On the Existence of Pareto Efficient and Envy-Free Allocations *†

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September 3, 2020

Abstract

Envy-freeness and Pareto Efficiency are two major goals in welfare economics. The existence of an allocation that satisfies both conditions has been studied for a long time. Whether items are indivisible or divisible, it is impossible to achieve envy-freeness and Pareto Efficiency ex post even in the case of two people and two items. In contrast, in this work, we prove that, for any cardinal utility functions (including complementary utilities for example) and for any number of items and players, there always exists an ex ante mixed allocation which is envy-free and Pareto Efficient, assuming the allowable assignments satisfy an anonymity property. The problem remains open in the divisible case.

We also investigate the communication complexity for finding a Pareto Efficient and envy-free allocation.

Keywords: Pareto Efficient, envy-free, fair allocation, communication complexity

Classification codes: D630; D610

1 Introduction

How can one partition a collection of indivisible goods so as to achieve both efficiency and fairness, when the recipients may have non-additive utilities,

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and further the possible allocations may be restricted? By efficiency we mean Pareto Efficiency and by fairness we intend envy-freeness. We first note that there is no deterministic allocation achieving both Pareto Efficiency and envy-freeness, as can be seen by considering the case of allocating one item among two players, one of whom must end up being envious.

Instead, we allow for randomization in the allocations, and measure the outcome in terms of expected, i.e. Von Neumann–Morgenstern utility. Of course, randomness is commonly used in many resource allocation settings. These include school settings, when it is used to break ties for places in over-demanded schools; at universities, where it is used to provide an order or ranking by which students choose over-demanded goods including university housing and seats in courses; it is used for assigning potential bads such as jury service and military call ups. Randomization is used in these settings because monetary transfers are considered undesirable and because the goods are indivisible.

Hylland and Zeckhauser [17] solved this problem in the case that the legal allocations are matchings. Their solution was based on computing a CEEI equilibrium to allocate fractional shares of the goods, followed by a suitable randomized rounding procedure that maintained expected utilities. A subsequent generalization by Budish et al. [8] considered the allocation of multiple goods subject to restrictions as would be needed for example in course assignment, where the restrictions correspond to bounds on the number of courses per student, the room capacities, etc. They showed that only certain types of restrictions could be managed. In addition, Budish et al. [7] gave an implementation of a different scheme for course scheduling based on an approximate CEEI notion.

Hylland and Zeckhauser's solution determines item prices. This does not suffice in general, as was noted in [8] and as we illustrate in Example 1.1 below. However, a pricing solution does exist for this example if one uses bundle prices. Bundle prices tend to be harder to compute, and in what circumstances they exist is not clear. Our second example below introduces restrictions on the allowable allocations, and shows that with these restrictions there need not be a bundle pricing either. But in both cases Pareto Efficient and envy-free allocations exist.

Example 1.1. *The setting has* 3 *items,* $\{A, B, C\}$*, and* 4 *players,* $\{1, 2, 3, 4\}$*.*

The utility for each player is as follows. For any subset $T \in \{A, B, C\}$,

$$u_1(T) = \begin{cases} 1 & \textit{if } \{A, B\} \subseteq T \\ 0 & \textit{otherwise,} \end{cases}$$

$$u_2(T) = \begin{cases} 1 & \textit{if } \{B, C\} \subseteq T \\ 0 & \textit{otherwise,} \end{cases}$$

$$u_3(T) = \begin{cases} 1 & \textit{if } \{A, C\} \subseteq T \\ 0 & \textit{otherwise,} \end{cases}$$

$$u_4(T) = |T|.$$

There are many Pareto Efficient and envy-free allocation. For example, Player 1 receives $\{A, B\}$ with probability $\frac{1}{3}$, Player 2 receives $\{B, C\}$ with probability $\frac{1}{3}$, Player 3 receives $\{A, C\}$ with probability $\frac{1}{3}$, and in each case Player 4 receives the remaining item. Another example has Player 1 receiving $\{A, B\}$ with probability $\frac{1}{2}$, Player 2 receiving $\{B, C\}$ with probability $\frac{1}{2}$, Player 3 receiving nothing, and Player 4 receiving the remaining item in each case. [1]

However there is no CEEI allocation based on item pricing. To see this, suppose each player comes with 1 unit of money. We start by considering equilibria in which the price of a bundle of items is just the sum of the prices of the individual items. Then, as we show in Lemma A.1 in the appendix, the only market equilibrium sets the prices of the items to all be equal to $\frac{4}{3}$, and allocates bundle $\{A, B\}$ to Player 1, bundle $\{B, C\}$ to Player 2, bundle $\{C, A\}$ to Player 3, each with $\frac{3}{8}$ probability, and everything else to Player 4. Therefore, the total amount of items received by Player 4, in expectation, is $\frac{3}{4}$.

Next, we show there does not exist a mixed allocation corresponding to this distribution of bundles. This is because in this distribution, all items are fully allocated to players and each of Players 1, 2, and 3 receives their favorite bundle with probability $\frac{3}{8}$ and nothing else. However, in any deterministic allocation, if all items are fully allocated to players, Player 4 will receive at least 1 item when the other three players receive either their preferred bundle or the empty set. Consequently, in a mixed allocation, which is a combination of deterministic allocations, Player 4 will receive at least 1 item in expectation, which contradicts the fact that Player 4 receives $\frac{3}{4}$ items in expectation in the CEEI allocation.

[1] The second example allocation was suggested to us by an anonymous referee.

But if bundle pricing is allowed, the above allocation is achievable: the individual item prices remain at 1, while each 2-item bundle is priced at 3. Now, however, the bundle can be purchased only by paying the bundle price.[2]

Our construction allows the feasible allocations to be constrained, if we impose an anonymity limitation on the allowable allocations. This arises because our construction, when faced with a candidate allocation in which there is envy, will seek to reduce the envy by performing partial swaps of the agents' individual allocations. For this to be possible, every permutation of a feasible allocation must also be feasible, and this is what we mean by anonymity.

It is helpful to specify allocations in terms of partitions of the set of items. Suppose there are n players. Then a partition is a particular division of the items into n disjoint bundles, but without an allocation of the bundles to the players. An allocation based on a partition $\mathcal{P}$ is a distribution of the n bundles in $\mathcal{P}$ to the n players. Because of the anonymity limitation, as there are n players, a single partition yields $n!$ possible allocations.

The next example shows that even with partition-independent bundle pricing, if the collection of allowable allocations is constrained, then there may be no pricing supporting a Pareto Efficient and envy-free allocation.

Example 1.2. *There are* 4 *players,* $\{1, 2, 3, 4\}$*,* 5 *items,* $\{A, B, C, D, E\}$*, and two legal partitions:* $\{\{A\}, \{B\}, \{C\}, \{D, E\}\}$ *and* $\{\{A\}, \{B\}, \{C, D\}, \{E\}\}$*. Players* 1 *and* 2 *are interested in items* A *and* B*. Player* 3 *is interested in item* C *or bundle* $\{C, D\}$*. Player* 4 *is interested in bundle* $\{D, E\}$ *or item* E*. The utilities are defined as follows: if the partition is* $\{\{A\}, \{B\}, \{C\}, \{D, E\}\}$*, then*

	$\{A\}$	$\{B\}$	$\{C\}$	$\{D, E\}$
Player 1	14	6	0	0
Player 2	11	9	0	0
Player 3	0	0	10	0
Player 4	0	0	0	10

and if the partition is $\{\{A\}, \{B\}, \{C, D\}, \{E\}\}$*, then*

[2] This solution was suggested to us by an anonymous referee.

	$\{A\}$	$\{B\}$	$\{C, D\}$	$\{E\}$
Player 1	9	11	0	0
Player 2	4	16	0	0
Player 3	0	0	10	0
Player 4	0	0	0	10

The role of Players 3 and 4 is to create partitions in which the utilities of each of Players 1 and 2 change. As we show in Claim A in the appendix, in any Pareto Efficient and envy-free allocation, Player 1 always receives item A and Player 2 always receives item B. To achieve envy-freeness, both partitions need to occur with non-zero probability, e.g. each with probability 1/2.

We show in Lemma A.2 in the appendix that there is no partition-independent bundle pricing supporting any Pareto Efficient and envy-free allocation. We now explain in more detail what this means. By partition-independent, we mean that the prices are the same in both partitions. Also, we envisage a price-based allocation of the following form. The solution specifies prices and allocations, and thus implicitly also specifies the probabilities p_1 *and* $p_2 = 1 - p_1$ *of each of the two partitions. Also each player receives an allocation that maximizes its utitility over all the allocations the player could obtain, given these partition probabilities. More precisely, Player* i *has a probability* p_{1ij} *of obtaining bundle* j *in partition 1 and a probability* p_{2ij} *of obtaining bundle* j *in partition 2. These probabilities satisfy* $\sum_j p_{1ij} = p_1$ *and* $\sum_j p_{2ij} = p_2$*; in addition, the expected cost of Player* i*'s bundle will be at most 1. These are the only constraints on the bundles Player* i *could receive and hence on its maximum possible utility.*

However, if we were to allow distinct prices in the different allocations, then we could provide a pricing supporting a Pareto Efficient and envy-free allocation.

In contrast, we show that for any *partition-based* utility functions, including negative-valued utilities, one can obtain a Pareto Efficient and envy-free allocation. Here, partition-based utility functions mean the players' utilities can depend on both the allocations they receive and the whole partition. In addition, we can impose restrictions on the allocations so long as they remain anonymous.

Our solution works by constructing a mapping from the space of mixed allocations and weight vectors to itself. We then apply the Kakutani fixed-point theorem [19] to obtain a fixed point. Finally, we prove that the fixed point corresponds to a mixed Pareto Efficient and envy-free allocation. The proof is inspired by [34, 2, 38].

We conclude the paper by asking how readily one can calculate a Pareto Efficient and envy-free allocation, in terms of the unavoidable communication cost, referred to as the communication complexity of the problem. In particular, we show that even with two players, if their utility functions are submodular, in general, calculating such an allocation will require $\Omega(2^{\frac{m}{2}})$ bits to be communicated between the players, where m is the number of items to be allocated, which rapidly becomes infeasible as m increases.

2 Related Work

A detailed survey on fairness and further background can be found in [4, 5, 27].

Research on fair allocation research dates back to at least [33]. A fair allocation is defined as a Pareto Efficient allocation in which everyone prefers their own bundle to other players' bundles, which is exactly the notion of envy-freeness proposed in [13].

The existence of Pareto Efficient and envy-free allocations has been studied in both the divisible and indivisible cases.

When items are divisible, previous work showed that Pareto Efficient and envy-free allocations exist under a variety of assumptions, including that utility functions are strongly monotone [36, 34, 11], continuously differentiable [35], or convex [37]. In contrast, Vohra [37] showed that when the economy has increasing-marginal-returns, there exist cases such that no Pareto Efficient and envy-free allocation exists. Also, Maniquet [25] gave an example with two items and three players for which there is no Pareto Efficient and envy-free allocation.

In the indivisible setting, for the case of mixed allocations, Bogomolnaia and Moulin [3] introduced the Probabilistic Serial mechanism and showed this new mechanism results in an *ordinally efficient* expected matching which is envy-free in their setting. Ordinal efficiency is a notion which is slightly weaker than Pareto Efficiency. Budish et al. [8] gave a Pareto Efficient and envy-free allocation when the allocation constraints satisfy a bihierarchy assumption which applies to multi-item allocation problems with possibly non-linear utility functions.

For the deterministic case, because of the simple counter-example mentioned above, researchers have proposed many other notions of fairness. The two most closely related notions are *EF1* (*envy-free up to one good*) [7] and

EFX (*envy-free up to any good*) [9]. Recall that the idea in the definition of envy-freeness is that each player will compare their bundle to those of the other players. These alternate notions also have players compare their bundle to the other players' bundles, but in *EF1*, players delete their favorite item from the other bundle before doing the comparison, and in *EFX*, players will not envy another bundle after deleting their least favorite item. Lipton et al. [24] showed that an *EF1* allocation always exists. For the *EFX* allocation, Plaut and Roughgarden [29] showed that in some situations (utility functions are identical or additive) existence is guaranteed, while for general utility functions, there exist examples such that no *EFX* allocation is Pareto Efficient.

In addition, Dickerson et al. [12] showed that if the number of items is at least a logarithmic factor larger than the number of players, then with high probability, an envy-free allocation exists.

Recently, Richter and Rubinstein [32] introduced the Normative Equilibrium. They considered when deterministic Pareto Efficient and envy-free allocations exist in this setting.

Other fairness notions include maximizing the product of individual utilities [21, 26] which has been variously called Proportional Fairness, Nash Social Welfare and Maximum Nash Welfare, max-min fairness [18, 22], and CEEI [36].

Communication complexity provides a lower bound on the cost of computing game theoretic solutions. The first such study, by Nisan and Segal [28], analyzed the cost of maximizing social welfare and the supporting prices for indivisible goods, and showed that it could be exponential in the number of goods. Hart and Mansour [16] showed exponential lower bounds on the cost of finding Nash Equilibria for n-player games. In another early work, Conitzer and Sandholm [10] analyzed the cost of various solution concepts, including Nash Equilibria, for 2-player games. Other studies considered the costs of auctions [15], of approximate 2-player Nash Equilibria [14, 1], and of cake cutting [6]. Recently, Plaut and Roughgarden [30] looked at the communication complexity of finding each of a deterministic envy-free and a proportional division with indivisible goods, identifying settings with respectively polynomial and exponential costs as a function of the number of goods.

3 Notation, Results and Examples

There are m items, n players, and T feasible partitions of the set of items: $\{D_t\}_{t=1,\cdots,T}$ (in Example 1.2 there are two feasible partitions). Each partition consists of n disjoint subsets of items: $D_t = \{D_{t1}, D_{t2}, \cdots, D_{tn}\}$, such that for all $b \neq c$, $D_{tb} \cap D_{tc} = \emptyset$ and $\cup_c D_{tc} \subseteq \{1, 2, \cdots, m\}$.[3] Each partition can be distributed to the players in any way, so long as each agent gets a distinct single element of the partition. Thus each partition yields $n!$ possible allocations, and consequently there are $k = T \cdot n!$ feasible allocations in total. We let $A^{(1)}, A^{(2)}, \cdots, A^{(k)}$ denote these allocations.

Given a feasible partition D_t, a player i who receives element D_{tc} of the partition obtains utility $u_i(D_t; D_{tc})$, which can be any real number. We note that the utility can depend on the partition, but not on the allocations of the other players. We also note that this allows utilities which are negative or altruistic. We call this genre of utilities *partition-based utilities*.

In order to specify utilities when the allocations of two players are swapped, we introduce the following notation. Suppose allocation $A^{(j)}$ is produced by D_t; we let $A_h^{(j)}$ denote the element of D_t allocated to player h. With a slight abuse of notation, we define $u_i(A^{(j)}; i) \triangleq u_i(D_t; A_i^{(j)})$ and $u_i(A^{(j)}; h) \triangleq u_i(D_t; A_h^{(j)})$.

We define a mixed allocation to be a probability distribution on the possible allocations: $\mathbf{p} = (p_1, p_2, \cdots, p_k) \in P$ such that $\sum_j p_j = 1$ and $p_j \geq 0$ for any j. Given $\mathbf{p}$, player i's expected utility is $\sum_j p_j u_i(A^{(j)}; i)$.

A mixed allocation $\mathbf{p}$ is *Pareto Efficient* (PE) if there is no other mixed allocation p' such that for all i, $\sum_j p'_j u_i(A^{(j)}; i) \geq \sum_j p_j u_i(A^{(j)}; i)$ and there exist at least one i for which this inequality is strict.

A mixed allocation $\mathbf{p}$ is *envy-free* (EF) if for every pair i and h of players, $\sum_j p_j u_i(A^{(j)}; i) \geq \sum_j p_j u_i(A^{(j)}; h)$.

Now, we present our main theorem.

Theorem 3.1. *For any set of partitions and any partition-based utility functions, there always exists a Pareto Efficient and envy-free mixed allocation.*

The mixed allocation in Theorem 3.1 creates n mixed bundles. One way to understand envy-freeness is that each player gets a favorite bundle.

[3] The union of all subsets in one partition need not be the full set.

3.1 Examples

We give some examples of settings in which a Pareto Efficient and envy-free allocation is desired.

Example 3.2 (Scheduling Delivery Volunteers). *As described in [23], the foodbanks in the Pittsburgh area need to collect food gifts from supermarkets and other donors for delivery to their warehouses. These deliveries are carried out by volunteers using personal vehicles. The foodbanks use a shared, cooperative system to schedule the volunteers. Volunteers express their preferences via a website and they are then given individual schedules.*

The primary goal is to schedule all deliveries. Assuming a sufficiency of volunteers, one then wants to maximize volunteer satisfaction. It is natural to seek Pareto Optimal and envy-free solutions w.r.t. expected utility, particularly given that volunteers will mostly be repeatedly participating and so their actual utility over time is likely close to their expected utility. We note that in practice, the scheduling system will be implicitly or explicitly using an inferred utility.

In this case, we give a concrete example to explain the terms, items and partitions, used above. Suppose there are two donors, $\{DN_1, DN_2\}$, two warehouses, $\{WH_1, WH_2\}$, and two volunteers. Each donor wants to donate one box of food; each warehouse wants to receive one box of food; and volunteers can deliver the food from any one of the two donors to any one of the two warehouses. In this example, volunteers are the players, and items are the possible deliveries from donors to warehouses, $\{Del_{11}, Del_{12}, Del_{21}, Del_{22}\}$, where Del_{st} represents the delivery from Donor s to Warehouse t. Since we want all the food on the donor side to be delivered to the warehouse side by volunteers, there are 4 partitions: $D_1 = \{\{Del_{11}\}, \{Del_{22}\}\}$, $D_2 = \{\{Del_{12}\}, \{Del_{21}\}\}$, $D_3 = \{\{Del_{11}, Del_{22}\}, \emptyset\}$, and $D_4 = \{\{Del_{12}, Del_{21}\}, \emptyset\}$. Here D_1 represents one volunteer delivering food from Donor 1 to Warehouse 1 and another volunteer delivering food from Donor 2 to Warehouse 2. D_2 is similar to D_1. D_3 represents one volunteer delivering food from Donor 1 to Warehouse 1 and delivering food from Donor 2 to Warehouse 2 and another volunteer doing nothing; D_4 is similar to D_3. Each volunteer will have a utility which is a function of their schedule. Our main result proves that there exists a randomized schedule for the deliveries which is Pareto Efficient and envy-free for all volunteers.

We finish with a brief discussion of some types of utility functions that seem natural. Volunteers may have preferences based on their total travel time (i.e. the time to get from their start point to the relevant donor location, the time for each delivery, and the time from the relevant warehouse to their *final destination). Also, depending on their available time, volunteers may prefer to make a single delivery or to make several deliveries back to back. (One might expect that in real settings there would be considerably more than two donors and more than two warehouses.)*

Our next example shows that we can relax the anonymity constraint at the cost of somewhat limiting the envy-free condition.

Example 3.3 (Allocating Places in Universities)**.** *This example is intended to capture the scenario in which there are two populations, men and women, and each has a strong preference that there be a sufficiently large number of the population of the other sex, say that they constitute at least 45% of the total.*

Let's suppose there are two universities, U_a and U_b, each with 1000 places (this is just for concreteness; we could have different numbers of places in the two universities). Suppose there are m men and w women seeking these places. We suppose that $m + w \leq 2000$ (we discuss later how to handle the case $m + w > 2000$).

Suppose each student's utility for a place in each university is a function of (i) the number of men at the university, (ii) the number of women at the university, and (iii) the university. Note that these utility functions are not partitioned-based in general: for a single student's utility for the seat they have been allocated may change on permuting the seats allocated to other students, as this can alter the numbers of men and women in each university. Note that here there is a single partition, comprising the partitioning of the set of 2000 places into 2000 subsets each comprising one place; if the utility function were partition-based, the utility of one student for a given place would be independent of the allocations of the other students.

We can show that there is a Pareto Efficient allocation in which there is no envy among the men and no envy among the women, but a man could envy a woman and a women could envy a man. This is achieved by means of a reduction to an anonymous setting, which is a larger game to which our game is mapped in a way that preserves allocations.

The larger game has the following form. There are two universities A' and B'. Now there are 2000 places in each university, of which 1000 are men-places and 1000 are women-places. We have the same m men and w women as before, and intuitively they have the same utilities as before — more on this in a moment. We introduce two more players, H_a and H_b (H is short for Huge). H_a desires one subset of $1000 + y$ places in university U_a, with $0 \leq y \leq 1000$, and all such subsets are equally attractive; similarly, H_b desires one subset of $1000 + z$ places in university U_b, $0 \leq z \leq 1000$. The *subset taken by H_a determines a partition of university U_a's places into a mix of x men-places and $1000 - x - y$ women-places, plus a set of the remaining $1000 + y$ places. In addition, each allowable partition must include subsets of the university U_a and university U_b places so that between them they leave exactly m men-places and w women-places; i.e. $m + w = 1000 - y - z$ and there are m men-places and w women-places available. As each distinct pair of subsets specifies a distinct partition, this construction allows the students to have utilities that are a function of the male and female populations at their university while achieving anonymity in the larger game.*

We extend the utility functions for the men to have large negative values, $-r$ say, on women-places and on each large subset, and similarly for the women. Finally, we set the utilities of H_a and H_b to be very negative except on their desired subsets, where their utilities are set to 1.

In Lemma A.4 in the appendix we show that all Pareto Efficient and envy-free allocations in the larger game have all men allocated to men-places and women to women-places. Consequently, there is a 1–1 correspondence between Pareto Efficient and envy-free allocations in the two games.

To handle the case the $m + w > 2000$ we introduce a third university in the larger game, corresponding to not being assigned a place in the two actual universities; we then also need a third player H_c. We leave the details to the interested reader. Also, this example is readily extended to having more than two groups of agents, with utilities that can depend on the group sizes, but without the ability to exclude cross-group envy.

As we shall see, our construction requires all utilities to be strictly positive; however, it is straightforward to extend it to allow negative utilities: simply add a sufficiently large constant to each original utility so that the resulting utilities are all strictly positive. Note that if the empty set could be allocated, then it now has a strictly positive utility. Thus our construction can handle the allocation of bads.

Example 3.4 (Allocating Bads)**.** *Some or all players have negative utility; examples includes jury duty assignments and military service.*

3.2 Communication Complexity

Next, we will look at the communication complexity of finding a Pareto Efficient and envy-free allocation. For this result, we will assume that the set of partitions includes all possible partitions of the items and players' utilities depend only on the bundle they receive. Accordingly, for simplicity, in the communication setting, we let $u_i(S)$ denote player i's utility when receiving bundle S.

The setting We assume that the players only know their own utility functions; they have no knowledge of others' utility functions. The problem is to determine the minimum number of bits they need to communicate with each other in order to calculate a Pareto Efficient and envy-free allocation, which means that, after communication and individual calculation, every player will agree on one allocation which is both Pareto Efficient and envy-free.

In this setting, we assume that for every S and i, $u_i(S)$ can be represented with a polynomial number of bits. Given this assumption, one easy conclusion is that there always exists a protocol which uses an exponential number of bits (exponential in the number of items) and outputs an allocation which is Pareto Efficient and envy-free. This is true because there would be no difficulty for all players to calculate one Pareto Efficient and envy-free allocation if each player knew everyone else's utility function. An exponential number of bits of communication suffices to achieve this.

Our question is whether there exists a protocol that needs only a polynomial number of bits. We show the following negative result.

Theorem 3.5. *Any protocol that calculates Pareto Efficient and envy-free allocation for two players with submodular uility functions needs an exponential number of bits.*

The definition of a submodular utility function follows.

Definition 3.6. *$u(\cdot)$ is a submudular function if and only if for any $X \subseteq Y$ and any element e,*

$$u(X \cup \{e\}) - u(X) \geq u(Y \cup \{e\}) - u(Y).$$

4 Proof of Theorem 3.1

WLOG, we assume that $1 \leq u_i(x_i) \leq 2$ for all i and x_i.[4] We will use a fixed point argument. To this end, we construct a correspondence Φ between $P \times W_\epsilon$ and itself. Here, P is the set of mixed allocations and W_ϵ is the set of weighted vectors $\{\mathbf{w} = (w_1, w_2, \cdots, w_n) | \sum_i w_i = 1 \text{ and } w_i \geq \epsilon\}$, where $\epsilon > 0$ is a sufficiently small value that is a function of the utilities. We will

[4] Note that every utility function can be rescaled into $[1, 2]$ without affecting the definition of Pareto Efficiency and envy-freeness.

specify ϵ precisely later. Φ maps $(\mathbf{p}, \mathbf{w})$ to $(\mathcal{P}(\mathbf{w}), \varpi(\mathbf{p}, \mathbf{w}))$, where $\mathcal{P}(\mathbf{w})$ is a subset of P and $\varpi(\mathbf{p}, \mathbf{w}) \in W_\epsilon$.

$$\mathcal{P}(\mathbf{w}) = \left\{\mathbf{p}' \,\middle|\, \mathbf{p}' \in P \text{ and } \mathbf{p}' \in \arg\max \textstyle\sum_i w_i \sum_j p'_j u_i(A^{(j)}; i)\right\};$$

$$\varpi(\mathbf{p}, \mathbf{w}) = \texttt{proj}_{W_\epsilon}(\nu(\mathbf{p}, \mathbf{w})); \tag{1}$$

$$\text{where } \nu_i(\mathbf{p}, \mathbf{w}) = w_i + \frac{\max_h \sum_j p_j u_i(A^{(j)}; h)}{\sum_{i'} \max_h \sum_j p_j u_{i'}(A^{(j)}; h)} - \frac{\sum_j p_j u_i(A^{(j)}; i)}{\sum_{i'} \sum_j p_j u_{i'}(A^{(j)}; i')}. \tag{2}$$

Note that if $\mathbf{p}$ is envy free, then $\max_h \sum_j p_j u_{i'}(A^{(j)}; h) = \sum_j p_j u_{i'}(A^{(j)}; i')$ for all i, and so $\nu_i(\mathbf{p}, \mathbf{w}) = w_i$ for all i.

We first show that a fixed point exists.

Lemma 4.1. *There exists a fixed point,* $(\mathbf{p}^*, \mathbf{w}^*)$*, such that* $\mathbf{p}^* \in \mathcal{P}(\mathbf{w}^*)$ *and* $\varpi(\mathbf{p}^*, \mathbf{w}^*) = \mathbf{w}^*$*.*

It's not hard to see that, for any fixed point $(\mathbf{p}^*, \mathbf{w}^*)$, $\mathbf{p}^*$ is a Pareto Efficient allocation.

Claim 4.2. *If* $(\mathbf{p}^*, \mathbf{w}^*)$ *is a fixed point of* Φ*, then* $\mathbf{p}^*$ *is a Pareto Efficient allocation.*

Proof. Note that the fact that $\mathbf{p}^* \in \mathcal{P}(\mathbf{w}^*)$ means $\mathbf{p}^*$ maximizes $\sum_i w_i^* \sum_j p_j^* u_i(A^{(j)}; i)$. So there cannot be another $\mathbf{p}$ such that for every i, $\sum_j p_j u_i(A^{(j)}; i) \geq \sum_j p_j^* u_i(A^{(j)}; i)$, with the inequality being strict for some i. □

The following two lemmas imply that, for any fixed point $(\mathbf{p}^*, \mathbf{w}^*)$, $\mathbf{p}^*$ is also envy-free.

Lemma 4.3. *If* $(\mathbf{p}^*, \mathbf{w}^*)$ *is a fixed point of* Φ *and* $\nu(\mathbf{p}^*, \mathbf{w}^*) \in W_\epsilon$*, then* $\mathbf{p}^*$ *is an envy-free allocation.*

Lemma 4.4. *If* ϵ *is small enough (the threshold being defined in terms of* u_i *only) and* $(\mathbf{p}^*, \mathbf{w}^*)$ *is a fixed point of* Φ*, then* $\nu(\mathbf{p}^*, \mathbf{w}^*)$ *is in* W_ϵ*.*

With these lemmas, Theorem 3.1 follows readily.

Proof of Theorem 3.1. By Lemma 4.1, the mapping Φ has a fixed point $(\mathbf{p}^*, \mathbf{w}^*)$. By Lemma 4.4, $\nu(\mathbf{p}^*, \mathbf{w}^*) \in W_\epsilon$. The theorem now follows from Claim 4.2 and Lemma 4.3. □

4.1 Proof of Lemma 4.1, the Existence of a Fixed Point

We prove Lemma 4.1 via the following two claims.

Claim 4.5. *Let $\mathcal{A}(\mathbf{w}) = \{j | A^{(j)}$ maximizes $\sum_i w_i u_i(A^{(j)}; i)$ over all allocations$\}$. Then, $\mathcal{P}(\mathbf{w})$ is a simplex on $A(\mathbf{w})$; i.e. for any $\mathbf{p}'' \in \mathcal{P}(\mathbf{w})$, $p''_j > 0$ only if $j \in \mathcal{A}(\mathbf{w})$ (and, of course $\sum_j p''_j = 1$ and $p''_j \geq 0$ for all j):*

$$\mathcal{P}(\mathbf{w}) = \left\{\mathbf{p}'' \,\middle|\, \left(p''_j > 0 \to j \in \mathcal{A}(\mathbf{w})\right) \cap \left(\forall j.\ p''_j \geq 0\right) \cap \left(\textstyle\sum_j p''_j = 1\right)\right\}.$$

Furthermore, $\mathcal{A}(\mathbf{w})$ is non-empty, and $\mathcal{P}(\mathbf{w})$ is non-empty and convex.

Proof. It's not hard to see that in the definition of $\mathcal{P}(\mathbf{w})$, we can rewrite $\sum_i w_i \sum_j p'_j u_i(A^{(j)}; i)$ as $\sum_j p'_j \sum_i w_i u_i(A^{(j)}; i)$. So any probability $p'_j > 0$ on an allocation $A^{(j)}$ that does not maximize $\sum_i w_i u_i(A^{(j)}; i)$ will contradict the definition of $\mathcal{P}(\mathbf{w})$.

$\mathcal{A}(\mathbf{w})$ is non-empty because it is a set of maximizers, and thus $\mathcal{P}(\mathbf{w})$ is also non-empty. Finally, a simplex is a convex set, and so $\mathcal{P}(\mathbf{w})$ is convex. $\square$

We use the Maximum Theorem to prove the next claim.

Theorem 4.6 (Maximum)**.** *Let X and Θ be topological spaces, $f : X \times \Theta \to \mathbb{R}$ be a continuous function on the product $X \times \Theta$, and $C : \Theta \rightrightarrows X$ be a compact-valued correspondence such that $C(\theta) \neq \emptyset$ for all $\theta \in \Theta$. Define the marginal function (or value function) $f^* : \Theta \to \mathbb{R}$ by*

$$f^*(\theta) = \sup\{f(x, \theta) : x \in C(\theta)\}$$

and the set of maximizers $C^* : \Theta \rightrightarrows X$ by

$$C^*(\theta) = \arg\sup\{f(x, \theta) : x \in C(\theta)\} = \{x \in C(\theta) : f(x, \theta) = f^*(\theta)\}.$$

If C is continuous at θ, then $\forall \theta^t \in \Theta$, $\forall x \in X$ and $\forall x^t \in C^*(\theta^t)$

$$\lim_{t\to\infty} \theta^t = \theta, \lim_{t\to\infty} x^t = x \implies x \in C^*(\theta).$$

Claim 4.7. *For any series $(\mathbf{w}^{(t)}, \mathbf{p}^{(t)})$ with $\lim_{t\to\infty} \mathbf{w}^{(t)} = \mathbf{w}$ and $\lim_{t\to\infty} \mathbf{p}^{(t)} = \mathbf{p}$, if for every t, $\mathbf{p}^{(t)} \in \mathcal{P}(\mathbf{w}^{(t)})$, then $\mathbf{p} \in \mathcal{P}(\mathbf{w})$. In other words, $\mathcal{P}(\mathbf{w})$ has a closed graph.*

Proof. Recall that P denotes the entire space of mixed allocations. We set $\Theta = W_\epsilon$, $X = P$, $f(x,\theta) = f(\mathbf{p},\mathbf{w}) = \sum_i w_i \sum_j p_j u_i(A^{(j)}; i)$ and $C(\theta) = C(\mathbf{w}) = P$. Then it's clear that at any point $\mathbf{w} \in W$, C is compact valued and continuous, $C(\mathbf{w}) \neq \emptyset$, and $C^*(\mathbf{w}) = P(\mathbf{w})$. The claim follows on applying the Maximum Theorem. □

Proof of Lemma 4.1. For each $(\mathbf{p},\mathbf{w})$, the mapping $\Phi(\mathbf{p},\mathbf{w}) = (\mathcal{P}(\mathbf{w}), \varpi(\mathbf{p},\mathbf{w}))$ is a convex set as $\mathcal{P}(\mathbf{w})$ is convex by Claim 4.5 and $\varpi(\mathbf{p},\mathbf{w})$ is a single point. It is non-empty as $\mathcal{A}(\mathbf{w})$ is non-empty, and by Claim 4.5 the corresponding $\mathcal{P}(\mathbf{w})$ is non-empty. By Claim 4.7, Φ has a closed graph. Hence Kakutani's fixed point theorem can be applied. □

4.2 Proof of Lemma 4.3: a Sufficient Condition for Envy Freeness

We begin by constructing an *envy graph* (V, E) based on $\mathbf{p}^*$. V is the set of players and there is a directed edge (i, i') from i to i' if and only if i envies i', which means that $\sum_j p^*_j u_i(A^{(j)}; i) < \sum_j p^*_j u_i(A^{(j)}; i')$.

The following claim uses the anonymity property, namely that all permutations of an allocation are feasible and changing the allocations of two players does not alter anyone else's utilities.

Claim 4.8. *Suppose* $\mathbf{p}^*$ *is a Pareto Efficient mixed allocation. The corresponding envy graph is acyclic.*

Proof. If the graph has a cycle, then we can improve everyone's utility functions in this cycle by exchanging the allocations along the cycle, contradicting Pareto Efficiency. □

Given the Pareto Efficient mixed allocation $\mathbf{p}^*$, we define the set of envy-free players to be $\mathcal{F}(\mathbf{p}^*) = \{i \,|\, \textit{player } i \textit{ does not envy player } h \textit{ for all } h\}$.

Claim 4.9. *Suppose* $\mathbf{p}^*$ *is a Pareto Efficient mixed allocation. Then* $\mathcal{F}(\mathbf{p}^*)$ *is not empty.*

Proof. This follows from the fact that the envy graph is acyclic. □

For the following claim, recall (2).

Claim 4.10. *Let* $\mathbf{p}^*$ *be a Pareto Efficient mixed allocation. Then, for any* i *in* $\mathcal{F}(\mathbf{p}^*)$, $\nu_i(\mathbf{p}^*, \mathbf{w}^*) \leq w^*_i$, *or equivalently,*

$$\frac{\max_h \sum_j p^*_j u_i(A^{(j)}; h)}{\sum_{i'} \max_h \sum_j p^*_j u_{i'}(A^{(j)}; h)} \leq \frac{\sum_j p^*_j u_i(A^{(j)}; i)}{\sum_{i'} \sum_j p^*_j u_{i'}(A^{(j)}; i')},$$

and equality holds for at least one $i \in \mathcal{F}(\mathbf{p}^*)$ if and only if $\mathbf{p}^*$ is envy-free.

[34] gives a result similar to Claim 4.10. Here we provide a simple proof for completeness.

Proof of Claim 4.10. The inequality follows from the following two facts:

- for all players i', $\max_h \sum_j p_j^* u_{i'}(A^{(j)}; h) \geq \sum_j p_j^* u_{i'}(A^{(j)}; i')$;
- for any $i \in \mathcal{F}(\mathbf{p}^*)$, $\max_h \sum_j p_j^* u_i(A^{(j)}; h) = \sum_j p_j^* u_i(A^{(j)}; i)$.

Equality holds if and only if for all players, $\max_h \sum_j p_j^* u_{i'}(A^{(j)}; h)$ $= \sum_j p_j^* u_{i'}(A^{(j)}; i')$ and thus no one envies anyone else. $\square$

Proof of Lemma 4.3. It is not hard to see that if $\nu(\mathbf{p}^*, \mathbf{w}^*) \in W_\epsilon$ and $(\mathbf{p}^*, \mathbf{w}^*)$ is a fixed point, then $\mathbf{w}^* = \varpi(\mathbf{p}^*, \mathbf{w}^*) = \nu(\mathbf{p}^*, \mathbf{w}^*)$, and consequently for all i, $\frac{\max_h \sum_j p_j^* u_i(A^{(j)};h)}{\sum_{i'} \max_h \sum_j p_j^* u_{i'}(A^{(j)};h)} = \frac{\sum_j p_j^* u_i(A^{(j)};i)}{\sum_{i'} \sum_j p_j^* u_{i'}(A^{(j)};i')}$. In addition, by Claim 4.2, $\mathbf{p}^*$ is a Pareto Efficient allocation and so by Claim 4.10, $\mathbf{p}^*$ is envy-free. $\square$

4.3 Lemma 4.4: The Sufficient Condition Holds at Every Fixed Point

We will show Lemma 4.4 by a proof by contradication: if $(\mathbf{p}^*, \mathbf{w}^*)$ is a fixed point and $\nu(\mathbf{p}^*, \mathbf{w}^*)$ is not in W_ϵ, then we deduce that $\mathbf{w}^* \neq \varpi(\mathbf{p}^*, \mathbf{w}^*)$, which causes a contradiction. In particular, we will show that there exists an i such that $\varpi_i(\mathbf{p}^*, \mathbf{w}^*) < w_i^*$.

The fact that $\nu(\mathbf{p}^*, \mathbf{w}^*) \notin W_\epsilon$ does not immediately yield a contradiction. Recall that the construction of $\nu(\mathbf{p}^*, \mathbf{w}^*)$ ensures that if p^* is envy-free, then $\nu(\mathbf{p}^*, \mathbf{w}^*) = \mathbf{w}^*$. As $\nu(\mathbf{p}^*, \mathbf{w}^*) \notin W_\epsilon$ implies $\nu(\mathbf{p}^*, \mathbf{w}^*) \neq \mathbf{w}^*$, one immediate conclusion is that p^* is not envy-free. We can argue that there must be a player in the set $\mathcal{F}(\mathbf{p}^*)$, the set of envy-free players, as the envy graph is acyclic. In addition, for this player i, the inequality in Claim 4.10 is strict as $\mathbf{p}^*$ is not envy-free, which implies $\nu_i(\mathbf{p}^*, \mathbf{w}^*) < w_i^*$. However, as $\varpi(\mathbf{p}^*, \mathbf{w}^*) = \mathtt{proj}_W(\nu(\mathbf{p}^*, \mathbf{w}^*))$, conceivably $\varpi_i(\mathbf{p}^*, \mathbf{w}^*) = w_i^*$. In fact, this is not possible. We deduce this conclusion via the following two claims.

Claim 4.11. $\varpi_i(\mathbf{p}^*, \mathbf{w}^*) \leq \max\{\nu_i(\mathbf{p}^*, \mathbf{w}^*), \epsilon\}$ *for all i.*

Claim 4.12. *If ϵ is sufficiently small, then there is a player $i \in \mathcal{F}(\mathbf{p}^*)$ with $w_i^* > \epsilon$.*

As we shall see, we set $\epsilon = \rho^{n+1}/n$, where ρ is defined in Claim 4.13 below.

Proof of Lemma 4.4. If $\nu(\mathbf{p}^*, \mathbf{w}^*)$ is not in W_ϵ, then $\nu(\mathbf{p}^*, \mathbf{w}^*) \neq \mathbf{w}^*$, which implies $\max_h \sum_j p_j^* u_i(A^{(j)}; h) \neq \sum_j p_j^* u_i(A^{(j)}; i)$ for some i. So $\mathbf{p}^*$ is not an envy-free allocation, which implies that for all $i \in \mathcal{F}(\mathbf{p}^*)$,

$$\frac{\max_h \sum_j p_j^* u_i(A^{(j)}; h)}{\sum_{i'} \max_h \sum_j p_j^* u_{i'}(A^{(j)}; h)} < \frac{\sum_j p_j^* u_i(A^{(j)}; i)}{\sum_{i'} \sum_j p_j^* u_{i'}(A^{(j)}; i')}. \tag{3}$$

By Claim 4.12, we know that there is a player $i^* \in \mathcal{F}(\mathbf{p}^*)$ with $w_{i^*}^* > \epsilon$. Therefore, by (3) and (2), $\nu_{i^*}(\mathbf{p}^*, \mathbf{w}^*) < w_{i^*}^*$. By Claim 4.11, $\varpi_{i^*}(\mathbf{p}^*, \mathbf{w}^*) \leq \max\{\nu_{i^*}(\mathbf{p}^*, \mathbf{w}^*), \epsilon\} < w_{i^*}^*$, contradicting $\varpi_{i^*}(\mathbf{p}^*, \mathbf{w}^*) = w_{i^*}^*$. The result follows. □

The proof of Claim 4.11 follows readily from the definition of `proj`.

Proof of Claim 4.11. For simplicity, let $x^* = \varpi(\mathbf{p}^*, \mathbf{w}^*)$ and $y^* = \nu(\mathbf{p}^*, \mathbf{w}^*)$. So we need to show $x_i^* \leq \max\{y_i^*, \epsilon\}$ for all i.

By (1), $x^* = \mathtt{proj}_{W_\epsilon} y^*$ means x^* is the result of the following optimization program: $\min_x \frac{1}{2}\|x - y^*\|^2$ such that $x \in W_\epsilon$. Note that $W_\epsilon = \{(w_1, \cdots, w_n) | \sum_i w_i = 1 \text{ and } w_i \geq \epsilon\}$. So, x^* is the optimal solution for the following convex program:

$$\begin{aligned} &\min_x \frac{1}{2}\|x - y^*\|^2 \\ &\text{s.t. } \textstyle\sum_i x_i = 1 \quad \text{ and for all } i,\ x_i \geq \epsilon \end{aligned}$$

The Lagrange form is $\frac{1}{2}\|x - y^*\|^2 - \lambda(\sum_i x_i - 1) - \sum_i \beta_i(x_i - \epsilon)$. From the KKT conditions, we know that $x_i^* - y_i^* - \lambda - \beta_i = 0$, $\beta_i \geq 0$, $x_i^* \geq \epsilon$, and $\beta_i(x_i^* - \epsilon) = 0$; also $\sum_i x_i^* = 1$. By (2), the definition of $\nu(\mathbf{p}^*, \mathbf{w}^*)$, we know that $\sum_i y_i^* = \sum_i w_i^* = 1$. By summing the KKT condition $x_i^* - y_i^* - \lambda - \beta_i = 0$ over all i, we obtain $\lambda = -\frac{1}{n}\sum_i \beta_i \leq 0$. If $x_i^* = \epsilon$ then the result holds. Otherwise, $\beta_i = 0$ as $\beta_i(x_i^* - \epsilon) = 0$; this implies $x_i^* = y_i^* + \lambda \leq y_i^*$, as $\lambda \leq 0$, and again the result holds. □

The proof of Claim 4.12 uses the following additional claim.

Claim 4.13. *Let $\mathbf{p} \in \mathcal{P}(\mathbf{w})$ be a Pareto Efficient allocation. Let*

$$\rho = \frac{1}{2} \min_{\substack{h,i,j \\ u_i(A^{(j)};i) < u_i(A^{(j)};h) \text{ and} \\ u_h(A^{(j)};i) < u_h(A^{(j)};h)}} \frac{u_i(A^{(j)}; h) - u_i(A^{(j)}; i)}{u_h(A^{(j)}; h) - u_h(A^{(j)}; i)},$$

$$\textit{if such } h,\ i, \textit{ and } j \textit{ exist; otherwise, } \rho = \frac{1}{2}.$$

Then, if $w_h \leq \rho w_i$, player i will not envy player h.

The intuition for this claim is that if player i's weight is sufficiently larger than player h's weight, and if we maximize the sum of the weighted utilities, then player i will not envy player h. The proof will use the anonymity property, namely that all permutations of an allocation are feasible and switching the allocations of two players doesn't alter anyone else's utilities. One point to note is that ρ is well defined and positive as the total number of allocations is finite.

Proof. Consider $\mathcal{A}(\mathbf{w})$ as defined in Claim 4.5. We show by contradiction that for any pure allocation in this set, player i will not envy player h. The result follows, for then player i will not envy player h in the mixed allocation $\mathbf{p} \in \mathcal{P}(\mathbf{w})$.

So suppose player i envies player h in an allocation $A^{(j)}$. Then, since $j \in \mathcal{A}(\mathbf{w})$,

$$w_i \cdot u_i(A^{(j)}; i) + w_h \cdot u_h(A^{(j)}; h) \geq w_i \cdot u_i(A^{(j)}; h) + w_h \cdot u_h(A^{(j)}; i).$$

Thus

$$w_h \left[u_h(A^{(j)}; h) - u_h(A^{(j)}; i) \right] \geq w_i \left[u_i(A^{(j)}; h) - u_i(A^{(j)}; i) \right]. \tag{4}$$

Since player i envies player h,

$$u_i(A^{(j)}; i) < u_i(A^{(j)}; h). \tag{5}$$

From (4), (5) and the fact that $\mathbf{w}$ is strictly positive,

$$u_h(A^{(j)}; i) < u_h(A^{(j)}; h).$$

Therefore

$$\rho = \frac{1}{2} \min_{\substack{h,i,j \\ u_i(A^{(j)};i) < u_i(A^{(j)};h) \text{ and} \\ u_h(A^{(j)};i) < u_h(A^{(j)};h)}} \frac{u_i(A^{(j)}; h) - u_i(A^{(j)}; i)}{u_h(A^{(j)}; h) - u_h(A^{(j)}; i)},$$

as the minimum is over a non-empty set of (h, i, j).

Since $w_h \leq \rho w_i$, by (4),

$$\rho \big[u_h(A^{(j)}; h) - u_h(A^{(j)}; i) \big] \geq u_i(A^{(j)}; h) - u_i(A^{(j)}; i),$$

which contradicts the definition of ρ. $\square$

Given Claim 4.13, Claim 4.12 can be shown by making ϵ small enough.

Proof of Claim 4.12. We set $\epsilon = \frac{\rho^{n+1}}{n}$. Therefore, there is a ξ such that $\frac{1}{n} \geq \xi > \epsilon$ and, for every i, w_i lies outside the interval $[\xi, \frac{\xi}{\rho})$. Consequently, we can separate all the players into two sets $D = \{i|w_i^* < \xi\}$ and $U = \{i|w_i^* \geq \frac{\xi}{\rho}\}$. So for every player i in U, $w_i^* > \epsilon$. Note that $D \cap U = \emptyset$, $D \cup U = \{1, 2, \cdots, n\}$, and U is not empty since there exists one player i with $w_i \geq \frac{1}{n}$. Since $\mathbf{p}^* \in \mathcal{P}(\mathbf{w}^*)$, by Claim 4.13, we know that players in U will not envy players in D, and by Claim 4.8 we know that the envy graph is acyclic, thus there must be a player in U that is envy-free, i.e. in $\mathcal{F}(\mathbf{p}^*)$. The result follows. □

5 The Communication Complexity Lower Bound

In order to prove Theorem 3.5, we make a reduction to the Disjointness problem.

Disjointness problem Suppose there are two players. Each player has a string of bits of length L. Suppose x_i is player i's string. The players want to determine if there exists a j such that $x_{1j} = x_{2j} = 1$. The complexity question asks how many bits they need to communicate in order to determine the result.

Theorem 5.1 ([20, 31])**.** *Any protocol for the Disjointness problem needs to communicate $\Omega(L)$ bits.*

Proof of Thereom 3.5. We follow a similar approach to [30]. We give a reduction to the Disjointness problem.

Suppose there are $m = 2q$ items and two players. Each player receives a string of bits. Let x_1 and x_2, respectively, be the strings Player 1 and Player 2 receive and suppose they satisfy $|x_1| = |x_2| = \frac{1}{2}\begin{pmatrix} 2q \\ q \end{pmatrix}$.

The feasible allocations comprise all possible allocations. The utility functions we define mostly depend just on the size of the allocation, but for the following subset $\mathcal{T}$ of allocations the dependence is more delicate. $\mathcal{T} = \{T_1, T_2, \cdots, T_r\}$ is a set of allocations which allocate q items to each player, where $r = \frac{1}{2}\begin{pmatrix} 2q \\ q \end{pmatrix}$ and $T_j = (T_{j1}, T_{j2})$. Informally speaking, T_{j1} is a subset of q items which includes item 1 and T_{j2} contains the remaining q items. It's easy to verify that in total there are $\frac{1}{2}\begin{pmatrix} 2q \\ q \end{pmatrix}$ possible choices of T_{j1}; this is the reason that $r = \frac{1}{2}\left(\begin{array}{c} 2q \\ q \end{array}\right)$. Formally, $T_j = (T_{j1}, T_{j2})$ is an allocation such that $T_{j1} \cup T_{j2} = \{1, 2, \cdots, 2q\}$, $T_{j1} \cap T_{j2} = \emptyset$, $|T_{j_1}| = |T_{j_2}| = q$, $1 \in T_{i1}$ and for any $j \neq l$, $T_{j1} \neq T_{l1}$.

Then, given x_i, player i's utility function is defined as follows:

$$u_i(S) = \begin{cases} 3|S| & \text{if } |S| < q \\ 3q & \text{if } |S| > q \\ 3q & \text{if } S = T_{ji} \text{ for some } j \text{ and } x_{ij} = 1 \\ 3q - 1 & \text{otherwise.} \end{cases}$$

In the next few paragraphs, we prove the following lemma.

Lemma 5.2. *If (x_1, x_2) is a no-instance of the Disjointness problem, then for all Pareto Efficient and envy-free allocations, the sum of utilities, $u_1 + u_2$, will equal $6p$. Otherwise, if (x_1, x_2) is a yes-instance, then for all Pareto Efficient and envy-free allocations, the sum of utilities will be no larger than $6p - 1$.*

This lemma actually shows that the communication cost of calculating the utility of the Pareto Efficient and envy-free allocation is at least the communication cost of the Disjointness problem with $L = \frac{1}{2}\left(\begin{array}{c} 2q \\ q \end{array}\right)$, which is exponential in m, the number of items, as $m = 2q$. Note that the communication cost of calculating the utility of an allocation is no larger than the communication cost of calculating the allocation, as, without additional communication cost, one can easily calculate the utility from the allocation. Therefore, this statement proves that the communication cost of calculating the Pareto Efficient and envy-free allocation is exponential in m.

Proof of Lemma 5.2. If (x_1, x_2) is a no-instance of the Disjointness problem, then there exists a j such that $x_{1j} = x_{2j} = 1$. This implies there exists a Pareto Efficient and envy-free allocation: Player 1 gets T_{j1} and Player 2 gets T_{j2}. In this case, both players will have utility $3p$. Since the most utility any player can get is $3p$, any allocation which is Pareto Efficient must give both players $3p$ utility. This implies that for any allocation which is Pareto Efficient and envy-free, the sum of the utilities, $u_1 + u_2$, will be $6p$.

If (x_1, x_2) is a yes-instance of the Disjointness problem, then for any j, $x_{1j} = 0$ or $x_{2j} = 0$. A simple observation is that any allocation in which the items are not fully allocated is not Pareto Efficient. Then we look at any deterministic fully allocated allocation (A_1, A_2). For any j, if $A_j = T_{j1}$ and

$x_{1j} = 0$ then $u_1(A_1) = 3q - 1$ and $u_2(A_2) \leq 3q$. So $u_1(A_1) + u_2(A_2) \leq 6q - 1$. The same is true if $A_2 = T_{j2}$ and $x_{2j} = 0$. Otherwise, either $|A_1| < q$ or $|A_2| < q$ and then $u_1(A_1) + u_2(A_2) \leq 6q - 3$.

$\square$

The final thing to do is to check that this utility function is a submodular utility function. It's easy to see that our utility function satisfies the submodular condition that for any $X \subseteq Y$ and any element e, $u(X \cup \{e\}) - u(X) \geq u(Y \cup \{e\}) - u(Y)$.

The result follows. $\square$

6 Discussion

In this paper, we showed that for any partition-based utility functions, a Pareto Efficient and envy-free allocation always exists if each possible allocation remains feasible under any permutation. We also showed by example that we could weaken the anonymity constraint at the cost of some limitation on the envy-free property. The problem remains open in the divisible case. For our proof cannot be simply generalized to the divisible case. This is because our ϵ might equal 0 as ρ might be 0 in the divisible case, in which case $\mathcal{P}(\mathbf{w})$ will no longer ensure Pareto Efficiency as for some i, w_i could be 0, whereas in the indivisible case we knew $w_i \geq \epsilon$.

One issue is that an envy-free allocation may not appear particularly fair. For example, in Example 1.1, the second Pareto Efficient envy-free allocation seems less fair than the first one. One natural notion of fairness is provided by an allocation given by a CEEI equilibrium; but we don't know whether such an equilibrium always exists. Other fairness notions, for example maximizing Nash Social Welfare (the product of the agents' utilitilies) or max-min fairness, are guaranteed to always exist, but then the solutions need not be envy free (as we show in Lemma A.5 in the appendix). One refinement of the Pareto Efficient envy-free criteria is to ask for a solution among all Pareto Efficient envy-free allocations that maximizes a fairness notion such as Nash Social Welfare or max-min fairness. In the case of Example 1.1, the resulting Pareto Efficient envy-free allocation is the first one given in the example.

Another open problem concerns communication complexity. One simple fact is that if all players have linear utility functions, then there exists a protocol which requires a polynomial number of bits. However, the communication complexity is unknown for larger classes of utilities, such as gross substitutes.

7 Acknowledgements

We thank Hervé Moulin, whose comments and suggestions have greatly improved the paper.
Funding: This work was supported in part by NSF grants CCF-1527568 and CCF-1909538.

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A Missing Proofs

Lemma A.1. *The CEEI equilibrium in Example 1.1 is unique.*

Proof. Suppose the prices for the items are p_A, p_B, and p_C; and suppose there is an equilibrium other than $p_A = p_B = p_C = \frac{4}{3}$. WLOG, we assume $p_C \geq p_B \geq p_A$ and $p_A + p_B + p_C = M$ in this equilibrium, where M is the total money spent in the market.

We first prove that $M = 4$. If $M < 4$, then one player must buy their favorite bundle (the bundle with highest value) with probability 1, in which case he or she doesn't want to spend more money. Since $0 \leq p_A \leq p_B \leq p_C$, Player 1 must be one of the people who obtain their favourite bundle with probability 1. When Player 1 obtains his/her favourite bundle $\{A, B\}$ in full, Players 2, 3 and 4 will only be able to divide item C in this CEEI allocation. The only possible solution in this case is to set $p_C = \infty$, which violates the market equilibrium condition.

Therefore, $M = 4$. In this case, if there is another equilibrium, then it satisfies $p_C > p_A$. An immediate conclusion is that

$$p_C > \frac{4}{3} \tag{6}$$

as the total money in the market is $M = 4$. Since Player 4 cannot obtain item A or B with probability 1, as Player 1 is only interested in the bundle $\{A, B\}$, Player 4 should put no money on item C, as the price for item C is higher than the price of item A. This implies that the total money spent on items A and B will be at least 2, for Player 1, who wants bundle $\{A, B\}$, will also put all his/her money on items A and B. Thus,

$$p_A + p_B \geq 2.$$

Additionally, If $p_A < 1$, then $p_B > p_A$ as $p_A + p_B \geq 2$. Then, Player 4 will hold item A with probability 1 as $p_A < p_B \leq p_C$. But Players 1 and 3 will also seek to buy some of item A, causing the demand to be greater than 1, meaning this is not an equilibrium. Therefore,

$$p_B \geq p_A \geq 1. \tag{7}$$

Now we look at item C. Player 3, who wants bundle $\{A, C\}$, will spend $\frac{p_C}{p_A+p_C}$ on item C, and Player 2, who wants bundle $\{B, C\}$, will spend $\frac{p_C}{p_B+p_C}$ on item C. As the total spending on item C equals its price in every equilibrium, it follows that $\frac{p_C}{p_A+p_C} + \frac{p_C}{p_B+p_C} = p_C$, which implies $1 = \frac{1}{p_A+p_C} + \frac{1}{p_B+p_C}$.

By (6) and (7), the RHS is smaller than $\frac{1}{1+\frac{4}{3}} + \frac{1}{1+\frac{4}{3}} < 1$, which is a contradiction.

Therefore the prices are unique and in addition the allocation is unique. $\square$

Lemma A.2. *There does not exist a bundle pricing supporting a Pareto Efficient and envy-free allocation for the setting in Example 1.2.*

Proof. We start by observing that any Pareto Efficient and envy-free allocation is some combination of the following four allocations:

- Allocation 1: Player 1 receives $\{A\}$, Player 2 receives $\{B\}$, Player 3 receives $\{C\}$, and Player 4 receives $\{D, E\}$;
- Allocation 2: Player 1 receives $\{B\}$, Player 2 receives $\{A\}$, Player 3 receives $\{C\}$, and Player 4 receives $\{D, E\}$;
- Allocation 3: Player 1 receives $\{A\}$, Player 2 receives $\{B\}$, Player 3 receives $\{C, D\}$, and Player 4 receives $\{E\}$;
- Allocation 4: Player 1 receives $\{B\}$, Player 2 receives $\{A\}$, Player 3 receives $\{C, D\}$, and Player 4 receives $\{E\}$.

We let p_1, p_2, p_3 and p_4, resp., denote the probabilities of Allocations 1, 2, 3 and 4 in a Pareto Efficient and envy-free allocation.

Claim A.3. *For any Pareto Efficient and envy-free allocation, $p_2 = p_4 = 0$, $p_1 \geq \frac{1}{5}$ and $p_3 \geq \frac{1}{7}$.*

Proof. The fact that $p_2 = p_4 = 0$ is easy to see as Allocations 2 and 4 are Pareto dominated by the following random allocation: Allocation 1 with $\frac{1}{2}$ probability and Allocation 3 with $\frac{1}{2}$ probability.

Since $p_2 = p_4 = 0$, $p_1 + p_3 = 1$. By the envy-free condition:

$$14p_1 + 9(1 - p_1) \geq 6p_1 + 11(1 - p_1) \text{ (Player 1 doesn't envy Player 2)}$$
$$9p_1 + 16(1 - p_1) \geq 11p_1 + 4(1 - p_1) \text{ (Player 2 doesn't envy Player 1)}$$

By calculation, $\frac{1}{5} \leq p_1 \leq \frac{6}{7}$. The result follows. $\square$

Note that Allocation 1 uses the first partition and Allocation 3 uses the second partition. So, both partitions appear with non-zero probability. One important observation is in both Allocations 1 and 3, Player 1 always receives item A and Player 2 always receives item B.

Now, we show there do not exist prices supporting any PEEF allocation. We prove this by contradiction. Suppose there exists a Pareto Efficient and envy-free allocation and there exists a bundle price which supports this allocation.

The first observation is that Player 1 and Player 2 both spend all their money. For if Player 1 still had money in hand, then in the second partition, Player 1 would always want to buy item B instead of item A. An analogous argument applies to Player 2. Therefore, the prices for item A and item B will be the same and equal to 1, the budget of each player.

However, if the prices are the same, then Player 1 will always want item B instead of item A in the second partition, which is a contradiction. □

Lemma A.4. *All Pareto Efficient and envy-free allocations in the larger game in Example 3.3 have all men allocated to men-places and women to women-places.*

Proof. First, note that if H_a is allocated some set other than one of its preferred sets, then performing a swap strictly improves its utility without harming that of the other player. And similarly for H_b. Thus, in any Pareto Efficient allocation, men and women are allocated men-places or women-places only.

If a man is allocated a woman-place, then there is a woman who is allocated a man-place. This is because the number of places provided to men and women matches the number of men and women. Therefore, swapping their allocations strictly improves their utility without harming the utility of any other player.

The lemma follows. □

Lemma A.5. *There exist allocation problems such that an optimal solution under either the Nash Social Welfare or max-min fairness criteria are not envy free.*

Proof. For the Nash Social Welfare there are two items A and B and two players, 1 and 2. Player 1 has utilities 2 and 1 for A and B resp., and Player 2 has utilities 1 and 0. Then the optimal allocation is to give item A to Player 2 and item B to Player 1. Clearly Player 2 envies Player 1.

For max-min fairness, there is one item and two players. Player 1 has value 2 for the item and Player 2 has value 1. The optimal allocation gives the item to Player 1 with $\frac{1}{3}$ probability and to Player 2 with $\frac{2}{3}$ probability. Clearly Player 1 envies Player 2. $\square$

Discussion of Lemma A.5 One might argue that the max-min construction is contrived. If one rescaled so that each player had utility 1 for this item, then the optimal solution would give each of them the item with probability $\frac{1}{2}$, and then there would be no envy. However, this is just side-stepping the issue of the meaning of inter-agent comparison of utilities which is needed for the max-min fairness notion.