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CLIFFORD AND WEYL SUPERALGEBRAS AND SPINOR REPRESENTATIONS

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Abstract. We construct a family of twisted generalized Weyl algebras which includes Weyl–Clifford superalgebras and quotients of the enveloping algebras of $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$. We give a condition for when a canonical representation by differential operators is faithful. Lastly, we give a description of the graded support of these algebras in terms of pattern-avoiding vector compositions.

1. Introduction

Twisted generalized Weyl algebras (TGWAs) were introduced by Mazorchuk and Turowska in [20], [21] in an attempt to include a wider range of examples than Bavula's generalized Weyl algebras (GWAs) [1]. Their structure and representations have been studied in [20], [21], [19], [24], [12], [13], [14], [15], [10]. Known examples of TGWAs include multiparameter quantized Weyl algebras [21], [12], [10], the Mickelsson–Zhelobenko step algebras associated to $(\mathfrak{gl}_{n+1}, \mathfrak{gl}_n \oplus \mathfrak{gl}_1)$ [19] and some primitive quotients of enveloping algebras [16].

In this paper we take a step further by proving that supersymmetric analogs of some classical algebras are also examples of TGWAs. Specifically, we show that Weyl–Clifford superalgebras and some quotients of the enveloping algebras of $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ can be realized as twisted generalized Weyl (TGW) algebras. This suggests that much of the general representation theory from [21], [19], [12] could be applied to the study of certain families of superalgebras. In addition our new algebras provide a large supply of consistent but non-regular TGW algebras (i.e., certain elements t_i are zero-divisors). This motivates future development of the theory to include such algebras.

It is also worth mentioning that, as a special case, we show that Clifford algebras can be presented as TGW algebras. This shows that TGW algebras can be finite-dimensional.

To summarize the contents of the present paper, in Section 2 we recall the definition of TGW algebras from [21] which includes certain scalars μ_{ij} that in our case will be ± 1. Some known results that will be used are also stated. In Section 3

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we prove that the Weyl–Clifford superalgebra from [23] can be realized as a TGW algebra.

The main object of the paper is introduced in Section 4, in which we define a family of TGW algebras $\mathcal{A}(\gamma)^{\pm}$ which depend on a certain matrix γ with integer entries. These algebras naturally come with an algebra homomorphism φ from $\mathcal{A}(\gamma)^{\pm}$ to a Clifford-Weyl algebra. This is a generalization of the construction in [16]. A sufficient condition for φ_γ to be injective is given in Section 4.2. This condition is related to the graded support of the algebra $\mathcal{A}(\gamma)^{\pm}$ which is combinatorially characterized in Section 4.3.

Lastly, these results are applied in Section 5 to prove that for appropriate γ, the TGW algebras $\mathcal{A}(\gamma)^{\pm}$ fit into commutative diagrams involving the spinor representation π of $U(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ studied by Nishiyama [23] and Coulembier [9]. As a corollary we obtain that $U(\mathfrak{g})/J$ are examples of TGW algebras for such $\mathfrak{g}$ as well as for classical Lie algebras. These results generalize previous realizations in [16]. We end with some open problems regarding exceptional types.

Notation

Throughout, we work over an algebraically closed field $\Bbbk$ of characteristic zero. Associative algebras are assumed to have a multiplicative identity. $[\![a, b]\!]$ denotes the set of integers x with $a \le x \le b$.

2. Twisted generalized Weyl algebras

We recall the definition of TGW algebras and some of their useful properties.

2.1. Definitions

Let I be a set.

Definition 1 (TGW Datum)**.** A *twisted generalized Weyl datum over* $\Bbbk$ *with index set* I is a triple (R, σ, t) where

- R is an associative $\Bbbk$-algebra,
- $\sigma = (\sigma_i)_{i \in I}$ is a sequence of commuting $\Bbbk$-algebra automorphisms of R,
- $t = (t_i)_{i \in I}$ is a sequence of central elements of R.

Let $\mathbb{Z}I$ denote the free abelian group on I, with basis denoted $\{\mathbf{e}_i\}_{i \in I}$. For $g = \sum g_i \mathbf{e}_i \in \mathbb{Z}I$ put $\sigma_g = \prod \sigma_i^{g_i}$. Then $g \mapsto \sigma_g$ defines an action of $\mathbb{Z}I$ on R by $\Bbbk$-algebra automorphisms.

Definition 2 (TGW Construction)**.** Let

- (R, σ, t) be a TGW datum over $\Bbbk$ with index set I,
- μ be an $I \times I$-matrix without diagonal, $\mu = (\mu_{ij})_{i \ne j}$, with $\mu_{ij} \in \Bbbk \setminus \{0\}$.

The *twisted generalized Weyl construction* associated to μ and (R, σ, t), denoted $\mathcal{C}_\mu(R, \sigma, t)$, is defined as the free R-ring on the set $\{X_i, Y_i \mid i \in I\}$ modulo the two-sided ideal generated by the following elements:

$$X_i r - \sigma_i(r) X_i, \qquad Y_i r - \sigma_i^{-1}(r) Y_i, \qquad \forall r \in R,\ i \in I, \tag{1a}$$

$$Y_i X_i - t_i, \qquad X_i Y_i - \sigma_i(t_i), \qquad \forall i \in I, \tag{1b}$$

$$X_i Y_j - \mu_{ij} Y_j X_i, \qquad \forall i, j \in I,\ i \ne j. \tag{1c}$$

The algebra $\mathcal{C}_\mu(R,\sigma,t)$ has a $\mathbb{Z}I$-gradation given by requiring $\deg X_i = \mathbf{e}_i$, $\deg Y_i = -\mathbf{e}_i$, $\deg r = 0\ \forall r \in R$. Let $\mathcal{I}_\mu(R,\sigma,t) \subseteq \mathcal{C}_\mu(R,\sigma,t)$ be the sum of all graded ideals $J \subseteq \mathcal{C}_\mu(R,\sigma,t)$ such that $\mathcal{C}_\mu(R,\sigma,t)_0 \cap J = \{0\}$. It is easy to see that $\mathcal{I}_\mu(R,\sigma,t)$ is the unique maximal graded ideal having zero intersection with the degree zero component.

Definition 3 (TGW Algebra)**.** The *twisted generalized Weyl algebra* $\mathcal{A}_\mu(R,\sigma,t)$ associated to μ and (R,σ,t) is defined as the quotient

$$\mathcal{A}_\mu(R,\sigma,t) := \mathcal{C}_\mu(R,\sigma,t)/\mathcal{I}_\mu(R,\sigma,t).$$

Since $\mathcal{I}_\mu(R,\sigma,t)$ is graded, $\mathcal{A}_\mu(R,\sigma,t)$ inherits a $\mathbb{Z}I$-gradation from $\mathcal{C}_\mu(R,\sigma,t)$. The images in $\mathcal{A}_\mu(R,\sigma,t)$ of the elements X_i, Y_i will also be denoted by X_i, Y_i.

Example 1. For an index set I, the I:th Weyl algebra over $\Bbbk$, $A_I = A_I(\Bbbk)$ is the $\Bbbk$-algebra generated by $\{x_i, \partial_i \mid i \in I\}$ subject to defining relations

$$[x_i, x_j] = [\partial_i, \partial_j] = [\partial_i, x_j] - \delta_{ij} = 0, \quad \forall i, j \in I.$$

There is a $\Bbbk$-algebra isomorphism $\mathcal{A}_\mu(R,\tau,u) \to A_n$ where $\mu_{ij} = 1$ for all $i \neq j$, $R = \Bbbk[u_i \mid i \in I]$, $\tau_i(u_j) = u_j - \delta_{ij}$, given by $X_i \mapsto x_i$, $Y_i \mapsto \partial_i$, $u_i \mapsto \partial_i x_i$.

2.2. Regularity and consistency

Definition 4 (Reduced and monic monomials)**.** A *monic monomial* in a TGW algebra is any finite product of elements from the set $\{X_i\}_{i\in I} \cup \{Y_i\}_{i\in I}$. A *reduced monomial* is an element of the form $Y_{i_1}\cdots Y_{i_k} X_{j_1}\cdots X_{j_l}$ where $\{i_1,\ldots,i_k\} \cap \{j_1,\ldots,j_l\} = \varnothing$.

Lemma 1. [12, Lem. 3.2] *$\mathcal{A}_\mu(R,\sigma,t)$ is generated as a left* (*and as a right*) *R-module by the reduced monomials.*

Since a TGW algebra $\mathcal{A}_\mu(R,\sigma,t)$ is a quotient of an R-ring, it is an R-ring itself with a natural map $\rho : R \to \mathcal{A}_\mu(R,\sigma,t)$. By Lemma 1, the degree zero component of $\mathcal{A}_\mu(R,\sigma,t)$ (with respect to the $\mathbb{Z}I$-gradation) is equal to the image of ρ.

Definition 5 (Regularity)**.** A TGW datum (R,σ,t) is called *regular* if t_i is regular (i.e., not a zero-divisor) in R for all i.

Due to Relation (1b), the canonical map $R \to \mathcal{C}_\mu(R,\sigma,t)$ is not guaranteed to be injective, and indeed sometimes it is not [10]. It is injective if and only if the map $R \to \mathcal{A}_\mu(R,\sigma,t)$ is injective.

Definition 6 (μ-Consistency)**.** A TGW datum (R,σ,t) is *μ-consistent* if the canonical map $\rho : R \to \mathcal{A}_\mu(R,\sigma,t)$ is injective.

Abusing language we say that a TGW algebra $\mathcal{A}_\mu(R,\sigma,t)$ is regular (respectively consistent) if (R,σ,t) is regular (respectively μ-consistent).

Theorem 2 ([10])**.** *A regular TGW algebra $\mathcal{A}_\mu(R,\sigma,t)$ is consistent if and only if*

$$\sigma_i\sigma_j(t_it_j) = \mu_{ij}\mu_{ji}\sigma_i(t_i)\sigma_j(t_j), \quad \forall i \neq j; \tag{2a}$$

$$\sigma_i\sigma_k(t_j)t_j = \sigma_i(t_j)\sigma_k(t_j), \quad \forall i \neq j \neq k \neq i. \tag{2b}$$

That relation (2a) is necessary for consistency of a regular TGW datum was known already in [20], [21]. If (R,σ,t) is not regular, sufficient and necessary conditions for μ-consistency are not known (see Problem 2). In this paper we produce many examples of consistent but non-regular TGW algebras.

Conversely, for consistent TGW algebras one can characterize regularity as follows:

Theorem 3 ([15, Thm. 4.3])**.** *Let $A = \mathcal{A}_\mu(R,\sigma,t)$ be a consistent TGW algebra. Then the following are equivalent*

(i) *(R,σ,t) is regular.*
(ii) *Each monic monomial in A is non-zero and generates a free left* (*and right*) *R-module of rank one.*
(iii) *A is* regularly graded, *i.e., for all $g \in \mathbb{Z}I$, there exists a nonzero regular element in A_g.*
(iv) *If $a \in A$ is a homogeneous element such that $bac = 0$ for some monic monomials $b, c \in A$, then $a = 0$.*

2.3. Non-degeneracy of the gradation form

For a group G, any G-graded ring $A = \bigoplus_{g\in G} A_g$ can be equipped with a $\mathbb{Z}$-bilinear form $\gamma : A \times A \to A_e$ called the *gradation form*, defined by

$$\gamma(a,b) = \mathfrak{p}_e(ab)$$

where $\mathfrak{p}_e$ is the projection $A \to A_e$ along the direct sum $\bigoplus_{g\in G} A_g$, and $e \in G$ is the neutral element.

Theorem 4 ([15, Cor. 3.3])**.** *The ideal $\mathcal{I}_\mu(R,\sigma,t)$ is equal to the radical of the gradation form γ of $\mathcal{C}_\mu(R,\sigma,t)$* (*with respect to the $\mathbb{Z}I$-gradation*)*, and thus the gradation form on $\mathcal{A}_\mu(R,\sigma,t)$ is non-degenerate.*

2.4. *R*-rings with involution

Definition 7. Let R be a commutative ring.

(i) An *involution* on a ring A is a $\mathbb{Z}$-linear map $* : A \to A, a \mapsto a^*$ satisfying $(ab)^* = b^*a^*$, $(a^*)^* = a$ for all $a, b \in A$.
(ii) An *R-ring with involution* is a ring A equipped with a ring homomorphism $h_A : R \to A$ and an involution $* : A \to A$ such that $h(r)^* = h(r)$ for all $r \in R$.
(iii) If A and B are two R-rings with involution, then a *map of R-rings with involution* is a ring homomorphism $k : A \to B$ such that $k \circ h_A = h_B$ and $k(a^*) = (k(a))^*$ for all $a \in A$.

When R is commutative, any TGW algebra $A = \mathcal{A}_\mu(R,\sigma,t)$ for which $\mu_{ij} = \mu_{ji}$ for all i, j, can be equipped with an involution $*$ given by $X_i^* = Y_i$, $Y_i^* = X_i$ $\forall i \in I$, $r^* = r$ $\forall r \in R$. Together with the canonical map $\rho : R \to A$ this turns A into an R-ring with involution. In particular we regard the Weyl algebra A_I as an R-ring with involution in this way, where $R = \Bbbk[u_i \mid i \in I]$ as in Example 1.

3. The Clifford/Weyl superalgebras

In this section let $\pm \in \{+,-\}$ and put $\mp = -\pm$. Let p and q be non-negative integers and put $n = p+q$. We consider supersymmetric analogs $A^{\pm}_{p|q}$ of Clifford and Weyl algebras and prove that they can be presented as TGW algebras.

3.1. Definition and properties

Definition 8. The *Clifford/Weyl superalgebra of degree* $p|q$, denoted $A^{\pm}_{p|q}$, is defined as the superalgebra with even generators x_i, ∂_i ($i \in [\![1,p]\!]$) and odd generators x_i, ∂_i ($i \in [\![p+1,n]\!]$) and relations

$$[\partial_i, x_j]_{\pm} - \delta_{ij} = [x_i, x_j]_{\pm} = [\partial_i, \partial_j]_{\pm} = 0 \qquad \text{for all } i,j \in [\![1,n]\!], \tag{3}$$

where $[\cdot\,,\cdot]_{\pm}$ denotes the super(anti-)commutator

$$[a,b]_{\pm} = ab \pm (-1)^{p(a)p(b)} ba.$$

Thus $A^{+}_{p|q}$ (respectively $A^{-}_{p|q}$) is a supersymmetric analog of the Clifford (respectively Weyl) algebra.

We will need the following result.

Lemma 5. *The subalgebra R of $A^{\pm}_{p|q}$ generated by $\{\partial_i x_i \mid i \in [\![1,n]\!]\}$ is maximal commutative.*

Proof. The algebra $A^{\pm}_{p|q}$ has a $\mathbb{Z}^n$-gradation determined by $\deg(x_i) = \boldsymbol{e}_i$ and $\deg(\partial_i) = -\boldsymbol{e}_i$ where $\{\boldsymbol{e}_i\}_{i=1}^n$ is a $\mathbb{Z}$-basis for $\mathbb{Z}^n$. We have $[\mp\partial_i x_i, x_j]_- = \delta_{ij} x_j$ and $[\mp\partial_i x_i, \partial_j]_- = -\delta_{ij}\partial_j$. In other words, $\big\{[\mp x_i\partial_i, -]_-\big\}_{i=1}^n$ is a set of commuting (even) derivations on $A^{\pm}_{p|q}$ whose common eigenspaces coincide with the graded homogeneous components. Thus the centralizer of R is the subalgebra A_0 of $A^{\pm}_{p|q}$ consisting of elements of degree $0 \in \mathbb{Z}^n$. Clearly $R \subseteq A_0$. The converse inclusion is straightforward to check using the commutation relations (3) and induction on the length of a monomial of degree zero. □

By the defining relations, $A^{\pm}_{p|q}$ is a graded algebra with respect to the free abelian group $\mathbb{Z}^n$. In addition $A^{\pm}_{p|q}$ has an involution $*$ given by $x_i^* = \partial_i$, $\partial_i^* = x_i$. Since $(\partial_i x_i)^* = \partial_i x_i$, $A^{\pm}_{p|q}$ is an R-ring with involution. Even though $A^{\pm}_{p|q}$ is not a domain in general, the following graded regularity property still holds.

Lemma 6. *Let $a \in A^{\pm}_{p|q}$ be homogeneous of degree $g \in \mathbb{Z}^n$. If $a^* \cdot a = 0$ then $a = 0$.*

Proof. We give a proof for $A = A^{-}_{p|q}$, the other case being analogous. Write $a = rx^{(g)}$ where $r \in R$ and $x^{(g)} = x_1^{(g_1)} \cdots x_n^{(g_n)}$ where for $s > 0$, $x_i^{(s)} = x_i^s$, $x_i^{(-s)} = \partial_i^s$. By reordering the indices, we may assume that the first k elements of the tuple $(g_{p+1}, \ldots, g_n)$ are zero, and the rest are nonzero. Put $u_i = \partial_i x_i$. For $i > p$ we have $u_i x_i = \partial_i x_i^2 = 0$ and $u_i \partial_i = \partial_i x_i \partial_i = (1 - x_i\partial_i)\partial_i = \partial_i$. Thus we may assume that r lies in the subalgebra of R generated by $\{u_1, \ldots, u_{p+k}\}$. If $a \cdot a^* = 0$ then we have

$$0 = a \cdot a^* = rx^{(g)}x^{(-g)}r = r^2 cb \tag{4}$$

where

$$b = x_1^{(g_1)} \cdots x_p^{(g_p)} \cdot x_p^{(-g_p)} \cdots x_1^{(-g_1)}$$

which can be written as a polynomial in u_i, $i \leq p$, and

$$c = x_{p+k+1}^{(g_{p+k+1})} \cdots x_n^{(g_n)} \cdot x_n^{(-g_n)} \cdots x_{p+k+1}^{(-g_{p+k+1})}$$

which can be written as a polynomial in u_i, $i > p$. Since b is regular in A, (4) implies $r^2 c = 0$. We have the following isomorphisms of algebras

$$A \simeq A_{p|0}^- \otimes_{\Bbbk} A_{0|q}^- \simeq A_{p|0}^- \otimes_{\Bbbk} M_{2^q}(\Bbbk) \simeq M_{2^k}(A_{p|0}^-) \otimes_{\Bbbk} M_{2^{q-k}}(\Bbbk).$$

Under this isomorphism, $r^2 c$ is mapped to $r^2 \otimes c$. That this is zero implies $r^2 = 0$. But R is isomorphic to $(\Bbbk[u_i \mid i \in [\![1,p]\!]])^{2^q}$ which is a direct product of domains, hence $r = 0$. This proves $a = 0$. $\square$

3.2. Realization as TGW algebras

To realize $A_{p|q}^{\pm}$ as TGW algebras, consider the commutative $\Bbbk$-algebra

$$R_{p|q}^{\pm} := \Bbbk\big[u_1, u_2, \ldots, u_n\big]/J^{\pm} \tag{5}$$

where $J^{\pm}$ is the ideal generated by $u_i^2 - u_i$ for all i is such that $(-1)^{p(i)} = \pm 1$. There is an injective homomorphism

$$\begin{aligned} \iota : R_{p|q}^{\pm} &\to A_{p|q}^{\pm}, \\ u_i &\mapsto \partial_i x_i. \end{aligned} \tag{6}$$

We will often use ι to identify $R_{p|q}^{\pm}$ with its image in $A_{p|q}^{\pm}$. One checks that the image of ι coincides with the degree zero subalgebra, $(A_{p|q}^{\pm})_0$, of $A_{p|q}^{\pm}$ with respect to the $\mathbb{Z}^n$-gradation $A_{p|q}^{\pm} = \bigoplus_{d \in \mathbb{Z}^n} (A_{p|q}^{\pm})_d$ given by $\deg(x_i) = \mathbf{e}_i$, $\deg(\partial_i) = -\mathbf{e}_i$, $\forall i \in [\![1,n]\!]$.

For $i, j \in [\![1,n]\!]$, put

$$\lambda_{ij} = \mp(-1)^{p(i)p(j)} \tag{7}$$

and for $i \in [\![1,n]\!]$, define $\tau_i \in \mathrm{Aut}_{\Bbbk}(R_{p|q}^{\pm})$ by

$$\tau_i(u_j) = \begin{cases} \lambda_{ii}(u_i - 1), & \text{if } i = j, \\ u_j, & \text{otherwise.} \end{cases} \tag{8}$$

One checks that τ_i preserves the relations $u_j^2 - u_j = 0$ for j with $(-1)^{p(j)} = \pm 1$. Let $\tau = (\tau_i)_{i=1}^n$ and $u = (u_i)_{i=1}^n$. Let $\mathcal{A}_{\lambda}(R_{p|q}^{\pm}, \tau, u)$ be the corresponding TGW algebra.

Theorem 7. *There is an isomorphism of* $\Bbbk$*-algebras*

$$\begin{aligned}\chi : \mathcal{A}_\lambda(R^\pm_{p|q}, \tau, u) &\xrightarrow{\sim} A^\pm_{p|q},\\ X_i &\mapsto x_i,\\ Y_i &\mapsto \partial_i.\end{aligned}$$

In particular, $\mathcal{A}_\lambda(R^\pm_I, \tau, u)$ *is consistent* (*i.e., the natural map*

$$\rho : R^\pm_{p|q} \to \mathcal{A}_\lambda(R^\pm_{p|q}, \tau, u)$$

is injective).

Proof. Put $R = R^\pm_{p|q}$ and $A = A^\pm_{p|q}$ and $[\cdot\,,\cdot] = [\cdot\,,\cdot]_\pm$. The identities $x_i(\partial_i x_i) = (x_i\partial_i)x_i$, $x_i\partial_j = \lambda_{ij}\partial_j x_i$ for $i \neq j$, and $x_i\partial_i = \lambda_{ii}(\partial_i x_i + 1)$ imply that relations (1) are preserved. Thus we have a map $\mathcal{C}_\lambda(R, \tau, u) \to A$ of R-rings given by $X_i \mapsto x_i$, $Y_i \mapsto \partial_i$. Furthermore, for each $i, j \in [\![1, n]\!]$, it can be checked, using Theorem 4 that $[X_i, X_j]$ and $[Y_i, Y_j]$ lie in the radical of the gradation form on $\mathcal{C}_\lambda(R, \tau, u)$. For example, if $i \neq j$ then by Lemma 1 the homogeneous component of degree $-\boldsymbol{e}_i - \boldsymbol{e}_j$ is equal to $RY_iY_j + RY_jY_i$ so by symmetry it suffices to show that $\gamma(Y_iY_j, [X_i, X_j]_\pm) = 0$. For simplicity, say $i, j \leq p$ and that $\pm = -$. Then we get $\gamma(Y_iY_j, [X_i, X_j]_-) = Y_iY_j(X_iX_j - X_jX_i) = \lambda_{ij}^{-1}u_iu_j - \tau_i^{-1}(u_j)u_i = 0$. The other cases are checked similarly. In fact the elements $[X_i, X_j]$ and $[Y_i, Y_j]$ generate the radical. To see this, let $\mathcal{J}'$ be the ideal of $\mathcal{C}_\lambda(R, \tau, u)$ generated by all $[X_i, X_j]$ and $[Y_i, Y_j]$. It suffices to show that $B := \mathcal{C}_\lambda(R, \tau, u)/\mathcal{J}'$ has a non-degenerate gradation form. By Lemma 1, any nonzero homogeneous component of B is a free cyclic left R-module. If $\pm = -$, say, then for $a = (a_1, \ldots, a_n) \in \mathbb{Z}^p \times \{-1, 0, 1\}^q$ the monomial $Z = X_1^{(a_1)} \cdots X_n^{(a_n)}$ (where $X_i^{(k)} = (X_i)^k$ for $k \geq 0$ and $X_i^{(k)} = (Y_i)^{|k|}$ for $k < 0$) and its dual $Z^* = X_n^{(-a_n)} \cdots X_1^{(-a_1)}$ satisfy $\gamma(Z^*, Z) = Z^*Z$ which simplifies to a nonzero element of R. Since $B_a = RZ$ this shows that γ is nondegenerate on B. Hence the commutators generate the ideal $\mathcal{I}_\lambda(R, \tau, u)$ by Theorem 4. Since $[x_i, x_j] = [\partial_i, \partial_j] = 0$ in A this shows that we have a well-defined map $\chi : \mathcal{A}_\lambda(R, \tau, u) \to A$ of R-rings given by $X_i \mapsto x_i$, $Y_i \mapsto \partial_i$. Since x_i and ∂_i generate A, the map χ is surjective. It remains to prove it is injective. Since χ is a map of R-rings, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A}_\lambda(R, \tau, u) & \xrightarrow{\chi} & A \\ {\scriptstyle\rho}\uparrow & \overset{\iota}{\nearrow} & \\ R & & \end{array} \quad . \tag{9}$$

Since ι is injective, ρ is injective. That is, $\mathcal{A}_\lambda(R, \tau, u)$ is consistent. Identifying R with the images under ρ and ι, the map $\chi|_R$ is the identity map. Both $\mathcal{A}_\lambda(R, \tau, u)$ and A are $\mathbb{Z}^n$-graded algebras and χ is a graded homomorphism. Therefore $J = \ker \chi$ is a graded ideal of $\mathcal{A}_\lambda(R, \tau, u)$. If $J \neq 0$ then, since $\mathcal{A}_\lambda(R, \tau, u)$ is a consistent TGW algebra, $J \cap R \neq 0$. However that contradicts that $\chi|_R$ is injective. Hence $J = 0$ which completes the proof that χ is an isomorphism. □

Remark 1. When $q > 0$, Theorem 7 implies that $A^-_{p|q}$ is a consistent TGW algebra which is not regularly graded. Indeed, if $j > p$, then u_j is not regular in R because $u_j(u_j + 1) = 0$. Thus, by Theorem 3, $\mathcal{A}_\lambda(R, \tau, u)$ is not regularly graded. Note that for non-regularly graded TGW algebras, it is not known if relations (2) are sufficient (or even necessary) for it to be consistent.

Remark 2. The algebra $A^-_{0|q}$ is finite-dimensional (an even Clifford algebra). Hence Theorem 7 shows that TGW algebras can be finite-dimensional.

Remark 3. Theorem 7 suggests that the class of TGW algebras already contains not only quantum deformations of many algebras (see, e.g., [21, Ex. 2.2.3]), but also supersymmetric analogues of certain algebras, without modifying the definition of TGW algebras.

4. A new family of TGW algebras $\mathcal{A}(\gamma)^\pm$

In this section we define a family of TGW algebras that depend on a matrix. This construction is a supersymmetric generalization of the one in [16].

4.1. Construction via monomial maps

In this section we use the Clifford/Weyl superalgebras $A^\pm_{p|q}$ to construct new TGW algebras denoted $\mathcal{A}(\gamma)^\pm$. Our method is to look for maps

$$\varphi : \mathcal{A}_\mu(R, \sigma, t) \to A^\pm_{p|q}$$

of R-rings with involution. Here $R = R^\pm_{p|q}$. The motivation is threefold. First it generalizes the construction from [16] which corresponds to the case $A^-_{p|0}$. Second, the TGW algebras obtained in this way automatically come with φ, which may be thought of as a representation by differential operators. Thirdly we show in Section 5 that certain quotients of enveloping algebras of Lie superalgebras are TGW algebras of exactly this form.

As in [16] we restrict attention to monomial embeddings

$$\varphi(X_i) = x_1^{(\gamma_{1i})} x_2^{(\gamma_{2i})} \cdots x_n^{(\gamma_{ni})}. \tag{10}$$

Here $n = p + q$, $\gamma_{ji} \in \mathbb{Z}$ and we use the notation

$$x_j^{(k)} = \begin{cases} x_j^k, & k \geq 0, \\ \partial_j^{-k}, & k < 0. \end{cases}$$

In the case of [16], under mild assumptions on φ the form (10) was in fact shown to be necessary. Here in our more general setting we shall be content with showing how the assumption that φ is a homomorphism of R-rings with involution such that (10) holds, naturally gives rise to conditions on γ_{ji} and also specifies the TGW datum (automorphisms σ_i, elements $t_i \in R$ and scalars μ_{ij}).

First, since φ is supposed to be a map of rings with involution, we necessarily have

$$\varphi(Y_i) = \varphi(X_i^*) = \varphi(X_i)^* = x_n^{(-\gamma_{ni})} \cdots x_2^{(-\gamma_{2i})} x_1^{(-\gamma_{1i})}.$$

Second, since φ is a map of R-rings and $t_i \in R$ we have

$$\begin{aligned} t_i &= \varphi(t_i) = \varphi(Y_i X_i) = \varphi(Y_i)\varphi(X_i) \\ &= x_n^{(-\gamma_{ni})} \cdots x_2^{(-\gamma_{2i})} x_1^{(-\gamma_{1i})} \cdot x_1^{(\gamma_{1i})} x_2^{(\gamma_{2i})} \cdots x_n^{(\gamma_{ni})} \\ &= x_1^{(-\gamma_{1i})} x_1^{(\gamma_{1i})} \cdot x_2^{(-\gamma_{2i})} x_2^{(\gamma_{2i})} \cdots x_n^{(-\gamma_{ni})} x_n^{(\gamma_{ni})}. \end{aligned}$$

In the last step we used the TGW algebra realization of $A^{\pm}_{p|q}$ which in particular has $\tau_i(u_j) = u_j$ for $j \neq i$. To obtain an explicit formula for t_i we compute $x_j^{(-\gamma_{ji})} x_j^{(\gamma_{ji})}$.

If $\gamma_{ji} > 0$ we have

$$x_j^{(-\gamma_{ji})} x_j^{(\gamma_{ji})} = \partial_j^{\gamma_{ji}} x_j^{\gamma_{ji}} = \tau_j^{(-\gamma_{ji}+1)}(u_j) \cdots \tau_j^{-1}(u_j) u_j.$$

Here we see that this is zero if $\lambda_{jj} = -1$ and $\gamma_{ji} > 1$ because then $\tau_j^{-1}(u_j)u_j = \tau_j^{-1}(u_j \lambda_{jj}(u_j - 1)) = 0$ due to $u_j^2 = u_j$ in R. To avoid this scenario (having $t_i = 0$ in a TGW algebra leads to degenerate behaviour such as $X_i = Y_i = 0$) we make our first assumption on γ_{ji}:

$$|\gamma_{ji}| \leq 1 \quad \text{for all } i, j \text{ such that } \lambda_{jj} = -1. \tag{11}$$

Under this assumption we can proceed and obtain the formula

$$x_j^{(-\gamma_{ji})} x_j^{(\gamma_{ji})} = (u_j + \gamma_{ji} - 1) \cdots (u_j + 1) u_j.$$

We used that $\tau_j(u_j) = \lambda_{jj}(u_j - 1)$, so the formula is clear when $\lambda_{jj} = 1$ while if $\lambda_{jj} = -1$ there is at most one factor (empty product is interpreted as 1.) The case $\gamma_{ji} < 0$ is handled analogously (which is why we put absolute value in (11)).

The final formula for t_i is

$$\begin{aligned} t_i &= u_{1i} u_{2i} \cdots u_{ni}, \\ u_{ji} &= \begin{cases} (u_j + \gamma_{ji} - 1) \cdots (u_j + 1) u_j, & \gamma_{ji} > 0, \\ 1, & \gamma_{ji} = 0, \\ (u_j - |\gamma_{ji}|) \cdots (u_j - 2)(u_j - 1), & \gamma_{ji} < 0. \end{cases} \end{aligned} \tag{12}$$

Similarly σ_i can be deduced as follows. We have

$$\varphi(X_i u_j) = \varphi(\sigma_i(u_j) X_i).$$

Since φ is a homomorphism of R-rings, we have

$$\varphi(X_i) u_j = \sigma_i(u_j) \varphi(X_i).$$

Substituting (10) we immediately obtain the sufficient condition

$$\sigma_i = \tau_1^{\gamma_{1i}} \tau_2^{\gamma_{2i}} \cdots \tau_n^{\gamma_{ni}}. \tag{13}$$

What remains is to ensure that for $i \neq j$,

$$X_i Y_j = \mu_{ij} Y_j X_i$$

holds for appropriate scalars μ_{ij}, under suitable assumptions on γ_{kl}. We have

$$\varphi(X_i)\varphi(Y_j) = x_1^{(\gamma_{1i})} \cdots x_n^{(\gamma_{ni})} \cdot x_n^{(-\gamma_{nj})} \cdots x_1^{(-\gamma_{1j})}.$$

First we observe that if there exists $k \in \{1, 2, \ldots, n\}$ with $\lambda_{kk} = -1$ and $\gamma_{ki}\gamma_{kj} < 0$ then $\varphi(X_i)\varphi(Y_j) = 0 = \varphi(Y_j)\varphi(X_i)$. If no such k exists we want to move all factors on the right $x_l^{(-\gamma_{lj})}$ to the left of all factors $x_k^{(\gamma_{ki})}$. The only problem is when $k = l$. A natural assumption for it to be possible is that actually $\gamma_{ki}\gamma_{kj} \leq 0$, because then the two factors are either both powers of x_k or both powers of ∂_k.

To summarize, we make the following second assumption on γ_{ji}:

$$\forall i \neq j : \text{Either } \gamma_{ki}\gamma_{kj} < 0 \text{ for some } k \text{ with } \lambda_{kk} = -1, \text{ or } \gamma_{ki}\gamma_{kj} \leq 0 \text{ for all } k.$$

Under this assumption we then have for all k, l:

$$x_k^{(\gamma_{ki})} x_l^{(\gamma_{lj})} = \lambda_{kl}^{\gamma_{ki}\gamma_{lj}} x_l^{(\gamma_{lj})} x_k^{(\gamma_{ki})}.$$

Thus we finally obtain that

$$\varphi(X_i)\varphi(Y_j) = \mu_{ij}\varphi(Y_j)\varphi(X_i)$$

holds, provided

$$\mu_{ij} = \prod_{1 \leq k, l \leq n} \lambda_{kl}^{\gamma_{ki}\gamma_{lj}}.$$

Using that $\lambda_{kl} = (\mp 1)(-1)^{p(k)p(l)}$ this can be written

$$\mu_{ij} = \mu_{ij}^{\pm} = (\mp 1)^{p'(i)p'(j)} \cdot (-1)^{p(i)p(j)}, \tag{14}$$

where the parities are defined by

$$p(i) = \sum_{k=1}^{n} \bar{\gamma}_{ki} p(k), \tag{15}$$

$$p'(i) = \sum_{k=1}^{n} \bar{\gamma}_{ki} \tag{16}$$

($\bar{x} \in \mathbb{Z}/2\mathbb{Z}$ is the image of $x \in \mathbb{Z}$ under the canonical projection).

Note that (15) expresses that the matrix γ, when regarded as a $\mathbb{Z}$-module map $\mathbb{Z}^m \to \mathbb{Z}^n$, is an even map, with respect to the parity $p(a_1, \ldots, a_n) = \sum_k \bar{a}_k p(k)$.

Theorem 8. *Let p, q, m be non-negative integers, put $n = p + q$. Let $\gamma = (\gamma_{ji})$ be a $n \times m$-matrix with integer entries satisfying the following two conditions:*

(i) $|\gamma_{ji}| \leq 1$ *whenever* $\lambda_{ii} = -1$,

(ii) $\forall i \neq j$: *either* $\gamma_{ki}\gamma_{kj} < 0$ *for some* k *with* $\lambda_{kk} = -1$, *or* $\gamma_{ki}\gamma_{kj} \leq 0$ *for all* k.

Then there exist a TGW algebra $\mathcal{A}(\gamma)^{\pm} = \mathcal{A}_\mu(R, \sigma, t)$ *with index set* $[\![1, m]\!]$, *and a homomorphism of* R*-rings with involution*

$$\varphi : \mathcal{A}_\mu(R, \sigma, t) \to A^{\pm}_{p|q}. \tag{17}$$

The homomorphism is uniquely determined by the condition

$$\varphi(X_i) = x_1^{(\gamma_{1i})} x_2^{(\gamma_{2i})} \cdots x_n^{(\gamma_{ni})},$$

and the TGW algebra is given by the following data:

$$R = R^{\pm}_{p|q} = \Bbbk[u_1, \ldots, u_n]/(u_i^2 - u_i \mid \lambda_{ii} = -1), \tag{18}$$

where $\lambda_{ij} = \mp(-1)^{p(i)p(j)}$ *and* $t = (t_1, \ldots, t_m)$ *where*

$$\begin{aligned} t_i &= u_{1i} u_{2i} \cdots u_{ni} \\ u_{ji} &= \begin{cases} (u_j + \gamma_{ji} - 1) \cdots (u_j + 1) u_j, & \gamma_{ji} > 0, \\ 1, & \gamma_{ji} = 0, \\ (u_j - |\gamma_{ji}|) \cdots (u_j - 2)(u_j - 1), & \gamma_{ji} < 0. \end{cases} \end{aligned} \tag{19}$$

Lastly, $\sigma = (\sigma_1, \ldots, \sigma_m)$, *where*

$$\sigma_i = \tau_1^{\gamma_{1i}} \tau_2^{\gamma_{2i}} \cdots \tau_n^{\gamma_{ni}} \tag{20}$$

where

$$\tau_i(u_j) = \begin{cases} \lambda_{ii}(u_i - 1), & \text{if } i = j, \\ u_j, & \text{otherwise}, \end{cases} \tag{21}$$

and $\mu = (\mu_{ij})_{1 \leq i,j \leq m}$ *where*

$$\mu_{ij} = (\mp 1)^{p'(i)p'(j)} \cdot (-1)^{p(i)p(j)}$$

where $p(i)$ *and* $p'(i)$ *were defined in* (15)–(16).

Proof. The discussion preceding the theorem proves that there exists a homomorphism of R-rings with involution

$$\varphi' : \mathcal{C} = \mathcal{C}_\mu(R, \sigma, t) \to A^{\pm}_{p|q}.$$

All that remains is to show that $\varphi'(\mathcal{I}) = 0$ where $\mathcal{I}$ is the unique maximal $\mathbb{Z}^m$-graded ideal trivially intersecting the degree zero component of $\mathcal{C}$. If a is a homogeneous element of $\mathcal{I}$ then $a^* \cdot a = 0$ hence, $\varphi'(a)^* \cdot \varphi'(a) = 0$. By Lemma 6, it follows that $\varphi(a) = 0$. $\square$

Remark 4. Theorem 8 provides a large family of consistent non-regular TGW algebras.

Remark 5. Let $p = 3$, $q = 2$, $m = 4$ and

$$\gamma = \left[\begin{array}{cccc} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ \hdashline & & -1 & 1 \\ & & & -1 \end{array}\right].$$

(The dashed line separates even from odd rows.) The corresponding TGW algebra $\mathcal{A}(\gamma)^-$ is a quotient of $U\big(\mathfrak{gl}(3|2)\big)$ (see Section 5).

4.2. Injectivity of φ

We prove a theorem which gives equivalent conditions for φ defined in (17) to be injective. This result will be used in Section 5.

Lemma 9 (Weak injectivity of φ). *If $g \in \mathbb{Z}^m$ and $a \in \mathcal{A}(\gamma)^{\pm}_g$, $a \neq 0$, then $\varphi(a) \neq 0$.*

Proof. Suppose $a \neq 0$. Then, by the non-degeneracy of the gradation form of a TGW algebra, $ba \neq 0$ for some $b \in \mathcal{A}(\gamma)^{\pm}_{-g}$. Applying φ we get $\varphi(ba) \neq 0$ since $\varphi|_{R_E}$ is injective. Hence $\varphi(b)\varphi(a) \neq 0$, so in particular $\varphi(a) \neq 0$. $\square$

Let $* : A^{\pm}_{p|q} \to A^{\pm}_{p|q}$, $a \mapsto a^*$, be the unique $\Bbbk$-linear map satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ for all $a, b \in A^{\pm}_{p|q}$, and $x_i^* = \partial_i$ for all i.

Lemma 10. *Let γ be a matrix satisfying the conditions of Theorem 8 and let $\mathcal{A}(\gamma)^{\pm}$ be the corresponding TGW algebra. Let $a \in \mathcal{A}(\gamma)^{\pm}$ be a homogeneous element of degree $g \in \mathbb{Z}^m$. If $a^* \cdot a = 0$ then $a = 0$.*

Proof. Suppose $a \neq 0$. By Lemma 9, $\varphi(a) \neq 0$. So, by Lemma 6, $\varphi(a)^* \cdot \varphi(a) \neq 0$. Since φ is a map of rings with involution, $\varphi(a^* \cdot a) \neq 0$. Hence $a^* \cdot a \neq 0$. $\square$

Remark 6. If $\lambda_{ii} = 1$ for all i then $R^{\pm}_{p|q}$ defined in (18) is a domain. Then, by [10, Prop. 2.9], $\mathcal{A}(\gamma)^{\pm}$ is also a domain. Hence Lemma 10 holds trivially in this case.

For a $\mathbb{Z}I$-graded algebra $A = \bigoplus_{g \in \mathbb{Z}I} A_g$ we define the (*graded*) *support* of A to be $\mathrm{Supp}(A) := \big\{g \in \mathbb{Z}I \mid A_g \neq \{0\}\big\}$.

Lemma 11. *Let $\mathcal{A}(\gamma)^{\pm}$ be a TGW algebra as constructed in Theorem 8. Let $S^{\pm} \subseteq \mathbb{Z}^m$ be the support of $\mathcal{A}(\gamma)^{\pm}$. Then, regarding γ as a $\mathbb{Z}$-linear map from $\mathbb{Z}^m$ to $\mathbb{Z}^n$ we have*

$$\gamma(S^+) \subseteq \{-1, 0, 1\}^p \times \mathbb{Z}^q,$$
$$\gamma(S^-) \subseteq \mathbb{Z}^p \times \{-1, 0, 1\}^q.$$

Proof. We consider the case S^-. The other case is analogous. Let $g \in S^-$. Since any TGW algebra is generated as a left R module by the reduced monomials (Lemma 1), there exist sequences $(i_1, i_2, \ldots, i_k)$ and $(j_1, j_2, \ldots, j_l)$ of elements from $\{1, 2, \ldots, m\}$ with $\{i_1, i_2, \ldots, i_k\} \cap \{j_1, j_2, \ldots, j_l\} = \varnothing$ such that

$$a = Y_{i_1} Y_{i_2} \cdots Y_{i_k} \cdot X_{j_1} X_{j_2} \cdots X_{j_l}$$

is a nonzero element in $\mathcal{A}(\gamma)_g^-$. By Lemma 9, $\varphi(a)\neq 0$. We have

$$\varphi(a)=\prod_{r=1}^m x_r^{(-\gamma_{ri_1})}\cdots x_r^{(-\gamma_{ri_k})}\cdot x_r^{(\gamma_{rj_1})}\cdots x_r^{(\gamma_{rj_l})}.$$

For $r>p$, a product of the form

$$x_r^{(-\gamma_{ri_1})}\cdots x_r^{(-\gamma_{ri_k})}\cdot x_r^{(\gamma_{rj_1})}\cdots x_r^{(\gamma_{rj_l})}$$

can only be nonzero if the factors $x_r^{(\beta)}$ alternate between x_r and ∂_r (ignoring factors where $\beta=0$). In particular, the number of x_r's must differ from the number of ∂_r's by at most one. □

To prove that homomorphisms from TGW algebras are injective, the following result is useful.

Theorem 12 ([15, Thm. 3.6])**.** *If $A=\mathcal{A}_\mu(R,\sigma,t)$ is consistent, then the centralizer $C_A(R)$ of R in A is an essential subalgebra of A, in the sense that $J\cap C_A(R)\neq\{0\}$ for any nonzero ideal J of A.*

Theorem 13. *Let γ be a matrix as in Theorem 8 and $A=\mathcal{A}(\gamma)^\pm$ be the corresponding TGW algebra. Put $R=R^\pm_{p|q}$. The following statements are equivalent.*

(i) *R is a maximal commutative subalgebra of A.*
(ii) *If $g\in\mathrm{Supp}(A)$ is such that $\sigma_g:=\prod_{i=1}^m\sigma_i^{g_i}=\mathrm{Id}_R$, then $g=0$.*
(iii) *Put*

$$\mathbb{Z}_-^{p|q}=\mathbb{Z}^p\times(\mathbb{Z}/2\mathbb{Z})^q,\qquad \mathbb{Z}_+^{p|q}=(\mathbb{Z}/2\mathbb{Z})^p\times\mathbb{Z}^q.$$

Then the composition

$$\mathrm{Supp}(A)\to\mathbb{Z}^m\xrightarrow{\gamma}\mathbb{Z}^n=\mathbb{Z}^p\times\mathbb{Z}^q\xrightarrow{P}\mathbb{Z}_\pm^{p|q}$$

is injective (the first map is inclusion and the last is canonical projection).
(iv) *The restriction of $\gamma:\mathbb{Z}^m\to\mathbb{Z}^n$ to $\mathrm{Supp}(A)$ is injective.*
(v) *The map φ defined in (17) is injective.*

Proof. (i)⇒(ii): Suppose $g\in\mathrm{Supp}(A)$ with $\sigma_g=\mathrm{Id}_R$. Then for any $a\in A_g$ and $r\in R$ we have $ar=\sigma_g(r)a=ra$ which means that $A_g\subseteq C_A(R)$. But $C_A(R)=R$ by (i). Thus, since $A_g\neq\{0\}$, this means that g must be 0 and $A_g=R$.

(ii)⇒(iii): Suppose $P\circ\gamma(g)=0$ in $\mathbb{Z}_\pm^{p|q}$ for some $g\in\mathrm{Supp}(A)$. Then $\sigma_g=\prod_{r=1}^n\tau_r^{\gamma(g)_r}=\mathrm{Id}_R$ because $\tau_r^2=\mathrm{Id}_R$ for $r>p$ when $\pm=-$ and for $r\leq p$ when $\pm=+$. By (ii) this implies $g=0$.

(iii)⇒(i): For simplicity we assume $\pm=-$. The other case is symmetric. Suppose $a\in C_A(R)$, $a\neq 0$. Since $C_A(R)$ is a graded subalgebra of A we may without loss of generality suppose there exists $g\in\mathbb{Z}^m$ such that $a\in A_g\cap C_A(R)$. Since $a\neq 0$, this implies $g\in\mathrm{Supp}(A)$. For all $r\in R$ we have $(\sigma_g(r)-r)a=ar-ra=0$. Taking $r=u_j$ we get

$$0=(\sigma_g(u_j)-u_j)a=(\tau_j^{\gamma(g)_j}(u_j)-u_j)a=\begin{cases}-\gamma(g)_ja, & j\leq p,\\ 0, & j>p,\gamma(g)_j=0\text{ in }\mathbb{Z}/2\mathbb{Z},\\ (1-2u_j)a, & j>p,\gamma(g)_j=1+2\mathbb{Z}.\end{cases}$$

Since $a \neq 0$, we get $\gamma(g)_j = 0$ for all $j \leq p$. Suppose $j > p$ and $\gamma(g)_j = 1 + 2\mathbb{Z}$, then $0 = u_j(1-2u_j)a = -u_j a$ since $u_j^2 = u_j$. Combining this with $(1-2u_j)a = 0$ we get $a = 0$, a contradiction. Therefore, for $j > p$ we must have $\gamma(g)_j = 0$ in $\mathbb{Z}/2\mathbb{Z}$. This proves that $\gamma(g) = 0$ in $\mathbb{Z}^p \oplus (\mathbb{Z}/2\mathbb{Z})^q$.

(iii)⇒ (iv): Trivial.

(iv)⇒ (iii): Suppose $P \circ \gamma(g) = 0$ for some $g \in \mathrm{Supp}(A)$. By Lemma 11 we get $\gamma(g) = 0$ so by (iv), $g = 0$.

(i)⇒(v): Let $K = \ker(\varphi)$. If $K \neq \{0\}$, then by Theorem 12, $K \cap C_A(R) \neq \{0\}$. By (i), $C_A(R) = R$. Hence $K \cap R \neq \{0\}$. But by Theorem 8, φ is a map of R-rings with involution and thus in particular $\varphi|_R = \mathrm{Id}_R$ (where we used the injective maps ρ and ι to identify R with its image in A and $A_E(\Bbbk)$ respectively). This contradiction shows that $K = \{0\}$.

(v)⇒(i): If $a \in C_A(R)$ then $\varphi(a) \in C_{A^{\pm}_{p|q}}(R)$ which equals R by Lemma 5. By (v) this implies $a \in R$. □

Example 2. Let p, q be non-negative integers and $n = p + q > 0$. Consider the matrices

$$\alpha = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & & \\ & & \ddots & 1 \\ & & & -1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & & & \\ & & \ddots & 1 & \\ & & & -1 & 1 \end{bmatrix},$$

$$\gamma = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & & & \\ & & \ddots & 1 & \\ & & & -1 & 2 \end{bmatrix}.$$

These are $n \times m$ matrices (where $m = n - 1$ in the case of α and $m = n$ for β, γ) and define $\mathbb{Z}$-linear maps $\mathbb{Z}^m \to \mathbb{Z}^n$. In each case the top p rows are defined to be even and the remaining q rows are odd. It is easy to see that these maps are injective, hence by Theorem 13(iv)⇒(v), the homomorphism $\varphi : \mathcal{A}(\zeta)^{\pm} \to A^{\pm}_{p|q}$ is injective for $\zeta = \alpha, \beta, \gamma$.

4.3. A description of the graded support of $\mathcal{A}(\gamma)^-$

Although sufficient for the application to Lie superalgebras, the characterization in Theorem 13 of the injectivity of the map (17) is not completely satisfactory because we lack a good description of the support of $\mathcal{A}(\gamma)^{\pm}$. In this section we give a combinatorial description of the support of $\mathcal{A}(\gamma)^-$ in terms of certain pattern-avoiding vector compositions of the columns of γ. A similar analysis applies to $\mathcal{A}(\gamma)^+$. This allows us to compute the support in the certain cases. In addition, it shows that that this is a non-trivial problem for a general (non-regular) TGW algebra.

Put $W = \mathbb{Z}^d$. A *d-dimensional vector composition* of $w \in W$ is a tuple $c = (c_1, c_2, \ldots, c_\ell) \in W^\ell$ such that $c_1 + c_2 + \cdots + c_\ell = w$. The non-negative integer

ℓ is the *length* of c. The c_i are called the *parts* of the composition c. A given vector $u \in W$ *appears with multiplicity* m *(in* c*)* if $c_j = u$ for exactly m choices of $j \in [\![1, \ell]\!]$.

Example 3. $\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right)$ is a 3-dimensional vector composition of $\begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$.

Theorem 14. *Let* $A = \mathcal{A}(\gamma)^-$ *be a TGW algebra constructed as in Theorem* 8. *The following are equivalent for* $g \in \mathbb{Z}^m$*:*

(i) $g \in \mathrm{Supp}(A)$.

(ii) *There exists an* n*-dimensional vector composition of* $\gamma(g)$ *of length* $|g| = \sum_{i \in V} |g_i|$ *such that*

(a) *each part is of the form* $\mathrm{sgn}(g_i)\gamma(\mathbf{e}_i)$ *for* $i \in V$ *which appears with multiplicity* $|g_i|$,

(b) *for each* $r > p$ *the sequence* $\big(\mathrm{sgn}(g_{i_1})\gamma_{ri_1}, \ldots, \mathrm{sgn}(g_{i_{|g|}})\gamma_{ri_{|g|}}\big)$ *contains no consecutive subsequence of the form*

$$(1, 0, \ldots, 0, 1) \quad \text{or} \quad (-1, 0, \ldots, 0, -1)$$

where there are zero or more 0*'s.*

Proof. By Lemma 1, $g \in \mathrm{Supp}(A)$ if and only if A_g contains a reduced monomial $a = Z_{i_1} Z_{i_2} \cdots Z_{i_{|g|}}$ (where each $Z_{i_k} \in \bigcup_{j \in V}\{X_j, Y_j\}$) such that $a \neq 0$, which by Lemma 9 is equivalent to $\varphi(a) \neq 0$. Put $\varepsilon_k = \mathrm{sgn}(g_{i_k})$. We have

$$\varphi(a) = \varphi(Z_{i_1}) \cdots \varphi(Z_{i_{|g|}}) = \pm \prod_{r \in E} x_r^{(\varepsilon_1 \gamma_{ri_1})} \cdots x_r^{(\varepsilon_{|g|} \gamma_{ri_{|g|}})}$$

which is nonzero if and only if property (b) in the theorem holds. □

Example 4. If $q = 0$ then $\mathrm{Supp}\big(\mathcal{A}(\gamma)^-\big) = \mathbb{Z}^m$ because condition (b) is void.

Example 5. Let $m = 3, p = 1, q = 2$ and $\gamma = \begin{bmatrix} 1 & 3 & 0 \\ \hdashline 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. Example 3 shows that $(1, 2, 1)$ belongs to the graded support of the TGW algebra $\mathcal{A}(\gamma)^-$. On the other hand $(2, 1, 0)$ does not, because there is no vector composition of length 3 with two parts equal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and one part equal to $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ which avoids the pattern $(1, 0, \ldots, 0, 1)$ in the second row.

Example 6. Let $m = 2, p = 0, q = 1$ and $\gamma = \begin{bmatrix} 1 & -1 \end{bmatrix}$. Then

$$\mathrm{Supp}\big(\mathcal{A}(\gamma)\big) = \{(g_1, g_2) \in \mathbb{Z}^2 \mid |g_1 - g_2| \leq 1\}.$$

Example 7. Let $m = 2, p = 0, q = 2$, $\gamma = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$, then

$$\mathrm{Supp}\big(\mathcal{A}(\gamma)\big) = \{(0,0), \pm(0,1), \pm(1,0), \pm(1,1), \pm(1,2)\}.$$

5. Relation to $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$

Irreducible completely pointed weight modules have been classified and realized by differential operators in the case of simple finite-dimensional complex Lie algebras $\mathfrak{g}$ in [6], [4] and over $U_q(\mathfrak{sl}_n)$ in [11]. In [9, Sect. 6], Coulembier classified all irreducible completely pointed highest weight modules over the orthosymplectic Lie superalgebras $\mathfrak{osp}(m|2n)$, and realized them by differential operators on supersymmetric Grassmann algebras. See also [23] for a uniform treatment of spinor representations of orthosymplectic Lie superalgebras. In this section we show that, analogously to the Lie algebra case [16], the realization of $\mathfrak{osp}(m|2n)$ by differential operators factors through a corresponding twisted generalized Weyl algebra of the form $\mathcal{A}(\alpha)$.

Recall that the Lie superalgebra $\mathfrak{gl}(m|n)$ is the Lie superalgebra of all linear transformations of $(m|n)$-dimensional vector superspace, and $\mathfrak{osp}(m|2n)$ is the subalgebra of $\mathfrak{gl}(m|2n)$ preserving a non-degenerate even symmetric bilinear form on an $(m|2n)$-dimensional vector superspace or, equivalently, the subalgebra of $\mathfrak{gl}(2n|m)$ preserving a non-degenerate even skew-symmetric bilinear form on an $(2n|m)$-dimensional vector superspace. The even part of $\mathfrak{osp}(m|2n)$ is the direct sum $\mathfrak{so}(m)\oplus\mathfrak{sp}(2n)$. The Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ are Kac–Moody superalgebras and can be described by Chevally generators and relations; see [17], as follows. Let p, q be nonnegative integers, $n = p+q > 0$. The Chevalley generators of $\mathfrak{gl}(p|q)$ are $e_1, \dots, e_{n-1}$, $h_1, \dots, h_n$, $f_1, \dots, f_{n-1}$, with the convention that e_p, f_p are odd and all other generators are even. They satisfy the relations

$$[h_i, h_j] = 0, \quad [h_i, e_j] = \delta_{i,j}e_j - \delta_{i,j+1}e_j, \quad [h_i, f_j] = -\delta_{i,j}f_j + \delta_{i,j+1}f_j,$$
$$[e_i, f_j] = \delta_{i,j}(h_i - (-1)^{\delta_{ip}}h_{i+1}).$$

The Lie superalgebra $\mathfrak{gl}(p|q)$ is the quotient of the infinite-dimensional Lie algebra with the above relations by the maximal ideal which intersects trivially the Cartan subalgebra generated by $h_1, \dots, h_n$. The Chevalley generators of $\mathfrak{osp}(2p+1|2q)$ are obtained from those for $\mathfrak{gl}(p|q)$ by adding odd generators e_n, f_n and relations

$$[h_i, e_n] = \delta_{i,n}e_n, \quad [h_i, f_n] = -\delta_{i,n}f_n, \quad [e_n, f_n] = h_n,$$
$$[e_i, f_n] = [e_n, f_i] = 0 \ \text{ if } n \neq i.$$

The Chevalley generators of $\mathfrak{osp}(2p|2q)$ are obtained from those for $\mathfrak{gl}(p|q)$ by adding even generators e_n^2, f_n^2. From the above description it is not difficult to see that we have an embedding of Lie superalgebras

$$\mathfrak{gl}(p|q) \subset \mathfrak{osp}(2p|2q) \subset \mathfrak{osp}(2p+1|2q).$$

5.1. Weyl superalgebra and $\mathfrak{osp}(2p|2q)$

Let V be a vector superspace equipped with even skew-symmetric form $\omega : V \times V \to \Bbbk$. We define the Weyl superalgebra $W(V, \omega)$ as the quotient of the tensor superalgebra $T(V)$ by the relations

$$v \otimes w - (-1)^{p(v)p(w)} w \otimes v = \omega(v, w).$$

Lemma 15. *Let $\mathfrak{g}$ denote the span of the elements of the form $vw+(-1)^{p(v)p(w)}wv$ for all $v, w \in V$. Then $\mathfrak{g}$ is closed under the supercommutator and the adjoint action of $\mathfrak{g}$ on V preserves the form ω.*

Proof. Note that

$$vw + (-1)^{p(v)p(w)}wv = 2vw - \omega(v, w)$$

and

$$[vw, u] = v[w, u] + (-1)^{p(w)p(u)}[v, u]w = \omega(w, u)v + (-1)^{p(w)p(u)}\omega(v, u)w.$$

The super Jacobi identity ensures that ω is ad_{vw}-invariant. Indeed,

$$\begin{aligned}\omega([vw, u_1], u_2) &+ (-1)^{p(vw)p(u_1)}\omega(u_1, [vw, u_2]) \\ &= [[vw, u_1], u_2] + (-1)^{p(vw)p(u_1)}[u_1, [vw, u_2]] = [vw, [u_1, u_2]] = 0.\end{aligned}$$

Finally, $\mathfrak{g}$ is closed under supercommutator as

$$[vw, xz] = [vw, x]z + (-1)^{p(vw)p(x)}x[vw, z] = [vw, x]z + (-1)^{p(vw)p(xz)}[vw, z]x. \qquad \square$$

Corollary 16. *If ω is non-degenerate then $\mathfrak{g}$ constructed in the previous lemma is isomorphic to $\mathfrak{osp}(r|s)$ where $r = \dim V_1$ and $s = \dim V_0$.*

Let us assume that the ω is non-degenerate and both r and s are even. Set $r = 2p$, $s = 2q$ and $n = p + q$. Choose basis $x_1, \dots, x_n, y_1, \dots, y_n$ in V such that

$$\omega(x_i, x_j) = \omega(y_i, y_j) = 0, \quad \omega(y_i, x_j) = \delta_{i,j}.$$

The parity is defined by

$$p(x_i) = p(y_i) = \begin{cases} 1 & \text{if } i \le p \\ 0 & \text{if } i > p. \end{cases}$$

In this case the Weyl algebra is isomorphic to $A^-_{q|p}$ since the defining relations are

$$\begin{aligned}&x_i x_j - (-1)^{p(i)p(j)}x_j x_i = y_i y_j - (-1)^{p(i)p(j)}y_j y_i = 0,\\ &y_i x_j - (-1)^{p(i)p(j)}x_j y_i = \delta_{ij}.\end{aligned}$$

Let $\mathfrak{g} = \mathfrak{gl}(p|q)$, or $\mathfrak{osp}(2p|2q)$ and identify $\mathbb{Z}^m$ with the root lattice of $\mathfrak{g}$ with basis consisting of the distinguished simple roots of $\mathfrak{g}$. Let $\zeta : \mathbb{Z}^m \to \mathbb{Z}^n$ be the $\mathbb{Z}$-linear maps given by the matrices

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & & \\ & & \ddots & 1 \\ & & & -1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & & & \\ & & \ddots & 1 & \\ & & & -1 & 2 \end{bmatrix}$$

respectively. Let $A^-_{q|p}$ be the Weyl superalgebra.

Theorem 17. *Let p, q be nonnegative integers, $n = p + q > 0$. Let $\mathfrak{g} = \mathfrak{gl}(p|q)$, or $\mathfrak{osp}(2p|2q)$ and let ζ be as above. Then there is a commutative triangle of associative algebras with involution*

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\quad\pi\quad} & A^-_{q|p} \\ & \searrow^{\psi} \quad {}^{\varphi}\nearrow & \\ & \mathcal{A}(\zeta)^- & \end{array} \tag{22}$$

where φ is given by Theorem 8, $\psi(e_i) = X_i$, $\psi(f_i) = Y_i$, $\psi(h_{ii}) = \lambda_{ii}(u_i - 1)$, and

$$\pi(e_i) = \begin{cases} x_i\partial_{i+1}, & i < n, \\ x_n^2, & i = n, \end{cases} \qquad \pi(f_i) = \pi(e_i)^*, \ \pi(h_i) = x_i\partial_i + (-1)^{p(i)}\tfrac{1}{2}.$$

Proof. First, the existence of π follows from Corollary 16. We need to check that $\widetilde{\pi}(\mathfrak{j}) = 0$. This follows immediately from the fact that $\widetilde{\pi}(\mathfrak{h})$ is the self-centralizing subalgebra of $\widetilde{\pi}(\widetilde{\mathfrak{g}})$. Therefore we have a map $\widetilde{\pi} : \mathfrak{g} \to A^-_{q|p}$ which extends to the homomorphism $\pi : U(\mathfrak{g}) \to A^-_{q|p}$ of associative algebras. By Theorem 13, φ is injective. Moreover, the image of φ coincides with the image of π. This immediately proves the existence of a unique map ψ such that the diagram commutes. $\square$

5.2. Clifford superalgebra and $\mathfrak{osp}(2p + 1|2q)$

Let V be a vector superspace equipped with even symmetric form $\beta : V \times V \to \Bbbk$. We define the Clifford superalgebra $\mathrm{Cliff}(V, \beta)$ as the quotient of the tensor superalgebra $T(V)$ by the relations

$$v \otimes w + (-1)^{p(v)p(w)} w \otimes v = \beta(v, w).$$

Note that $\mathrm{Cliff}(V, \beta)$ is finite-dimensional iff V is purely even. As any associative superalgebra $\mathrm{Cliff}(V, \beta)$ has the associated Lie superalgebra structure defined by $[x, y] = xy - (-1)^{p(x)p(y)}yx$. Let $\mathfrak{g}$ denote the Lie subalgebra of $\mathrm{Cliff}(V, \beta)$ generated by V.

Lemma 18. *We have the decomposition $\mathfrak{g} = V \oplus [V, V]$ such that $[[V, V], V] \subset V$. As a vector space $[V, V]$ is isomorphic to $\Lambda^2 V$ and concides with the span of $2vw - \beta(v, w)$ for all $v, w \in V$.*

Proof. First, we compute the commutator

$$[v, w] = vw - (-1)^{p(v)p(w)}wv = 2vw - \beta(v, w).$$

Next we compute the commutator between $[v, w]$ and u using super Leibniz identity

$$\begin{aligned} [u, [v, w]] &= 2[u, vw] = 2([u, v]w + (-1)^{p(u)p(v)}v[u, w]) \\ &= 2(2uvw - \beta(u, v)w + (-1)^{p(u)p(v)}2vuw - (-1)^{p(u)p(v)}\beta(u, w)v). \end{aligned}$$

Using $vu = -(-1)^{p(u)p(v)}uv + \beta(v, u)$ and the symmetry of β we obtain

$$[u, [v, w]] = 2(\beta(u, v)w - (-1)^{p(u)p(v)}\beta(u, w)v).$$

Hence we have obtained $[[V, V], V] \subset V$ and by Jacobi identity $[[V, V], [V, V]] \subset [V, V]$. $\square$

We concentrate on the case when β is non-degenerate and $\dim V = (2p|2q)$, let $n = p + q$ and choose a basis $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n$ such that

$$\beta(\xi_i, \xi_j) = \beta(\eta_i, \eta_j) = 0, \quad \beta(\eta_i, \xi_j) = \delta_{i,j}.$$

The parity is defined by

$$p(\xi_i) = p(\eta_i) = \begin{cases} 0 & \text{if } i \leq p, \\ 1 & \text{if } i > p. \end{cases}$$

The corresponding Clifford superalgebra is isomorphic to $A^+_{p|q}$. The defining relations are

$$\begin{aligned} &\xi_i\xi_j + (-1)^{p(i)p(j)}\xi_j\xi_i = \eta_i\eta_j + (-1)^{p(i)p(j)}\eta_j\eta_i = 0, \\ &\eta_i\xi_j + (-1)^{p(i)p(j)}\xi_j\eta_i = \delta_{ij}. \end{aligned}$$

Lemma 19. *The Lie subsuperalgebra of $A^+_{p|q}$ generated by ξ_i, η_i for $i = 1, \dots, n$ is isomorphic to $\mathfrak{osp}(2p + 1|2q)$.*

Proof. In notations of Lemma 18, consider the adjoint action of $[V, V]$ on V. The Leibniz rule implies that the form β is invariant under this action. Hence $[V, V]$ is isomorphic to $\mathfrak{osp}(2p, 2q)$ and V is its natural representation. Since obviously $V \oplus [V, V]$ is simple, it must be isomorphic to $\mathfrak{osp}(2p + 1|2q)$. $\square$

Corollary 20. *There exist homomorphisms of associative superalgebras*

$$\pi_1 : U(\mathfrak{osp}(2p|2q)) \to A^+_{p|q} \quad \textit{and} \quad \pi_2 : U(\mathfrak{osp}(2p + 1|2q)) \to A^+_{p|q}.$$

Let $q \neq 0$. Let us assume that $e_1, \dots, e_n$ and $f_1, \dots f_n$ are the Chevalley generators of $\mathfrak{osp}(2p+1|2q)$ such that e_p, f_p, e_n, f_n are odd and all other generators are even. Then we have

$$\pi_2(e_i) = \begin{cases} \xi_i\eta_{i+1} & \text{if } i < n, \\ \xi_n & \text{if } i = n, \end{cases} \qquad \pi_2(f_i) = \begin{cases} \xi_{i+1}\eta_i & \text{if } i < n, \\ \eta_n & \text{if } i = n, \end{cases}$$

and π_1 is obtained from π_2 by restriction.

Let $\mathfrak{g} = \mathfrak{gl}(p|q)$, $\mathfrak{osp}(2p|2q)$ or $\mathfrak{osp}(2p+1|2q)$ and identify $\mathbb{Z}^m$ with the root lattice of $\mathfrak{g}$ with basis consisting of the distinguished simple roots of $\mathfrak{g}$. Let $\zeta : \mathbb{Z}^m \to \mathbb{Z}^n$ be the $\mathbb{Z}$-linear maps given by the matrices

$$\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & & & \\ & & \ddots & 1 & \\ & & & -1 & \end{bmatrix}, \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & & & \\ & & \ddots & 1 & \\ & & & -1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & & & \\ & & \ddots & 1 & \\ & & & -1 & 1 \end{bmatrix} \tag{23}$$

respectively. Let $A^+_{p|q} = A_I$ be the Weyl algebra with index superset I, $I_{\bar{0}} = [\![1, p]\!]$, $I_{\bar{1}} = [\![p + 1, p + q]\!]$.

Theorem 21. *Let p, q be nonnegative integers, $n = p + q > 0$. Let $\mathfrak{g} = \mathfrak{gl}(p|q)$, $\mathfrak{osp}(2p|2q)$ or $\mathfrak{osp}(2p+1|2q)$. Then there is a commutative triangle of associative algebras with involution*

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\quad\pi\quad} & A^+_{p|q} \\ & {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle\varphi} & \\ & \mathcal{A}(\zeta)^+ & \end{array} \tag{24}$$

where φ is given by Theorem 8, $\psi(e_i) = X_i$, $\psi(f_i) = Y_i$, $\psi(h_{ii}) = \lambda_{ii}(u_i - 1)$, and

$$\pi(e_i) = \begin{cases} x_i\partial_{i+1}, & i < n, \\ x_n, & i = n,\ \mathfrak{g} = \mathfrak{osp}(2q+1|2p), \\ x_n^2, & i = n,\ \mathfrak{g} = \mathfrak{osp}(2q|2p), \end{cases}$$

$$\pi(f_i) = \pi(e_i)^*,\ \pi(h_i) = x_i\partial_i - (-1)^{p(i)}\tfrac{1}{2}.$$

The proof is similar to Theorem 17 and we leave it to the reader.

5.3. On $A^+_{p|q}$ versus $A^-_{q|p}$

If we disregard $\mathbb{Z}_2$-grading, then we have an isomorphism of associative algebras $A^\pm_{p|0} \simeq A^\mp_{0|p}$. We suspect that $A^+_{p|q}$ and $A^-_{q|p}$ are not isomorphic in general. Note also that $A^-_{p|q}$ is isomorphic to the tensor product $M_{2^q} \otimes (A^-_{p|0})$, while $A^+_{q|p}$ is isomorphic to the supertensor product $M_{2^q} \otimes (A^+_{0|p})$. However, we do have the following result.

Corollary 22. *Consider the sublattice*

$$\Gamma = \{(a_1, \dots, a_n) \,|\, a_1 + \cdots + a_n \in 2\mathbb{Z}\}$$

in $\mathbb{Z}^n$. Let $C^\pm_{p|q}$ denote the subsuperalgebra of elements of $A^\pm_{p|q}$ with the support in Γ. Then $C^+_{p|q}$ and $C^-_{q|p}$ are isomorphic superalgebras.

Proof. Theorems 17 and 21 provide the homomorphisms from $U(\mathfrak{osp}(2p|2q))$ to $A^-_{q|p}$ and $A^+_{p|q}$ respectively. It follows from formulas defining these isomorphisms that $C^-_{q|p}$ and $C^+_{p|q}$ are respective images. Consider the modules

$$M^- := A^-_{q|p} \otimes_{\Bbbk[\partial_1,\dots,\partial_n]} \Bbbk, \quad M^+ := A^+_{p|q} \otimes_{\Bbbk[\eta_1,\dots,\eta_n]} \Bbbk,$$

and let

$$N^- = C^-_{q|p}(1 \otimes 1), \quad N^+ = C^+_{p|q}(1 \otimes 1).$$

Note that $N^\pm$ is a simple module over $C^+_{p|q}$ and $C^-_{q|p}$, respectively, hence both N^+ and N^- are simple $U(\mathfrak{osp}(2p|2q))$-modules. Furthemore if $v = 1 \otimes 1$, then

$$f_i v = 0, \quad h_i v = -(-1)^{p(i)} v.$$

Thus both N^+ and N^- are simple lowest weight modules with the same lowest weight. Thus, N^+ and N^- are isomorphic, therefore they have the same annihilator $J \subset U(\mathfrak{osp}(2p|2q))$ and we obtain

$$C^+_{p|q} \simeq U(\mathfrak{osp}(2p|2q)/J \simeq C^-_{q|p}. \qquad \square$$

5.4. Consequence for classical Lie algebras

Taking $q = 0$ in Theorem 21 we immediately get the following result.

Corollary 23. *For $\mathfrak{g} = \mathfrak{sl}_n, \mathfrak{so}_{2n+1}$, or $\mathfrak{so}_{2n}$, there is a corresponding γ and a commutative triangle of associative algebras with involution*

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\quad\pi\quad} & A^+_{n|0} \\ & \psi \searrow \quad \nearrow \varphi & \\ & \mathcal{A}(\gamma)^+ & \end{array} \tag{25}$$

We can now prove that further primitive quotients of enveloping algebras of classical Lie algebras are examples of TGWAs. This extends previous results by the authors [16], where a condition for $U(\mathfrak{g})/J$ to be a not-necessarily abelian TGW algebra (i.e., we allowed $\sigma_i\sigma_j \neq \sigma_j\sigma_i$) was given.

Theorem 24. *If $\mathfrak{g} = \mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}$ or $\mathfrak{sp}_{2n}$ and M be a finite-dimensional completely pointed simple $\mathfrak{g}$-module and let $J = \operatorname{Ann}_{U(\mathfrak{g})} M$. Then $U(\mathfrak{g})/J$ is graded isomorphic to a TGWA of the form $\mathcal{A}(\gamma)^+$. The same is true for any fundamental representation of $\mathfrak{sl}_n$.*

Proof. The problem is to show that we can choose σ_i so that the group G generated by σ_i is abelian.

If $\mathfrak{g} = \mathfrak{so}_{2n}$ or $\mathfrak{so}_{2n+1}$ and M is a spinor representation, then $U(\mathfrak{g})/J$ is isomorphic to a subalgebra in the Clifford algebra with abelian G as follows from Corollary 23.

Let $\mathfrak{g} = \mathfrak{sl}_n$. Consider the embedding $\mathfrak{sl}_n \subset \mathfrak{so}_{2n+1}$ induced by the embedding of the corresponding Dynkin diagrams. The restriction of the spinor representation to $\mathfrak{sl}_n$ contains all fundamental representations. Let γ be the rightmost matrix in (23) and consider the subalgebra in $\mathcal{C} \subset \mathcal{A}(\gamma)^+$ generated by $X_1, \dots, X_{n-1}, Y_1, \dots, Y_{n-1}$. Let $I = \operatorname{Ann}_{\mathcal{C}} M$ and $\mathcal{B} = \mathcal{C}/I \simeq \operatorname{End}(M)$. Then $\mathcal{B}$ is a direct summand in the semisimple algebra $\mathcal{C}$. Hence σ_i for $i = 1, \dots, n-1$ preserve $\mathcal{B} \cap R$ and the statement follows.

Let Γ denote the set of weights of M. Note that σ_i must permute projectors E_β, hence it is defined by a permuation of Γ.

Let M be the standard representation of $\mathfrak{sp}_{2n}$. Then $\Gamma = \{\pm\varepsilon_i\}$. Let $\sigma_1 = \sigma_2 = \dots = \sigma_{n-1}$ be defined by the permutation $\kappa = (\varepsilon_1, \dots, \varepsilon_n)(-\varepsilon_n, \dots, -\varepsilon_1)$ and σ_n be defined by the permutation $\tau = (\varepsilon_1, -\varepsilon_1) \cdots (\varepsilon_n, -\varepsilon_n)$.

If $\mathfrak{g} = \mathfrak{so}_{2n}$ and M is the standard representation, then we choose $\sigma_1 = \dots = \sigma_{n-1}$ as in the previous case and let σ_n be given by the permutation $\kappa\tau$.

Finally, if $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and M is the standard representation, then $\Gamma = \{\pm\varepsilon_i, 0\}$ and we define $\sigma_1 = \dots = \sigma_n$ by the permutation $(\varepsilon_1, \dots, \varepsilon_n, 0, -\varepsilon_n, \dots, -\varepsilon_1)$. □

6. Open problems

Problem 1. For a simple Lie algebra $\mathfrak{g}$, list all finite-dimensional irreducible $\mathfrak{g}$-modules M for which there exists a graded isomorphism between $U(\mathfrak{g})/\operatorname{Ann}_{U(\mathfrak{g})} M$ and a TGW algebra (equivalently, for which there is a choice of commuting σ_i).

We believe none of the non-fundamental representations of $\mathfrak{sl}_n$ for $n > 2$ are in this list. The remaining cases to consider are the 27-dimensional representation of E_6 and 56-dimensional representation of E_7.

Problem 2. Find necessary and sufficient conditions for a not necessarily regular TGW algebra $\mathcal{A}_\mu(R, \sigma, t)$ to be consistent, generalizing the main result of [10].

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