Simple spectrum of $c\bar{c}c\bar{c}$ states in the dynamical diquark model

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We develop the spectroscopy of $c\bar{c}c\bar{c}$ and other all-heavy tetraquark states in the dynamical diquark model. In the most minimal form of the model (e.g., each diquark appears only in the color-triplet combination; the nonorbital spin couplings connect only quarks within each diquark), the spectroscopy is extremely simple. Namely, the S-wave multiplets contain precisely three degenerate states (0^{++}, 1^{+-}, 2^{++}) and the seven P-wave states satisfy an equal-spacing rule when the tensor coupling is negligible. When comparing numerically to the recent LHCb results, we find the best interpretation is assigning $X(6900)$ to the $2S$ multiplet, while a lower state suggested at about 6740 MeV fits well with the members of the $1P$ multiplet. We also predict the location of other multiplets ($1S$, $1D$, etc.) and discuss the significance of the cc open-flavor threshold.

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I. INTRODUCTION

The LHCb Collaboration has recently presented evidence [1] for the observation of at least one resonance in the J/ψ-pair spectrum at about 6900 MeV and the likely presence of at least one additional resonance lying below this mass but above the 6200 MeV J/ψ-pair threshold. Such states are naturally assigned the valence-quark content $c\bar{c}c\bar{c}$, making them the first all-heavy multiquark exotic candidates claimed to date in the experimental literature.

Theoretical studies of $c\bar{c}c\bar{c}$ states have a much longer history, dating indeed to a time only two years after the discovery of the J/ψ [2] and followed by a smattering of papers in the 1980s [3–5]. The current interest in $c\bar{c}c\bar{c}$ states, starting in 2011 [6,7] and particularly ramping up since 2016 [8–27], emerged from the expectation of dedicated searches at the LHC.

A notable feature of the all-heavy multiquark exotics $Q_1\bar{Q}_2Q_3\bar{Q}_4$ ($Q_i = c$ or b), in contrast to the known exotics $Q\bar{Q}q\bar{q}'$ [28] ($q, q' \in \{u, d\}$), is the lack of a plausible molecular structure for the states. The lightness of the quarks $q, \bar{q}'$ in the $Q\bar{Q}q\bar{q}'$ case suggests the possibility of $(Q\bar{q}')(\bar{Q}q)$ molecules, bound by the exchange of light mesons with valence content $(q\bar{q}')$ and possessing a spatial extent at least as large as the light-meson wave function, of order $1/\Lambda_{\rm QCD} \simeq O(1)$ fm. If the state lies especially close to the $(Q\bar{q}')(\bar{Q}q)$ threshold [e.g., $X(3872)$], then its spatial extent is determined by the inverse of the binding energy and can be quite substantial, possibly as large as several fm. Moreover, Yukawa-like light-meson binding exchanges as an explanation for such near-threshold states begin to appear implausibly fine-tuned, and instead *threshold rescattering effects* (loop exchanges of virtual particles between the constituent mesons that numerically enhance the amplitude near the threshold) provide a mechanism for binding the state. In contrast, the case of all-heavy $Q_1\bar{Q}_2Q_3\bar{Q}_4$ states lacks a light-meson exchange mechanism, both for Yukawa-type exchanges and for threshold effects. The $X(6900)$ is noted [1] to lie in the vicinity of the $\chi_{c0}\chi_{c0}$ and $\chi_{c1}\chi_{c0}$ thresholds, but to our knowledge no calculation has yet suggested the ability of such a threshold rescattering to produce a strong resonance.

In general, one expects the lowest-lying $Q_1\bar{Q}_2Q_3\bar{Q}_4$ states to exhibit comparable distances between all four heavy quarks. If, say, the $Q_1\bar{Q}_2$ and $Q_3\bar{Q}_4$ pairs are formed with substantially smaller internal separations than the distance between the two pairs, then one expects the immediate formation of two free conventional quarkonium states rather than a single resonance, even if both pairs are in color octets and require gluon exchange (which has a range comparable to that of light-meson exchange) in order for $Q_1\bar{Q}_2$ and $Q_3\bar{Q}_4$ to hadronize as color singlets.

As a result, the most common models for $Q_1\bar{Q}_2Q_3\bar{Q}_4$ states assume a diquark-antidiquark $[(Q_1Q_3)(\bar{Q}_2\bar{Q}_4)]$ structure, typically exploiting the attractive color-antitriplet quark-quark coupling. One should keep in mind, however, that if all four quarks have comparable separations (as is

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anticipated for the ground states), then a combination of different color structures should be expected to appear for those states (e.g., as in the lattice simulation of Ref. [12]).

Beyond the ground states, the separations between the quarks can become differentiated. As noted above, closer association of the $Q\bar{Q}$ pairs is expected to lead to an immediate dissociation into quarkonium pairs, while the configuration $(Q_1Q_3)(\bar{Q}_2\bar{Q}_4)$ with color-triplet diquarks becomes the only one that features an attractive interaction between the component constituents (the quarks within the diquarks), but must still remain bound due to confinement, independent of the exchange of any number of gluons. These features define the *dynamical diquark picture* of multiquark exotics [29,30]. In the original picture, the diquark separation is a consequence of the production process; for example, $c\bar{c}q\bar{q}'$ tetraquarks can be manifested due to the large momentum release between the $c\bar{c}$ pair in B-meson decays into a $(cq)(\bar{c}\bar{q}')$ structure. To be more precise, the diquark-antidiquark state couples most strongly to the portion of the four-quark momentum-space wave function for which the relative momentum between the quasiparticles $\delta \equiv (Q_1Q_3)$ and $\bar{\delta} \equiv (\bar{Q}_2\bar{Q}_4)$ is significantly larger than the relative momenta within them.

The dynamical diquark picture is elevated to a full model by identifying its mass eigenstates with those of the gluon field connecting the diquarks [31]. Explicitly, confinement limits the eventual separation of the δ-$\bar{\delta}$ pair even though they may form with a large relative momentum, and the specific stationary states of the full system are supplied by the quantized modes of the gluon field stretching between the two heavy, (eventually) nearly stationary sources δ, $\bar{\delta}$. This approach uses the Born-Oppenheimer (BO) approximation in precisely the same manner as is done for simulations of heavy-quark hybrids on the lattice (e.g., Refs. [32–36]). Indeed, the specific form of the static potential $V_\Gamma(r)$ between the heavy sources for a particular BO glue configuration Γ is precisely the same one computed in each lattice simulation just referenced. The corresponding coupled Schrödinger equations were first numerically solved for $c\bar{c}q\bar{q}'$ states in Ref. [37].

Typical diquark models approximate the quasiparticles δ, $\bar{\delta}$ to be pointlike, even though they are expected to have spatial extents comparable to those of mesons carrying the same valence-quark flavor content. Nevertheless, model calculations in Ref. [38] for $c\bar{c}q\bar{q}$ states show that finite diquark size has a surprisingly mild effect on the spectrum for a $\delta = (cq)$ radius as large as 0.4 fm.

The dynamical diquark model also selects a very specific set of spin-dependent couplings as the ones deemed most physically significant. In this model, the δ, $\bar{\delta}$ pair form distinguishable, separate entities within the full state, so that the dominant spin-spin couplings are taken to be the ones between quarks within each diquark [39], while typical existing models for $c\bar{c}c\bar{c}$ states (e.g., Refs. [6,7]) treat all quark spin-spin interactions on equal footing, or consider only couplings to full diquark spins (e.g., Ref. [18]). The more restrictive paradigm used here leads to very simple predictions for the spectrum of $c\bar{c}c\bar{c}$ states, particularly in S-wave multiplets, which will become immediately testable once the quantum numbers of the $c\bar{c}c\bar{c}$ states are known.

On the other hand, the dominant operators in this model for $c\bar{c}c\bar{c}$ states carrying orbital angular momentum dependence (relevant to P- and higher-wave states) are taken to couple only to the diquarks as units, since δ, $\bar{\delta}$ are assumed to have no internal orbital excitation for all low-lying $c\bar{c}c\bar{c}$ states.[1] The resultant spin-orbit and tensor operators for the low-lying spectrum are the same as those used in Ref. [18], but differ from those used in Ref. [20], which instead are chosen to couple to all individual quark spins. Again, a very simple spectrum arises in this model for the P-wave states, the degree of validity for which will become immediately apparent with further data.

Our purpose in this paper is therefore not to compete with detailed calculations of spectra that are based upon assuming specific forms for all operators contributing to the Hamiltonian of $c\bar{c}c\bar{c}$ states (e.g., using a one-gluon-exchange potential to obtain an explicit functional form for the coefficient for every operator, as in Ref. [18]). Rather, we describe the most significant features in the spectrum parametrically, identifying particular spin-spin, spin-orbit, or tensor terms to pinpoint their origin, while remaining agnostic as to the precise dynamical origin of these operators. We nevertheless also present an initial fit to the $c\bar{c}c\bar{c}$ spectrum, using numerical values for the Hamiltonian parameters obtained from the analogous operators in other sectors of exotics to which the model has previously been applied. Specifically, the strength of the spin-spin operator is obtained from a recent fit to $c\bar{c}s\bar{s}$ candidates [41], and the spin-orbit and tensor strengths are taken from a recent fit to P-wave $c\bar{c}q\bar{q}'$ candidates [40].

This paper is organized as follows. In Sec. II, we review the spectroscopy of the model for S- and P-wave $Q_1\bar{Q}_2Q_3\bar{Q}_4$ states, identifying quantum-number restrictions arising from spin statistics. Section III presents the Hamiltonian and tabulates all matrix elements for the allowed states, and we identify features of the spectrum that appear based upon their parametric analysis. In Sec. IV, we present a numerical prediction for the $c\bar{c}c\bar{c}$ spectrum, using as described above the results of previous work; and in Sec. V we conclude.

[1]In contrast, the tensor operator for P-wave $c\bar{c}q\bar{q}'$ states in Ref. [40], owing its origin to a pionlike exchange within the state, was chosen to couple only to the light-quark spins within the diquarks. Nevertheless, the matrix elements for an alternative tensor operator that couples only to the full diquark spins (as to be used here) are also computed in that work.

II. SPECTROSCOPY OF $Q\bar{Q}Q\bar{Q}$ EXOTICS

The spectroscopy of δ-$\bar{\delta}$ states in which the diquarks δ, $\bar{\delta}$ contain no internal orbital angular momentum, but that allows for arbitrary orbital excitation and gluon-field excitation between the δ-$\bar{\delta}$ pair, is presented in Ref. [31]. For the all-heavy states with distinguishable quarks in δ and $\bar{\delta}$ (i.e., $b\bar{b}c\bar{c}$, or for that matter, $c\bar{c}s\bar{s}$), precisely the same enumeration of states occurs. The core states, expressed in the basis of good diquark-spin eigenvalues with labels such as 1_δ, are given by

$$\begin{aligned}
J^{PC} = 0^{++}&: \; X_0 \equiv |0_\delta, 0_{\bar{\delta}}\rangle_0, \quad X_0' \equiv |1_\delta, 1_{\bar{\delta}}\rangle_0, \\
J^{PC} = 1^{++}&: \; X_1 \equiv \frac{1}{\sqrt{2}}(|1_\delta, 0_{\bar{\delta}}\rangle_1 + |0_\delta, 1_{\bar{\delta}}\rangle_1), \\
J^{PC} = 1^{+-}&: \quad Z \equiv \frac{1}{\sqrt{2}}(|1_\delta, 0_{\bar{\delta}}\rangle_1 - |0_\delta, 1_{\bar{\delta}}\rangle_1), \\
&\qquad Z' \equiv |1_\delta, 1_{\bar{\delta}}\rangle_1, \\
J^{PC} = 2^{++}&: \; X_2 \equiv |1_\delta, 1_{\bar{\delta}}\rangle_2,
\end{aligned} \tag{1}$$

with the outer subscripts on the kets indicating total quark spin S. On their own, these six states fill the lowest multiplet $\Sigma_g^+(1S)$ within the BO approximation for the gluon-field potential connecting the δ-$\bar{\delta}$ pair. Higher BO potentials (like Σ_u^-, where standard BO quantum-number labels such as these are defined in Ref. [31]) produce the multiquark analogues to hybrid mesons and thus are expected to lie about 1 GeV above the $\Sigma_g^+(1S)$ ground states. For phenomenological reasons to be discussed in Sec. IV, we do not discuss such states further here.

The diquarks δ, $\bar{\delta}$ in this model transform as color (anti) triplets, which are antisymmetric under quark-color exchange. If the quarks within δ or $\bar{\delta}$ are identical, then the space-spin wave function of the corresponding diquark must be symmetric in order to satisfy Fermi statistics for the complete δ or $\bar{\delta}$ wave function; however, since the model assumes no orbital excitation within the diquarks, their spatial wave function and hence also their spin wave function alone must be symmetric, which thus requires the corresponding diquark spin to equal unity: only 1_δ and $1_{\bar{\delta}}$ survive. In the $c\bar{c}c\bar{c}$ or $b\bar{b}b\bar{b}$ case, one immediately sees from Eq. (1) that the states X_0, X_1, and Z are forbidden by spin statistics.[2] The ground-state multiplet $\Sigma_g^+(1S)$ is thus halved: only the three states X_0' (0^{++}), Z' (1^{+-}), and X_2 (2^{++}) survive. An identical analysis applies to all radial-excitation multiplets $\Sigma_g^+(nS)$.

One immediate conclusion of this model becomes evident: if the full state wave function contains a component that allows either diquark to appear in the (symmetric) color sextet, then that diquark in the low-lying states must appear in the antisymmetric spin-0 combination 0_δ or $0_{\bar{\delta}}$. In that case, the full spectrum of six states from Eq. (1), most notably a state with $J^{PC} = 1^{++}$, must appear. The observation of a 1^{++} $c\bar{c}c\bar{c}$ state in the lowest multiplet (or any S-wave multiplet) would provide direct evidence of dynamics lying outside the most restrictive diquark models.

The addition of a nonzero orbital-excitation quantum number L is now straightforward. Since the intrinsic parity factor (-1) for an antiquark appears twice, the parity eigenvalue of the full state is just given by the usual spatial factor $(-1)^L$. All S-wave, D-wave, etc. states therefore have $P = +$, and all P-wave, F-wave, etc. states have $P = -$. Starting with the S-wave "core" states X_0', Z', and X_2 of Eq. (1), one invokes the usual angular momentum addition rules to produce states of good total J (indicated by a superscript "(J)," using the notation developed in Ref. [31]). Explicitly, the seven P-wave $c\bar{c}c\bar{c}$ states, accompanied by their J^{PC} eigenvalues, are

$$\begin{aligned}
&X_{0P}'^{(1)}(1^{--}), \quad Z_P'^{(0)}(0^{-+}), \quad Z_P'^{(1)}(1^{-+}), \quad Z_P'^{(2)}(2^{-+}), \\
&X_{2P}^{(1)}(1^{--}), \quad X_{2P}^{(2)}(2^{--}), \quad X_{2P}^{(3)}(3^{--}).
\end{aligned} \tag{2}$$

For completeness, we note that each of the D-wave, F-wave, etc. multiplets contain precisely nine $c\bar{c}c\bar{c}$ states. In particular, the $\Sigma_g^+(1D)$ multiplet is the lowest one to contain a 1^{++} state, $X_{2D}^{(1)}$.

III. MASS HAMILTONIAN

The full mass spectrum of all states in the dynamical diquark model is computed by the following procedure. First, a particular BO potential Γ $(= \Sigma_g^+, \Pi_u$, etc.) that gives rise to a multiplet of states $[\Sigma_g^+(1P), \Pi_u(2P)$, etc.] is specified. The corresponding potentials $V_\Gamma(r)$ have been computed numerically on the lattice [32–36]. One specifies a diquark mass $m_{\delta,\bar{\delta}}$ (or in the case of pentaquarks, a color-triplet *triquark* mass as well) and solves the resulting Schrödinger equation for this Hamiltonian H_0 numerically [37],[3] giving rise to a multiplet-average mass eigenvalue $M_0(nL)$ for particular radial (n) and orbital (L) quantum numbers attached to the particular BO potential Γ. In this paper, we are interested only in the Σ_g^+ potential and primarily in the levels within the lowest multiplets $\Sigma_g^+(1S)$, $\Sigma_g^+(1P)$, and $\Sigma_g^+(2S)$.

The next step is to identify and compute fine-structure corrections to the spectrum of each such multiplet. In the

[2]One may also consider truly exotic states like $b\bar{b}b\bar{c}$, in which 0_δ is forbidden but $0_{\bar{\delta}}$ is allowed, in which case only the state X_0 is eliminated. For such states, C also ceases to be a good quantum number, so that X_1 and Z become the same 1^+ state, thus leaving a total of four states in the multiplet $\Sigma_g^+(1S)$. In contrast, the case $b\bar{b}c\bar{c}$ (considered in, e.g., Ref. [18]) retains the C quantum number and all six $\Sigma_g^+(1S)$ states.

[3]In some cases, the BO potentials mix, leading to coupled Schrödinger equations that require a more involved numerical solution technique.

dynamical diquark model, the dominant spin-dependent, isospin-independent operator is taken to be the spin-spin coupling between quarks in the diquark, and between the antiquarks in the antidiquark. In the case of $Q\bar{Q}q\bar{q}'$ states (where $q, q' \in \{u, d\}$), the model also includes a spin-dependent, isospin-dependent operator that mimics the form present in pion exchange. The analysis of the $\Sigma_g^+(1S)$ multiplet of $c\bar{c}q\bar{q}'$ states in Ref. [38] uses a Hamiltonian consisting only of H_0 and the two operators thus described,

$$H = H_0 + 2\kappa_{qQ}(\mathbf{s}_q \cdot \mathbf{s}_Q + \mathbf{s}_{\bar{q}'} \cdot \mathbf{s}_{\bar{Q}}) + V_0 \boldsymbol{\tau}_q \cdot \boldsymbol{\tau}_{\bar{q}'} \boldsymbol{\sigma}_q \cdot \boldsymbol{\sigma}_{\bar{q}'}, \quad (3)$$

where of course $Q = c$, and κ_{qQ} is assumed to be isospin symmetric. This very simple Hamiltonian is used to great effect in Ref. [38], where it provides a natural explanation for the 1^{++} $X(3872)$ being the lightest observed state in the $\Sigma_g^+(1S)$ multiplet and for the appearance of the preferential decay patterns $Z_c(3900) \to J/\psi$ and $Z_c(4020) \to h_c$. In the intermediate case of $c\bar{c}s\bar{s}$ states in Ref. [41], as well as in the all-heavy case $Q\bar{Q}Q\bar{Q}$ considered here (or more generally, $Q_1\bar{Q}_2Q_3\bar{Q}_4$), the isospin-dependent term V_0 is absent. In addition, the coefficients κ_{qQ}, κ_{sQ}, and κ_{QQ} refer to spin couplings within diquarks containing increasingly heavy quarks, and therefore the diquarks are expected to be increasingly spatially compact. Since the fundamental quark spins thus interact at increasingly close range, one may expect the numerical size of these couplings to increase for heavier quark combinations, a point to which we return in Sec. IV.

The S-wave Hamiltonian for $Q\bar{Q}Q\bar{Q}$ therefore contains only one new parameter,

$$H = H_0 + 2\kappa_{QQ}(\mathbf{s}_Q \cdot \mathbf{s}_Q + \mathbf{s}_{\bar{Q}} \cdot \mathbf{s}_{\bar{Q}}), \quad (4)$$

where the two factors of $\mathbf{s}_Q$ and of $\mathbf{s}_{\bar{Q}}$ are each understood to apply to a separate heavy quark. The eigenvalues of H are trivially computed in the basis of good diquark spin,

$$M = M_0 + \kappa_{QQ}[s_\delta(s_\delta + 1) + s_{\bar{\delta}}(s_{\bar{\delta}} + 1) - 3]. \quad (5)$$

Since as noted above, $s_\delta = s_{\bar{\delta}} = 1$ in any state for which diquarks have negligible coupling to the color-sextet channel, we immediately obtain a strong result. The three states of each $\Sigma_g^+(nS)$ multiplet, 0^{++}, 1^{+-}, and 2^{++}, are degenerate in this model, with a common mass eigenvalue given by

$$M(nS) = M_0 + \kappa_{QQ}, \quad (6)$$

where of course both M_0 and κ_{QQ} may vary with the radial excitation number n. The measurement of nonzero mass splittings between these three states would therefore provide direct evidence that the quarks within different diquarks have nonnegligible spin-spin couplings between them.[4] In comparison, one does not expect this degeneracy in the Ξ_{cc} ground states, since although s_δ is still constrained to equal 1, the (light) third quark is not spatially separated from δ, so that one still expects distinct couplings to the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ ground states.

Turning now to $L > 0$ states, the new operators appearing in the Hamiltonian are pure orbital [$\mathbf{L}^2$, which is the same for all states in the $\Sigma_g^+(nL)$ multiplet and therefore provides a contribution to M_0], spin-orbit, and tensor operators. Both of the latter operators are considered in Ref. [40] for P-wave $c\bar{c}q\bar{q}'$ states.

The spin-orbit operator in this model appears as

$$\Delta H_{LS} = V_{LS}\mathbf{L} \cdot (\mathbf{s}_\delta + \mathbf{s}_{\bar{\delta}}) = V_{LS}\mathbf{L} \cdot \mathbf{S}, \quad (7)$$

where S is the total spin carried by the quarks [the state subscripts in Eqs. (1), or 1 for $Z^{(\prime)}$], which trivially gives the matrix elements

$$\Delta M_{LS} = \frac{V_{LS}}{2}[J(J+1) - L(L+1) - S(S+1)]. \quad (8)$$

Note that according to Eq. (7) the model treats all four quarks on the same footing, each interacting with the same total $\mathbf{L}$ operator since the individual diquarks are assumed to have no internal excitation. Thus, only one separation coordinate $(\mathbf{r}_\delta - \mathbf{r}_{\bar{\delta}})$ and only one orbital angular momentum operator $\mathbf{L}$ is relevant.[5]

The final operator in the model for $L > 0$ states is the tensor coupling S_{12} between the δ-$\bar{\delta}$ pair, defined by

$$\Delta H_T = V_T S_{12}, \quad (9)$$

where

$$S_{12} \equiv 3\boldsymbol{\sigma}_1 \cdot \boldsymbol{r}\boldsymbol{\sigma}_2 \cdot \boldsymbol{r}/r^2 - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2. \quad (10)$$

$\boldsymbol{\sigma}$ here and below denotes twice the canonically normalized spin operator of the full entity coupling to the tensor force. In the study of P-wave $c\bar{c}q\bar{q}'$ states in Ref. [40], the tensor operator is assumed to originate as an analogue to the corresponding operator in nucleon-nucleon interactions arising from pion exchange, and therefore $\boldsymbol{\sigma}$ couples only to the light quarks within δ and $\bar{\delta}$, just as for the spin-spin V_0 operator in Eq. (3). The assumption of coupling only to the light quarks rather than to the full δ, $\bar{\delta}$ as units is viable in the dynamical diquark model because again, the diquarks are *not* treated as completely pointlike. Nevertheless, the

[4]This result is parametrically apparent from the first equations of Sec. IIB in Ref. [7] (setting their $\kappa_+ = 0$). However, since all spin-spin couplings are numerically comparable in their model, this feature was not commented upon there.

[5]Alternate $c\bar{c}c\bar{c}$ tetraquark models (e.g., Refs. [5,20]) have been presented in which all four quarks and their three relative separations are significant for a full description of the state.

alternative hypothesis of coupling the isospin-dependent spin-spin and tensor operators to δ, $\bar{\delta}$ as units was also studied in Refs. [38,40] and found to be incompatible with known phenomenology [e.g., in predicting a degenerate $I = 1$ partner to the $X(3872)$, which is known not to exist].

In the all-heavy case, one not only expects that δ, $\bar{\delta}$ are more compact than in the $Q\bar{Q}q\bar{q}'$ case, but also notes that the privileged position of light quarks with respect to isospin no longer occurs. In this case, the spin operators $\boldsymbol{\sigma}$ in the tensor operator of Eq. (10) refer to the full QQ or $\bar{Q}\,\bar{Q}$ diquark spins. The matrix elements in that case are computed in Appendix A of Ref. [40],

$$\langle L', S', J|S_{12}|L, S, J\rangle = (-1)^{S+J}\sqrt{30[L][L'][S][S']}\begin{Bmatrix} J & S' & L' \\ 2 & L & S \end{Bmatrix}\begin{pmatrix} L' & 2 & L \\ 0 & 0 & 0 \end{pmatrix} \times \begin{Bmatrix} s_\delta & s_{\bar{\delta}} & S \\ s_{\delta'} & s_{\bar{\delta}'} & S' \\ 1 & 1 & 2 \end{Bmatrix}\langle s_{\delta'}||\boldsymbol{\sigma}_1||s_\delta\rangle\langle s_{\bar{\delta}'}||\boldsymbol{\sigma}_2||s_{\bar{\delta}}\rangle, \tag{11}$$

where $[j] \equiv 2j + 1$. The reduced matrix elements of the angular momentum generators are given by

$$\langle j'||\mathbf{j}||j\rangle = \sqrt{j(2j+1)(j+1)}\,\delta_{j'j}. \tag{12}$$

The tensor operator of Eq. (10) does not change individual diquark spins [as is evident from Eq. (12)] and vanishes if $s_\delta = 0$ or $s_{\bar{\delta}} = 0$ [as is evident from the $9j$ symbol in Eq. (11)]. It does however allow the total quark spin S to change, as well as the orbital excitation L.

In summary, the full Hamiltonian of the dynamical diquark model for all-heavy states $Q\bar{Q}Q\bar{Q}$ (and with small modifications, for general all-heavy states $Q_1\bar{Q}_2Q_3\bar{Q}_4$) is given by the sum of Eqs. (4), (7), and (9),

$$H = H_0 + 2\kappa_{QQ}(\mathbf{s}_Q\cdot\mathbf{s}_Q + \mathbf{s}_{\bar{Q}}\cdot\mathbf{s}_{\bar{Q}}) + V_{LS}\mathbf{L}\cdot\mathbf{S} + V_T S_{12}^{(\delta\bar{\delta})}. \tag{13}$$

Only the first two terms are required for $\Sigma_g^+(nS)$ states, while the latter two terms are needed for $L > 0$ states. The matrix elements (i.e., mass eigenvalues) for the three S-wave states are degenerate and are given in Eq. (6), while those for the seven P-wave states are presented in Table I. The latter are listed in a particular order that recognizes another interesting feature of this model: If $V_{LS} \gg V_T$, then the P-wave states fill an equal-spaced multiplet. Assuming that $V_{LS} > 0$ (as occurs in Ref. [40]) means that the states in Table I may be expected to appear in order of increasing mass. This ordering almost precisely matches the corresponding (unmixed) numbers in Ref. [20], despite the fact that the latter calculation includes not only tensor terms, but also couplings between all of the quarks.[6]

TABLE I. Mass eigenvalues of the seven $\Sigma_g^+(nP)$ states, which assume the simple forms $M = M_0 + \kappa_{QQ} + \Delta M_{LS} + \Delta M_T$. The two 1^{--} states $X_2^{(1)}$, $X_1^{\prime(0)}$ also have an off-diagonal mixing term given by Eq. (14).

State	J^{PC}	ΔM_{LS}	ΔM_T
$X_2^{(1)}$	1^{--}	$-3V_{LS}$	$-\frac{28}{5}V_T$
$Z^{\prime(0)}$	0^{-+}	$-2V_{LS}$	$-8V_T$
$Z^{\prime(1)}$	1^{-+}	$-V_{LS}$	$+4V_T$
$X_2^{(2)}$	2^{--}	$-V_{LS}$	$+\frac{28}{5}V_T$
$X_0^{\prime(1)}$	1^{--}	$0V_{LS}$	$0V_T$
$Z^{\prime(2)}$	2^{-+}	$+V_{LS}$	$-\frac{4}{5}V_T$
$X_2^{(3)}$	3^{--}	$+2V_{LS}$	$-\frac{8}{5}V_T$

The only $\Sigma_g^+(1P)$ states degenerate in J^{PC} are the 1^{--} pair $X_2^{(1)}$ and $X_0^{\prime(1)}$. In that case, for $V_T \neq 0$, the states form a 2×2 mass matrix whose diagonal values are given in Table I and whose off-diagonal element is

$$\Delta M_{X_2^{(1)}-X_0^{\prime(1)}} = +\frac{8}{\sqrt{5}}V_T. \tag{14}$$

IV. NUMERICAL ANALYSIS

LHCb analyzes the results of their observations [1] by providing fits to two model scenarios which are as follows:

(I) $X(6900)$ has $m = 6905 \pm 11$ MeV and $\Gamma = 80 \pm 19$ MeV The second resonance, hereinafter labeled $X(6500)$, lies at 6490 ± 15 MeV.[7] The mass splitting between these states is $\Delta m_{\mathrm{I}} = 415 \pm 19$ MeV.

(II) $X(6900)$ has $m = 6886 \pm 11$ MeV and $\Gamma = 168 \pm 33$ MeV The second resonance, hereinafter labeled $X(6740)$, has $m = 6741 \pm 6$ MeV and $\Gamma = 288 \pm 16$ MeV The mass splitting between these states is $\Delta m_{\mathrm{II}} = 145 \pm 15$ MeV.

We now show that the scenario of Model II appears to support a much more favorable interpretation within the dynamical diquark model.

For this analysis, we first assume that $X(6900)$ is not a $1S$ state, because it would then lie 700 MeV above the J/ψ-pair threshold, which would represent an astonishing mass gap for the appearance of the lowest $c\bar{c}c\bar{c}$ resonances. Similar conclusions appear in Refs. [20–26]. We discuss the fate of the $1S$ states in our model later in this section; the subsequent multiplets in order of increasing mass turn out to be $1P$, $2S$, $1D$, $2P$, and $2D$, as confirmed below.

[6] In their full calculation, Ref. [20] also includes color-sextet combinations.

[7] This value is not stated in Ref. [1], but rather is estimated by us using their Fig. 3(b).

The next required input of the analysis is a reliable value of the internal diquark spin-spin coupling κ_{cc} appearing in Eqs. (4)–(6). The closest available analogue to $c\bar{c}c\bar{c}$ state is found with $c\bar{c}s\bar{s}$ candidates such as $X(4140)$, which have been analyzed using this model very recently in Ref. [41]. In that work, κ_{cs} is found to be quite large (114.2 MeV) compared to the fit value for κ_{cq} or κ_{bq} (17.9–22.5 MeV). We observed in Ref. [41] that this pattern is explained by the diquark coupling being strongly dependent upon the lighter quark flavor (κ_{cs} vs κ_{cq}) and much less sensitive to the heavy-quark flavor (κ_{cq} vs κ_{bq}). We argued that the s quark, being much heavier than u or d, has less Fermi motion within δ, permitting δ to be substantially more compact and thus enhancing the strength of spin couplings within it. Therefore, it is reasonable to assume that the (cc) diquark has a similarly large spin-spin coupling (and possibly even larger, if s is insufficiently heavy to reach the point of flavor independence for the lighter quark in δ). Hence, for all states in this fit, we take the spin-spin coupling to be

$$\kappa_{cc} = 114.2 \text{ MeV}. \tag{15}$$

Note from Eq. (6) or Table I that such a large value of κ_{cc} leads to the interesting consequence of predicting M_0, and hence the diquark mass m_δ, to be rather smaller than in fits from other works.

We now possess sufficient information to study S-wave multiplet masses, as well as P-wave multiplet masses ignoring for the moment the spin-orbit and tensor terms. Two natural assignments for $X(6900)$ may be considered: as a $\Sigma_g^+(1P)$ or as a $\Sigma_g^+(2S)$ state. One then calculates for each case the mass splittings to lower multiplets, in order to confirm whether one or both of these assignments matches the mass splittings $\Delta m_{\rm I}$ and/or $\Delta m_{\rm II}$ between peaks from LHCb's Model I or II, respectively.

First, we investigate the possibility that $X(6900)$ is a $\Sigma_g^+(1P)$ state. Since the J/ψ pair has $C = +$, Table I suggests that the lightest allowed candidate (assuming $V_{LS}, V_T > 0$, as is used below) is $Z'^{(0)}(0^{-+})$. To be quantitative, we adopt the numerical results obtained from the P-wave $c\bar{c}q\bar{q}$ states in Ref. [40]. Specifically, we use values obtained from Cases 3 and 5 of Ref. [40] for V_{LS} and V_T, which are

$$V_{LS} = 42.9 \text{ MeV}, \qquad V_T = 5.5 \text{ MeV} \tag{16}$$

and

$$V_{LS} = 49.0 \text{ MeV}, \qquad V_T = 3.8 \text{ MeV}, \tag{17}$$

respectively. These cases were deemed in Ref. [40] to be the ones most likely to accurately represent the true P-wave $c\bar{c}q\bar{q}'$ spectrum. Their application to the $c\bar{c}c\bar{c}$ system deserves some discussion. The spin-orbit term in this model connects two separated heavy diquarks in either case [(cq) or (cc)], and therefore we assume the size of the coupling V_{LS} to depend upon the source only through its spin and not its flavor content, so long as the diquarks are heavy. The tensor term, on the other hand, is an entirely different matter. In Ref. [40], the tensor operator was chosen to couple only to light-quark spins [see the discussion below Eq. (10)], while the $c\bar{c}q\bar{q}'$ analogue to the form of Eq. (9) used here for $c\bar{c}c\bar{c}$ was found to be phenomenologically irrelevant. We therefore take as our final assumption that V_T for $c\bar{c}c\bar{c}$ is numerically no larger than the V_T values obtained from $c\bar{c}q\bar{q}'$.

Using the values for κ_{cc}, V_{LS}, V_T in Eqs. (15)–(17), one then needs only the mass expressions in Table I and Eqs. (6) and (14). Fixing the $Z'^{(0)}$ mass eigenvalue to the (Model I) $X(6900)$ mass, we implement the Schrödinger equation-solving numerical techniques applied to lattice-calculated potentials, as described in Ref. [37]. We thus obtain

$$M_0(1P) = 6931.3 \text{ MeV} \quad \text{and} \quad 6954.0 \text{ MeV}, \tag{18}$$

using the inputs of Eqs. (16) and (17), respectively.[8] Further computing $M_0(1S)$ and $M_0(2S)$ in the same calculation, we obtain the M_0 mass differences,

$$\begin{aligned} \Delta m_{1P-1S} &= +343.3 \text{ MeV}, \\ \Delta m_{1P-2S} &= -156.9 \text{ MeV}, \end{aligned} \tag{19}$$

using Eq. (16). The corresponding values obtained using Eq. (17) are hardly changed, being $+343.2$ and -156.7 MeV, respectively. In comparison with the LHCb results, the first of Eq. (19) is too small to match Model I (i.e., $\Delta m_{1P-1S} < \Delta m_{\rm I}$), especially since $M_0(1P)$ lies rather higher than the $Z'^{(0)}$ mass we fix to $X(6900)$, while the second has the right magnitude but the wrong sign to match Model II (i.e., $\Delta m_{\rm II} \approx -\Delta m_{1P-2S}$), since we predict that $2S$ states lie above $1P$ states. We therefore conclude that the assignment of $X(6900)$ as a $\Sigma_g^+(1P)$ state is heavily disfavored in the dynamical diquark model.

We therefore turn to the alternate possibility that $X(6900)$ is one of the states in the multiplet $\Sigma_g^+(2S)$ (which again, are degenerate in this model). Then using Eqs. (6), (15), and the Model-II mass value, we obtain

$$M_0(2S) = 6771.8 \text{ MeV}. \tag{20}$$

Once again implementing the techniques developed in Ref. [37], we calculate the M_0 mass differences.

[8]The variation of these particular eigenvalues with the lattice potentials obtained in Refs. [32–36] amounts to only about 0.07 MeV. The specific values presented here use Ref. [35].

$$\Delta m_{2S-1P} = 160.4 \text{ MeV},$$
$$\Delta m_{2S-1S} = 505.7 \text{ MeV}. \quad (21)$$

In this case, we observe that the latter mass splitting is too large to agree with Model I (i.e., $\Delta m_{2S-1S} > \Delta m_{\mathrm{I}}$), but the former agrees very well with Model II (i.e., $\Delta m_{2S-1P} \approx \Delta m_{\mathrm{II}}$). Therefore, assuming that LHCb's Model II is confirmed to be the correct interpretation of the data, we find that $X(6900)$ is favored in the dynamical diquark model to be a $\Sigma_g^+(2S)$ state and $X(6740)$ a $\Sigma_g^+(1P)$ state.

Concluding from these calculations that $X(6900)$ is indeed a $\Sigma_g^+(2S)$ state with $M_0(2S)$ given by Eq. (20), the corresponding diquark masses are computed to be

$$m_\delta = m_{\bar{\delta}} = 3126.4 - 3146.4 \text{ MeV}, \quad (22)$$

which is only slightly larger than $m_{J/\psi}$. Using this value of m_δ, we further obtain

$$M_0(1S) = 6264.0 - 6266.1 \text{ MeV},$$
$$M_0(1P) = 6611.4 \text{ MeV},$$
$$M_0(1D) = 6860.5 - 6862.4 \text{ MeV},$$
$$M_0(2P) = 7010.8 - 7013.0 \text{ MeV}. \quad (23)$$

The variation here arises from using the differing lattice results of Refs. [32–36]. The prediction for $M_0(1S)$ deserves special discussion, because the expected spatial size of a $1S$ state according to this model is calculated to be $\langle r \rangle \approx 0.3$ fm, the same magnitude as (or even smaller than) J/ψ states. In this scenario, all four of the quarks have comparable spatial separation, a configuration that runs afoul of the original separated-diquark motivation of the dynamical diquark model. At present, the LHCb data in the ~6300 MeV mass region are not yet sufficiently resolved to discern particular structures, so it will be interesting to see how well the model works even in situations for which it is expected to fail.

Having identified $X(6900)$ with one of the (degenerate) $\Sigma_g^+(2S)$ states, we use the values of V_{LS} and V_T given by Eqs. (16) and (17) and the expressions in Table I and Eq. (14) to compute the full $\Sigma_g^+(1P)$ spectrum. The results are presented in Table II. One notes that the variation in mass for any given state between the two fits [excepting $X_2^{(2)}(2^{--})$] is $\lesssim 13$ MeV, and that the ordering of the states in mass is nearly identical to the one expected parametrically from the equal-spacing rule identified in Table I, even though the equal spacing itself is numerically not so well supported. Since the values of V_T in Eqs. (16) and (17) are based upon a naive assumption, the equal-spacing rule might turn out to be much better in practice if the actual V_T value is smaller.

An interesting feature of LHCb Model II is the enormous width $\Gamma = 288$ MeV given for $X(6740)$ (twice the width of ρ, for example). From Table II, we note that all P-wave states that could decay to a J/ψ pair ($C = +$) have masses consistent with appearing within this wide peak, meaning that the broad $X(6740)$ peak could easily turn out to be a superposition of several narrower $1P$-state peaks.

TABLE II. Mass eigenvalues (in MeV) of the 7 $\Sigma_g^+(1P)$ states, using the expressions given in Table I and Eq. (14). $M_0(1P)$ is obtained from the same numerical fit identifying $X(6900)$ as a $\Sigma_g^+(2S)$ state (specifically, using the lattice simulation of Ref. [35]), κ_{cc} is given in Eq. (15), and the columns represent two different choices for V_{LS} and V_T values.

State	J^{PC}	Equation (16)	Equation (17)
$X_2^{(1)}$	1^{--}	6563.70	6556.22
$Z'^{(0)}$	0^{-+}	6595.79	6597.19
$Z'^{(1)}$	1^{-+}	6704.69	6691.79
$X_2^{(2)}$	2^{--}	6713.49	6687.87
$X_0'^{(1)}$	1^{--}	6727.98	6726.68
$Z'^{(2)}$	2^{-+}	6764.09	6771.55
$X_2^{(3)}$	3^{--}	6802.59	6817.51

Finally, a notable enhancement in the LHCb data appears slightly above 7200 MeV. This value coincides with the Ξ_{cc}-$\bar{\Xi}_{cc}$ threshold 7242.4 MeV, at which sufficient energy becomes available to create the lightest hadronic state containing both $c\bar{c}c\bar{c}$ and an additional light $q\bar{q}$ valence pair, namely, the baryon pair $(ccq)(\bar{c}\,\bar{c}\,\bar{q})$. Above this threshold, one expects no further narrow resonances decaying dominantly to J/ψ pairs, since new open-flavor decay channels become kinematically available. This prediction is particularly easy to see in the dynamical diquark model; it is the point at which the gluon flux tube connecting the δ-$\bar{\delta}$ pair gains enough energy to fragment through $q\bar{q}$ pair creation and was anticipated in Ref. [29] for $c\bar{c}q\bar{q}$ states to occur at the Λ_c^+-$\bar{\Lambda}_c^-$ threshold. Interestingly, we find the $2D$ states to have a common multiplet mass of

$$M_0(2D) = 7213.3 - 7216.7 \text{ MeV}, \quad (24)$$

meaning that the enhancement in the data above 7200 MeV may be a combination of some $2P$ and/or $2D$ $c\bar{c}c\bar{c}$ states [not forgetting the large mass offset due to κ_{cc} from Eqs. (13) and (15)] with threshold effects in the form of rescattering of Ξ_{cc}-$\bar{\Xi}_{cc}$ pairs to J/ψ pairs. In addition, the $c\bar{c}c\bar{c}$ states in higher BO multiplets than Σ_g^+ (i.e., analogues to hybrid mesons) would also occur at or above the Ξ_{cc}-$\bar{\Xi}_{cc}$ threshold.

V. CONCLUSIONS

The recent LHCb discovery of resonancelike structures in the J/ψ-pair spectrum opens a whole new arena for hadronic spectroscopy. The $X(6900)$ represents the first

clear candidate for a multiquark exotic hadron that contains only heavy valence quarks. This paper and multiple prior works referenced here suggest that numerous other such states, carrying a variety of quantum numbers, await discovery as experimental observations are refined. Furthermore, the all-heavy sector is particularly interesting from a theoretical point of view, since the molecular binding paradigm popular for light-flavor containing multiquark states like $X(3872)$ is much less viable (particularly for states that lie so far above the J/ψ-pair threshold), leaving a diquark-antidiquark binding structure as the leading candidate.

This paper has explored the basic spectroscopic properties of the all-heavy four-quark states $Q_1\bar{Q}_2Q_3\bar{Q}_4$ in the dynamical diquark model. Its defining features for this system are (1) the dominance of the color-triplet binding between $\delta \equiv Q_1Q_3$ and between $\bar{\delta} \equiv \bar{Q}_2\bar{Q}_4$, which for the identical-quark cases $c\bar{c}c\bar{c}$ or $b\bar{b}b\bar{b}$ leads to the absence of 1^{++} S-wave states; (2) the dominance of spin-spin couplings within δ and within $\bar{\delta}$, but not between quarks and antiquarks, which leads to the degeneracy of all three states in each $Q\bar{Q}Q\bar{Q}$ S-wave multiplet; and (3) a spin-orbit coupling for $L > 0$ that couples to all quarks with the same strength. If the strength of the tensor coupling is substantially smaller than the spin-orbit coupling, then the seven states of the P-wave $Q\bar{Q}Q\bar{Q}$ multiplet exhibit a remarkable equal-spacing spectrum. These features clearly provide simple and immediate tests of various aspects of the model.

We have also produced numerical predictions of the full spectrum for the $1S$, $1P$, and $2S$ multiplets, and multiplet-averaged masses for $1D$, $2P$, and $2D$, using lattice-calculated confining potentials, the spin-spin coupling obtained from $c\bar{c}s\bar{s}$ candidate states, and the spin-orbit and tensor couplings obtained from P-wave $c\bar{c}q\bar{q}'$ states, all using this model. In attempting different assignments for the $X(6900)$, we find that the only one compatible with the model is to identify $X(6900)$ with a state or states within the $2S$ multiplet, and the lower structure at about 6740 MeV from LHCb's "Model II" being some combination of the $C = +$ states within the $1P$ multiplet. Evidence for the $1S$ multiplet is obscure, possibly because it is predicted to occur at masses at which the δ-$\bar{\delta}$ structure is no longer viable, since all interquark distances become comparable not far above the J/ψ-pair threshold, while $1D$ states could easily be obscured by the large $X(6900)$ peak, and some $2P$ and $2D$ states are predicted to lie at or above the Ξ_{cc}-$\bar{\Xi}_{cc}$ threshold (which coincides with a structure in the LHCb results), at which point the $c\bar{c}c\bar{c}$ states are expected to become much wider.

The resolution of the newly observed J/ψ-pair structures (possibly into several peaks) and the measurement of specific J^{PC} quantum numbers will contribute immeasurably to an understanding of the structure of these states. Future studies of other charmonium-pair structures (including χ_c, h_c, and η_c) will be no less valuable in this regard.

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