COLORING USING COUPLED OSCILLATORS

The foundation of using coupled oscillators to color graphs lies in the equivalence between the eigenvalues of the coupled oscillator system in state space as and those of the adjacency matrix, A of the graph [3]:

$$\dot{x}(t) = (C_I + C_C + C_L)^{-1} [-G(x)x(t) + H(x)] \quad (1)$$

where, C_I , C_C , C_L are the internal, coupling and the load capacitances of the system and $x = \{x_1, x_2, \dots, x_n\}$ represents state of the oscillators. Further, to help map the oscillator dynamics to graph coloring, the problem is recast as an equivalent color sorting problem that aims to establish a circular ordering such that nodes of the same color appear consecutively. The oscillator phase ordering then corresponds to ordering of the colored nodes [3] which can subsequently be sorted (in polynomial time) to obtain the solution to the graph coloring problem.

In order to process a graph using coupled oscillators, the adjacency Matrix, A , of the input graph is used to construct a topologically equivalent oscillator circuit. Each node (represented by a column/row of A) is represented by an oscillator and every edge (A_{ij}) is represented by a coupling element; since we consider undirected graphs here, $A_{ij} = A_{ji}$. Fig. 1(a) shows a representative 10-node graph (with 36 edges) that is mapped to a topologically equivalent coupled oscillator circuit (experimental details are described in the following section). The corresponding time domain output of the oscillators (Fig. 1(b)) shows the relative phase difference and phase ordering among the oscillators which encodes the combinatorial solution to the problem. The oscillators exhibit a unique phase ordering such that clusters of nodes in the graph network without an edge (the nodes in a cluster can be assigned the same color since they have no edge) appear consecutively in the ordering. For instance, nodes 2 and 3, which do not have an edge, appear consecutively; similarly, nodes 8, 10; and 1, 9 etc. Subsequently, the cyclic ordering of nodes (Fig. 1(c)) can be sorted into the clusters of nodes having the same color using a simple sorting algorithm with the number of such clusters approximating a near-optimal/optimal solution ($=5$; solution is optimal for the considered graph) to the problem (Fig. 1(d)).

III. COUPLED OSCILLATOR HARDWARE

To explore this computational capability of the oscillators, we utilize an integrated circuit (die photo in Fig. 7) consisting of 30 oscillators with all-to-all reconfigurable coupling. Fig. 2(a) shows the architecture of the oscillator IC with the reconfigurable coupling network implemented using a capacitor (32.5 fF) in series with a transmission gate (T-gate) to turn ON/OFF the coupling between any two oscillators. There are 870 ($=P(30,2)$) coupling elements which facilitates any oscillator to be coupled/decoupled to any and all other oscillators in the network. Consequently, the IC can solve a graph with arbitrary connectivity among the oscillator nodes including NP-hard non-planar graphs. The relaxation oscillator

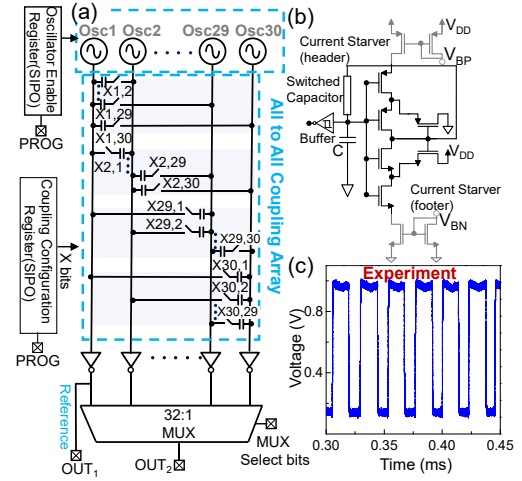


Fig. 2. (a) Architecture of the oscillator IC with reconfigurable all-to-all coupling. (b) Circuit schematic; and (c) Measured output of a single oscillator in the IC.

is implemented using an inverting Schmitt-trigger with a negative RC feedback. The feedback resistor (tuned to $R=402 \text{ M}\Omega$) is realized using a switched capacitor ($C=1.66 \text{ fF}$; $f=1.5 \text{ MHz}$) (Fig. 2(b)). The output of each oscillator is buffered, and the corresponding time domain waveform is shown in Fig. 2(c). The binary output of the oscillator simplifies the measurement and read-out of the oscillator phases while preserving the critical phase information which is essential for computing the graph coloring solution.

The oscillators are selected using the oscillator enable register. The coupling network is programmed according to the values of A_{ij} such that the T-gate is ON when $A_{ij}=1$; else it is turned OFF. The adjacency matrix is formulated into a binary bit-stream and passed on to a SIPO register (Coupling configuration register) to program the coupling elements. The solution which is encoded in the steady state phase ordering of the oscillators is computed by comparing the phase of each oscillator measured through a 32:1 MUX) with a reference oscillator (oscillator 1) read directly from the buffer (i.e. without the MUX).

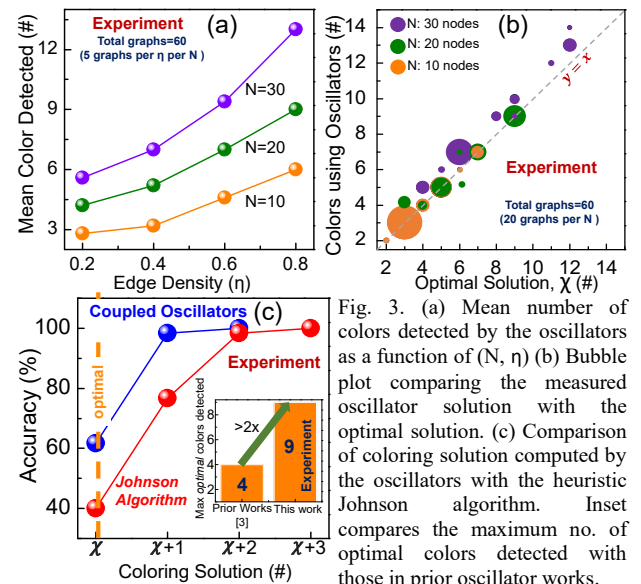


Fig. 3. (a) Mean number of colors detected by the oscillators as a function of (N, η) (b) Bubble plot comparing the measured oscillator solution with the optimal solution. (c) Comparison of coloring solution computed by the oscillators with the heuristic Johnson algorithm. Inset compares the maximum no. of optimal colors detected with those in prior oscillator works.

IV. COMPUTATIONAL PERFORMANCE OF COUPLED OSCILLATORS

To evaluate the computational properties of the oscillators, we experimentally measure multiple randomly generated graph instances (including non-planar graphs) of various size (N : 10, 20, 30) and edge density (η : 0.2-0.8); 5 graphs are evaluated for every combination of N , η (total 60 graphs). Fig. 3(a) shows the mean number of colors (averaged over 5 graphs for each N , η) detected by the oscillators as a function of the η for graphs of different sizes (10, 20, 30 nodes). It is evident that larger and denser graphs (high η) require larger number of colors making them more challenging to compute. Furthermore, the bubble plot in Fig. 3(b) compares the oscillator solutions for the measured graphs with the optimal coloring solutions. It can be observed that most solutions lie close to the identity line ($y=x$) implying the oscillators compute high quality (near-) optimal solutions; size of the bubble indicates the number of instances lying at the same (x,y) .

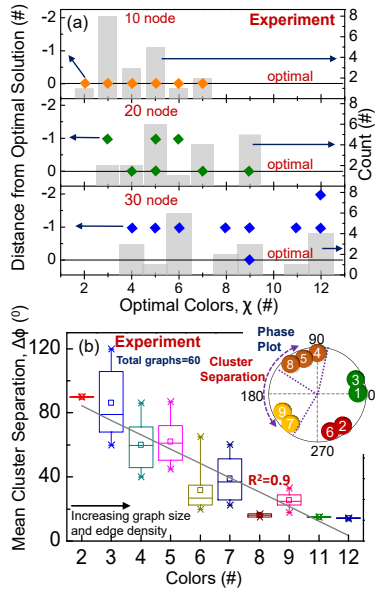


Fig. 4 (a) Distance of the measured solution from the optimal solution as a function of the number of colors. (b) Mean cluster separation vs. the number of colors detected; cluster separation decreases with increase in the coloring solution.

However, since computing the optimal solution to the graphs using an exact algorithm can take up to several days (especially in denser graphs), we also benchmark our results with the Johnson's heuristic algorithm which does not guarantee optimal solutions but is significantly faster. Fig. 3(c) shows the accuracy of the solution produced by the coupled oscillators and compares it with that obtained using the Johnson algorithm. The oscillators (with minimal post processing) produce a solution within 1 color of the optimal solution in 98% of the measured graphs in comparison to the 77% observed with the Johnson algorithm; the oscillators produce an optimal solution in 62% of the measured graphs in comparison to the 40% accuracy observed with the Johnson's algorithm. The inset in Fig. 3(c) shows the maximum number of optimal colors detected by the oscillators ($=9$) in comparison to prior oscillator works [3].

To further analyze how the input graph affects the computational performance of the oscillators, the bar plots in Fig. 4(a) measure the quality (optimality) of the oscillator solution (specified as the distance of the measured oscillator solution from the optimal value) as a function of the number of optimal colors. We observe that the deviation from the optimal solution is a strong function of the number of optimal colors required which increases with the size and edge density of the graph as shown in Fig. 3a. Bigger and denser graphs requiring larger number of colors for optimal coloring show increased deviation. This is also supported by the observed phase dynamics of the oscillator system evaluated using mean cluster separation, given by:

$$\Delta\phi = \frac{\sum_{i=1}^m |\Sigma \phi_{(i+1)\%m} / n_{(i+1)\%m} - \Sigma \phi_i / n_i|}{m} \quad (2)$$

where m : number of clusters, n_i : number of elements in i^{th} cluster, ϕ : phase difference of an oscillator with respect to the reference oscillator, and $\%$ represents modulo operation. Fig. 4(b) shows that the mean separation between the clusters (that are assigned a particular color each) decreases with increasing number of optimal colors making it more challenging for the system to settle to the phase optimal ordering in larger and denser graphs.

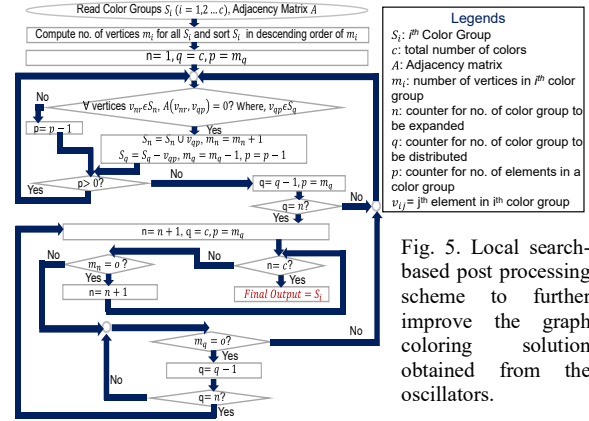


Fig. 5. Local search-based post processing scheme to further improve the graph coloring solution obtained from the oscillators.

V. POST PROCESSING SCHEME

The reduced cluster separation in larger and denser graphs increases the likelihood of the system settling to a sub-optimal phase ordering. Our empirical observations reveal that this due to a few nodes being misplaced in the ordering. As alluded to earlier, this dramatically impacts the optimal graph coloring solution even though it may not affect the solution to other problems like MIS. We therefore develop a polynomial time post processing scheme (based on local search) that *corrects* the oscillator ordering to reduce the number of colors. The underlying algorithm shown in Fig. 5 starts by sorting the color groups obtained from the oscillator dynamics in descending order of their size. Subsequently, a node from the smallest group is selected and added to a larger group if and only if it shares no common edge with the existing nodes in the group. This process is repeated for all the nodes during which it is observed that (nodes from) some smaller groups get

completely redistributed. Consequently, this reduces the number of colors and improves the solution.

VI. SCALABILITY

We also test using circuit simulations, the scalability of the oscillator approach to color larger graphs beyond 30 nodes. We use the SPICE-compatible Xyce computing platform (provided by Sandia National Labs [10]) to build and emulate the coupled oscillators and evaluate graphs from the DIMACS [11] as well as other randomly generated graph instances; for the latter, we consider graph sizes (N) up to 128 nodes and $\eta=0.2-0.8$; 2 graphs are analyzed for each N , η (24 graphs in total). Fig. 6(a)(b) shows the number of colors detected and compares the oscillator solution with that obtained from the Johnson's heuristic algorithm (finding optimal solutions is computationally intractable and no solution is available even after >24 hours); the table in Fig. 6(c) compares the graph coloring solutions obtained for the graph instances from the DIMACS database where the oscillators provide an equal or improved solution in 4 of the 6 graphs considered. More importantly the oscillators provide a $\sim 100\times$ speed up in the time to compute the solution (computation time includes time to settle into the circular phase ordering, and the time required for post processing) as shown in Fig. 6(d).

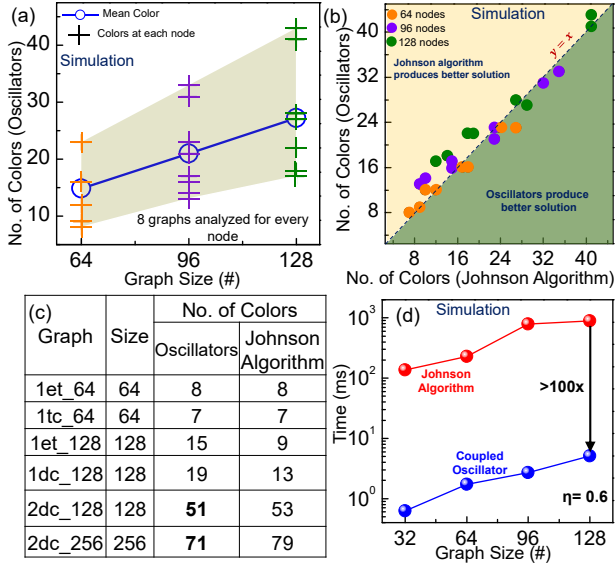


Fig. 6. (a) Number of colors detected by the oscillators as a function of graph size (N). (b) Bubble plot comparing the number of colors detected using the oscillators with that detected using the Johnson algorithm. (c) Simulations comparing coupled oscillators with the Johnson algorithm for larger DIMACS implementation challenge graphs. (d) Comparison of computing time for Johnson algorithm and the coupled oscillators. The oscillator computing time includes time for the system to reach steady-state as well as the post-processing scheme.

VII. COMPARISON

Fig. 7 shows the die photo and operating parameters of the oscillator IC ($1.2\times 1.2\text{ mm}^2$) used in this work. The IC is fabricated using the bulk CMOS 65nm technology. Fig. 8 compares our approach with other non-Boolean schemes (including other oscillator demonstrations) that have been used to solve the graph coloring problem (it is important to note that only [3] actually focuses on solving graph coloring; other demonstrations focus on solving other combinatorial

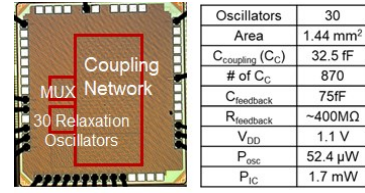


Fig. 7. Die photo and operating parameters of the coupled oscillator IC used in this work.

	This Work	Parihar et al. [3]	Ahmed et al. [6]	Hamerly et al. [12]	Takata et al. [13]	Chou et al. [14]
Architecture	Coupled Oscillators (Schmitt Trigger Oscillators)	Coupled Oscillators (VO ₂ Oscillators)	Coupled Oscillators (Ring Oscillators)	Qubits (D-Wave)	Optical Parametric Oscillators	Coupled Oscillators (LC Oscillators)
Technology	CMOS 65nm	IMT devices	CMOS 65nm	Super-conductors	Optical oscillators	Discrete LC Oscillators
Coupling	All-to-All	Discrete	Nearest Neighbor	Nearest Neighbor	All-to-all	Discrete
Maximum detected optimal coloring solution	9	4	4 All Planar graphs are 4 colorable [7]	-	Not reported	4
Solves Non-Planar Graph	Yes	No	No	-	Not reported	No
Peak Power	1.7 mW (52.4 μ W/oscillator)	Not Reported	23 mW (41 μ W/oscillator)	25 KW (Cryogenic Cooling)	1 W (16 Spins)	Not Reported

Fig. 8. Present work compared with other non-Boolean approaches.

optimization problems such as Max-Cut). It can be observed that the oscillator approach and hardware evaluated in this work not only enables a compact (IC technology compatible), low-power and highly reconfigurable platform (in comparison to other non-oscillator-based approaches) but also facilitates the evaluation of NP-hard non-planar graphs (in contrast to prior oscillator works).

VIII. CONCLUSION

In summary, we show that coupled oscillators provide an efficient non-Boolean approach for computing high quality (near-) optimal solutions to graph coloring; the oscillator hardware demonstrates a 98% success rate in comparison to 77% with state-of-the-art Johnson's algorithm. Our work marks a step forward towards realizing highly efficient and compact non-Boolean circuits that complement digital computers to solve NP-hard problems.

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