



# The matroid intersection cover problem

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## ABSTRACT

We consider the matroid intersection cover problem. This is a special case of set cover where the sets are derived from the intersection of matroids. We introduce a technique for computing matroid intersection covers. We give polynomial-time algorithms to compute partition decompositions for matroids that commonly arise in combinatorial optimization problems. We then give a polynomial-time algorithm for computing matroid intersection covers given the partition decompositions of the matroids. Combining these algorithms, we obtain an  $O(1)$ -approximation algorithm when each of the  $O(1)$  matroids is of a standard type.

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## 1. Introduction

In the canonical set cover problem the input is a set system  $M = (X, \mathcal{I})$ , where  $X$  is a collection of  $n$  elements, and  $\mathcal{I} \subseteq 2^X$  is a collection of subsets of  $X$ . A sub-collection  $S \subseteq \mathcal{I}$  is said to be a cover of  $Y \subseteq X$  if each element of  $Y$  is in at least one set in  $S$ . A feasible solution is a sub-collection  $S \subseteq \mathcal{I}$  that covers  $X$ . The objective is to minimize  $|S|$ . The optimal objective value is called the **cover number** of  $M$ , which we denote as  $\alpha(M)$ . In this paper we are interested in the following special case of the set cover problem, which we call the matroid intersection cover problem:

**Matroid Intersection Cover Problem:** The input is a collection of  $k$  matroids  $M_1 = (X, \mathcal{I}_1), M_2 = (X, \mathcal{I}_2), \dots, M_k = (X, \mathcal{I}_k)$ . This input represents the set system  $M = (X, \cap_{i=1}^k \mathcal{I}_i)$ . A feasible solution  $S \subseteq \cap_{i=1}^k \mathcal{I}_i$  is a collection of sets that covers  $X$ . Each set in  $S$  must be independent in every matroid. The objective is to minimize  $|S|$ .

This problem can alternatively be viewed as the Matroid Coloring Problem, where one can color elements with the same color if they are independent in all the given matroids and the goal is to color all elements with the minimum number of colors.

In this paper we consider the approximation ratio achievable by polynomial-time algorithms for natural matroid intersection cover problems in which the number  $k$  of matroids is small. In particular, we are interested in determining whether one can achieve  $O(1)$ -approximation when  $k$  is a constant, which would

be better than what one can achieve for general set cover instances, assuming  $P \neq NP$ . Before stating our results, let us give some background, and discuss the relationship of matroid intersection cover with other matroid problems.

### 1.1. Background

Matroid intersection cover is a natural extension of the matroid intersection problem, as well as a generalization of the matroid cover problem (also known as the matroid partitioning problem) [11,25]. The matroid cover problem is exactly our problem, matroid intersection cover, when  $k = 1$ .

The maximum coverage version of matroid intersection has been more widely studied in the algorithms community beginning with the work of Nemhauser and Wolsey [20]. In this problem, the goal is to find a single set of maximum size that is independent in each of the  $k$  matroids. The natural greedy algorithm has approximation ratio  $k$  [24]. An approximation ratio of  $k - 1 + \epsilon$  is achievable using local search [18]. No polynomial-time algorithm can have approximation ratio  $O(\frac{k}{\log k})$  unless  $P = NP$  [20]. The natural repeated application of these maximum coverage algorithms to matroid intersection cover would yield approximation ratios that are  $O(k \log n)$ . Extensions to this maximum coverage problem to other submodular maximum coverage problems are considered in [18,24].

The natural greedy algorithm will compute the maximum profit independent set in a matroid. In fact, this is exactly equivalent to a hereditary set system being a matroid [22,28]. Edmonds showed that computing the minimum cover of a single matroid can be reduced in polynomial-time to computing the maximum independent set in the intersection of two matroids, which in turn

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can be computed in polynomial time [11,25]. One can compute in polynomial-time the optimal matroid intersection cover for two partition matroids, since this is just the problem of edge coloring a bipartite graph [25]. The cover number of the intersection of two arbitrary matroids is at most twice the larger of the cover numbers of the two matroids [2, Theorem 8.9]; This proof is *non-constructive*. Matroid cover intersection is related to Rota's basis conjecture [12], which corresponds to the exact (non-approximate) version of our problem when one matroid corresponds to a partition matroid and the other is a linear matroid. It is known that matroid intersection cover is NP-Hard for  $k \geq 3$ . Indeed, it is known that it is NP-hard to determine if a 3-partite hypergraph of degree 3 is 3 edge colorable [21]. This corresponds to the matroid intersection cover problem when  $k = 3$ , the matroids are three partition matroids and the goal is to determine if the cover number is 3. As an immediate consequence, it is also NP-hard to approximate the cover number for three partition matroids within a factor of  $4/3 - \epsilon$ . It is also known that a super polynomial number of independence queries are required when  $k = 2$  for two arbitrary matroids when independence of a set is given via an oracle [5].

Matroid intersection cover is related to scheduling coflows in a network. In coflow scheduling problems there are some constraints on the flows that may be simultaneously routed in the network. Often these constraints are matroid related. Chowdhury and Stoica [8] cited information gathering at a central location in the network as a common communication pattern. Motivated by this, Im et al. [13] considered the problem of scheduling flows with gammoidal (and more generally matroidal) constraints. The bulk of the algorithmic literature on co-flow scheduling has been on scheduling matchings in a bipartite graph [1,3,14,15,23]. Bipartite matchings can be expressed as the intersection of two partition matroids. The version of the problem with the makespan objective is a special case of matroid intersection cover problem where  $k = 2$  and both matroids are partition matroids.

## 1.2. Our results

Our motivation for considering approximation algorithms for matroid intersection cover was sparked by our interest in more general coflow scheduling problems and the observation that other natural algorithmic problems can be also modeled as matroid cover intersection problems. Some examples are given in Section 5.

We give a general algorithmic technique for computing matroid intersection covers. The first step of this algorithmic technique is to compute what we call a partition decomposition of each of the matroids. Roughly speaking a partition decomposition of a matroid  $M$  is a partition of the elements into sets of size  $O(\alpha(M))$  such that any set, that contains no more than one element from any partition, can be covered by  $b = O(1)$  independent sets. A partition decomposition generalizes the concept of a weak map into a partition matroid [22]. A weak map from a general matroid  $M$  to a partition matroid is a partition decomposition in which  $b = 1$ . Except for gammoids, all of our decompositions will be weak maps.

This paper gives polynomial-time algorithms to compute such partition decompositions for many of the types of matroids that commonly arise in combinatorial optimization problems. Namely, partition matroids, graphic matroids, laminar matroids, transversal matroids, and gammoids. We then give a polynomial-time algorithm for computing matroid intersection covers given the partition decompositions of the matroids. Combining these algorithms, we obtain an  $O(1)$ -approximation algorithm when each of the  $O(1)$  matroids is of a standard type.

Formally, we define a partition decomposition as follows.

**Definition 1.** A matroid  $M = (X, \mathcal{I})$  is  $(b, c)$ -decomposable if  $X$  can be partitioned into sets  $X_1, X_2, \dots, X_t$  such that:

- For all  $i \in [t]$ , it is the case that  $|X_i| \leq c \cdot \alpha(M)$ .
- Any set  $Y = \{r_1, \dots, r_t\}$ , consisting of one representative element  $r_i$  from each  $X_i$ , can be covered by  $b$  sets from  $\mathcal{I}$ .

A class of matroids is partition decomposable if there exist constants  $b$  and  $c$  such that every matroid in that class is  $(b, c)$ -decomposable.

Although several types of matroid decomposition exist [26], this type of matroid decomposition does not seem to have been previously considered. We believe this type of decomposition is of independent interest, as it seems likely it will be useful for related problems.

Section 3 is devoted to proving the following theorem:

**Theorem 2.** *There are polynomial-time algorithms to compute:*

- A  $(1, 2)$ -decomposition of a graphic matroid given the underlying graph as input.
- A  $(1, 3)$ -decomposition of a laminar matroid given the laminar matroid as input.
- A  $(1, 1)$ -decomposition of a transversal matroid given the underlying bipartite graph as input.
- A  $(18, 1)$ -decomposition of a gammoid given the underlying directed graph as input.

For graphic, transversal and laminar matroids, the construction of the partition can be constructed from the natural underlying structure of the matroid. For gammoids, the construction of the partition can be done by leveraging known results on unsplitably routing flow to a single source [16].

Section 4 will be devoted to proving the following theorem:

**Theorem 3.** *Consider  $k$  matroids  $M_1, M_2, \dots, M_k$  defined over a common ground set  $X$ , where matroid  $M_i$  has cover number  $\alpha_i$ . There is a polynomial-time algorithm that, given a  $(b_i, c_i)$ -decomposition of each matroid  $M_i$ , computes a cover of size at most  $\prod_{i \in [k]} b_i \cdot (1 + \sum_{i \in [k]} (c_i \alpha_i - 1))$ , thus with cover number at most  $(\prod_{i \in [k]} b_i) \cdot (\sum_{i \in [k]} c_i) \alpha^*$  where  $\alpha^* = \max_{i \in [k]} \alpha_i$ .*

As  $\alpha^*$  is an obvious lower bound to the optimal cover number, an immediate consequence is that this algorithm has approximation ratio at most  $(\prod_{i \in [k]} b_i) \cdot (\sum_{i \in [k]} c_i)$ . Conceptually the algorithm is based on a natural greedy algorithm for hyperedge coloring  $k$ -partite hypergraphs, and the bound on the cover number follows directly from a simple analysis of the approximation ratio of this coloring algorithm.

Combining Theorems 2 and 3, we obtain the following corollaries:

**Corollary 4.** *For instances of the matroid intersection problem consisting of  $k$  matroids that are either graphic, laminar, or transversal matroids, there is a polynomial-time algorithm that has approximation ratio  $O(k)$ .*

**Corollary 5.** *For instances of the matroid intersection problem consisting of  $k$  matroids that are either graphic, laminar, or transversal matroids and  $\ell$  matroids that are gammoids, there is a polynomial-time algorithm that has approximation ratio  $O((k + \ell) \cdot 18^\ell)$ .*

In Section 5 we explain the approximation ratios that our technique yields for some particular matroid intersection cover problems.

## 2. Related results

Set Cover is a canonical problem in the field of approximation algorithms [27,29]. The greedy algorithm has an approximation ratio of  $H_n \approx \ln n$  on  $n$  element instances [27,29], and this is essentially optimal assuming  $P \neq NP$  [10]. Polynomial-time algorithms with approximation ratio  $o(\log n)$  are known for two types of special instances. The first type is where some parameter is known to be small. An example of this type is vertex cover, where no element can be in more than two sets. Many different 2-approximation algorithms are known for vertex cover [27,29]. The second type is geometric based/inspired. An example of this type is covering points in the plane by a minimum number of discs, for which a polynomial time approximation scheme is known [19]. As another example, it is known that constant approximation is possible if the set system has bounded VC dimension [7]. VC dimension is at least arguably a geometrically inspired concept. One interesting aspect of the results in this paper is that they give  $o(\log n)$  approximations for a different type of natural set cover instances. There are of course multitudinous papers dealing with generalizations and variations on set cover, e.g. weighted covers, capacitated covers, and multicovers.

Computing unsplittable flows in single-source networks is considered in [4,9,16,17], with the main take-away message being that for many natural problems the optimal objective value does not degrade too much if flows are required to be routed unsplittably. Our partition decomposition of gammoids is based on techniques from [16].

Recently and independently of this work, [6] gave upper bounds on the cover number of the intersection of two matroids for common combinatorial matroids by giving partition decompositions. In particular, among other results, [6] also gave a  $(1, 1)$ -decomposition of traversal matroids, and a  $(1, 2 - 2/\alpha)$ -decomposition for graphic matroids. The main result in [6] is a proof that gammoids are  $(1, 2 - 2/\alpha)$ -decomposable. The results in [6] are not explicitly algorithmic. Although presumably many of the existential proofs could be readily converted into efficient algorithms, this is not true for gammoids as [6] states finding an efficient algorithm to compute such a partition decomposition of a gammoid as an open problem.

## 3. Computing partition decompositions

In this section, we show how to compute  $(O(1), O(1))$ -decompositions for many types of matroids that commonly arise in the combinatorial optimization literature. This includes graphic matroids, transversal matroids, laminar matroids, and gammoids.

### 3.1. Graphic matroids are $(1, 2)$ -decomposable

We begin by considering the graphic matroid. Let  $M$  be a graphic matroid on a ground set  $X$ . In this case,  $X$  corresponds to edges of an undirected graph  $G = (V, X)$ . Independent sets of the matroid  $M$  correspond to acyclic subsets of edges. Let  $\alpha := \alpha(M)$  be the value of the minimum cover. The goal of this section is to show the following theorem.

**Theorem 6.** *Graphic matroids are  $(1, 2)$ -decomposable and this decomposition can be computed in polynomial time.*

To prove the theorem, we begin by defining the decomposition. Consider the graph  $G$ . Let  $v$  be the vertex in  $G$  that has the smallest degree that is non-zero. Let  $X_v$  be a set in the decomposition that contains all of the edges adjacent to  $v$ . Remove  $v$  and its adjacent edges from  $G$ . Then recurse on the resulting graph to obtain the next set in the partition. The procedure stops when

there are no remaining edges. Notice that the sets  $X_v$  constructed are a partition of the ground set  $X$  and these correspond to the sets of the decomposition.

Our goal is to show that this is a valid decomposition. To do so, we first show that as edges are removed from the graph, the size of the optimal cover of the remaining edges never increases.

**Lemma 7.** *Consider any undirected graph  $G = (V, X)$  and any vertex  $v \in V$ . Let  $M$  be the graphic matroid corresponding to  $G$ . Let  $G'$  be the graph obtained from  $G$  by deleting a vertex  $v$  and all of its adjacent edges and let  $M'$  be the corresponding graphic matroid for  $G'$ . Then we have  $\alpha(M') \leq \alpha(M)$ .*

**Proof.** Let  $X_1^*, X_2^*, \dots, X_\alpha^*$  denote the optimal cover of  $M$ . Consider removing the edges adjacent to  $v$  from the sets  $X^*$ . The resulting sets are a cover of  $M'$  and are independent in  $M'$ . This is because each set  $X_i^*$  will correspond to an acyclic set of edges in  $G'$  since they were acyclic in  $G$ .  $\square$

We can use the previous lemma to show every part has size at most  $2\alpha$ .

**Lemma 8.** *Every partition  $X_v$  constructed has size at most  $2\alpha$ .*

**Proof.** Consider any set  $X_v$  and let  $G' = (V', X')$  be the graph that remains in the algorithm just before  $X_v$  is constructed. We know that the edges of  $G'$  can be partitioned into at most  $\alpha$  sets of acyclic edges by Lemma 7. Notice that any acyclic subgraph of  $G'$  has at most  $|V'| - 1$  edges. This implies that  $G'$  has at most  $\alpha(|V'| - 1)$  edges and the aggregate degrees of the vertices is at most  $2\alpha(|V'| - 1)$ . By definition of the algorithm  $v$  is the minimum degree vertex in  $V'$  and therefore has at most  $2\alpha$  adjacent edges. Hence  $|X_v| \leq 2\alpha$ .  $\square$

To complete the proof of the theorem, it will be shown that any set  $S$  containing up to one element from each partition  $X_v$  is independent in the original matroid  $M$ . This is established in the following lemma.

**Lemma 9.** *Let  $X_{v_1}, X_{v_2}, \dots, X_{v_\ell}$  be the partition of  $X$  constructed by the algorithm. For any set  $S$  such that  $|S \cap X_{v_i}| \leq 1$  it is the case that  $S$  is an acyclic set of edges in  $G$ .*

**Proof.** For the sake of contradiction, suppose  $S$  contains a cycle  $C$  for some  $S$  such that  $|S \cap X_{v_i}| \leq 1$ . Assume wlog that  $X_{v_1}, X_{v_2}, \dots, X_{v_\ell}$  were constructed in this order. Notice the vertices of the cycle will be the vertices  $v_i$  where  $|X_{v_i} \cap S| = 1$ . If edge  $(u, v)$  is part of  $C$ , then either  $(u, v) \in X_u$  or  $(u, v) \in X_v$ . Since  $C$  can collect at most one edge from each partition, this implies that either  $(v_{i(1)}, v_{i(2)}) \in X_{v_{i(1)}}$ ,  $(v_{i(2)}, v_{i(3)}) \in X_{v_{i(2)}}$ ,  $\dots$ ,  $(v_{i(|C|)}, v_{i(1)}) \in X_{v_{i(|C|)}}$  or  $(v_{i(1)}, v_{i(2)}) \in X_{v_{i(2)}}$ ,  $(v_{i(2)}, v_{i(3)}) \in X_{v_{i(3)}}$ ,  $\dots$ ,  $(v_{i(|C|)}, v_{i(1)}) \in X_{v_{i(1)}}$  for some  $1 \leq i(1) < i(2) < \dots < i(|C|) \leq \ell$ . For the first case, we have a contradiction since  $X_{v_{i(1)}}$  was constructed before  $X_{v_{i(|C|)}}$ , meaning that  $X_{v_{i(|C|)}}$  has no edges adjacent to  $v_{i(1)}$ . For the second case, we obtain a similar contradiction.  $\square$

Theorem 6 follows immediately from Lemmas 8 and 9.

### 3.2. Transversal matroids are $(1, 1)$ -decomposable

This section shows that transversal matroids are  $(1, 1)$ -decomposable. Let  $M$  be a transversal matroid defined over a ground set  $X$ . In this case,  $X$  corresponds to one side of a bipartite graph  $G = (X \cup Y, E)$ . A subset of  $X$  is independent if there exists a matching of  $G$  that matches all vertices in  $X$ . A partition matroid is a special case of a transversal matroid in which there is a partition  $X_1, \dots, X_k$  of the ground set  $X$  and a bound  $u_i$  for each  $X_i$ . A collection of elements is then independent if it does not contain more than  $u_i$  elements from any  $X_i$ .

**Theorem 10.** *Transversal matroids are (1, 1)-decomposable and this decomposition can be computed in polynomial time.*

**Proof.** Adopting the above notation, let  $\alpha := \alpha(M)$  be the minimum cover of  $X$  using sets independent in  $M$ . Let  $X_1^*, X_2^*, \dots, X_\alpha^*$  denote a minimum cover of all of the elements in  $X$  such that each of these sets is independent in  $M$ . Assume wlog that the sets are disjoint. For each set  $X_i^*$  let  $E_i^*$  be a matching saturating  $X_i^*$  in  $G$ . Let  $E^* = E_1^* \cup E_2^* \cup \dots \cup E_\alpha^*$  be the subset of  $E$  edges used in the matchings. Let the graph  $G^*$  be the graph induced by  $E^*$ . The partition will be induced by the vertices in  $Y$  and their neighbors in  $G^*$  that are in the ground set  $X$ . In particular, let  $X_v$  be all elements in  $X$  that are adjacent to  $v \in Y$  in  $G^*$ . This completes the definition. Notice that these sets are computable in polynomial time since the minimum cover of  $M$  and  $E^*$  are computable.

We need to show two properties to complete the proof. The first property is that  $|X_v| \leq \alpha$ . This follows because the construction of the graph  $G^*$  ensures that every vertex has degrees at most  $\alpha$  since  $E^*$  consists of  $\alpha$  matchings. The second property that needs to be shown is that every set  $S$  is independent in  $M$  if  $|S \cap X_v| \leq 1$  for all  $v \in Y$ . Indeed, consider any such set  $S$ . The definition of  $S$  implies that each vertex in  $S$  is adjacent to a unique vertex in  $G^*$ . Thus,  $S$  must correspond to a subset of  $X$  that can be saturated in a matching of  $G$  and therefore  $S$  is independent in  $M$ .  $\square$

### 3.3. Laminar matroids are (1, 3)-decomposable

In this section we show that laminar matroids are (1, 3)-decomposable. Let  $M$  be a laminar matroid on a ground set  $X$ . Let  $\mathcal{F}$  be the laminar family of subsets of  $X$  associated with the matroid – the family is said to be laminar if for any  $A, B \in \mathcal{F}$ , we have  $A \cap B = \emptyset$ ,  $A \subseteq B$ , or  $B \subseteq A$ . Further each set  $A \in \mathcal{F}$  is associated with a positive integer  $b(A)$ , which is called the *capacity* of the set. A set  $I \subset X$  is independent in  $M$  if  $|I \cap A| \leq b(A)$  for all  $A \in \mathcal{F}$ .

The goal of this section is to show the following theorem.

**Theorem 11.** *Laminar matroids are (1, 3)-decomposable and this decomposition can be computed in polynomial time.*

We say that a set  $A \in \mathcal{F}$  is a leaf set of the family  $\mathcal{F}$  if there is no  $B \in \mathcal{F}$  such that  $B \subset A$ . For any non-leaf set  $A \in \mathcal{F}$ , we can assume wlog  $b(A) \geq 2$ ; otherwise, we can remove any subsets of  $A$  from  $\mathcal{F}$  as they are redundant. If there is an element  $x$  in  $X$  that does not belong to any leaf set, we create a singleton set  $\{x\}$  with capacity 1 and add it to the family without loss of generality. We also assume that  $|X| > 3\alpha$  since if  $|X| \leq 3\alpha$ , then the set  $X$  itself is an obvious (1, 3)-decomposition of  $M$ .

We now describe how we obtain a (1, 3)-decomposition of  $M$ . Let  $\alpha := \alpha(M)$  be the min cover size of  $M$ .

1. For each leaf set  $A \in \mathcal{F}$  if  $|A| \leq \alpha$ , set  $b(A)$  to 1. Otherwise, partition  $A$  into disjoint subsets  $A_1, A_2, \dots, A_k$  so that  $|A_1| = |A_2| = \dots = |A_{k-1}| = \alpha$  and  $|A_k| \leq \alpha$ . Let  $b(A_1) = b(A_2) = \dots = b(A_k) = 1$ . Add these leaf sets to  $\mathcal{F}$ .
2. Linearly order elements  $x_1, x_2, \dots, x_n$  so that for any  $1 \leq u < v < w \leq n$ , if  $x_u, x_w \in A$ , where  $A \in \mathcal{F}$ , then  $x_v \in A$ .
3. From left to right, we repeat the following: collect the smallest possible collection of leaf sets and take the union of them, so that the union has least  $2\alpha$  elements. Let the resulting unions,  $X_1, \dots, X_t$  be our decomposition of  $X$ .

The following observation is immediate from the algorithm definition since after partitioning leaf sets, every leaf set has cardinality at most  $\alpha$ .

**Observation 12.** *For all  $i \in \{1, 2, \dots, t\}$ ,  $2\alpha \leq |X_i| < 3\alpha$ .*

To show  $X_1, X_2, \dots, X_t$  is a valid (1, 3)-decomposition of  $M$ , it suffices to show the following.

**Lemma 13.** *For any  $I \subseteq X$  such that  $|I \cap X_i| \leq 1$  for all  $i \in [t]$ , we have  $|I \cap A| \leq b(A)$  for all  $A \in \mathcal{F}$ .*

**Proof.** For the sake of contradiction suppose  $|I \cap A| > b(A)$  for some  $A \in \mathcal{F}$  that we did not create. Let  $X_{j(1)}, X_{j(2)}, \dots, X_{j(|I \cap A|)}$  be the sets that include an element from  $I \cap A$  – note that the number of such sets is exactly  $|I \cap A|$  since  $I$  can have at most one element from each partition. We consider three cases and show a contradiction for each case.

Case (i)  $b(A) \geq 3$ . Because of the second and third steps,  $X_{j(2)}, \dots, X_{j(|I \cap A| - 1)}$  must be subsets of  $A$ . Thus,  $|A| \geq \sum_{i \in \{|I \cap A| - 1\} \setminus \{1\}} |X_{j(i)}| \geq 2\alpha \cdot (b(A) - 1)$  by [Observation 12](#) and due to the assumption that  $|I \cap A| > b(A)$ . However, we have  $|A| \leq \alpha \cdot b(A)$  since otherwise the min cover size of  $M$  would be greater than  $\alpha$ . Combining these two inequalities gives us  $b(A) \leq 2$ , which is a contradiction.

Case (ii)  $b(A) = 2$ . As before, we have  $X_{j(2)} \subseteq A$ . Further,  $A \cap X_{j(1)} \neq \emptyset$ . By [Observation 12](#), we have  $|A| > 2\alpha$ . This is a contradiction since  $b(A) = 2$ .

Case (iii)  $b(A) = 1$ . In this case,  $A \subseteq X_{t'}$  for some  $t' \in [t]$ . This is because  $A$  must be a leaf set that we did not create. Since  $|I \cap X_{t'}| \leq 1$ , we have  $|I \cap A| \leq 1$ , which means  $I$  did not violate  $A$ 's capacity.  $\square$

Thus, we have shown [Theorem 11](#).

### 3.4. Gammoids are (18, 1)-decomposable

This section shows that gammoid matroids are (18, 1)-decomposable. Gammoids are defined as follows. Let  $M$  be a gammoid defined on the ground set  $X$  of  $n$  elements. The matroid is associated with a directed graph  $G = (V, E)$  with a sink  $t \in V$ . The ground set of  $X$  is a collection of sources,  $\{s_1, \dots, s_n\} \subseteq V$ . A subset  $S \subseteq X$  is independent in the matroid if there is routing of a unit of flow from each of the sources in  $S$  to the given sink  $t$  so that no more than one unit of flow is routed over any edge.

We note that this definition of gammoid is equivalent to another definition of gammoid based on vertex-disjoint paths. To see the equivalence, we assume wlog that every source has no incoming edges and has only one out-going edge by adding dummy edges: if  $s_i$  is a source vertex, we can add an edge  $(s'_i, s_i)$  and consider  $s'_i$  as the new corresponding source. We first consider the reduction from vertex-disjoint-path gammoids to edge-disjoint-path gammoids, where we split each vertex  $v$  into  $v_{in}$  and  $v_{out}$  and connect to  $v_{in}$  all vertices adjacent to  $v$  and  $v_{out}$  to all vertices adjacent from  $v$ , and connect  $v_{in}$  to  $v_{out}$ . It is easy to see that each path before the splitting has a uniquely corresponding path after the splitting, and two paths before the splitting are vertex-disjoint if and only if the corresponding paths are edge-disjoint after the splitting. The other direction can be shown by replacing each vertex  $v$  with two sets of vertices  $S(v)^+$  and  $S(v)^-$  where  $|S(v)^+| = \delta^+(v)$  and  $|S(v)^-| = \delta^-(v)$ . There is a complete bipartite graph from vertices in  $S(v)^-$  to vertices in  $S(v)^+$ . Each edge  $(v, u)$  is replaced with an edge going from a unique vertex in  $S^+(v)$  to a unique vertex in  $S^-(u)$ .

**Theorem 14.** *There is a polynomial time algorithm to compute (18, 1)-decomposition of a gammoid  $M$  given the representation  $(G, t, X)$  as input.*

**Proof.** Lemma 6.8 in [16] shows how to compute in polynomial time a collection  $\{T_1, \dots, T_h\}$  of reverse arborescences rooted at vertices  $\{\ell_1, \dots, \ell_h\}$ , and a partition  $X = \{X_1, \dots, X_h\}$  of the sources with the following properties:

- Each  $X_i$  has size at most  $\alpha(M)$ .
- For all  $i$ , all sources in  $X_i$  are connected to an edge in  $T_i$ .
- No edge in  $G$  is contained in more than two  $T_i$ .
- There is a feasible fractional routing  $F$  of one unit of flow from each  $\ell_i$  to  $t$  such that the flow through any edge is at most 7.

The collection  $X$  will be our partition decomposition.

Now consider a collection  $S$  of sources consisting of at most one arbitrary source  $s_i$  from each  $X_i$ . There must then be a fractional routing of one unit of flow from each source in  $S$  to  $t$  such that the flow through any edge is at most 9. One way to achieve this is to route flow from each  $s_i$  to  $\ell_i$  within  $T_i$  and then from  $\ell_i$  to the sink  $t$  following  $F$ . Each edge can have most two units routed through it due to the initial routing within  $T_i$ , and at most 7 units routed through it due to  $F$ .

Lemma 5.5 in [16] shows how to transform a fractional routing with maximum flow  $c$  on any edge into a collection of  $2c$  integral routings that collectively route all the flow, and such that no edge has more than one unit of flow routed through it in each of the  $2c$  routings. (Lemma 5.5 in [16] is only stated for undirected graphs, but it is noted later, right before Theorem 6.14, that it still holds for directed graphs.) Applying the lemma to our fractional flow, we get a decomposition of  $S$  into 18 independent sets.  $\square$

We note that using techniques from [16] one can also show that gammoids that arise from undirected graphs are (6, 1)-decomposable.

#### 4. Covering partitioned matroids

The purpose of this section is to prove Theorem 3. Let  $P_i$  be a  $(b_i, c_i)$ -decomposition of matroid  $M_i$ . Consider the  $k$ -partite  $k$ -uniform hypergraph  $H = (V_1, \dots, V_k, E)$  where there is one vertex in each  $V_i$  corresponding to each partition in  $P_i$ , and one hyperedge  $e$  corresponding to each element  $e$  in the ground set  $X$ . Hyperedge  $e$  will be incident to the vertex in  $V_i$  corresponding to the partition to which  $e$  belongs in  $P_i$ . Define a proper coloring of the hyperedges to be an assignment of colors to the hyperedges such that no pair of hyperedges incident on a common vertex are assigned the same color. Note for all  $i \in [k]$  that by the definition of  $(b_i, c_i)$ -decomposability the hyperedges colored a particular color in a proper coloring of  $H$  can be decomposed into  $b_i$  independent sets in  $M_i$ . And thus for all  $i \in [k]$  the color classes in a proper coloring of  $H$  correspond to a cover of  $X$  by sets that each be decomposed into  $b_i$  independent sets in  $M_i$ .

Our algorithm will first color the hyperedges in  $H$  in a greedy fashion. That is, the hyperedges are considered in arbitrary order, and hyperedge  $e$  is assigned the first color not already assigned to any edge that is incident on a common vertex with  $e$ . The number of colors used by this greedy algorithm will be at most one more than the maximum number of hyperedges that can be incident on a common vertex with a particular hyperedge  $e$ . This is at most  $\sum_{i \in [k]} (c_i \alpha_i - 1)$ . Thus this greedy algorithm produces a hyperedge coloring that can be interpreted as a cover  $\mathcal{C}$  of  $X$  of at most  $1 + \sum_{i \in [k]} (c_i \alpha_i - 1)$  sets.

Then for each  $S \in \mathcal{C}$  and each matroid  $M_i$ , our algorithm uses any polynomial time algorithm for matroid covering to partition each set  $S$  into a collection  $\mathcal{D}_i^S = \{D_i^S(1), \dots, D_i^S(b_j)\}$  parts such that each part  $D_i^S(j)$  is independent in  $M_i$ . Now let  $V$  be the collection  $k$  dimensional vectors where component  $i$  is an integer in the range  $[1, b_i]$ . So the cardinality of  $V$  is  $\prod_{i=1}^k b_i$ . Then for all

$v \in V$ , let  $S^v$  be the collection of  $s \in S$  such that for all  $i \in [k]$  it is the case that  $s \in D_i^S(v(i))$ , where  $v(i)$  is component  $i$  of  $v$ . In other words,  $S^v$  is the collection of all elements of  $S$  that are in part  $v(i)$  in  $\mathcal{D}_i^S$ . Our final cover will be the collection of all such  $S^v$ , namely:

$$\{S^v \mid S \in \mathcal{C} \text{ and } v \in V\}$$

#### 5. Applications of our techniques

The following problems can all be modeled as matroid intersection cover problems. We also remark that minimizing makespan of a co-flow of jobs that arrive all at the same time is a special case of our problem when  $k = 2$  and the input is two partition matroids.

1. The input is a  $k$ -partite hypergraph  $G$  in which each hyper-edge contains exactly one vertex from each of the  $k$  sets in the partition of the vertices. A feasible solution is a coloring of the hyperedges such that no two hyperedges that share a vertex receive the same color. The objective is to minimize the number of colors. As this problem plays a special role in our results, let us refer to this as the hypergraph coloring problem.
2. The input is a directed graph, where each vertex  $v$  has an associated bound  $b_v$ , each edge has a color, and each color  $c$  has an associated bound  $u_c$ . A feasible solution is a partition  $E_1, \dots, E_t$  of the edges such that:
  - Each  $E_i$  is the union of disjoint arborescences.
  - For all vertices  $v$  and for all  $E_i$  the out-degree of  $v$  in  $E_i$  is at most  $b_v$ .
  - For all colors  $c$  and for all  $E_i$  the number of edges colored  $c$  in  $E_i$  is at most  $u_c$ .

The objective is to minimize  $t$ .

3. The input is  $k = O(1)$  graphs/networks  $G_i = (V_i, E_i)$ ,  $i \in [k]$ , a sink vertex  $t_i$  in each  $V_i$ , and a collection  $S \subseteq \bigcap_{i=1}^k V_i$  of source vertices common to all networks (the rest of the networks are disjoint). Each source needs to be simultaneously connected by a dedicated path  $P_i$  in each  $G_i$  to each  $t_i$  for one consecutive unit of time. Multiple sources can be connected at the same time as long as their paths are disjoint. The problem is to minimize the time that it takes to achieve this.
4. The input is a layered graph  $G$  where the vertices are partitioned into  $k$  layers, and all edges are between vertices in adjacent layers. Further each layer has an associated laminar decomposition where each set  $s$  in the laminar decomposition has an associated bound  $b_s$ . A full path in  $G$  is a simple path from a vertex in the first layer to a vertex in the last layer. A layered covering of  $G$  is a collection  $P_1, \dots, P_t$  where each  $P_i$  consists of a disjoint collection of full paths such that:
  - Every full path is in some  $P_i$ , and
  - For every  $P_i$ , and for every layer and for every set  $s$  in the laminar decomposition of this layer, the number of paths in  $P_i$  that contain a vertex in  $s$  is at most  $b_s$ .

The problem is to find the layered covering that minimizes  $t$ .

We now explain how to obtain bounds on the approximation ratios of the polynomial-time algorithm for these problems that one can derive from Corollary 5:

1. A feasible color class can be represented as the intersection of  $k$  partition matroids. Since partition matroids are (1, 1)-decomposable, applying Theorem 3 we obtain a polynomial-time  $k$ -approximation algorithm.

2. The feasible  $A_i$  can be represented by the intersection of a graphic matroid guaranteeing that the edges are acyclic (ignoring the directions of the edges), a partition matroid guaranteeing that in-degree of each vertex is at most one, a partition matroid bounding the out-degree of each vertex, and a partition matroid bounding the number of colors. Thus as there are three partition matroids, each of which are  $(1, 1)$ -decomposable, and one graphic matroid, which is  $(1, 2)$ -decomposable, applying [Theorem 3](#) yields a polynomial-time 5-approximate algorithm.
3. The collection of sources that can be feasibly routed in a time step can be represented by the intersection of  $k$  gammoids, which we have shown are  $(18, 1)$ -decomposable. Thus applying [Theorem 3](#) we obtain a  $k18^k$ -approximation algorithm.
4. The feasible  $P_i$  can be represented as the intersection of  $k$  laminar matroids over the full paths. Each partition corresponds to a laminar matroid. As we have shown that laminar matroids are  $(1, 3)$ -decomposable, applying [Theorem 3](#) we obtain a polynomial time  $3k$ -approximation algorithm.

## 6. Conclusion and open problems

There are many natural interesting open questions that naturally arise from the results in this paper, including:

- Is there an efficient algorithm to find a  $(1, O(1))$  decomposition of a gammoid? If not, how about gammoids derived from undirected graphs?
- [\[6\]](#) conjectured that all matroids are  $(1, 2)$ -decomposable. Is this really the case? If not, how about  $(1, O(1))$  or even  $(O(1), O(1))$ ? If not, can the class of matroids that have such partition decompositions have nice characterizations?
- In the event that general matroids are not  $(O(1), O(1))$ -decomposable, is polynomial-time  $O(1)$ -approximation still possible for matroid intersection cover with  $O(1)$  general matroids?
- Or even stronger, is  $O(k)$ -approximation possible with  $k$  general matroids?
- If the sets have weights, is  $O(1)$ -approximation possible for the problem of computing the minimum weight matroid intersection cover?

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