

ON WEISSLER'S CONJECTURE ON THE HAMMING CUBE I

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ABSTRACT. Let $1 \leq p \leq q < \infty$, and let $w \in \mathbb{C}$. Weisler conjectured that the Hermite operator $e^{w\Delta}$ is bounded as an operator from L^p to L^q on the Hamming cube $\{-1, 1\}^n$ with the norm bound independent of n if and only if

$$|p - 2 - e^{2w}(q - 2)| \leq p - |e^{2w}|q.$$

It was proved by Bonami (1970), Beckner (1975), and Weisler (1979) in all cases except $2 < p \leq q < 3$ and $3/2 < p \leq q < 2$, which stood open until now. The goal of this paper is to give a full proof of Weisler's conjecture in the case $p = q$. Several applications will be presented.

1. INTRODUCTION

1.1. Complex hypercontractivity. Given $n \geq 1$, let $\{-1, 1\}^n$ be the Hamming cube of dimension n , i.e., the set of vectors $x = (x_1, \dots, x_n)$ such that $x_j = 1$ or -1 for all $j = 1, \dots, n$. For any $f : \{-1, 1\}^n \rightarrow \mathbb{C}$, define its average value $\mathbb{E}f$ and its L_p norm $\|f\|_p$, $p \geq 1$, to be

$$\mathbb{E}f = \frac{1}{2^n} \sum_{x \in \{-1, 1\}^n} f(x) \quad \text{and} \quad \|f\|_p = (\mathbb{E}|f|^p)^{1/p}.$$

Functions on the Hamming cube can be represented via Fourier–Walsh series. Namely, for any $f : \{-1, 1\}^n \rightarrow \mathbb{C}$, we have

$$(1.1) \quad f(x) = \sum_{S \subset \{1, \dots, n\}} a_S w_S(x), \quad \text{where} \quad w_S(x) = \prod_{j \in S} x_j$$

and a_S are the Fourier coefficients of f . It follows from (1.1) that $a_S = \mathbb{E}f w_S$ and $a_\emptyset = \mathbb{E}f$. For any $z \in \mathbb{C}$, the Hermite operator T_z is defined as

$$T_z f(x) = \sum_{S \subset \{1, \dots, n\}} z^{|S|} a_S w_S(x),$$

where $|S|$ denotes the cardinality of the set $S \subset \{1, \dots, n\}$. Weisler [17] made the following

Conjecture. *Let $1 \leq p \leq q < \infty$, and let $z \in \mathbb{C}$. We have*

$$\sup_{\|f\|_p=1, n \geq 1} \|T_z f\|_q = C(p, q, z) < \infty$$

if and only if

$$(1.2) \quad |p - 2 - z^2(q - 2)| \leq p - |z|^2 q.$$

Moreover, $C(p, q, z) < \infty$ implies $C(p, q, z) = 1$, i.e., that T_z is contractive.

In this paper we prove Weisler's conjecture for $p = q$. We intend to settle the general case in an upcoming manuscript. Our argument for the case $p < q$ requires checking the positivity of two polynomials with large integer coefficients, which is currently hard to present in a human verifiable way.

2010 *Mathematics Subject Classification.* 39B62, 42B35, 47A30.

¹When $z = e^{-t}$, $t \geq 0$, the traditional notation for the Hermite operator is $e^{-t\Delta}$ instead of $T_{e^{-t}}$. In the quantum field literature T_z is called the second quantization operator of z . In computer science T_z is referred to as the noise operator.

1.2. Development of hypercontractivity in mid 70's: Boolean and Gaussian. In 1970, Bonami [2] considered the case $z = r \in \mathbb{R}$. She showed that it is enough to verify that T_r is contractive on the Hamming cube of dimension $n = 1$. A little bit earlier, in 1966, Nelson [14] (independently of Bonami) showed that the Gaussian analog T_r^G of T_r maps boundedly $L^2(d\gamma)$ to $L^q(d\gamma)$, ($q > 2$, $r \in \mathbb{R}$, $d\gamma$ is the standard Gaussian measure on $\mathbb{R}^k$) with norm independent of k provided that r is sufficiently close to zero. Later, Glimm [6] proved that for r sufficiently close to 0, T_r^G is in fact contractive as an operator from $L^2(d\gamma)$ to $L^q(d\gamma)$. In 1970, Segal [16] obtained a result which implies that it suffices to check the boundedness of the operator T_r^G in dimension $k = 1$. Finally, in 1973, Nelson [15] showed that if $1 \leq p \leq q < \infty$, then $T_r^G : L^p(d\gamma) \rightarrow L^q(d\gamma)$ is contractive if and only if $|r| \leq \sqrt{\frac{p-1}{q-1}}$; and if $|r| > \sqrt{\frac{p-1}{q-1}}$, then T_r^G is not even bounded.

We should mention that Nelson's result easily follows via the central limit theorem from Bonami's *real* hypercontractivity on the Hamming cube.

In 1975, Gross in his celebrated paper [7] gave a simple proof of the real hypercontractivity on the Hamming cube by showing its equivalence to log-Sobolev inequalities. Inspired by works of Nelson and Gross [2], Beckner in [1] obtained the Hausdorff–Young inequality with sharp constants by showing that it follows from the contractivity of $T_{i\sqrt{p-1}}$ from L^p to $L^{p'}$, $p' = \frac{p}{p-1}$, on the Hamming cube when $p \in (1, 2]$. At that time, the proofs of the real hypercontractivity by Nelson, and later by Gross, were real valued, and they could not be extended directly to complex z . The main technical part of Beckner's paper [1] is the proof of Bonami's two-point inequality when $z = i\sqrt{p-1} \in \mathbb{C}$, $q = p'$, and $p \in (1, 2]$.

It became an open problem under what conditions on the triples (p, q, z) with $1 \leq p \leq q < \infty$ and $z \in \mathbb{C}$ the operator T_z^G is bounded from $L^p(d\gamma)$ to $L^q(d\gamma)$ with norm independent of the dimension of the Euclidean space. In 1979, Coifman, Cwikel, Rochberg, Sagher, and Weiss [3] proved that T_z^G is a contraction from $L^p(d\gamma)$ to $L^{p'}(d\gamma)$ if z satisfies (1.2). The same year, Weissler [17] proved the full version of the conjecture except when $2 < p \leq q < 3$, and $3/2 < p \leq q < 2$. Weissler writes in his paper that he believes the theorem should be true for all $1 \leq p \leq q < \infty$, $z \in \mathbb{C}$ that satisfy (1.2). The main open problem was to prove that the two-point inequality of Bonami–Beckner

$$(1.3) \quad \left(\frac{|a + zb|^q + |a - zb|^q}{2} \right)^{1/q} \leq \left(\frac{|a + b|^p + |a - b|^p}{2} \right)^{1/p}$$

holds for all $a, b \in \mathbb{C}$ if the triple (p, q, z) satisfies condition (1.2). In 1989, Epperson [5] proved the Gaussian counterpart of the conjecture by showing that condition (1.2) implies $\|T_z^G\|_{L^p(d\gamma) \rightarrow L^q(d\gamma)} \leq 1$. His proof avoided the verification of difficult two-point inequalities (1.3) and, thereby, did not imply the corresponding result on the Hamming cube.

In 1990, Lieb obtained a very general theorem [11], which, in particular, implied the result of Epperson. After the work of Lieb, in 1997, Janson [10] gave one more proof of the Gaussian complex hypercontractivity via Ito calculus. Janson's argument was later rewritten in terms of heat flows by Hu [9]. Since 1979 no progress has been made on Weissler's conjecture.

1.3. Applications. In the case $p = q > 1$, condition (1.2) can be simplified to

$$\left| z \pm i \frac{|p-2|}{2\sqrt{p-1}} \right| \leq \frac{p}{2\sqrt{p-1}},$$

see (2.8). In other words, the admissible domain for z is a lens domain, i.e., an intersection of two disks (see Fig. 1).

The bounding circles of these disks pass through the points $z = \pm 1$, which belong to the boundary of the lens domain. Let $\pi\alpha_p$ be the exterior angle between the two circles at the point $z = 1$. We have

$$\alpha_p = 1 + \frac{2}{\pi} \arctan \left(\frac{|p-2|}{2\sqrt{p-1}} \right).$$

²Apparently not knowing about Bonami.

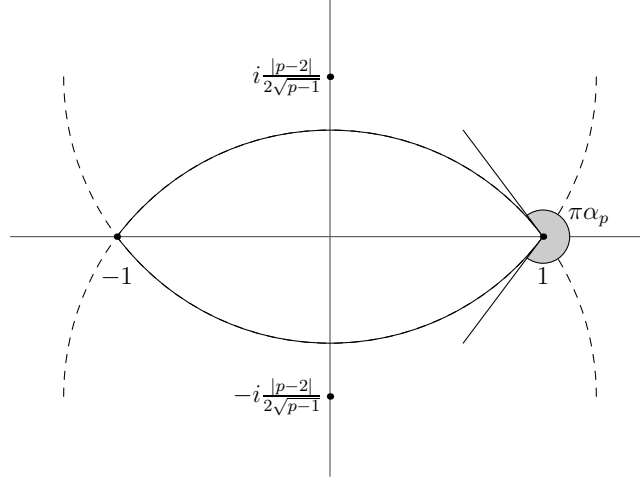


FIGURE 1. Lens domain.

Recently it was shown in [4] that the angle α_p plays an important role in Markov–Bernstein type estimates on the Hamming cube. We will list a series of corollaries that automatically follow from [4, 13] and the explicit knowledge of α_p . We remind that given $f : \{-1, 1\}^n \rightarrow \mathbb{C}$, its Laplacian Δf is defined by $\Delta f = \sum_{j=1}^n D_j f$ where

$$D_j f(x) = \frac{f(x_1, \dots, x_j, \dots, x_n) - f(x_1, \dots, -x_j, \dots, x_n)}{2}, \quad x = (x_1, \dots, x_n) \in \{-1, 1\}^n.$$

The operator Δ is linear, and $\Delta w_S(x) = |S|w_S(x)$. Also define the discrete gradient

$$|\nabla f|^2 = \sum_{j=1}^n (D_j f)^2.$$

Corollary 1.1. *For each $p > 1$, there exist finite $c_1, c_2, c_3 > 0$ depending on p such that for all $f = \sum_{S \subset \{1, \dots, n\}, |S| \geq d} a_S w_S$ and all $n \geq d$, we have*

$$(1.4) \quad \|e^{-t\Delta} f\|_p \leq c_1 e^{-c_2 d \min\{t, t^{\frac{1}{2-\alpha_p}}\}} \|f\|_p \quad \text{for all } t \geq 0,$$

and hence

$$(1.5) \quad \|\Delta f\|_p \geq c_3 d^{2-\alpha_p} \|f\|_p.$$

In [13] Mendel and Naor showed that if f takes values in a K -convex Banach space X , i.e., $a_S \in X$ for all $S \in \{1, \dots, n\}$, then inequality (1.4) holds with some $\alpha_p(X) \in (0, 1)$. They asked if $\alpha_p(X)$ in the inequalities (1.4) and (1.5) can be replaced by 1. The question is open even for $X = \mathbb{R}$ and is known as the “heat smoothing conjecture” (see [8], where the case $d = 1$ has been resolved, and [4], where the conjecture has been resolved for functions with “narrow” spectrum). In [4] the first author with A. Eskenazis showed that the conclusion of Corollary 1.1 holds for $p \in (1, 3/2) \cup (3, \infty)$ thanks to the theorem of Weisler [17]. Repeating the arguments in [4] verbatim and using the main result of this paper for $p = q \in (3/2, 3)$ we obtain Corollary 1.1 for all $p \in (1, \infty)$.

We know that the bounds (1.4) and (1.5) are not sharp in general; for example when $p > 2$ is such that $\alpha_p > 3/2$, i.e., $p > 4 + 2\sqrt{2}$, then better bounds are available due to Meyer.

Theorem 1.2 (Meyer [12]). *For each $p \geq 2$, there exist $C_p, c_p > 0$ such that for any $f = \sum_{S \subset \{1, \dots, n\}, |S| \geq d} a_S w_S$ and all $n \geq d$, we have*

$$\|e^{-t\Delta} f\|_p \leq e^{-c_p d \min\{t, t^2\}} \|f\|_p \quad \text{for all } t \geq 0,$$

and hence

$$\|\Delta f\|_p \geq C_p d^{1/2} \|f\|_p.$$

The next corollary is due to Eskenazis–Ivanisvili [4], proved for $p \in (1, 3/2) \cup (3, \infty)$ and now valid (by the main result of this paper) for all $p \in (1, \infty)$. It naturally extends Freud’s inequalities on the Hamming cube

Corollary 1.3. *For all $p > 1$, there exists finite $C_p > 0$ such that*

$$\begin{aligned} \|\Delta f\|_p &\leq 10d^{\alpha_p} \|f\|_p, \\ \|\nabla f\|_p &\leq \begin{cases} C_p d^{\alpha_p/p} \ln(d+1) \|f\|_p, & p \in (1, 2), \\ C_p d^{\alpha_p/2} \|f\|_p, & p \in [2, \infty], \end{cases} \end{aligned}$$

for all $f = \sum_{S \subset \{1, \dots, n\}, |S| \leq d} a_S w_S$ and all $n \geq d$.

1.3.1. Complex and real hypercontractivity. To illustrate the advantage of complex hypercontractivity over the real one let us outline a method described in [4] for obtaining bounds on the norms of Fourier multipliers on spaces of functions on the Hamming cube with restricted spectrum in a systematic way.

Let $1 < p \leq q$, and suppose we are interested in obtaining a bound of the type

$$(1.6) \quad \left\| \sum_{S \subset \{0, \dots, n\}, |S| \leq d} \varphi(|S|) a_S w_S \right\|_q \leq C_{p,q,d,\varphi} \left\| \sum_{S \subset \{0, \dots, n\}, |S| \leq d} a_S w_S \right\|_p$$

where φ is some fixed function, which is usually called a Fourier multiplier. The complex hypercontractivity yields the bound

$$(1.7) \quad \left\| \sum_{S \subset \{1, \dots, n\}, |S| \leq d} z^{|S|} a_S w_S \right\|_q \leq \left\| \sum_{S \subset \{1, \dots, n\}, |S| \leq d} a_S w_S \right\|_p,$$

which holds true for all $z \in \Omega_{p,q}$, where $\Omega_{p,q}$ is the domain described in (1.2). To obtain the bound of the type (1.6) it suffices to find a complex valued measure μ on $\Omega_{p,q}$ such that

$$\int_{\Omega_{p,q}} z^j d\mu(z) = \varphi(j) \quad \text{for all } j = 0, \dots, d.$$

Then the triangle inequality together with (1.7) gives

$$\begin{aligned} \left\| \sum_{S \subset \{1, \dots, n\}, |S| \leq d} \varphi(|S|) a_S w_S \right\|_q &\leq \int_{\Omega_{p,q}} \left\| \sum_{S \subset \{1, \dots, n\}, |S| \leq d} z^{|S|} a_S w_S \right\|_q d|\mu|(z) \stackrel{(1.7)}{\leq} \\ &\|\mu\| \left\| \sum_{S \subset \{1, \dots, n\}, |S| \leq d} a_S w_S \right\|_p, \end{aligned}$$

where $\|\mu\|$ stands for the total variation norm of μ .

To minimize the norm of μ , we may invoke the Hahn–Banach theorem together with the Riesz representation theorem. Let $\mathcal{P}^d \subset C(\Omega_{p,q})$ be a subspace consisting of all analytic polynomials of degree d . Let L be a linear functional on $\mathcal{P}^d$ such that $L(z^j) = \varphi(j)$ for all $0 \leq j \leq d$. Clearly its norm $\|L\|$ is the smallest positive constant $C_{p,q,d,\varphi}$ for which

$$(1.8) \quad \left| \sum_{j=0}^d \varphi(j) a_j \right| \leq C_{p,q,d,\varphi} \left\| \sum_{j=0}^d a_j z^j \right\|_{C(\Omega_{p,q})} \quad \text{for all } a_0, \dots, a_d \in \mathbb{C}.$$

By the Hahn–Banach theorem, there exists $\tilde{L} \in C^*(\Omega_{p,q})$ such that $\tilde{L}|_{\mathcal{P}^d} = L$, and $\|\tilde{L}\|_{C^*(\Omega_{p,q})} = C_{p,q,d,\varphi}$. By the Riesz representation theorem $\tilde{L}(h) = \int_{\Omega_{p,q}} h(z) d\mu(z)$ for some complex valued Radon measure $d\mu$ on $\Omega_{p,q}$ such that $\int_{\Omega_{p,q}} d|\mu|(z) = \|\tilde{L}\|_{C^*(\Omega_{p,q})} = C_{p,q,d,\varphi}$. Thus we obtain (1.6) with the constant $C_{p,q,d,\varphi}$ that solves the extremal problem (1.8).

If we use only the real hypercontractivity in this argument, then the space $C(\Omega_{p,q})$ on the right hand side of (1.8) will be replaced by $C(\Omega_{p,q}^{\mathbb{R}})$ where

$$\Omega_{p,q}^{\mathbb{R}} = \left[-\sqrt{\frac{p-1}{q-1}}, \sqrt{\frac{p-1}{q-1}} \right] \subset \mathbb{R},$$

in which case one gets (1.8) with a constant $C_{p,q,d,\varphi}^{\mathbb{R}} \geq C_{p,q,d,\varphi}$ due to the fact that $\Omega_{p,q}^{\mathbb{R}} \subset \Omega_{p,q}$.

Thus we see that “the amount of improvement” the complex hypercontractivity gives over the real one is determined by how small the norm of the functional L on $\mathcal{P}^d$ equipped with the $C(\Omega_{p,q})$ norm can be compared to the case when the norm on $\mathcal{P}^d$ is replaced by the $C(\Omega_{p,q}^{\mathbb{R}})$ one.

Some more applications of complex hypercontractivity are given in [4].

2. THE PROOF

We start with several observations (Sections 2.1 – 2.5) that are well known to experts. We decided to include them here for the reader's convenience.

2.1. Tensor power trick and induction on dimension. In this section we show the equivalence of several inequalities.

Lemma 2.1. *Let $1 \leq p \leq q < \infty$ and $z \in \mathbb{C}$ be fixed. The following are equivalent:*

- (i) $\|T_z f\|_q \leq C(p, q, z) \|f\|_p$ for some $C(p, q, z) < \infty$, all $f : \{-1, 1\}^n \rightarrow \mathbb{C}$, and all $n \geq 1$.
- (ii) $\|T_z f\|_q \leq \|f\|_p$ for all $f : \{-1, 1\}^n \rightarrow \mathbb{C}$ and all $n \geq 1$.
- (iii) $\|T_z f\|_q \leq \|f\|_p$ for all $f : \{-1, 1\} \rightarrow \mathbb{C}$.

Clearly it follows from the lemma that the conjecture will be proved once the equivalence of the two-point inequality (iii) and the condition (1.2) is verified.

Proof. Obviously (ii) implies (i). To show that (i) implies (ii), consider $F(x) = f(x^1) \cdots f(x^k)$ where $x = (x^1, \dots, x^k) \in \{-1, 1\}^{nk}$. Then $\|F\|_p = \|f\|_p^k$, $\|T_z F\|_q = \|T_z f\|_q^k$. Therefore, (i) implies $\|T_z F\|_q \leq C(p, q, z) \|F\|_p$, which in turn implies $\|T_z f\|_q \leq (C(p, q, z))^{1/k} \|f\|_p$. Letting $k \rightarrow \infty$ we obtain (ii).

Obviously (ii) implies (iii). Next, we show that (iii) implies (ii). Let

$$(2.1) \quad f(x_1, \dots, x_n) = \sum_{S \subset \{1, \dots, n\}} a_S \prod_{j \in S} x_j.$$

Let us extend the domain of definition of f to be all $\mathbb{R}^n$ by considering the right hand side of (2.1) as a multivariate polynomial of variables $x_1, \dots, x_n$. Then we can write

$$T_z f(x_1, \dots, x_n) = f(zx_1, \dots, zx_n) \quad \text{for all } (x_1, \dots, x_n) \in \{-1, 1\}^n.$$

By (iii), for any complex numbers $A, B \in \mathbb{C}$, we have

$$(2.2) \quad \mathbb{E}_{x_1} |A + zx_1 B|^q \leq (\mathbb{E}_{x_1} |A + x_1 B|^p)^{q/p}$$

where $\mathbb{E}_{x_1}$ means that we take the expectation with respect to the symmetric ± 1 Bernoulli random variable x_1 . Since

$$f(zx_1, zx_2, \dots, zx_n) = A(zx_2, \dots, zx_n) + zx_1 B(zx_2, \dots, zx_n),$$

we can write

$$\begin{aligned} \|T_z f\|_q^p &= (\mathbb{E}_{x_n} \dots \mathbb{E}_{x_1} |A(zx_2, \dots, zx_n) + zx_1 B(zx_2, \dots, zx_n)|^q)^{p/q} \stackrel{(2.2)}{\leq} \\ &\left(\mathbb{E}_{x_n} \dots \mathbb{E}_{x_2} (\mathbb{E}_{x_1} |A(zx_2, \dots, zx_n) + x_1 B(zx_2, \dots, zx_n)|^p)^{q/p} \right)^{p/q} \stackrel{\text{Minkowski}}{\leq} \\ &\mathbb{E}_{x_1} (\mathbb{E}_{x_n} \dots \mathbb{E}_{x_2} |A(zx_2, \dots, zx_n) + x_1 B(zx_2, \dots, zx_n)|^q)^{p/q} \stackrel{\text{induction}}{\leq} \mathbb{E} |f|^p. \end{aligned}$$

Notice that in the second inequality we used the condition $1 \leq q/p$. □

In the next section we explain where the condition (1.2) comes from.

2.2. From global to local: the necessity part. The condition (iii) of Lemma 2.1 is equivalent to the following two-point inequality

$$(2.3) \quad \left(\frac{|1 + wz|^q + |1 - wz|^q}{2} \right)^{1/q} \leq \left(\frac{|1 + w|^p + |1 - w|^p}{2} \right)^{1/p},$$

which should hold true for all $w \in \mathbb{C}$. Let us explain that condition (1.2) is in fact an “infinitesimal” form of the inequality (2.3) when $w \rightarrow 0$. Indeed, let $w = \varepsilon v$ where $v \in \mathbb{C}$, $|v| = 1$, is fixed and $\varepsilon > 0$. As $\varepsilon \rightarrow 0$, we have

$$\begin{aligned} |1 + w|^p &= (1 + 2\varepsilon \Re v + \varepsilon^2 |v|^2)^{p/2} \\ &= 1 + \frac{p}{2} (2\varepsilon \Re v + \varepsilon^2 |v|^2) + \frac{p}{4} \left(\frac{p}{2} - 1 \right) 4\varepsilon^2 (\Re v)^2 + O(\varepsilon^3) \\ &= 1 + \varepsilon p \Re v + \varepsilon^2 \frac{p}{2} (|v|^2 + (p-2)(\Re v)^2) + O(\varepsilon^3). \end{aligned}$$

Therefore, comparing the second order terms, we see that the two-point inequality (2.3), in particular, implies that

$$(2.4) \quad |vz|^2 + (q-2)(\Re vz)^2 \leq |v|^2 + (p-2)(\Re v)^2 \quad \text{for all unit vectors } v \in \mathbb{C}.$$

The last inequality can be rewritten as

$$(p-2) \left(\frac{v + \bar{v}}{2} \right)^2 - (q-2) \left(\frac{vz + \bar{v}z}{2} \right)^2 \geq |z|^2 - 1.$$

Multiplying by 2 and opening the parentheses, we obtain

$$(p-2) - (q-2)|z|^2 + \Re[(p-2) - (q-2)z^2]v^2 \geq 2|z|^2 - 2,$$

i.e.,

$$-\Re[(p-2) - (q-2)z^2]v^2 \leq p - q|z|^2,$$

which, since v is an arbitrary unit vector, is equivalent to (1.2).

2.3. The inf representation. It will be helpful to describe the domain $\Omega_{p,q}$ of all z 's satisfying (1.2) in polar coordinates. Notice that if $q = 1$, then $p = 1$ and, therefore, (1.2) implies that $z \in [-1, 1]$. In this case (1.3) trivially holds by convexity. In what follows we assume that $q > 1$.

Let $z = re^{it} \in \Omega_{p,q}$ and $v = e^{i\beta}$. Then (2.4) takes the form

$$r^2 (1 + (q-2) \cos^2(t + \beta)) \leq 1 + (p-2) \cos^2(\beta).$$

Dividing both sides of this inequality by $1 + (q-2) \cos^2(t + \beta)$ and taking the infimum over all $\beta \in \mathbb{R}$, we obtain

$$r \leq \inf_{\beta \in \mathbb{R}} \sqrt{\frac{1 + (p-2) \cos^2(\beta)}{1 + (q-2) \cos^2(t + \beta)}}.$$

Given $t \in \mathbb{R}$, let $z = r(t)e^{it}$, $r(t) \geq 0$ be such that z lies on the boundary of $\Omega_{p,q}$. Then

$$(2.5) \quad r(t) = \inf_{\beta \in \mathbb{R}} \sqrt{\frac{1 + (p-2) \cos^2(\beta)}{1 + (q-2) \cos^2(t + \beta)}}.$$

It follows from (2.5) that $r(t)$ is an even π -periodic function.

Throughout the rest of the paper we assume that $p = q > 1$.

2.4. Lens domain. For $p = q$ the two-point inequality (2.3) takes the form

$$(2.6) \quad |1 + wz|^p + |1 - wz|^p \leq |1 + w|^p + |1 - w|^p \quad \text{for all } w \in \mathbb{C},$$

and (1.2) takes the form

$$(2.7) \quad 1 - |z|^2 \geq \frac{|p-2|}{p} |1 - z^2|.$$

Squaring and subtracting $(1 - |z|^2)^2 \frac{(p-2)^2}{p^2}$ from both sides of the inequality (2.7), we obtain

$$(1 - |z|^2)^2 \frac{4(p-1)}{p^2} \geq \frac{(p-2)^2}{p^2} (|1 - z^2|^2 - (1 - |z|^2)^2) = \frac{(p-2)^2}{p^2} \left(\frac{z - \bar{z}}{i} \right)^2.$$

From (2.7) we see that $|z| \leq 1$, so taking the square root in the last inequality we obtain

$$1 - |z|^2 \geq \frac{|p-2|}{\sqrt{p-1}} |\Im z|.$$

The last condition can be rewritten as

$$(2.8) \quad \left| z \pm i \frac{|p-2|}{2\sqrt{p-1}} \right| \leq \frac{p}{2\sqrt{p-1}}.$$

The set (2.8) represents a “lens” domain, i.e., the intersection of two discs centered at points $\pm \frac{i|p-2|}{2\sqrt{p-1}}$ of radii $\frac{p}{2\sqrt{p-1}}$ whose boundary circles pass through the points 1 and -1 .

Recall that the boundary of $\Omega_{p,p}$ is described in polar coordinates by the equation $z = r(t)e^{it}$. It is easy to see from (2.8) that $r(t)$ is a decreasing function on $[0, \pi/2]$,

$$(2.9) \quad r(0) = 1, \quad r(\pi/2) = \min \left\{ \sqrt{p-1}, \frac{1}{\sqrt{p-1}} \right\}.$$

In the next section we explain that it is enough to prove the two-point inequality (2.6) only for $p \geq 2$ and a certain family of points z and w .

2.5. Duality, symmetry, and convexity. Since $(T_z)^* = T_{\bar{z}}$, and $\|T_z\|_{L^p \rightarrow L^p} = \|(T_z)^*\|_{L^{p'} \rightarrow L^{p'}}$ (as usual $p' = \frac{p}{p-1}$), we may assume without loss of generality that $p \geq 2$. The reader may also verify that the condition (2.8) is invariant under the replacement of p by p' .

Next, we claim that it suffices to check (2.6) for $z \in \partial\Omega_{p,p}$. Indeed, every interior point $z \in \Omega_{p,p}$ can be written as a convex combination of two boundary points z_1, z_2 of $\Omega_{p,p}$. Since the function

$$z \mapsto \frac{|1 + wz|^p + |1 - wz|^p}{2}$$

is convex on $\mathbb{C}$, its value at z does not exceed the maximum of its values at z_1 and z_2 .

In what follows, we set $z = r(t)e^{it}$, $t \in \mathbb{R}$. Let us rewrite (2.6) as

$$(2.10) \quad \left| 1 + \frac{w}{z} \right|^p + \left| 1 - \frac{w}{z} \right|^p \geq |1 + w|^p + |1 - w|^p.$$

Let

$$(2.11) \quad c(t) = \frac{1}{r(t)} \in [1, \sqrt{p-1}] \quad (\text{recall that } p \geq 2)$$

and let $w = ye^{ia}$ with $y \geq 0, a \in \mathbb{R}$. Notice that $w/z = c(t)ye^{i(-t+a)} = c(-t)ye^{i(-t+a)}$. Changing the variable $-t$ back to t , we can rewrite the inequality (2.10) as follows

$$(2.12) \quad |1 + c(t)ye^{i(t+a)}|^p + |1 - c(t)ye^{i(t+a)}|^p \geq |1 + ye^{ia}|^p + |1 - ye^{ia}|^p.$$

Lemma 2.2. *It is enough to check (2.12) for $0 \leq a \leq a+t \leq \frac{\pi}{2}$.*

Proof. Denote for brevity $c = c(t)$, and rewrite (2.12) as

$$(2.13) \quad (c^2 y^2 + 1 + 2cy \cos(a+t))^{p/2} + (c^2 y^2 + 1 - 2cy \cos(a+t))^{p/2} \geq (y^2 + 1 + 2y \cos(a))^{p/2} + (y^2 + 1 - 2y \cos(a))^{p/2}.$$

The map $s \mapsto |1 + sye^{i(t+a)}|^p$ is convex. Therefore the map

$$s \mapsto |1 + sye^{i(t+a)}|^p + |1 - sye^{i(t+a)}|^p = (s^2 y^2 + 1 + 2sy \cos(a+t))^{p/2} + (s^2 y^2 + 1 - 2sy \cos(a+t))^{p/2}$$

is increasing for $s \geq 0$. Since $c \geq 1$, we have

$$(c^2 y^2 + 1 + 2cy \cos(a+t))^{p/2} + (c^2 y^2 + 1 - 2cy \cos(a+t))^{p/2} \geq (y^2 + 1 + 2y \cos(a+t))^{p/2} + (y^2 + 1 - 2y \cos(a+t))^{p/2}.$$

Also notice that since $p \geq 2$, the map $s \mapsto (A + Bs)^{p/2}$ is convex and, thereby, the map $s \mapsto (A + Bs)^{p/2} + (A - Bs)^{p/2}$ is increasing for $s \geq 0$ as long as $A \pm Bs \geq 0$. Thus, if $|\cos(a+t)| \geq |\cos(a)|$, then

$$(y^2 + 1 + 2y \cos(a+t))^{p/2} + (y^2 + 1 - 2y \cos(a+t))^{p/2} \geq (y^2 + 1 + 2y \cos(a))^{p/2} + (y^2 + 1 - 2y \cos(a))^{p/2},$$

i.e., inequality (2.12) trivially holds whenever $|\cos(a+t)| \geq |\cos(a)|$.

By the 2π -periodicity of $\cos(x)$ and $c(t)$ we can assume that $a, t \in [0, 2\pi)$.

(i) Suppose $a \in [0, \pi/2]$. The assumption $|\cos(a+t)| < |\cos(a)|$ implies that $a+t \in [a, \pi - a] \cup [\pi + a, 2\pi - a]$. Consider first the case when $a+t \in [a, \pi - a]$. If $a+t \in [\pi/2, \pi - a]$, then $t^* = \pi - t - 2a \geq 0$. Clearly t^* satisfies $a+t^* \leq \pi/2$, $|\cos(a+t)| = |\cos(a+t^*)|$. Since $t, t+2a \in [0, \pi]$ and $t+a \geq \pi/2$, the inequality $|\pi/2 - t| \leq |\pi/2 - (2a+t)| \leq \pi/2$ holds and, thereby, $c(t^*) = c(t+2a) \leq c(t)$ (we remind that $c(s) = 1/r(s)$, $s \mapsto r(s)$ is decreasing on $[0, \pi/2]$ and $r(s) = r(\pi - s)$ for $s \in [0, \pi/2]$, so the closer $s \in [0, \pi]$ is to $\pi/2$, the larger $c(s)$ is). Thus (2.13) becomes stronger if we replace t by t^* .

If $a+t \in [a+\pi, 2\pi - a]$, then we consider $t^* = t - \pi$ and use the fact that $c(t^*) = c(t)$ and $|\cos(a+t)| = |\cos(a+t^*)|$. On the other hand, $a+t^* \in [a, \pi - a]$, which reduces this subcase to the previously considered one.

(ii) Suppose $a \in (\pi/2, \pi)$. Consider $a^* = \pi - a \in (0, \pi/2)$ and $t^* = 2\pi - t$. Then $|\cos(a^*)| = |\cos(a)|$, $|\cos(a^*+t^*)| = |\cos(a+t)|$, $c(t) = c(t^*)$, and the inequality reduces to case (i).

(iii) Suppose $a \in [\pi, 2\pi)$. Then replace a by $a^* = a - \pi$ and reduce the inequality in question to the previous two cases. The lemma is proved. $\square$

2.6. Proof of (2.13) when $p \geq 3$ via “mock log-Sobolev inequality”. Let us give a proof of (2.13) for $p \geq 3$. This case was proved by Weissler [17]. His argument is similar to the proof of the equivalence of the log-Sobolev inequality and the real hypercontractivity. Indeed, let us briefly mention the connection. The real hypercontractivity is equivalent (see [2]) to the following two-point inequality

$$(2.14) \quad \left(\frac{|a + \sqrt{\frac{p-1}{q-1}} b|^q + |a - \sqrt{\frac{p-1}{q-1}} b|^q}{2} \right)^{1/q} \leq \left(\frac{|a + b|^p + |a - b|^p}{2} \right)^{1/p}$$

for all $1 < p \leq q < \infty$ and all $a, b \in \mathbb{R}$. The factor $\sqrt{\frac{p-1}{q-1}}$ is a ratio of the values of the same function $s \mapsto \sqrt{s-1}$ at $s = p$ and $s = q$. Therefore (2.14) is equivalent to the statement that the mapping

$$(2.15) \quad p \mapsto \left(\frac{\left| a + \frac{x}{\sqrt{p-1}} \right|^p + \left| a - \frac{x}{\sqrt{p-1}} \right|^p}{2} \right)^{1/p}$$

is decreasing on $(1, \infty)$ for all fixed $a, x \in \mathbb{R}$. Differentiating (2.15) with respect to p , we arrive at what is called “the log-Sobolev inequality on the two-point space” (see [7]).

Ideally, we would like to use the same idea when proving (2.13). The first obstacle is that the factor $c(t)$ does not have the desired quotient structure. Nevertheless, using (2.11) and the representation (2.5), we can fix this issue estimating $c(t)$ from below as

$$(2.16) \quad c(t) = \sup_{\beta \in \mathbb{R}} \frac{\sqrt{1 + (p-2)\cos^2(t+\beta)}}{\sqrt{1 + (p-2)\cos^2(\beta)}} \stackrel{\beta=-a-t}{\geq} \frac{\sqrt{1 + (p-2)\cos^2(a)}}{\sqrt{1 + (p-2)\cos^2(a+t)}}.$$

Therefore, (2.13) with fixed $p > 2$ is implied by (but no longer equivalent to) the statement that the mapping

$$(2.17) \quad s \mapsto \left(1 + \frac{x^2}{1 + (p-2)\cos^2(s)} + \frac{2x\cos(s)}{\sqrt{1 + (p-2)\cos^2(s)}} \right)^{p/2} + \left(1 + \frac{x^2}{1 + (p-2)\cos^2(s)} - \frac{2x\cos(s)}{\sqrt{1 + (p-2)\cos^2(s)}} \right)^{p/2}$$

is increasing on $[0, \pi/2]$ for all $x \geq 0$ (and the fact that for $p > 2$ the left hand side of (2.13) is increasing in c). This statement seems to be a right substitute for the log-Sobolev inequality in the complex contractivity case. The next lemma shows that, unfortunately, this monotonicity holds only for $p \geq 3$.

Lemma 2.3 (“Mock log-Sobolev inequality”). *Let $p > 2$. The map (2.17) is increasing on $[0, \pi/2]$ for all fixed $x \in \mathbb{R}$ if and only if $p \geq 3$.*

Proof. Denote $b = \frac{1}{1+(p-2)\cos^2(s)} \in [\frac{1}{p-1}, 1]$ and $u = \frac{x}{\sqrt{p-2}} \in \mathbb{R}$. We want to show that the map

$$b \mapsto \left(1 + bu^2(p-2) + 2u\sqrt{1-b} \right)^{p/2} + \left(1 + bu^2(p-2) - 2u\sqrt{1-b} \right)^{p/2}$$

is increasing on $[\frac{1}{p-1}, 1]$. Without loss of generality, assume $u \geq 0$. After taking the derivative with respect to b , we end up with showing that

$$(2.18) \quad \left(\frac{1 + bu^2(p-2) - 2u\sqrt{1-b}}{1 + bu^2(p-2) + 2u\sqrt{1-b}} \right)^{p/2-1} - \frac{1 - u(p-2)\sqrt{1-b}}{1 + u(p-2)\sqrt{1-b}} \geq 0.$$

If $b = \frac{1}{p-1}$, then (2.18) takes the form

$$(2.19) \quad \left| \frac{1 - u\sqrt{\frac{p-2}{p-1}}}{1 + u\sqrt{\frac{p-2}{p-1}}} \right|^{p-2} - \frac{1 - u\sqrt{\frac{p-2}{p-1}}(p-2)}{1 + u\sqrt{\frac{p-2}{p-1}}(p-2)} \geq 0.$$

Denote $u\sqrt{\frac{p-2}{p-1}} = k \geq 0$ and $p-2 = \alpha \geq 0$. If $\alpha \in (0, 1)$ and $k \in (0, 1)$, then

$$(2.20) \quad \left| \frac{1-k}{1+k} \right|^\alpha \leq \frac{1-k\alpha}{1+k\alpha}.$$

Indeed, it follows from the following general principle: if $a_k \geq 0$ and the function $g(x) = 1 - \sum_{k \geq 1} a_k x^k > 0$ on $(-1, 1)$, then

$$\frac{g(x)}{g(-x)} = \frac{1 - a_1 x - \sum_{k \geq 2} a_k x^k}{1 + a_1 x - \sum_{k \geq 2} a_k (-1)^k x^k} \leq \frac{1 - a_1 x}{1 + a_1 x}, \quad x \in [0, 1)$$

because of the inequality $\sum_{k \geq 2} a_k x^k \geq \sum_{k \geq 2} a_k (-1)^k x^k$ and the fact that the mapping $s \mapsto \frac{A+s}{B+s}$ is increasing when $A \leq B$ and both the numerator and the denominator are nonnegative. Note that $g(k) = (1-k)^\alpha$ satisfies the assumptions of this principle when $\alpha \in (0, 1)$. Thus for $p \in (2, 3)$ we obtain the inequality which is reverse to (2.19).

Let now that $p \geq 3$. In (2.18) we can assume that $u \geq 0$ is such that $1 - u(p-2)\sqrt{1-b} > 0$, otherwise there is nothing to prove. We have

$$\begin{aligned} \left[\left(\frac{1 - u(p-2)\sqrt{1-b}}{1 + u(p-2)\sqrt{1-b}} \right)^{\frac{1}{p-2}} \right]^2 &\stackrel{(2.20)}{\leq} \left[\frac{1 - u\sqrt{1-b}}{1 + u\sqrt{1-b}} \right]^2 = \frac{1 - 2u\sqrt{1-b} + u^2(1-b)}{1 + 2u\sqrt{1-b} + u^2(1-b)} \\ &\leq \frac{1 - 2u\sqrt{1-b} + bu^2(p-2)}{1 + 2u\sqrt{1-b} + bu^2(p-2)}, \end{aligned}$$

where in the last inequality we used the above observation about the monotonicity of the mapping $s \mapsto \frac{A+s}{B+s}$ again (note that $1-b \leq b(p-2)$ since $b \geq \frac{1}{p-1}$). The obtained inequality is the same as (2.18). The lemma is proved. $\square$

Clearly the lemma proves inequality (2.13) in the case $p \geq 3$. In particular, we just reproved the $p = q$ case of the theorem of Weissler, i.e., showed that the conjecture holds for all $p \geq 1$ except $p \in (3/2, 2) \cup (2, 3)$. The reader may think that the argument presented in this section is different from the one of Weissler [17] because, for example, Weissler uses non-trivial estimates for a certain implicitly defined function. Nevertheless, we should say that both arguments are essentially the same because they use the inequality (2.16) and the monotonicity expressed by the mock log-Sobolev inequality.

Before we move to the case $p \in (2, 3)$, let us explain in the next section that the monotonicity approach we just presented cannot be adapted to that case.

2.7. Why is the case $p \in (2, 3)$ difficult? Uniqueness lemma. Weissler writes in his paper (see a remark on page 117 in [17]) “Even though Proposition 7 is false without the condition $p \geq 3$, one should not give hope for (3.9)”. Without going into the details, this remark says that the reason the monotonicity argument (2.17) fails when $p \in (2, 3)$ is because the estimate (2.16) was too rough. One could hope that there might be a better substitute for (2.16). However, we will now show that this is not the case and thus one should give up on the chase for “monotone quantities” when $p \in (2, 3)$.

Let $f \in C^1([0, \pi/2])$, $f > 0$, be such that

$$(2.21) \quad c(t) \geq \frac{f(a)}{f(a+t)} \quad \text{for all } 0 \leq a \leq a+t \leq \pi/2$$

and the map

$$(2.22) \quad \psi(s) = \left(1 + \frac{x^2}{f^2(s)} + \frac{2x \cos(s)}{f(s)} \right)^{p/2} + \left(1 + \frac{x^2}{f^2(s)} - \frac{2x \cos(s)}{f(s)} \right)^{p/2}$$

is increasing on $[0, \pi/2]$ for all $x \geq 0$. Clearly, as we have seen in the previous section, if such f exists, then the two-point inequality (2.13) follows.

Lemma 2.4 (Uniqueness of the mock log-Sobolev inequality). *If (2.21) holds and the map given by (2.22) is increasing on $[0, \pi/2]$ for all $x \geq 0$, then necessarily $f(s) = C\sqrt{1 + (p-2)\cos^2(s)}$ on $[0, \pi/2]$ for some constant $C > 0$.*

In other words, the lemma says that one needs to come up with a different approach to prove (2.13) when $p \in (2, 3)$.

Proof. Notice that when $x \approx 0$, the map (2.22) behaves as

$$\psi(s) = 2 + \frac{p(1 + (p-2)\cos^2(s))}{f(s)^2} x^2 + O(x^4).$$

Therefore, the map

$$(2.23) \quad h(s) = \frac{1 + (p-2)\cos^2(s)}{f(s)^2}$$

should be increasing on $[0, \pi/2]$. The latter together with (2.21) implies that

$$(2.24) \quad c(t) \geq \frac{f(a)}{f(a+t)} \geq \frac{\sqrt{1+(p-2)\cos^2(a)}}{\sqrt{1+(p-2)\cos^2(a+t)}}.$$

Next we claim that h is constant on $[0, \pi/2]$. To prove the claim, we notice that the monotonicity of h , i.e., the condition $h'(s) \geq 0$, can be written as

$$(2.25) \quad \frac{d}{ds} \ln(f(s)) \leq \frac{d}{ds} \ln(g(s))$$

where $g(s) = \sqrt{1+(p-2)\cos^2(s)}$. Integrating (2.25) over the interval $[0, \pi/2]$ with respect to s , we obtain

$$\frac{1}{c(\pi/2)} \stackrel{(2.24)}{\leq} \frac{f(\pi/2)}{f(0)} \stackrel{(2.25)}{\leq} \frac{g(\pi/2)}{g(0)} = \frac{1}{\sqrt{p-1}} \stackrel{(2.9)}{=} r(\pi/2) \stackrel{(2.11)}{=} \frac{1}{c(\pi/2)},$$

which means that (2.25) must be an equality for all $s \in (0, \pi/2)$ and, thereby, $f(s) = Cg(s)$ on $(0, \pi/2)$. The claim, and hence the lemma is proved. $\square$

2.8. Self-improvement and hidden invariance in the two-point inequality.

Lemma 2.5. *It is enough to check (2.12) for $0 \leq c(t)y \leq 1$.*

Proof. Assuming that the inequality (2.12) holds with some fixed a and t for all y satisfying $0 \leq cy \leq 1$ where $c = c(t)$, we show that it also holds with the same a, t for the case when $cy > 1$. Fix y such that $cy > 1$. Dividing both sides of the inequality by $(cy)^p$, we can rewrite (2.12) as

$$(2.26) \quad \left| \frac{1}{cy} + e^{i(t+a)} \right|^p + \left| \frac{1}{cy} - e^{i(t+a)} \right|^p \geq \left| \frac{1}{cy} + e^{ia} \frac{1}{c} \right|^p + \left| \frac{1}{cy} - e^{ia} \frac{1}{c} \right|^p.$$

Using the identities $\left| \frac{1}{cy} \pm e^{i(t+a)} \right|^p = \left| \frac{1}{cy} e^{i(t+a)} \pm 1 \right|^p$ and $\left| \frac{1}{cy} \pm e^{ia} \frac{1}{c} \right|^p = \left| \frac{1}{cy} e^{ia} \pm \frac{1}{c} \right|^p$ we can rewrite (2.26) as

$$\left| \frac{1}{cy} e^{i(t+a)} + 1 \right|^p + \left| \frac{1}{cy} e^{i(t+a)} - 1 \right|^p \geq \left| \frac{1}{cy} e^{ia} + \frac{1}{c} \right|^p + \left| \frac{1}{cy} e^{ia} - \frac{1}{c} \right|^p.$$

To verify the latter inequality let $\tilde{y} = \frac{1}{c^2 y}$. Then $c\tilde{y} < 1$, i.e., we are in the range in which we assumed the validity of the estimate (2.12) for the pair $c, \tilde{y}$. Applying (2.12) to $c, \tilde{y}$ we obtain

$$\left| \frac{1}{cy} e^{i(t+a)} + 1 \right|^p + \left| \frac{1}{cy} e^{i(t+a)} - 1 \right|^p \geq \left| \frac{1}{c^2 y} e^{ia} + 1 \right|^p + \left| \frac{1}{c^2 y} e^{ia} - 1 \right|^p.$$

Next, we claim that

$$\left| \frac{1}{c^2 y} e^{ia} + 1 \right|^p + \left| \frac{1}{c^2 y} e^{ia} - 1 \right|^p \geq \left| \frac{1}{cy} e^{ia} + \frac{1}{c} \right|^p + \left| \frac{1}{cy} e^{ia} - \frac{1}{c} \right|^p.$$

Indeed, after multiplying both sides of the inequality by c^p , we can rewrite the latter estimate as

$$(2.27) \quad \left(c^2 + \frac{1}{c^2 y^2} + \frac{2}{y} \cos(a) \right)^{p/2} + \left(c^2 + \frac{1}{c^2 y^2} - \frac{2}{y} \cos(a) \right)^{p/2} \geq \left(1 + \frac{1}{y^2} + \frac{2}{y} \cos(a) \right)^{p/2} + \left(1 + \frac{1}{y^2} - \frac{2}{y} \cos(a) \right)^{p/2}.$$

Next, notice that

$$c^2 + \frac{1}{c^2 y^2} - \left(1 + \frac{1}{y^2} \right) = \frac{(c^2 - 1)(c^2 y^2 - 1)}{c^2 y^2} \geq 0,$$

where we have used the fact that $c \geq 1$ and $cy > 1$. Therefore (2.27) follows from the fact that the mapping $s \mapsto (s + A)^{p/2} + (s - A)^{p/2}$ is increasing on $[A, \infty)$. Thus (2.12) holds for all $y \geq 0$. $\square$

2.9. From multiplicativity to additivity: chasing the fourth order terms. Let $p \in (2, 3)$. We only need to verify (2.12) in the regime when $0 \leq a \leq a+t \leq \frac{\pi}{2}$ and $0 \leq cy \leq 1$ where $c = c(t)$. Indeed, assuming that (2.12) is proved for such a, t, y , Lemma 2.5 allows us to extend the range of y to $[0, +\infty)$ keeping the restriction on a, t only. On the other hand Lemma 2.2 says that if (2.12) holds for some $y \geq 0$ and all a, t such that $0 \leq a \leq a+t \leq \pi/2$ then it holds for the same y and all a, t .

We would like to prove the inequality

$$(2.28) \quad (1 + c^2 y^2 + 2cy \cos(a+t))^s + (1 + c^2 y^2 - 2cy \cos(a+t))^s \geq (1 + y^2 + 2y \cos(a))^s + (1 + y^2 - 2y \cos(a))^s.$$

where

$$(2.29) \quad s = \frac{p}{2} \in \left(1, \frac{3}{2}\right), \quad c = c(t) = \frac{1}{r(t)} \in [1, \sqrt{p-1}].$$

Dividing both sides of (2.28) by $2(1+y^2)^s$ and expanding both sides into power series, we can rewrite (2.28) as

$$\left(\frac{1+c^2 y^2}{1+y^2}\right)^s \sum_{\ell=0}^{\infty} \left(\frac{2cy \cos(a+t)}{1+c^2 y^2}\right)^{2\ell} \binom{s}{2\ell} \geq \sum_{\ell=0}^{\infty} \left(\frac{2y \cos(a)}{1+y^2}\right)^{2\ell} \binom{s}{2\ell}.$$

We can estimate the left hand side as

$$\begin{aligned} LHS &= \left(\frac{1+c^2 y^2}{1+y^2}\right)^s \sum_{\ell=0}^{\infty} \left(\frac{2cy \cos(a+t)}{1+c^2 y^2}\right)^{2\ell} \binom{s}{2\ell} \geq \\ &\left(\frac{1+c^2 y^2}{1+y^2}\right)^s + \left(\frac{1+c^2 y^2}{1+y^2}\right)^s \left(\frac{2cy \cos(a+t)}{1+c^2 y^2}\right)^2 \binom{s}{2} + \sum_{\ell=2}^{\infty} \left(\frac{2cy \cos(a+t)}{1+c^2 y^2}\right)^{2\ell} \binom{s}{2\ell} \geq \\ &\left(\frac{1+c^2 y^2}{1+y^2}\right)^s + \left(\frac{1+c^2 y^2}{1+y^2}\right)^s \left(\frac{2cy \cos(a+t)}{1+c^2 y^2}\right)^2 \binom{s}{2} + \sum_{\ell=2}^{\infty} \left(\frac{2y \cos(a+t)}{1+y^2}\right)^{2\ell} \binom{s}{2\ell}. \end{aligned}$$

In the first inequality we used the fact that $\frac{1+c^2 y^2}{1+y^2} \geq 1$ and $\binom{s}{2\ell} \geq 0$. In the second inequality we used the fact that

$$\frac{cy}{1+c^2 y^2} - \frac{y}{1+y^2} = \frac{y(c-1)(1-cy^2)}{(1+c^2 y^2)(1+y^2)} \geq 0$$

which is true because $0 \leq cy \leq 1$ and $c \geq 1$.

The right hand side can be rewritten as

$$RHS = 1 + \left(\frac{2y \cos(a)}{1+y^2}\right)^2 \binom{s}{2} + \sum_{\ell=2}^{\infty} \left(\frac{2y \cos(a)}{1+y^2}\right)^{2\ell} \binom{s}{2\ell}.$$

Thus, it suffices to prove the inequality

$$(2.30) \quad \begin{aligned} &\left(\frac{1+c^2 y^2}{1+y^2}\right)^s + \left(\frac{1+c^2 y^2}{1+y^2}\right)^s \left(\frac{2cy \cos(a+t)}{1+c^2 y^2}\right)^2 \binom{s}{2} - 1 - \left(\frac{2y \cos(a)}{1+y^2}\right)^2 \binom{s}{2} \\ &\geq \sum_{\ell=2}^{\infty} \left(\frac{2y}{1+y^2}\right)^{2\ell} (\cos^{2\ell}(a) - \cos^{2\ell}(a+t)) \binom{s}{2\ell}. \end{aligned}$$

2.10. Contribution of the infinite series. In this section we prove the following key lemma which gives the upper bound for the infinite series on the right hand side of (2.30).

Lemma 2.6. *We have*

$$\begin{aligned} &\sum_{\ell=2}^{\infty} \left(\frac{2y}{1+y^2}\right)^{2\ell} (\cos^{2\ell}(a) - \cos^{2\ell}(a+t)) \binom{s}{2\ell} \leq \\ &\frac{\sqrt{3}}{4} \cdot \frac{s(s-1)(s-2)(s-3)}{2} \left(\frac{2y}{1+y^2}\right)^2 y^2 \sin(t) \end{aligned}$$

for all $0 \leq a \leq a+t \leq \pi/2$, $y \geq 0$, and $s \in (1, 3/2)$.

Proof. Let us denote $w = \frac{2y}{1+y^2}$. Then $y = \frac{1-\sqrt{1-w^2}}{w}$, and therefore

$$\left(\frac{2y}{1+y^2}\right)^2 y^2 = (1 - \sqrt{1-w^2})^2 = 2 - 2\sqrt{1-w^2} - w^2 = 2 \sum_{k=2}^{\infty} w^{2k} \left| \binom{1/2}{k} \right| = \sum_{k=2}^{\infty} a_k w^{2k},$$

where $a_k = 2 \left| \binom{1/2}{k} \right|$ for $k \geq 2$. Clearly $a_2 = \frac{1}{4}$ and

$$\frac{a_{k+1}}{a_k} = \frac{\left| \binom{1/2}{k+1} \right|}{\left| \binom{1/2}{k} \right|} = \frac{k - \frac{1}{2}}{k+1}.$$

Thus it suffices to show that

$$\sum_{\ell=2}^{\infty} \left(\cos^{2\ell}(a) - \cos^{2\ell}(a+t) \right) \binom{s}{2\ell} w^{2\ell} \leq \frac{\sqrt{3}}{4} \cdot \frac{s(s-1)(s-2)(s-3)}{2} \sin(t) \sum_{\ell=2}^{\infty} a_{\ell} w^{2\ell}.$$

We have

$$\cos^{2\ell}(a) - \cos^{2\ell}(a+t) = \int_a^{a+t} -\frac{d}{dx} \cos^{2\ell}(x) dx = 2\ell \int_a^{a+t} \cos^{2\ell-1}(x) \sin(x) dx.$$

By Lemma [2.7](#) proved below, the right hand side can be estimated from above by

$$2\ell \sup_{x \in \mathbb{R}} \left(\cos^{2\ell-1}(x) \sin(x) \right) \cdot \sin(t) = \sqrt{2\ell} \left(\frac{2\ell-1}{2\ell} \right)^{\frac{2\ell-1}{2}} \cdot \sin(t).$$

Therefore

$$\begin{aligned} & \sum_{\ell=2}^{\infty} \left(\cos^{2\ell}(a) - \cos^{2\ell}(a+t) \right) \binom{s}{2\ell} w^{2\ell} \leq \\ & \sin(t) \sum_{\ell=2}^{\infty} \sqrt{2\ell} \left(\frac{2\ell-1}{2\ell} \right)^{\frac{2\ell-1}{2}} \binom{s}{2\ell} w^{2\ell} = \sin(t) \sum_{\ell=2}^{\infty} b_{\ell} w^{2\ell}, \end{aligned}$$

where $b_{\ell} = \sqrt{2\ell} \left(\frac{2\ell-1}{2\ell} \right)^{\frac{2\ell-1}{2}} \binom{s}{2\ell}$ for $\ell \geq 2$. We have

$$\frac{b_{\ell+1}}{b_{\ell}} = \frac{\sqrt{2(\ell+1)} \left(\frac{2\ell+1}{2\ell+2} \right)^{\frac{2\ell+1}{2}} \binom{s}{2\ell+2}}{\sqrt{2\ell} \left(\frac{2\ell-1}{2\ell} \right)^{\frac{2\ell-1}{2}} \binom{s}{2\ell}} = \sqrt{\frac{\ell+1}{\ell}} \cdot \frac{(2\ell-s)(2\ell+1-s)}{(2\ell+1)(2\ell+2)} \cdot \frac{\left(\frac{2\ell+1}{2\ell+2} \right)^{\frac{2\ell+1}{2}}}{\left(\frac{2\ell-1}{2\ell} \right)^{\frac{2\ell-1}{2}}}.$$

We claim that $\frac{b_{\ell+1}}{b_{\ell}} \leq \frac{\ell-\frac{1}{2}}{\ell+1}$. Indeed, $\sqrt{\frac{\ell+1}{\ell}} \leq 1 + \frac{1}{2\ell}$ and, since the mapping $n \mapsto (1 - 1/n)^{n-1}$ is decreasing for $n > 1$, we have $\left(\frac{2\ell+1}{2\ell+2} \right)^{\frac{2\ell+1}{2}} \leq \left(\frac{2\ell-1}{2\ell} \right)^{\frac{2\ell-1}{2}}$. Next, we notice that $(2\ell-s)(2\ell+1-s) \leq (2\ell-1)2\ell$. Finally, combining these three estimates, we obtain that

$$\frac{b_{\ell+1}}{b_{\ell}} = \sqrt{\frac{\ell+1}{\ell}} \cdot \frac{(2\ell-s)(2\ell+1-s)}{(2\ell+1)(2\ell+2)} \cdot \frac{\left(\frac{2\ell+1}{2\ell+2} \right)^{\frac{2\ell+1}{2}}}{\left(\frac{2\ell-1}{2\ell} \right)^{\frac{2\ell-1}{2}}} \leq \left(1 + \frac{1}{2\ell} \right) \frac{(2\ell-1)2\ell}{(2\ell+1)(2\ell+2)} = \frac{\ell - \frac{1}{2}}{\ell+1}.$$

Telescoping the product of $\frac{b_{k+1}}{b_k} \leq \frac{a_{k+1}}{a_k}$ ($k = 2, \dots, \ell$), we obtain

$$b_{\ell+1} \leq a_{\ell+1} \frac{b_2}{a_2} = a_{\ell+1} 8 \left(\frac{3}{4} \right)^{\frac{3}{2}} \binom{s}{4} = a_{\ell+1} \frac{s(s-1)(s-2)(s-3)}{2} \cdot \frac{\sqrt{3}}{4}.$$

Therefore

$$\begin{aligned} \sin(t) \sum_{\ell=2}^{\infty} b_{\ell} w^{2\ell} &\leq \sin(t) \cdot \frac{s(s-1)(s-2)(s-3)}{2} \cdot \frac{\sqrt{3}}{4} \sum_{\ell=2}^{\infty} a_{\ell} w^{2\ell} = \\ &\sin(t) \cdot \frac{s(s-1)(s-2)(s-3)}{2} \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{2y}{1+y^2} \right)^2 y^2. \end{aligned}$$

Thus it remains to prove the following lemma. □

Lemma 2.7. *For all $0 \leq a \leq a+t \leq \frac{\pi}{2}$ and all $\ell \geq 2$, we have*

$$\int_a^{a+t} \cos^{2\ell-1}(x) \sin(x) dx \leq \sup_{x \in \mathbb{R}} \left(\cos^{2\ell-1}(x) \sin(x) \right) \cdot \sin(t).$$

Proof. First, we need the following

Lemma 2.8 (Cap lemma). *Let f and g be two continuous unimodal nonnegative real valued functions defined on $\mathbb{R}$ such that $f = 0$ on $\mathbb{R} \setminus (a, b)$, and $g = 0$ on $\mathbb{R} \setminus (a', b')$. Assume that $a' \leq a < b' \leq b$ and $x_0 \in (a, b')$ is the point of the common global maximum of f and g with $f(x_0) = g(x_0)$. Suppose also that there exists $c \in (x_0, b')$ such that $g(x) \geq f(x)$ on $[a', c]$ and $g(x) \leq f(x)$ on $[c, b]$ (see Fig. 2).*

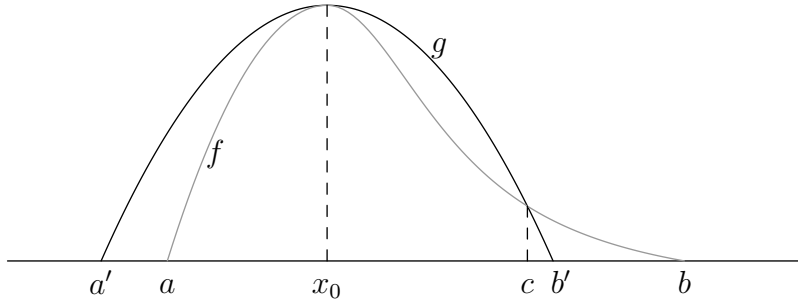


FIGURE 2. The functions f and g .

At last assume that

$$(2.31) \quad \int_{x_0}^{b'} g \geq \int_{x_0}^b f.$$

Then for all $t \in [0, b-a]$, we have

$$(2.32) \quad \max_{\substack{|I|=t \\ I \subset [a,b] \text{ is an interval}}} \int_I f \leq \max_{\substack{|I|=t \\ I \subset \mathbb{R} \text{ is an interval}}} \int_I g.$$

Proof. It follows from the unimodality of f that the maximum on the left hand side of (2.32) is attained when $I = [\alpha, \beta]$ with $\alpha \leq x_0$ and $\beta \geq x_0$. Consider two cases. If $\beta \in [x_0, c]$, then there is nothing to prove because $f \leq g$ on I . If $\beta \geq c$, then we have

$$\begin{aligned} \int_I f &= \int_{\alpha}^{x_0} f + \int_{x_0}^{\beta} f \leq \int_{\alpha}^{x_0} g + \int_{x_0}^{\beta} f = \int_I g + \int_{x_0}^c (f - g) + \int_c^{\beta} (f - g) \\ &\leq \int_I g + \int_{x_0}^c (f - g) + \int_c^b (f - g) = \int_I g + \int_{x_0}^b (f - g) \stackrel{(2.31)}{\leq} \int_I g. \end{aligned}$$

The second inequality follows from the fact that $f \geq g$ on $[\beta, b]$. Lemma 2.8 is proved. □

Next, fix any integer $\ell \geq 2$. Take $a = 0$, $b = \frac{\pi}{2}$, $f(x) = \cos^{2\ell-1}(x) \sin(x)$, $x_0 = \arcsin \frac{1}{\sqrt{2\ell}}$. Redefine f to be 0 outside $[0, \pi/2]$. To construct an appropriate g , we calculate the derivatives

of f . For $x \in (0, \pi/2)$, we have

$$\begin{aligned} f'(x) &= -(2\ell - 1) \cos^{2\ell-2}(x) \sin^2(x) + \cos^{2\ell}(x) = -(2\ell - 1) \cos^{2\ell-2}(x) + 2\ell \cos^{2\ell}(x); \\ f''(x) &= (2\ell - 1)(2\ell - 2) \cos^{2\ell-3}(x) \sin(x) - 4\ell^2 \cos^{2\ell-1}(x) \sin(x) \\ (2.33) \quad &= f(x) \left(\frac{(2\ell - 1)(2\ell - 2)}{\cos^2(x)} - 4\ell^2 \right). \end{aligned}$$

In particular, we see that f''/f is increasing on $[0, \pi/2]$. We have

$$\frac{f''(x_0)}{f(x_0)} = \frac{(2\ell - 1)(2\ell - 2)}{1 - \frac{1}{2\ell}} - 4\ell^2 = -4\ell.$$

This suggests that we should take

$$g(x) = \begin{cases} f(x_0) \cos(2\sqrt{\ell}(x_0 - x)), & x \leq x_0, \\ f(x_0) \cos(A_\ell(x - x_0)), & x \geq x_0, \end{cases}$$

where A_ℓ satisfies $\frac{1}{A_\ell} + \frac{1}{2\sqrt{\ell}} = 1$. Next, let $a' \leq x_0$ be the largest number such that $g(a') = 0$, i.e., $a' = x_0 - \frac{\pi}{2\sqrt{\ell}}$. Let $b' \geq x_0$ be the smallest number such that $g(b') = 0$, i.e., $b' = \frac{\pi}{2A_\ell} + x_0$. Redefine g to be zero outside (a', b') .

Note that by the choice of A_ℓ , g is equimeasurable with the mapping $s \mapsto f(x_0) \cos(s)$, $s \in [0, \pi/2]$, i.e.,

$$|\{x \in \mathbb{R} : g(x) > \lambda\}| = |\{s \in [0, \pi/2] : f(x_0) \cos(s) > \lambda\}|$$

for all $\lambda > 0$. Thereby, for every $t \in (0, \pi/2)$, we have

$$\begin{aligned} (2.34) \quad \max_{\substack{|I|=t \\ I \text{ is an interval}}} \int_I g &\leq \max_{\substack{|E|=t \\ E \text{ is measurable}}} \int_E g = \max_{\substack{|E'|=t, E' \subset [0, \pi/2] \\ E' \text{ is measurable}}} \int_{E'} f(x_0) \cos(s) ds \\ &= \int_0^t f(x_0) \cos(s) ds = f(x_0) \sin(t). \end{aligned}$$

Lemma 2.9. *Functions f and g satisfy the conditions of the cap lemma.*

Proof. Clearly both f and g are unimodal functions, x_0 is the point of the global maximum for f and g , and $f(x_0) = g(x_0)$. Since $\arcsin(s) < \frac{\pi}{2}s$ for every $s \in (0, 1)$, we conclude that $a' = \arcsin(\frac{1}{2\sqrt{\ell}}) - \frac{\pi}{2} \cdot \frac{1}{2\sqrt{\ell}} < 0$. The choice of A_ℓ implies that $b' - a' = \pi/2$. Thereby $a' < 0 = a < x_0 < b' < \pi/2 = b$.

Next, we need to check that

$$\begin{aligned} \int_{x_0}^{b'} g &= f(x_0) \frac{1}{A_\ell} = \frac{1}{\sqrt{2\ell}} \left(\frac{2\ell - 1}{2\ell} \right)^{\frac{2\ell-1}{2}} \left(1 - \frac{1}{2\sqrt{\ell}} \right) \geq \\ \int_{x_0}^{\pi/2} f &= \frac{1}{2\ell} \cos^{2\ell}(x) \Big|_{\frac{\pi}{2}}^{x_0} = \frac{1}{2\ell} \left(\frac{2\ell - 1}{2\ell} \right)^\ell, \end{aligned}$$

i.e., that $1 - \frac{1}{2\sqrt{\ell}} \geq \frac{1}{\sqrt{2\ell}} \sqrt{1 - \frac{1}{2\ell}}$, which is indeed true even for $\ell \geq 1$.

In order to show that $g(x) \geq f(x)$ for $x \in [a', x_0]$, it suffices to check the claim $g(x) \geq f(x)$ on $[0, x_0]$. The claim follows from the fact that $f(x_0) = g(x_0) > 0$, $f'(x_0) = g'(x_0) = 0$, and $\frac{f''}{f} < \frac{g''}{g}$ on $[0, x_0]$. Indeed, we calculate

$$\begin{aligned} \lim_{t \rightarrow x_0^-} g''(t) &= f''(x_0) = -4\ell f(x_0); \\ f'''(x_0) &= f(x_0) \left(\frac{2(2\ell - 1)(2\ell - 2) \sin(x_0)}{\cos^3(x_0)} \right) > 0 = \lim_{t \rightarrow x_0^-} g'''(t) \end{aligned}$$

(to calculate $f'''(x_0)$ quickly, use (2.33) and the fact that $f'(x_0) = 0$). Therefore $g(x) > f(x)$ when $x \in (x_0 - \varepsilon, x_0)$ provided that $\varepsilon > 0$ is sufficiently small. It follows from the piece-wise analyticity of f and g that the equation $f(x) = g(x)$ can have only finite number of solutions on

$(0, x_0)$. Let $x_1 \in (0, x_0)$ be the largest point (if it exists) such that $g \geq f$ on (x_1, x_0) and $g < f$ on $(x_1 - \delta, x_1)$ for a sufficiently small $\delta > 0$. Clearly $f(x_1) = g(x_1)$ and $g'(x_1) \geq f'(x_1)$, so we have

$$0 \leq (f'g - fg')|_{x_1}^{x_0} = \int_{x_1}^{x_0} \left(\frac{f''}{f} - \frac{g''}{g} \right) fg < 0,$$

which is a contradiction. Thus there is no such x_1 , which implies that $g \geq f$ on $(0, x_0)$ and we are done.

Next, we show that there exists $c \in (x_0, b')$ such that $g \geq f$ on (x_0, c) and $g \leq f$ on (c, b') (on $[b', \pi/2)$ we clearly have $g = 0 \leq f$). Note that

$$\lim_{t \rightarrow x_0^+} g''(t) = -f(x_0)A_\ell^2 = -f(x_0) \frac{4\ell}{(2\sqrt{\ell} - 1)^2} > -4\ell f(x_0) = f''(x_0)$$

for $\ell \geq 2$. Thus $g > f$ on $(x_0, x_0 + \varepsilon)$ provided that $\varepsilon > 0$ is sufficiently small. By the piece-wise analyticity of f and g , the equation $f(x) = g(x)$ has finite number of solutions on $[x_0, b']$. Let $x_1 > x_0$ be the smallest number such that $g \geq f$ on (x_0, x_1) and $g < f$ on $(x_1, x_1 + \delta)$ for a sufficiently small $\delta > 0$. If there were no such point, we would have $g \geq f$ on $(x_0, b']$ and, in particular, $0 = g(b') \geq f(b') > 0$, which is a contradiction.

If the inequality $g \leq f$ is violated on $[x_1, b']$, there exists a point $x_2 \in (x_1, b')$ such that $g \leq f$ on (x_1, x_2) and $g > f$ on $(x_2, x_2 + \delta')$ for some sufficiently small $\delta' > 0$.

Note that $f(x_1) = g(x_1)$ and $f'(x_1) \geq g'(x_1)$, so

$$0 \leq (f'g - fg')|_{x_0}^{x_1} = \int_{x_0}^{x_1} \left(\frac{f''}{f} - \frac{g''}{g} \right) fg,$$

whence $\frac{f''}{f} - \frac{g''}{g} \geq 0$ somewhere on $[x_0, x_1]$ and, thereby, $\frac{f''}{f} - \frac{g''}{g} > 0$ on (x_1, x_2) (since f''/f is strictly increasing and g''/g is constant). On the other hand, we have $f(x_2) = g(x_2)$, $f'(x_2) \leq g'(x_2)$ and therefore

$$0 \geq (f'g - fg')|_{x_1}^{x_2} = \int_{x_1}^{x_2} \left(\frac{f''}{f} - \frac{g''}{g} \right) fg > 0,$$

which is a contradiction. Thus we can take $c = x_1$. □

Lemma 2.7 is now completely proved. □

2.11. Sharpening Bernoulli. Combining Lemma 2.6 and inequality (2.30), we see that it suffices to prove the inequality

$$(2.35) \quad \left(\frac{1 + c^2 y^2}{1 + y^2} \right)^s + \left(\frac{1 + c^2 y^2}{1 + y^2} \right)^s \left(\frac{2cy \cos(a+t)}{1 + c^2 y^2} \right)^2 \binom{s}{2} - 1 - \left(\frac{2y \cos(a)}{1 + y^2} \right)^2 \binom{s}{2} \geq \frac{\sqrt{3}}{4} \cdot \frac{s(s-1)(s-2)(s-3)}{2} \left(\frac{2y}{1 + y^2} \right)^2 y^2 \sin(t)$$

for all $0 \leq y \leq \frac{1}{c}$, $0 \leq a \leq a+t \leq \pi/2$, $s \in (1, 3/2)$, where $c = c(t)$ is defined by (2.29).

Let us estimate the left hand side from below. We have

$$\begin{aligned} LHS &= \left(\frac{1 + c^2 y^2}{1 + y^2} \right)^s - 1 + \binom{s}{2} \left(\frac{1 + c^2 y^2}{1 + y^2} \right)^{s-1} \frac{4c^2 y^2 \cos^2(a+t)}{(1 + c^2 y^2)(1 + y^2)} - \binom{s}{2} \frac{4y^2 \cos^2(a+t)}{(1 + y^2)^2} + \\ &\quad \binom{s}{2} \frac{4y^2}{(1 + y^2)^2} (\cos^2(a+t) - \cos^2(a)). \end{aligned}$$

Consider the map

$$h(x) = x^s - 1 + \rho x^{s-1}, \quad x > 1,$$

where $\rho \in [0, 1)$. Clearly $h''(x) = (s-1)x^{s-3}(sx + \rho(s-2)) > 0$ when $x \geq 1$. Therefore h is convex on $[1, \infty)$ and hence $h(x) \geq \rho + (s + \rho(s-1))(x-1)$ there. Let us apply the latter

inequality to the case when $x = \frac{1+c^2y^2}{1+y^2} \geq 1$ and

$$\rho = \binom{s}{2} \frac{4c^2y^2 \cos^2(a+t)}{(1+c^2y^2)(1+y^2)} < \frac{3/2 \cdot 1/2}{2} \cdot \frac{2cy}{1+c^2y^2} \cdot \frac{2y}{1+y^2} \cdot c \cdot \cos^2(a+t) \leq \frac{3}{8}\sqrt{2} < 1.$$

Then we can estimate

$$\begin{aligned} LHS &\geq \binom{s}{2} \frac{4c^2y^2 \cos^2(a+t)}{(1+c^2y^2)(1+y^2)} + \frac{y^2(c^2-1)}{1+y^2} \left(s + (s-1) \binom{s}{2} \frac{4c^2y^2 \cos^2(a+t)}{(1+c^2y^2)(1+y^2)} \right) - \\ &\quad \binom{s}{2} \frac{4y^2 \cos^2(a+t)}{(1+y^2)^2} + \binom{s}{2} \frac{4y^2}{(1+y^2)^2} (\cos^2(a+t) - \cos^2(a)) = \\ &\quad s \frac{y^2(c^2-1)}{1+y^2} + \binom{s}{2} \left[\left(1 + (s-1) \frac{y^2(c^2-1)}{1+y^2} \right) \frac{4c^2y^2 \cos^2(a+t)}{(1+c^2y^2)(1+y^2)} - \frac{4y^2 \cos^2(a+t)}{(1+y^2)^2} \right] + \\ &\quad \binom{s}{2} \frac{4y^2}{(1+y^2)^2} (\cos^2(a+t) - \cos^2(a)) = \\ &\quad s \frac{y^2(c^2-1)}{1+y^2} + \binom{s}{2} \left[(s-1)y^2c^2 \frac{4y^2 \cos^2(a+t)(c^2-1)}{(1+c^2y^2)(1+y^2)^2} + \frac{4y^2 \cos^2(a+t)(c^2-1)}{(1+c^2y^2)(1+y^2)^2} \right] + \\ &\quad \binom{s}{2} \frac{4y^2}{(1+y^2)^2} (\cos^2(a+t) - \cos^2(a)) = \\ &\quad s \frac{y^2(c^2-1)}{1+y^2} + \binom{s}{2} \left[((s-1)y^2c^2 + 1) \frac{4y^2 \cos^2(a+t)(c^2-1)}{(1+c^2y^2)(1+y^2)^2} \right] + \\ &\quad \binom{s}{2} \frac{4y^2}{(1+y^2)^2} (\cos^2(a+t) - \cos^2(a)) = \\ &\quad s \frac{y^2(c^2-1)}{(1+y^2)^2} \left\{ 1 + y^2 + \frac{(s-1)((s-1)y^2c^2 + 1)}{2} \frac{4 \cos^2(a+t)}{1+c^2y^2} \right\} + \\ &\quad \frac{s}{2}(s-1) \frac{4y^2}{(1+y^2)^2} (\cos^2(a+t) - \cos^2(a)) = \\ &\quad \frac{s}{2} \left(\frac{2y}{1+y^2} \right)^2 \times \left\{ \frac{1}{2}(c^2-1)(1+2(s-1)\cos^2(a+t)) - (s-1)(\cos^2(a) - \cos^2(a+t)) \right. \\ &\quad \left. + \frac{1}{2}(c^2-1)y^2 \left(1 - \frac{2(s-1)(2-s)c^2 \cos^2(a+t)}{1+c^2y^2} \right) \right\}. \end{aligned}$$

Combining the obtained lower bound and inequality (2.35), we see that it suffices to show that

$$\begin{aligned} (2.36) \quad &\frac{1}{2}(c^2-1)(1+2(s-1)\cos^2(a+t)) - (s-1)(\cos^2(a) - \cos^2(a+t)) + \\ &\frac{1}{2}(c^2-1)y^2 \left(1 - \frac{2(s-1)(2-s)c^2 \cos^2(a+t)}{1+c^2y^2} \right) \geq \frac{\sqrt{3}}{4} \cdot (s-1)(s-2)(s-3)y^2 \sin(t) \end{aligned}$$

for all $0 \leq y \leq \frac{1}{c}$, $0 \leq a \leq a+t \leq \frac{\pi}{2}$, and $s \in (1, 3/2)$.

2.12. Moving to the boundary and factoring: an interplay between Analysis and Algebra. We denote $c^2 = C \geq 1$. We multiply both sides of inequality (2.36) by 2 and estimate

the factor $-\frac{1}{1+c^2y^2}$ on the left hand side of (2.36) from below by -1 . After rearranging the terms, we see that it suffices to show the inequality

$$(2.37) \quad (C-1)(1+2(s-1)\cos^2(a+t)) - 2(s-1)(\cos^2(a) - \cos^2(a+t)) + y^2 \times \left\{ (C-1)(1-2(s-1)(2-s)C\cos^2(a+t)) - \frac{\sqrt{3}}{2} \cdot (s-1)(s-2)(s-3)\sin(t) \right\} \geq 0.$$

The left hand side of (2.37) is linear in $u = y^2 \in [0, \frac{1}{C}]$. If $y^2 = 0$, the inequality reduces to

$$C-1 \geq \frac{2(s-1)(\cos^2(a) - \cos^2(a+t))}{1+2(s-1)\cos^2(a+t)},$$

which, after adding 1 to both sides, reduces to (2.16). Therefore, by linearity it suffices to consider the case $y^2 = \frac{1}{C}$. After substituting $y^2 = \frac{1}{C}$, we can rewrite the left hand side of the inequality (2.37) as

$$\begin{aligned} & (C-1)(1+2(s-1)\cos^2(a+t)) - 2(s-1)(\cos^2(a) - \cos^2(a+t)) + \\ & (C-1)\left(\frac{1}{C} - 2(s-1)(2-s)\cos^2(a+t)\right) - \frac{\sqrt{3}}{2C} \cdot (s-1)(s-2)(s-3)\sin(t) = \\ & (C-1)\left(1 + \frac{1}{C}\right) + 2(C-1)(s-1)^2\cos^2(a+t) - 2(s-1)(\cos^2(a) - \cos^2(a+t)) \\ & - \frac{\sqrt{3}}{2C} \cdot (s-1)(s-2)(s-3)\sin(t) \geq \\ & (C-1)\left(1 + \frac{1}{C}\right) - 2(s-1)(\cos^2(a) - \cos^2(a+t)) - \frac{\sqrt{3}}{2C} \cdot (s-1)(s-2)(s-3)\sin(t). \end{aligned}$$

Next, notice that $\cos^2(a) - \cos^2(a+t) = \sin(t)\sin(2a+t) \leq \sin(t)$. Therefore it suffices to show the inequality

$$(2.38) \quad (C-1)\left(1 + \frac{1}{C}\right) \geq \left(2 + \frac{\sqrt{3}}{2C} \cdot (s-2)(s-3)\right)(s-1)\sin(t).$$

It follows from (2.8) and the cosine theorem (see Fig. 1) that

$$r(t)^2 + \left(\frac{s-1}{\sqrt{2s-1}}\right)^2 + 2r(t)\frac{s-1}{\sqrt{2s-1}}\sin(t) = \frac{s^2}{2s-1}.$$

Using the equality $C = \frac{1}{r(t)^2}$, we obtain

$$(s-1)\sin(t) = \frac{1-r(t)^2}{2r(t)}\sqrt{2s-1} = \frac{(C-1)\sqrt{2s-1}}{2\sqrt{C}}, \quad C \in [1, 2s-1].$$

Therefore, inequality (2.38) simplifies to

$$\sqrt{C}\left(1 + \frac{1}{C}\right) \geq \left[1 + \frac{\sqrt{3}}{4C}(s-2)(s-3)\right]\sqrt{2s-1}, \quad C \in [1, 2s-1], \quad s \in [1, 3/2].$$

Since $\sqrt{C}(1 + \frac{1}{C}) \geq \sqrt{C}\frac{2}{\sqrt{C}} = 2$, $\frac{1}{C} \leq 1$, and $\sqrt{2s-1} \leq s$, it suffices to show that

$$2 \geq \left[1 + \frac{\sqrt{3}}{4}(s-2)(s-3)\right]s.$$

Subtracting s from both sides of this inequality and dividing by $(2 - s)$, we get $1 \geq \frac{\sqrt{3}}{4}s(3 - s)$, i.e., $s^2 - 3s + \frac{4}{\sqrt{3}} = (s - \frac{3}{2})^2 + (\frac{4}{\sqrt{3}} - \frac{9}{4}) \geq 0$ to prove. It remains to notice that $\sqrt{3} \leq \frac{16}{9}$, i.e.,

$$(2.39) \quad 3 \leq \frac{256}{81}.$$

Acknowledgments. The authors are grateful to both anonymous referees for helpful suggestions. P.I. was partially supported by the NSF grants DMS-1856486 and CAREER-DMS-2052865. F.N. was partially supported by the NSF grant DMS-1600239.

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