THE CERESA CLASS: TROPICAL, TOPOLOGICAL, AND ALGEBRAIC

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ABSTRACT. The *Ceresa cycle* is an algebraic cycle attached to a smooth algebraic curve with a marked point, which is trivial when the curve is hyperelliptic with a marked Weierstrass point. The image of the Ceresa cycle under a certain cycle class map provides a class in étale cohomology called the *Ceresa class*. Describing the Ceresa class explicitly for non-hyperelliptic curves is in general not easy. We present a "combinatorialization" of this problem, explaining how to define a Ceresa class for a tropical algebraic curve, and also for a topological surface endowed with a multiset of commuting Dehn twists (where it is related to the Morita cocycle on the mapping class group). We explain how these are related to the Ceresa class of a smooth algebraic curve over $\mathbb{C}((t))$, and show that the Ceresa class in each of these settings is torsion.

1. INTRODUCTION

When X is a smooth algebraic curve with a marked point over a field, there is a canonical algebraic 1-cycle on the Jacobian of X called the *Ceresa cycle*. The Ceresa cycle is homologically trivial, but, as Ceresa showed in [6], it is not algebraically equivalent to zero for a very general curve of genus greater than 2. In some sense it is the simplest non-trivial canonical algebraic cycle "beyond homology" and as such it has found itself relevant to many natural problems in the geometry of curves and their Jacobians [10, 18, 20].

In recent years, many useful notions in algebraic geometry, and especially in the geometry of algebraic curves, have been seen to carry over to the tropical context, where they become interesting combinatorial notions. The motivation for the present paper is to understand whether the theory of the Ceresa cycle (or, more precisely, a cohomology class associated to that cycle) can be given a meaningful interpretation in the tropical setting. In particular, since a tropical curve is just a graph with positive real lengths assigned to the edges and integer weights assigned to the vertices, the Ceresa cycle would be a combinatorial invariant of such a graph. We define such an invariant in the present paper and begin to investigate its properties. We show, for example, that the Ceresa class of any hyperelliptic graph is zero (in conformity with the classical case) but that the Ceresa class of the complete graph on four vertices with all edges of length 1 is a nonzero class of order 16, see Proposition 4.6 and Remark 7.5, respectively. Moreover, we show in Example 7.2 that the Ceresa class is nonzero for *every* tropical curve whose underlying graph is the complete graph on four vertices.

Our approach is to model a tropical curve with integral edge lengths as the tropicalization of a curve that degenerates to a stable curve. We start by considering an algebraic family of smooth complex curves of genus g over a punctured disc $D\backslash *$, which degenerates to a stable curve over the central fiber $*$. There are several lenses through which one can view such a degeneration.

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- *Topology*: The family of complex genus-g curves over $D\backslash *$, considered as a manifold, is homotopic to a family of genus-g surfaces fibered over the circle, which we can think of as obtained by taking $\Sigma_g \times [0,1]$ and identifying $\Sigma_g \times 0$ with $\Sigma_g \times 1$ via a diffeomorphism of Σ_g defined up to homotopy, i.e., an element of the mapping class group. The stable reduction implies that this mapping class is a *multitwist*; that is, product of integral powers of commuting Dehn twists. Which twists they are can be read off the dual graph of the stable fiber at $*$, and which powers of each twist appear is determined by the multiplicity of the nodes in the degeneration.
- *Tropical geometry*: It is well known that a stable degeneration gives rise to a *tropical curve*, which is to say a vertex weighted metric graph; in this case, it will be the dual graph of the stable fiber, with edge-lengths determined by the multiplicity with which the family of curves strikes various boundary components of $\overline{\mathscr{M}}_g$.
- *Algebraic geometry over a local field*: We can also restrict the holomorphic family to an infinitesimal neighborhood of $*$, yielding an algebraic curve over $\mathbb{C}((t))$ with stable reduction.

In each case, there is a certain combinatorial datum which describes the degeneration: in the first case, the mapping class; in the second case, the tropical curve itself; and in the third case the action of the (pro-cyclic) absolute Galois group of $\mathbb{C}((t))$ on the étale fundamental group of $X_{\overline{\mathbb{C}((t))}}$ (or, as we shall see, just on the quotient of that fundamental group by the third term of its lower central series.) These three data agree in a sense made precise in §§3, 4.

The only one of these contexts in which there is a literal Ceresa class is the third one. But we shall see that we can in fact define the Ceresa cycle directly from the combinatorial datum. Thus we may now speak of the Ceresa class of a multitwist in the mapping class group, or the Ceresa class of a unipotent automorphism of the geometric étale fundamental group of a curve over $\mathbb{C}((t))$, or the Ceresa class of a tropical curve with integral edge lengths. (In this last case, our definition should be compared with that proposed by Zharkov in [21]; see Remark 7.3 for some speculations about this.) We explain in §4.3 how to extend the definition to non-integral edge lengths.

The topological definition rests fundamentally on the Morita cocycle on the mapping class group [14] (an extension of the Johnson homomorphism). For the algebraic story we use in a crucial way the work of Hain and Matsumoto [10] relating the Ceresa class in étale cohomology (over any field K) to the Galois action on the 2-nilpotent fundamental group. Indeed, we could just as well have described this paper as being about the "Morita class" rather than the "Ceresa class" — it is the ℓ-adic Harris-Pulte theorem of Hain and Matsumoto [10, §8] that relates the Morita class in group cohomology with the image of the Ceresa cycle under the cycle class map.

In fact, most of the proofs and theorems in the paper are carried out not in an algebro-geometric context but in the setting of the mapping class group, which seems to be the easiest to work with in practice. In this context, the Ceresa class of a multitwist can be described quite simply. Recall that an element of the mapping class group Γ_g is *hyperelliptic* if it commutes with some hyperelliptic involution $\tau \in \Gamma_g$. The Ceresa class of a multitwist $\gamma \in \Gamma_g$ is an obstruction to the existence of an element $\tilde{\tau}$ of Γ_g which acts as $-I$ on homology (a hyperelliptic involution being an example of such a $\tilde{\tau}$) such that the commutator $[\tilde{\tau}, \gamma]$ lies in the Johnson kernel. In shorter terms, one might say a multitwist has Ceresa class zero if it is "hyperelliptic up to the Johnson kernel."

Our main theorem, Theorem 6.7, is that the Ceresa class we define is torsion for any multitwist (and thus for any tropical curve with integral edge lengths.)

The paper is structured as follows. In §2 we define the Ceresa class of a multitwist. In §3 we explain the relation between the topological definition and the Ceresa class in algebraic geometry, and in §4 we explain how the definition extends to a tropical curve with arbitrary real edge lengths. In §§5-6 we prove Theorem 6.7 and describe a finite group in which the tropical Ceresa class naturally lies, a group which might be thought of as a sort of tropical intermediate Jacobian. Finally, in §7, we compute the Ceresa classes of several low-genus graphs. We close with a question: are there non-hyperelliptic tropical curves with Ceresa class zero?

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2. The topological Ceresa class

2.1. **The Mapping Class Group and the Symplectic Representation.** In this subsection, we recall some basic facts about the mapping class group, see [9] for a detailed treatment.

Throughout the paper, let Σ_g denote a closed genus g surface and Σ_g^1 denote a genus g surface with one puncture. Let Γ_g (resp. Γ_g^1) be the mapping class group of Σ_g (resp. Σ_g^1), i.e., the group of isotopy classes of orientation-preserving diffeomorphisms of the surface, and $\Pi_g = \pi_1(\Sigma_g)$. These groups fit into the Birman exact sequence:

$$1 \to \Pi_g \to \Gamma_g^1 \to \Gamma_g \to 1. \tag{2.1}$$

Given a simple closed curve a in Σ_g (resp. Σ_g^1), denote by T_a the left-handed *Dehn twist* of a. A *separating twist* is a Dehn twist T_a where a is a separating curve, and a *bounding pair map* is $T_aT_b^{-1}$ where a and b are homologous non-separating, non-intersecting, simple closed curves.

The singular homology group $H_1(\Sigma_g^1, \mathbb{Z}) \cong H_1(\Sigma_g, \mathbb{Z})$, which we denote by H, has a symplectic structure given by the the algebraic intersection pairing $\hat{\imath} : H \wedge H \to \mathbb{Z}$. The action of Γ_g (resp. Γ_g^1) on H respects this pairing. This yields the symplectic representation of Γ_g (resp. Γ_g^1), and we have the short exact sequence

$$1 \to \mathscr{I}_g \text{ (resp. } \mathscr{I}_g^1) \to \Gamma_g \text{ (resp. } \Gamma_g^1) \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1 \tag{2.2}$$

where $\mathscr{I}_g$ (resp. $\mathscr{I}_g^1$) is called the *Torelli group*. By [4, 16], the Torelli group is generated by separating twists and bounding pair maps.

The *Johnson homomorphism* was introduced by Johnson in [11] to study the action of the Torelli group on the third nilpotent quotient of a surface group. We provide the following characterization. Recall that for any symplectic free $\mathbb{Z}$-module V with symplectic basis α_i, β_i ($i = 1, \ldots, g$), the form

$$\omega_V = \left(\sum_{i=1}^{g} \alpha_i \wedge \beta_i \right). \tag{2.3}$$

does not depend on the choice of symplectic basis. When $V = H$, we simply write ω for this form. Set $L = \wedge^3 H$, and view H as a subgroup of L via the embedding $h \mapsto \omega \wedge h$. The Johnson homomorphism for a once-punctured surface is a group homomorphism $J : \mathscr{I}_g^1 \to L$; by the previous paragraph, it suffices to describe how J operates on separating twists and bounding pair maps. If T_a is a separating twist, then $J(T_a) = 0$. Suppose $T_a T_b^{-1}$ is a bounding pair map. The curves a and b separate Σ_g^1 into two subsurfaces, let S be the subsurface which does not contain the puncture. The inclusion $S \hookrightarrow \Sigma_g^1$ induces an injective map $H_1(S, \mathbb{Z}) \to H$ which respects the symplectic forms on these spaces. Denote the image of this map by W. Then ω restricts to ω_W on W, and

$$J(T_a T_b^{-1}) = \omega_W \wedge [a]. \tag{2.4}$$

The Johnson homomorphism for Σ_g is a homomorphism $J : \mathscr{I}_g \to L/H$ and operates on separating twists and bounding pair maps as above, except that S may be either of the two subsurfaces cut off by a and b.

2.2. **Construction of the Ceresa Class.** In this section, we construct a class in $H^1(\Gamma_g, L/H)$ whose restriction to $\mathscr{I}_g$ equals twice the Johnson homomorphism. By the work of Hain and Matsumoto [10], this class with ℓ-adic coefficients agrees with the universal Ceresa class over $\mathscr{M}_g$. We discuss this further in §3.2.

Let $F_{2g} = \pi_1(\Sigma_g^1)$, which is the rank-$2g$ free group, and

$$F_{2g} = L^1 F_{2g} \supseteq L^2 F_{2g} \supseteq L^3 F_{2g} \supseteq \dots$$

be the lower central series of F_{2g}, i.e., $L^{k+1}F_{2g} = [F_{2g}, L^k F_{2g}]$. The k-th nilpotent quotient of F_{2g} is $N_k = F_{2g}/L^k F_{2g}$. Note that $N_2 \cong H$. Set $\mathrm{gr}_L^k F_{2g} = L^k F_{2g}/L^{k+1}F_{2g}$, which is a central subgroup of N_{k+1}. We note that the N_k and the $\mathrm{gr}_L^k F_{2g}$ are all characteristic quotients of F_{2g} and thus carry natural actions of $\mathrm{Aut}(F_{2g})$; in particular, they carry actions of the mapping class group Γ_g^1. What's more, the action of $\mathrm{Aut}(F_{2g})$ on $\mathrm{gr}_L^k F_{2g}$ factors through the natural homomorphism $\mathrm{Aut}(F_{2g}) \to \mathrm{GL}(H)$.

By [14, Proposition 2.3], $\mathrm{Aut}(N_3)$ fits into an exact sequence of groups

$$1 \to \mathrm{Hom}(H, \mathrm{gr}_L^2 F_{2g}) \xrightarrow{\phi} \mathrm{Aut}(N_3) \xrightarrow{p} \mathrm{GL}(H) \to 1. \tag{2.5}$$

Here, for any $f \in \mathrm{Hom}(H, \mathrm{gr}_L^2 F_{2g})$ the action of $\phi(f)$ on N_3 is to send $\gamma \in N_3$ to $\gamma f([\gamma])$, where $[\gamma]$ is the image of γ under the natural projection to H. Because $\mathrm{Hom}(H, \mathrm{gr}_L^2 F_{2g})$ is abelian, we write the group operation additively. The group $\mathrm{Aut}(N_3)$ acts on $\mathrm{Hom}(H, \mathrm{gr}_L^2 F_{2g})$ by conjugation.

Let τ be an element of $\mathrm{Aut}(N_3)$ such that $p(\tau) = -I$. Since $p(\tau)$ is central in $\mathrm{GL}(H)$, any commutator in $\mathrm{Aut}(N_3)$ of the form $[x, \tau]$ lies in $\mathrm{Hom}(H, \mathrm{gr}_L^2 F_{2g})$. Define

$$\mu_\tau : \mathrm{Aut}(N_3) \to \mathrm{Hom}(H, \mathrm{gr}_L^2 F_{2g}) \qquad x \mapsto [x, \tau]. \tag{2.6}$$

Proposition 2.1. *The map μ_τ is a crossed homomorphism, and its cohomology class*

$$\mu \in H^1(\mathrm{Aut}(N_3), \mathrm{Hom}(H, \mathrm{gr}_L^2 F_{2g}))$$

is independent of the choice of τ.

Proof. That μ_τ is a crossed homomorphism follows from the fact that $[xy, \tau] = [y, \tau]^x [x, \tau]$, and hence

$$\mu_\tau(xy) = [xy, \tau] = \mu_\tau(x) + x \cdot \mu_\tau(y).$$

Now suppose we had made a different choice τ'; then $\tau' = t\tau$ for some $t \in \ker p$. One checks, using the fact that $\ker p$ is abelian, that

$$[x, t\tau] = t^x t^{-1} [x, \tau]$$

which is to say that

$$\mu_{\tau'}(x) = \mu_\tau(x) - t + x \cdot t$$

so $\mu_{\tau'}$ is cohomologous to μ_τ, as claimed. □

The preimage of $\ker p$ under the natural morphism $\Gamma_g^1 \to \mathrm{Aut}(N_3)$ is the Torelli group $\mathscr{I}_g^1$, and the restriction of this morphism to $\mathscr{I}_g^1$ is the Johnson homomorphism. By the work of Johnson [11], the homomorphism J is not surjective onto $\mathrm{Hom}(H, \mathrm{gr}_L^2 F_{2g})$; rather, its image is the natural $\mathrm{GSp}(H)$-equivariant inclusion of $L = \wedge^3 H$ into

$$\mathrm{Hom}(H, \mathrm{gr}_L^2 F_{2g}) = \mathrm{Hom}(H, \wedge^2 H).$$

We can thus inflate μ to Γ_g^1 to get a cohomology class $\mu \in H^1(\Gamma_g^1, L)$ represented by the cocycle

$$\mu_\tau(\gamma) = J([\gamma, \tau]) \tag{2.7}$$

where $\tau \in \Gamma_g^1$ acts on H as $-I$. We say that τ is a *hyperelliptic quasi-involution*; a hyperelliptic quasi-involution that is an honest involution is called a *hyperelliptic involution*. Following Proposition 2.1, the class μ is defined independent of the choice of τ.

Proposition 2.2. *The class μ is the unique element in $H^1(\Gamma_g^1, L)$ whose restriction to $\mathscr{I}_g^1$ is $2J \in H^1(\mathscr{I}_g^1, L)$.*

Proof. Pick an element $\gamma \in \mathscr{I}_g^1$ and fix a hyperelliptic quasi-involution $\tau \in \Gamma_g^1$. Because τ acts as $-I$ on H, it also acts as $-I$ on $\wedge^3 H$. Therefore, we have

$$J([\gamma, \tau]) = J(\gamma) + J(\tau\gamma^{-1}\tau^{-1}) = J(\gamma) + \tau \cdot J(\gamma^{-1}) = 2J(\gamma).$$

The uniqueness of μ follows from [10, Proposition 5.5]. □

Let $\nu \in H^1(\Gamma_g, L/H)$ denote the image of μ under the composition

$$H^1(\Gamma_g^1, L) \to H^1(\Gamma_g^1, L/H) \cong H^1(\Gamma_g, L/H).$$

The map $H^1(\Gamma_g, L/H) \to H^1(\Gamma_g^1, L/H)$ is induced by restriction, and is an isomorphism by [10, Proposition 10.3]. Similar to the once-punctured case, the class ν is represented by the cocycle $\gamma \mapsto J([\gamma, \tau])$ where $\tau \in \Gamma_g$ is a hyperelliptic quasi-involution.

Definition 2.3. The *Ceresa class* of $\gamma \in \Gamma_g$ (resp. Γ_g^1), denoted by $\nu(\gamma)$ (resp. $\mu(\gamma)$), is the restriction of ν (resp μ) to the cyclic group generated by γ, viewed as a cohomology class in $H^1(\langle\gamma\rangle, L/H)$ (resp. $H^1(\langle\gamma\rangle, L)$).

Remark 2.4. The justification for this notation is the ℓ-adic Harris-Pulte theorem of Hain and Matsumoto [10, §§8,10], which identifies the ℓ-adic analogue of the classes μ, ν defined above with the image of the Ceresa cycle under a cycle class map in étale cohomology. We discuss this aspect in detail in §3.2.

The Ceresa class $\nu(\gamma)$ lies in $H^1(\langle\gamma\rangle, L/H) \cong L/((\gamma-1)L+H)$. It is certainly trivial for any γ which commutes with τ, which is to say it is trivial for any γ in the *hyperelliptic mapping class group*. But the converse is not true; for instance, if γ is in the kernel of the Johnson homomorphism, then so is $[\gamma,\tau]$, so $\mu(\gamma) = J([\gamma,\tau]) = 0$; but such a γ certainly need not be hyperelliptic. More generally, any mapping class whose commutator with τ lands in the Johnson kernel has trivial Ceresa class.

When the action of γ on H has no eigenvalues that are roots of unity, the group $H^1(\langle\gamma\rangle, L/H)$ is finite and the Ceresa class is torsion. At the other extreme, if γ lies in the Torelli group, $H^1(\langle\gamma\rangle, L/H) = L/H$ and the Ceresa class is an element of this free $\mathbb{Z}$-module of positive rank and can be of infinite order.

The case of primary interest in the present paper is that where γ is a product of commuting Dehn twists, or a *positive multitwist*. In this case, the action of γ on H is unipotent, and $H^1(\langle\gamma\rangle, L/H)$ is infinite; however, in this case, as we shall prove in §6, the Ceresa class is still of finite order.

Theorem (Theorem 6.7)**.** *Let $\gamma \in \Gamma_g$ be a positive multitwist. Then the Ceresa class $\nu(\gamma)$ is torsion.*

In fact, we will show how the order of the Ceresa class can be explicitly computed, though the computation is somewhat onerous. We will, along the way, prove the analogous statement for multitwists in the punctured mapping class group Γ_g^1 as well.

Our method for proving Theorem 2.2 will be to show that the Ceresa class lies in a canonical finite subgroup of $H^1(\langle\gamma\rangle, L/H)$, which subgroup we might think of as the tropical intermediate Jacobian. A notion of tropical intermediate Jacobian was proposed by Mikhalkin and Zharkov in [13]; it would be interesting to know whether the two notions agree in the context considered here.

The next two sections will explain the relationship between the topological, tropical, and local-algebraic pictures; the reader whose interest is solely in the mapping class group can skip ahead to §5.

3. THE ℓ-ADIC CERESA CLASS

In the previous section, we defined the Ceresa class $\nu(\gamma)$ (resp. $\mu(\gamma)$) as a cohomological invariant of any element γ in the mapping class group Γ_g (resp. Γ_g^1). In this section, we discuss how the Ceresa class of multitwists arise in arithmetic geometry. We begin by recalling the monodromy action associated to a 1-parameter family of genus g surfaces degenerating to a stable curve.

3.1. **Monodromy.**

Our discussion on the monodromy of a degenerating family of stable curves mainly follows [1, §3.2] and [2, §1.1]. For details on the construction of a local universal family of a stable curve, see [19, §3]. Our goal is to recall the non-abelian Picard–Lefschetz formula [2, Theorem 2.2] which says that the monodromy action on the fundamental group of a smooth fiber is given by a multitwist.

Let X_0 be a stable complex curve of genus $g \geq 2$. Let $\mathscr{Y} \to \mathscr{D}$ be the local universal family for X_0 as was constructed in [19, Theorem 3.1.5]. The base $\mathscr{D}$ is homeomorphic to D^{3g-3} where D denotes a small open complex disc centered at 0. Let $\mathscr{B} \subset \mathscr{D}$ be the discriminant locus of $\mathscr{Y} \to \mathscr{D}$ and $\mathscr{D}^* = \mathscr{D} \setminus \mathscr{B}$; each fiber $\mathscr{Y}_p$ for $p \in \mathscr{D}^*$ is diffeomeorphic to a closed surface Σ_g. Choose a point $p_0 \in \mathscr{D}^*$ sufficiently close to 0 at which all loops in $\mathscr{D}^*$ will be based when we consider its fundamental group.

The combinatorial data of a stable curve is recorded in its *dual graph*, which is a connected vertex-weighted graph defined in the following way. Recall that a vertex-weighted graph $\mathbf{G}$ is a connected graph G, possibly with loops or multiple edges, together with a nonnegative integer w_v for each vertex v of G. The dual graph $\mathbf{G}$ of X_0 consists of

- a vertex v_C for each irreducible component C of X_0 whose weight is the arithmetic genus of C, and
- an edge e between v_C and $v_{C'}$ for each node n in the intersection of C and C'.

For each edge e_i of $\mathbf{G}$, choose a small loop ℓ_i' in the smooth locus of X_0 that goes around n_i. Shrinking $\mathscr{D}$ if necessary, the inclusion $X_0 \hookrightarrow \mathscr{Y}$ admits a retraction $\mathscr{Y} \to X_0$ so that $\mathscr{Y} \to X_0 \to \mathscr{Y}$ is homotopic to the identity. This gives rise to a specialization map $r_p : \mathscr{Y}_p \to X_0$ for each fiber $\mathscr{Y}_p$ over $p \in \mathscr{D}$. Then each $\ell_i = r_p^{-1}(\ell_i')$ defines a closed curve in $\mathscr{Y}_p$.

The discriminant locus $\mathscr{B}$ is a normal crossings divisor [2, Proposition 1.1] (see also [12, Theorem 2.7]). Following [2, Proposition 1.1(3)], choose coordinates $z_1, \ldots, z_{3g-3}$ on $\mathscr{D}$ so that B_i, the irreducible component of $\mathscr{B}$ consisting of those $p \in \mathscr{D}$ such that ℓ_i is contractible in $\mathscr{Y}_p$, has the form $B_i = \{z_i = 0\} \cap \mathscr{D}$. In particular, $\mathscr{Y}_0 = X_0$. Then $\mathscr{D}^*$ is homeomorphic to $(D^*)^N \times (D)^{3g-3-N}$ where N is the number of edges of G and $D^* = D \setminus 0$. Thus $\pi_1(\mathscr{D}^*)$ is isomorphic to $\oplus_{i=1}^N \mathbb{Z} \cdot \lambda_i$ where λ_i is a loop in $\mathscr{D}^*$ based at p_0 that goes around B_i. The monodromy action on Π_g is given by a non-abelian Picard-Lefschetz formula [2, Theorem 2.2]:

$$\rho_{\mathscr{Y}} : \pi_1(\mathscr{D}^*) \to \operatorname{Out}(\Pi_g) \qquad \lambda_i \mapsto T_{\ell_i}^{-1}. \tag{3.1}$$

Let $\mathfrak{Y} \to D$ be a 1-parameter degeneration such that $\mathfrak{Y}_0 = X_0$ is the only singular fiber. Suppose that the local equation in $\mathfrak{Y}$ near the node n_i corresponding to the edge e_i of the dual graph is $xy = t^{c_i}$ where t is the parameter on D and $c_i \in \mathbb{Z}_{>0}, i = 1, \ldots, N$. The following proposition is a variant of [15, Main Lemma].

Proposition 3.1. *The restriction of the monodromy map $\rho_{\mathscr{Y}}$ to $\pi_1(D^*) = \mathbb{Z} \cdot \gamma$ is given by*

$$\rho_{\mathfrak{Y}} : \pi_1(D^*) \to \operatorname{Out}(\Pi_g) \qquad \gamma \mapsto \prod_{i=1}^N T_{\ell_i}^{-c_i}.$$

Proof. The map $\mathfrak{Y} \to D$ is given by the pullback of $\mathscr{Y} \to \mathscr{D}$ under a map $j : D \to \mathscr{D}$, and c_i is the multiplicity at which $j(D)$ intersects the divisor B_i. Explicitly, with t being the local coordinate on D, the map j is given by

$$j(t) = (a_1 t^{c_1}, \ldots, a_N t^{c_N}, 0, \ldots, 0) + \text{higher order terms}$$

where $a_i \in \mathbb{C}^*$. Composing with the orthogonal projection to $B_i^\perp = \{z_k = 0 \,|\, k \neq i\}$ yields the map $D \to B_i^\perp$ given by $t \mapsto t^{c_i}$, and therefore $\pi_1(D^*) \to \pi_1(B_i^\perp \setminus 0)$ is given by $\gamma \mapsto c_i \cdot \lambda_i$. The proposition now follows from Equation (3.1). $\square$

3.2. **The ℓ-adic Ceresa class for algebraic curves over $\mathbb{C}((t))$.** In this subsection, we recall the definition of the Ceresa cycle associated with an algebraic curve and its induced class in Galois cohomology, following [10]. Using comparison theorems, this class agrees with the topological Ceresa class with ℓ-adic coefficients, justifying the definition of the Ceresa class ν in §2.

Let K be a field of characteristic 0, G_K its absolute Galois group, and ℓ a fixed prime number. Let X be a smooth, complete, genus $g \geq 3$ curve over K. For the moment, suppose X has a K-rational point ξ, which yields an embedding $\Phi_\xi : X \to \mathrm{Jac}(X)$. Define algebraic cycles in $\mathrm{CH}_1(\mathrm{Jac}(X))$ given by $X_\xi := (\Phi_\xi)_*(X)$ and $X_\xi^- := (\iota)_*(X_\xi)$ where ι is the inverse map on $\mathrm{Jac}(X)$. Because ι induces the identity map on singular cohomology groups $H_{2r}(\mathrm{Jac}(X), \mathbb{Z})$ for all $r \geq 0$, the cycle $X_\xi - X_\xi^-$ is null-homologous. Thus, the image of $X_\xi - X_\xi^-$ under the ℓ-adic Abel-Jacobi map produces a Galois cohomology class

$$\mu^{(\ell)}(X, \xi) \in H^1(G_K, H^{2g-3}_{\text{ét}}((\mathrm{Jac}\, X)_{\overline{K}}, \mathbb{Z}_\ell(g-1))).$$

Via Poincaré duality,

$$H^{2g-3}_{\text{ét}}((\mathrm{Jac}\, X)_{\overline{K}}, \mathbb{Z}_\ell(g-1)) \cong H^3_{\text{ét}}((\mathrm{Jac}\, X)_{\overline{K}}, \mathbb{Z}_\ell(1))(-1) \cong (\wedge^3 H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(1)))(-1).$$

Let $H_{\mathbb{Z}_\ell} = H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(1))$, $L_{\mathbb{Z}_\ell} = (\wedge^3 H_{\mathbb{Z}_\ell})(-1)$, and $\omega \in \wedge^2 H_{\mathbb{Z}_\ell}$ the polarization.

The map $h \mapsto \omega \wedge h$ yields an embedding $H_{\mathbb{Z}_\ell} \hookrightarrow L_{\mathbb{Z}_\ell}$. The *$\ell$-adic Ceresa class*, denoted by $\nu^{(\ell)}(X)$, is the image of $\mu^{(\ell)}(X, \xi)$ under the map $H^1(G_K, L_{\mathbb{Z}_\ell}) \to H^1(G_K, L_{\mathbb{Z}_\ell}/H_{\mathbb{Z}_\ell})$, where we view $\mu^{(\ell)}(X, \xi)$ as an element of $H^1(G_K, L_{\mathbb{Z}_\ell})$. By [10, §10.4], the class $\nu^{(\ell)}(X)$ only depends on the curve X/K and can be defined when X has no K-rational point. Hain and Matsumoto construct a universal characteristic class

$$\hat{n}^{(\ell)} \in H^1(\mathrm{Out}\, \Pi_g^{(\ell)}, L_{\mathbb{Z}_\ell}/H_{\mathbb{Z}_\ell})$$

which is the ℓ-adic analog of ν defined in §2. The class ν with $\mathbb{Z}_\ell$ coefficient corresponds to $(\hat{\rho}_X^{(\ell)})^*(\hat{n}^{(\ell)})$ under the comparison map

$$H^1(\Gamma_g, L/H) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^1_{\text{ét}}(\mathscr{M}_g \otimes \overline{K}, L_{\mathbb{Z}_\ell}/H_{\mathbb{Z}_\ell})$$

where

$$\hat{\rho}_X^{(\ell)} : \pi_1(\mathscr{M}_g, X_{\overline{K}}) \to \mathrm{Out}\, \pi_1^{(\ell)}(X_{\overline{K}})$$

is the universal monodromy representation. Let $n^{(\ell)}(X) \in H^1(G_K, L_{\mathbb{Z}_\ell}/H_{\mathbb{Z}_\ell})$ denote the pullback of $\hat{n}^{(\ell)}$ under the natural action $G_K \to \mathrm{Out}\, \pi_1^{(\ell)}(X_{\overline{K}})$. The ℓ-adic Harris-Pulte theorem [10, Theorem 10.5] asserts

$$n^{(\ell)}(X) = \nu^{(\ell)}(X). \tag{3.2}$$

We now show that the ℓ-adic Ceresa class of a curve over $\mathbb{C}((t))$ is torsion. We obtain this by showing that the ℓ-adic Ceresa class is, in a natural sense, the ℓ-adic completion of the Ceresa class of product of Dehn twists attached to the curve in the previous section. This fact justifies calling that topologically-defined class "the Ceresa class," and allows us to apply Theorem 6.7 to the ℓ-adic Ceresa class.

Theorem 3.2. *Suppose $K = \mathbb{C}((t))$ and X is a smooth curve over K. The ℓ-adic Ceresa class $\nu^{(\ell)}(X)$ is torsion.*

Proof. We begin by reducing to the semistable reduction case. By the semistable reduction theorem, there is a positive integer n such that the pullback X' of X by the map

$$\phi : \operatorname{Spec} K \to \operatorname{Spec} K \qquad \phi^{\#}(t) = t^n$$

has semistable reduction. The map ϕ induces an endomorphism of $H^1(G_K, L_{\mathbb{Z}_\ell}/H_{\mathbb{Z}_\ell})$ which is multiplication by n. This means that $\nu^{(\ell)}(X)$ is torsion if and only if $\nu^{(\ell)}(X')$ is torsion.

Therefore, we may assume that X has a semistable model $\mathfrak{X}$ defined over $\mathcal{O}_K$ with special fiber X_0. The étale local equation in $\mathfrak{X}$ of each node of X_0 is $xy = t^{c_i}$ for some $c_i \in \mathbb{Z}_{>0}$. Let

$$S = \operatorname{Spec} \mathbb{C}[\![x_1, \ldots, x_{3g-3}]\!], \qquad S' = \operatorname{Spec} \mathbb{C}[\![x_1^{\pm}, \ldots, x_N^{\pm}, x_{N+1}, \ldots x_{3g-3}]\!].$$

where N is the number of nodes of X_0. The map $\mathfrak{X} \to \operatorname{Spec} \mathcal{O}_K$ is the pullback of the (algebraic) local universal family $\mathscr{X} \to S$ by a morphism $i : \operatorname{Spec} \mathcal{O}_K \to S$ of the form

$$i^{\#}(x_k) = a_k t^{c_k} + \text{higher order terms}$$

where $a_k \in \mathbb{C}^*$. Define an analytic family $\mathfrak{Y} \to D$ by the pullback of $\mathscr{Y} \to \mathscr{D}$ under the map

$$j : D \to \mathscr{D} \qquad t \mapsto (a_1 t^{c_1}, \ldots, a_N t^{c_N}, 0, \ldots, 0).$$

Consider the following diagram.

$$\begin{tikzcd} \pi_1(D^*) \arrow[r, "j_*"] \arrow[d] & \pi_1(\mathscr{D}^*) \arrow[r, "\rho_{\mathscr{Y}}"] \arrow[d] & \operatorname{Out}(\Pi_g) \arrow[d] \\ G_K \arrow[r, "i_*"] & \pi_1^{\mathrm{alg}}(S') \arrow[r, "\rho_{\mathscr{X}}"] & \operatorname{Out}(\Pi_g^{(\ell)}) \end{tikzcd} \tag{3.3}$$

The left and middle vertical arrows are profinite completions, the right arrow is the ℓ-adic completion, and $\rho_{\mathscr{X}}$ is the monodromy map associated to $\mathscr{X} \to S$. The left square commutes because $G_K \to \pi_1^{\mathrm{alg}}(S')$ is the profinite completion of $\pi_1(D^*) \to \pi_1(\mathscr{D}^*)$, and the composition of the bottom two arrows may be identified with the natural action $G_K \to \operatorname{Out}(\Pi_g^{(\ell)})$. Commutativity of the right square follows from commutativity of the following diagram

$$\begin{tikzcd} 1 \arrow[r] & \Pi_g \arrow[r] \arrow[d] & \pi_1(\mathscr{Y}^*) \arrow[r] \arrow[d] & \pi_1(\mathscr{D}^*) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \widehat{\Pi}_g \arrow[r] & \pi_1^{\mathrm{alg}}(\mathscr{X}^*) \arrow[r] & \pi_1^{\mathrm{alg}}(S') \arrow[r] & 1 \end{tikzcd}$$

where $\mathscr{X}^*$ and $\mathscr{Y}^*$ are the smooth loci of $\mathscr{X} \to S$ and $\mathscr{Y} \to \mathscr{D}$, respectively. Here, the vertical arrows are profinite completions and the rows are exact. Let $\gamma \in \operatorname{Out}(\Pi_g)$ denote the image of the counterclockwise generator of $\pi_1(D^*)$ in $\operatorname{Out}(\Pi_g)$. Commutativity of the diagram in (3.3) yields the following commutative square

$$\begin{tikzcd} H^1(\mathrm{Out}(\Pi_g), L/H) \arrow[r] \arrow[d] & H^1(\langle\gamma\rangle, L/H) \arrow[d] \\ H^1(\mathrm{Out}(\widehat{\Pi}_g), L_{\mathbb{Z}_\ell}/H_{\mathbb{Z}_\ell}) \arrow[r] & H^1(G_K, L_{\mathbb{Z}_\ell}/H_{\mathbb{Z}_\ell}) \end{tikzcd}$$

Because the left arrow takes ν to $\hat{n}^{(\ell)}$, the right arrow takes $\nu(\gamma)$ to $\nu^{(\ell)}(X)$. By Proposition 3.1, γ acts as the multitwist $\prod_i T_{\ell_i}^{-c_i}$ on Π_g. The theorem now follows from Theorem 6.7. $\square$

Remark 3.3. We are indebted to Daniel Litt for the observation that it ought to be possible to prove directly, via arguments on weights [3], that the Ceresa class of a curve over $\mathbb{C}((t))$ is trivial, and to derive the topological theorems in this paper from this algebraic fact using the fact that every multitwist can be modeled by an algebraic degeneration. Our feeling is that modeling the paper this way could create the misleading impression that the topological statement was true for reasons involving hard theorems in algebraic geometry, while in fact, as we shall see, it is a topological fact with a topological proof.

4. THE TROPICAL CERESA CLASS

4.1. **Tropical curves.** A *tropical curve* Γ consists of a vertex weighted graph $\mathbf{G}$, together with a positive real-value c_e associated to each edge e, recording its length. The *genus* of Γ is

$$g(\Gamma) = |V(G)| - |E(G)| + 1 + |w|$$

where G is the underlying graph of Γ, and $|w|$ is the sum of the vertex weights. This is consistent with the interpretation of a weight on a vertex as an "infinitesimal loop." The *valence* of a vertex v, denoted by $\mathrm{val}(v)$, is the number of half-edges adjacent to v; in particular, a loop edge contributes 2 to the valence.

Given an arrangement of simple, closed, non-intersecting curves $\Lambda = \{\ell_1, \dots, \ell_N\}$ in Σ_g, its *dual graph* is the vertex weighted graph with:

- a vertex v_S for each connected component S of $\Sigma_g \setminus \bigcup_{i=1}^N \ell_i$ whose weight is the genus of S, and
- an edge e_i between v_S and $v_{S'}$ for each loop ℓ_i between S and S'.

Any vertex-weighted graph $\mathbf{G}$ of genus g may be realized as the dual graph to an arrangement of pairwise non-intersecting curves on Σ_g in the following way. For each vertex v, let Σ_v be a genus-w_v surface with $\mathrm{val}(v)$ boundary components. For each edge e of $\mathbf{G}$ between the (not necessarily distinct) vertices u and v, glue a boundary component of Σ_u to a boundary component of Σ_v; denote the glued locus in the resulting surface by ℓ_e. This process yields a genus-g surface, together with an arrangement of pairwise non-intersecting curves $\Lambda = \{\ell_e \,|\, e \in E(\Gamma)\}$ whose dual graph is $\mathbf{G}$. For an illustration, see Figure 4.1. If Γ is a tropical curve with integral edge lengths c_e, then we have a canonical multitwist

$$T_\Gamma = \prod_{e \in E(\Gamma)} T_{\ell_e}^{c_e}$$

and we let $\delta_\Gamma \in \mathrm{Sp}(H)$ denote the action of T_Γ on H. At this point, one may define the Ceresa class of Γ to be $\nu(T_\Gamma) \in H^1(\langle T_\Gamma \rangle, L/H) = H^1(\langle \delta_\Gamma \rangle, L/H)$. However, when the edge lengths of Γ are not integral, we cannot define δ_Γ and $\nu(\Gamma)$ in terms of a multitwist. We will define the Ceresa class for a tropical curve and what it means for it to be trivial in §4.3.

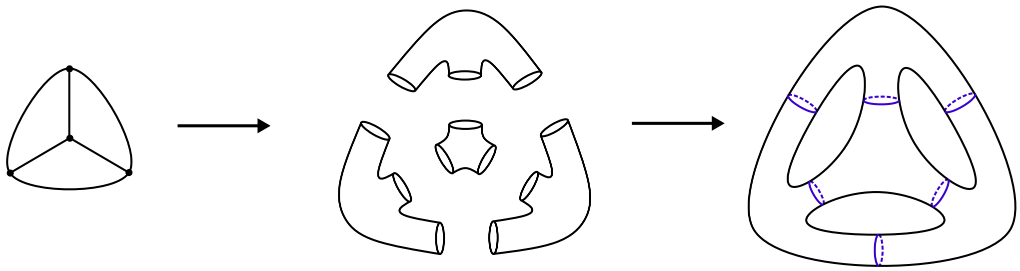

FIGURE 4.1. Passing from a tropical curve to a multitwist

4.2. **The tropical Jacobian.** Now suppose Γ has genus $g \geq 2$ and fix an orientation on the underlying graph G. Its *Jacobian* is the real g dimensional torus

$$\mathrm{Jac}(\Gamma) = (H_1(G,\mathbb{R}) \oplus \mathbb{R}^{|w|})/(H_1(G,\mathbb{Z}) \oplus \mathbb{Z}^{|w|})$$

together with the semi-positive quadratic form Q_Γ which vanishes on $\mathbb{R}^{|w|}$ and on $H_1(G,\mathbb{R})$ is equal to

$$Q_\Gamma\left(\sum_{e\in E(G)} \alpha_e \cdot e\right) = \sum_{e \in E(G)} \alpha_e^2 \cdot c_e.$$

The form Q_Γ is positive definite when all vertex weights are 0, and $\det(Q_\Gamma)$ is the *first Symanzik polynomial* of of G [1, Proposition 2.9]. That is,

$$\det(Q_\Gamma) = \sum_T c_T, \quad \text{where} \quad c_T = \prod_{e \notin E(T)} c_e \tag{4.1}$$

and the sum is taken over all spanning trees T of G.

When Γ has integral edge lengths, δ_Γ and Q_Γ are related in the following way. First, embed G into Σ_g so that each vertex v maps to a point in Σ_v, and each edge e maps to a simple arc intersecting the loop ℓ_e exactly one time, and no other $\ell_{e'}$. This embedding, which we denote by $\iota : G \hookrightarrow \Sigma_g$, induces an injective map on integral homology groups. Then

$$Q_\Gamma(\gamma) = \hat{\imath}([\iota(\gamma)], (\delta_\Gamma - I)[\iota(\gamma)]). \tag{4.2}$$

Here is a more explicit description of the relationship between Q_Γ and δ_Γ. Enumerate the edge set $E(G) = \{e_1, \ldots, e_N\}$ so that $E(G) \setminus \{e_1, \ldots, e_h\}$ are the edges of a spanning tree T. The graph $T \cup \{e_i\}$ has a unique cycle. The cycles $[\iota(\gamma_1)], \ldots, [\iota(\gamma_h)]$ form a basis for an isotropic subspace of H. Orient $\iota(\gamma_i)$ and ℓ_i so that $\hat{\imath}([\iota(\gamma_i)], [\ell_i]) = 1$, for $1 \leq i \leq h$. Setting $\alpha_i = [\iota(\gamma_i)]$ and $\beta_i = [\ell_i]$ yields a symplectic basis of a symplectic subspace of H.

This extends to a symplectic basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ on all of H, allowing us to identify Q_Γ with a symmetric $g \times g$ matrix. Then

$$\delta_\Gamma = \begin{pmatrix} I & 0 \\ Q_\Gamma & I \end{pmatrix}. \tag{4.3}$$

In particular, we may identify Q_Γ with the restriction $\delta_\Gamma - I : H/Y \to Y$, where Y is the $\mathbb{Z}$-submodule of H spanned by the β_i for $i = 1, \dots, h$.

4.3. **The tropical Ceresa class.** We saw in §4.1 how one may define the Ceresa class of a tropical curve with integral edge lengths in terms of a multitwist. When the edge lengths of Γ are not integral, then we do not have access to such a multitwist. Instead, we will define what it means for a tropical curve to be Ceresa trivial.

The kernel of the Johnson homomorphism, denoted by $\mathscr{K}_g$, is a normal subgroup of Γ_g, which allows us to form the quotient $\mathscr{G}_g = \Gamma_g/\mathscr{K}_g$. This follows from the fact that $\mathscr{K}_g$ is the kernel of the map $\Gamma_g \to \mathrm{Out}(N_3)$. Let $\mathbf{G}$ be a vertex-weighted graph, and denote by $\mathcal{B}(\mathbf{G})$ the subgroup of $\mathscr{G}_g$ generated by the twists T_{ℓ_e} for $e \in E(\mathbf{G})$. This is a free $\mathbb{Z}$-module because the ℓ_e's are non-intersecting, and it has rank $N - s$ where s is the number of separating edges in $\mathbf{G}$. Given a hyperelliptic quasi-involution $\tau \in \mathscr{G}_g$, define

$$\mathcal{B}_\tau(\mathbf{G}) = \mathcal{B}(\mathbf{G}) \cap C_{\mathscr{G}_g}(\tau) \leq \mathcal{B}(\mathbf{G})$$

where $C_{\mathscr{G}_g}(\tau)$ denotes the centralizer of τ in $\mathscr{G}_g$. Let

$$\nu(\Gamma) = \sum_{e \in E(\mathbf{G})} c_e T_e \in \mathcal{B}(\mathbf{G})_{\mathbb{R}}.$$

We say that Γ is *Ceresa trivial* if there exists a hyperelliptic quasi-involution τ such that $\nu(\Gamma) \in \mathcal{B}_\tau(\mathbf{G})_{\mathbb{R}}$. Proposition 4.2 below shows that this notion agrees with the Ceresa class associated to the multitwist T_Γ being trivial in the case where Γ has integral edge length, but first we will need the following Lemma.

Lemma 4.1. *The Ceresa class $\nu(T_\Gamma)$ is trivial if and only if there is a hyperelliptic quasi-involution τ such that $J([T_\Gamma, \tau]) = 0$.*

Proof. The "if" direction is clear. Suppose $\nu(T_\Gamma) = 0$. The class $\nu(T_\Gamma)$ is represented by the cocycle $\gamma \mapsto J([\gamma, \tau])$ for some hyperelliptic quasi-involution τ, and hence

$$J([T_\Gamma, \tau]) = T_\Gamma \cdot h - h$$

for some $h \in L/H$. Because the Johnson homomorphism is surjective, there is a $t \in \mathscr{I}_g$ such that $J(t) = h$. By rearranging the above equality, we see that $J([\gamma, t^{-1}\tau]) = 0$. The Lemma now follows from the fact that $t^{-1}\tau$ is also a hyperelliptic quasi-involution. □

Proposition 4.2. *Suppose Γ has integral edge lengths. Then Γ is Ceresa trivial if and only if $\nu(T_\Gamma) = 0$.*

Proof. The tropical curve Γ is Ceresa trivial if and only if there is a hyperelliptic quasi-involution τ such that $\nu(\Gamma) \in \mathcal{B}_\tau(\mathbf{G})$, if and only if $[T_\Gamma, \tau] = 1$ in $\mathscr{G}_g$, if and only if $J([T_\Gamma, \tau]) = 0$ in L/H, if and only if $\nu(T_\Gamma) = 0$. The last equivalence is Lemma 4.1. □

Proposition 4.3. *The following are equivalent:*

(1) Γ is Ceresa nontrivial for all positive real edge lengths;
(2) Γ is Ceresa nontrivial for all positive integral edge lengths;

(3) $\nu(T_\Gamma) \neq 0$ *for all positive integral edge lengths;*
(4) *for any hyperelliptic quasi-involution* τ, *the subgroup* $\mathcal{B}_\tau(\mathbf{G})$ *is contained in a coordinate hyperplane of* $\mathcal{B}(\mathbf{G})_\mathbb{R}$.

Proof. The implications (4) $\Rightarrow$ (1) $\Rightarrow$ (2) are clear, and (2) $\Rightarrow$ (3) follows from Proposition 4.2. Suppose there is a τ such that $\mathcal{B}_\tau(\mathbf{G})$ is not contained in a coordinate hyperplane of $\mathcal{B}(\mathbf{G})_\mathbb{R}$. This means that there is a lattice point in $\mathcal{B}_\tau(\mathbf{G})$ whose coordinates are all positive. This corresponds to a tropical curve Γ with underlying vertex-weighted graph $\mathbf{G}$ such that $\nu(T_\Gamma) = 0$. This proves (3) $\Rightarrow$ (4). $\square$

We end this section by showing that the Ceresa class vanishes for hyperelliptic tropical curves. First, we recall some terminology. Let $\mathbf{G}$ be a vertex-weighted graph. A vertex v of a vertex-weighted graph $\mathbf{G}$ is *stable* if

$$2w_v - 2 + \mathrm{val}(v) > 0.$$

and $\mathbf{G}$ is stable if all of its vertices are stable. A tropical curve is stable if its underlying weighted graph is stable. Two tropical curves are *tropically equivalent* if one can be obtained from the other via a sequence of the following moves:

- adding or removing a 1-valent vertex v with $w_v = 0$ and the edge incident to v, or
- adding or removing a 2-valent vertex v with $w_v = 0$, preserving the underlying metric space.

Every tropical curve Γ of genus $g \geq 2$ is tropically equivalent to a unique tropical curve whose underlying weighted graph is stable [5, Section 2].

Lemma 4.4. *If* Γ *and* Γ' *are tropically equivalent, then* $\nu(\Gamma) = \nu(\Gamma')$.

Proof. Let v be a vertex with $w_v = 0$. Suppose $\mathrm{val}(v) = 1$, and denote by e the adjacent edge. Then Σ_v is a disc, so ℓ_e is contractible, and hence $T_{\ell_e} = 1$. Now suppose $\mathrm{val}(v) = 2$, and denote by e, f the adjacent edges. Then ℓ_e is isotopic to ℓ_f, and hence $T_{\ell_e} = T_{\ell_f}$. We conclude that the Ceresa class of tropically equivalent tropical curves coincide. $\square$

Suppose Γ is a stable hyperelliptic tropical curve with underlying vertex-weighted graph $\mathbf{G}$, and σ the hyperelliptic involution of Γ. That is, $\sigma : \Gamma \to \Gamma$ is an isometry that induces an involution of $\mathbf{G}$ such that all vertices of positive weight are fixed and Γ/σ is a metric tree. By [8, Proposition 2.5] the edge set of Γ partitions into the subsets

- $\{e\}$ for separating edges e and $\sigma(e) = e$,
- $\{e, f\}$ where $e \neq f$ form a separating pair of edges and $\sigma(e) = f$, and
- $\{e\}$ where e is any other edge, and $\sigma(e) = e$.

If $\{e, f\}$ is a separating pair, then $c_e = c_f$ because σ is an isometry.

Lemma 4.5. *Suppose* Γ *is a hyperelliptic tropical curve, and* $\{\ell_e \mid e \in E(\Gamma)\}$ *is an arrangement of loops on* Σ_g *whose dual graph is* Γ. *There is a hyperelliptic involution* τ *of* Σ_g *such that* $\tau(\ell_e) = \ell_{\sigma(e)}$.

Proof. Let σ be a hyperelliptic involution on Γ. Fix homeomorphisms $\tau_{u,\sigma(u)} : \Sigma_u \to \Sigma_{\sigma(u)}$ such that, if u, v are vertices connected by the edge e, then $\tau_{u,\sigma(u)}(\ell_e) = \ell_{\sigma(e)}$. These homeomorphisms piece together to give a hyperelliptic involution τ on Σ_g with the requisite property. $\square$

Proposition 4.6. *If* Γ *is a hyperelliptic tropical curve, then* Γ *is Ceresa trivial.*

Proof. By Lemma 4.4, we may assume that Γ is stable. Let τ be the hyperelliptic involution from Lemma 4.5. If e is a separating edge, then T_{ℓ_e} is a separating twist, which is trivial in $\mathcal{B}(\mathbf{G})$. If $\{e, f\}$ is a pair of separating edges, then $T_{\ell_e} + T_{\ell_f} \in \mathcal{B}_\tau(\mathbf{G})$. If e is any other edge, then $T_{\ell_e} \in \mathcal{B}_\tau(\mathbf{G})$. Decompose $\nu(\Gamma)$ as

$$\nu(\Gamma) = \sum (c_e T_{\ell_e} + c_f T_{\ell_f}) + \sum c_e T_{\ell_e}$$

where the sum on the left is over all separating pairs, and the sum on the right is over all non-separating edges not in a separating pair. Because $c_e = c_f$ whenever $\{e, f\}$ is a separating pair, we have that $\nu(\Gamma) \in \mathcal{B}_\tau(\mathbf{G})_{\mathbb{R}}$, and hence Γ is Ceresa trivial. $\square$

5. A finite subgroup of $H^1(\mathbb{Z}, L)$

5.1. **Filtrations on** $H^1(\mathbb{Z}, \wedge^k H)$**.** In this section we set up a rather general framework for abelian groups with a unipotent automorphism, which we will apply in the case of the singular homology of a genus-g topological surface acted upon by a multitwist.

Let H be a finitely generated free $\mathbb{Z}$-module and $\delta \in \mathrm{SL}(H)$ be an element such that $(\delta - I)^2 = 0$. We consider the action of the cyclic group $\langle \delta \rangle \cong \mathbb{Z}$ on H, which induces an action of $\langle \delta \rangle$ on $\wedge^k H$ for any $k \geq 0$. Let Y be the saturation of $\mathrm{im}(\delta - I)$ in H. By the hypothesis on δ, we have $Y \leq \ker(\delta - I)$, i.e., δ acts trivially on Y. Consider the following descending filtration on $\wedge^k H$:

$$F_q \wedge^k H = (\wedge^q Y) \wedge (\wedge^{k-q} H). \tag{5.1}$$

Note that $F_q \wedge^k H$ is saturated in $\wedge^k H$, so the graded piece $\mathrm{gr}^F_q \wedge^k H := F_q \wedge^k H / F_{q+1} \wedge^k H$ is torsion-free. The following Lemma shows that $(\delta - I)(F_q \wedge^k H) \leq F_{q+1} \wedge^k H$ for any $k \geq q \geq 0$.

Lemma 5.1. *For $y \in \wedge^q Y$ and $h = h_1 \wedge \ldots \wedge h_{k-q} \in \wedge^{k-q} H$,*

$$(\delta - I)(y \wedge h) = \sum_{i=1}^{k-q} y \wedge h_1 \wedge \ldots \wedge (\delta - I)h_i \wedge \ldots \wedge h_{k-q} \mod F_{q+2} \wedge^k H.$$

In particular, $(\delta - I)(F_q \wedge^k H) \leq F_{q+1} \wedge^k H$.

Proof. Because $\delta y = y$, we can write

$$(\delta - I)(y \wedge h) = y \wedge (\delta - I + I)h_1 \wedge \ldots \wedge (\delta - I + I)h_{k-q} - y \wedge h$$

and expand the latter as

$$\sum_{i=1}^{k-q} y \wedge h_1 \wedge \ldots \wedge (\delta - I)h_i \wedge \ldots \wedge h_{k-q}$$
$$+ \sum_{1 \leq i < j \leq k-q} y \wedge h_1 \wedge \ldots \wedge (\delta - I)h_i \wedge \ldots \wedge (\delta - I)h_j \wedge \ldots \wedge h_{k-q} + \ldots \qquad \square$$

This means, in particular, that $\delta - I$ induces a map $\mathrm{gr}^F_{q-1} \wedge^k H \to \mathrm{gr}^F_q \wedge^k H$ for all q. While these maps are typically not surjective, what we will see in the lemma below is that at least half of them are surjective *rationally*; that is, their cokernels are finite.

Lemma 5.2. *The map*
$$(\delta - I)_{\mathbb{Q}} : \mathrm{gr}^F_{q-1} \wedge^k H_{\mathbb{Q}} \to \mathrm{gr}^F_q \wedge^k H_{\mathbb{Q}}$$
is surjective for $q > k/2$.

Proof. Let $y = y_1 \wedge \ldots \wedge y_q \in \wedge^q Y_{\mathbb{Q}}$ and $h = h_1 \wedge \ldots \wedge h_{k-q} \in \wedge^{k-q} H_{\mathbb{Q}}$. It suffices to show that $y \wedge h$ lies in the image of $(\delta - I)_{\mathbb{Q}}$ mod $F_{q+1} \wedge^k H_{\mathbb{Q}}$ for all such y, h. Choose $x_i \in H_{\mathbb{Q}}$ such that $(\delta - I)x_i = y_i$. For $I \in \binom{[q]}{p}$, let $\hat{y}_I$ be obtained from y by replacing y_i with x_i for each $i \in I$. Similarly, for $J \in \binom{[k-q]}{p}$ let $\hat{h}_J$ be obtained from h by replacing h_j with $(\delta - I)h_j$ for each $j \in J$. We define
$$\mathfrak{y}(a,b) = \sum_{I \in \binom{[q]}{a}, J \in \binom{[k-q]}{b}} \hat{y}_I \wedge \hat{h}_J \in F_{q-a+b} \wedge^k H_{\mathbb{Q}},$$
Note that $\mathfrak{y}(a,b) \neq 0$ only if $0 \leq a \leq q$ and $0 \leq b \leq k-q$, and $\mathfrak{y}(0,0) = y \wedge h$. By Lemma 5.1,
$$(\delta - I)_{\mathbb{Q}}(\hat{y}_I \wedge \hat{h}_J) \equiv \sum_{i \in I} \hat{y}_{I \setminus \{i\}} \wedge \hat{h}_J + \sum_{j \notin J} \hat{y}_I \wedge \hat{h}_{J \cup \{j\}} \bmod F_{q-a+b+2} \wedge^k H_{\mathbb{Q}}.$$
In particular,
(5.2)
$$(\delta - I)_{\mathbb{Q}}(\mathfrak{y}(a,b)) \equiv (q-a+1) \cdot \mathfrak{y}(a-1,b) + (b+1) \cdot \mathfrak{y}(a,b+1) \bmod F_{q-a+b+2} \wedge^k H_{\mathbb{Q}}.$$

Claim: For $0 \leq p \leq k-q$, $\mathfrak{y}(p,p)$ is in $\mathrm{im}(\delta - I)_{\mathbb{Q}}$ mod $F_{q+1} \wedge^k H_{\mathbb{Q}}$.

The case $p = 0$ is exactly the statement that $y \wedge h \in \mathrm{im}(\delta - I)_{\mathbb{Q}}$ mod $F_{q+1} \wedge^k H_{\mathbb{Q}}$. We will proceed by downward induction on p. Because $\mathfrak{y}(k-q+1, k-q+1) = 0$, Equation (5.2) yields
$$(\delta - I)_{\mathbb{Q}} \left(\mathfrak{y}(k-q+1, k-q)\right) \equiv (2q-k) \cdot \mathfrak{y}(k-q, k-q) \bmod F_{q+1} \wedge^k H_{\mathbb{Q}}$$
and therefore the claim holds when $p = q$, noting that $2q > k$. Assuming the Claim is true for $p+1$, we will show that it is true for p. Again by Equation 5.2,
$$(\delta - I)_{\mathbb{Q}} \left(\mathfrak{y}(p+1, p)\right) \equiv (q-p) \cdot \mathfrak{y}(p,p) + (p+1) \cdot \mathfrak{y}(p+1, p+1) \bmod F_{q+1} \wedge^k H_{\mathbb{Q}}.$$
By the inductive hypothesis, $\mathfrak{y}(p+1, p+1)$ is in the image of $(\delta - I)_{\mathbb{Q}}$ mod $F_{q+1} \wedge^k H_{\mathbb{Q}}$, and therefore so is $(q-p)\mathfrak{y}(p,p)$; by the hypothesis that $q > k/2$, we have $q - p \neq 0$, so $\mathfrak{y}(p,p)$ is in the image of $(\delta - I)_{\mathbb{Q}}$ mod $F_{q+1} \wedge^k H_{\mathbb{Q}}$ and we are done. □

Remark 5.3. In the case where $\dim H = 2 \dim Y$, the largest possible dimension of Y, the bound $q > k/2$ in Lemma 5.2 is sharp. Set $d = \dim Y$, $e = \dim H$, and $u(q) = \dim(\mathrm{gr}^F_q \wedge^k H)$. Then
$$u(q) = \binom{d}{q}\binom{e-d}{k-q}.$$
When $e = 2d$ and $0 \leq q \leq k/2$
$$\frac{u(q)}{u(q-1)} = \frac{(d-q+1)(k-q+1)}{q(d-k+q)} \geq \frac{(d-q+1)(q+1)}{q(d-q)} > 1$$
and therefore u is a strictly increasing function on this interval. This means that $(\delta - I)_{\mathbb{Q}}$ as in Lemma 5.2 cannot be surjective.

We denote by $A_q(\delta)$ and $B_q(\delta)$ the groups:

$$A_q(\delta) = \mathrm{im}(H^1(\langle\delta\rangle, F_q \wedge^{2q-1} H) \to H^1(\langle\delta\rangle, \wedge^{2q-1} H)),$$
$$B_q(\delta) = \mathrm{coker}(\delta - I : \mathrm{gr}^F_{q-1} \wedge^{2q-1} H \to \mathrm{gr}^F_q \wedge^{2q-1} H).$$

Proposition 5.4. *We have isomorphisms of groups*

$$A_q(\delta) \cong \frac{F_q \wedge^{2q-1} H}{((\delta - I) \wedge^{2q-1} H) \cap (F_q \wedge^{2q-1} H)}, \quad B_q(\delta) \cong \frac{F_q \wedge^{2q-1} H}{((\delta - I)F_{q-1} \wedge^{2q-1} H) + (F_{q+1} \wedge^{2q-1} H)}.$$

In particular, $A_q(\delta)$ and $B_q(\delta)$ are finite.

Proof. It is a standard fact that

$$H^1(\langle\delta\rangle, \wedge^k H) \to \wedge^k H/(\delta - I) \wedge^k H \qquad [\zeta] \mapsto \zeta(\delta)$$

is an isomorphism. This yields the isomorphism involving $A_q(\delta)$, and the one involving $B_q(\delta)$ is clear. Because each $(\delta - I)_{\mathbb{Q}} : \mathrm{gr}^F_{i-1} \wedge^{2q-1} H_{\mathbb{Q}} \to \mathrm{gr}^F_i \wedge^{2q-1} H_{\mathbb{Q}}$ is surjective for $i \geq q$ by Lemma 5.2, we see that $F_q \wedge^{2q-1} H_{\mathbb{Q}}$ is contained in $(\delta - I)(F_{q-1} \wedge^{2q-1} H_{\mathbb{Q}})$. Therefore, $A_q(\delta)$ and $B_q(\delta)$ are finite. □

Proposition 5.5. *If $\delta - I : \mathrm{gr}^F_{q-2} \wedge^k H \to \mathrm{gr}^F_{q-1} \wedge^k H$ is injective, then*

$$(\delta - I)(F_{q-2} \wedge^k H) \cap F_q \wedge^k H = (\delta - I)F_{q-1} \wedge^k H.$$

In particular, this means that if $\delta - I : \mathrm{gr}^F_{i-2} \wedge^{2q-1} H \to \mathrm{gr}^F_{i-1} \wedge^{2q-1} H$ is injective for all $i \leq q$, then $A_q(\delta) \cong F_q \wedge^{2q-1} H/(\delta - I)F_{q-1} \wedge^{2q-1} H$, and hence there is a natural surjection $A_q(\delta) \to B_q(\delta)$.

Proof. Suppose $y \in (\delta - I)(F_{q-2} \wedge^k H) \cap F_q \wedge^k H$ and $y = (\delta - I)x$. By injectivity of $\delta - I : \mathrm{gr}^F_{q-2} \wedge^k H \to \mathrm{gr}^F_{q-1} \wedge^k H$ and the fact that $y \in F_q \wedge^k H$, we see that $x \equiv 0 \mod F_{q-1} \wedge^k H$, i.e., $y \in (\delta - I)F_{q-1} \wedge^k H$. The other inclusion follows from Lemma 5.1. □

In the following section, we will need to show that certain classes in $H^1(\langle\delta\rangle, \wedge^k H)$ arising from topology are trivial, for which we will need the following explicit computation.

Proposition 5.6. *Any element of $\wedge^k H$ of the form $y \wedge z_1 \wedge \ldots \wedge z_{k-1}$ where $y \in \mathrm{im}(\delta - I)$ and $z_i \in \ker(\delta - I)$ lies in $(\delta - I) \wedge^k H$.*

Proof. Choose $x \in H$ such that $(\delta - I)x = y$. Then

$$\begin{aligned}(\delta - I)(x \wedge z_1 \wedge \ldots \wedge z_{k-1}) &= \delta x \wedge \delta z_1 \wedge \ldots \wedge \delta z_{k-1} - x \wedge z_1 \wedge \ldots \wedge z_{k-1} \\ &= \delta x \wedge z_1 \wedge \ldots \wedge z_{k-1} - x \wedge z_1 \wedge \ldots \wedge z_{k-1} \\ &= y \wedge z_1 \wedge \ldots \wedge z_{k-1}.\end{aligned}$$

□

The main application of Proposition 5.4 will be in the case $k = 3$ and we will denote $L = \wedge^3 H$ as before. For this reason, we simply write $A(\delta)$ and $B(\delta)$ for the subgroups $A_2(\delta)$ and $B_2(\delta)$, respectively. These groups are finite by Proposition 5.4, and in particular any element of $H^1(\langle\delta\rangle, L)$ which lies in $A(\delta)$ is torsion.

5.2. **The maximal rank case.** An important subcase is that where $\dim Y$ is as large as possible, namely $\dim Y = g = \frac{1}{2}\dim H$. Because $Y \subset H$ is saturated, there is a subgroup $X \subset H$ such that $H = X \oplus Y$. Let $Q : X \to Y$ denote the restriction of $\delta - I$ to X and $q_1 \leq \cdots \leq q_g$ its invariant factors, with $q_i | q_{i+1}$. Because Q is rationally surjective, each q_i is positive and

$$\operatorname{coker}(Q) \cong \prod_{i=1}^{g} \mathbb{Z}/(q_i).$$

is a finite group. Choose bases $\{x_1, \ldots, x_g\}$ of X and $\{y_1, \ldots, y_g\}$ of Y such that $Q(x_i) = q_i y_i$. To compute $A(\delta)$, we decompose $L = \wedge^3 H$ as the direct sum of

$$\begin{aligned}
V_1 &= \operatorname{span}\{x_i \wedge x_j \wedge x_k \,|\, i<j<k\}, & V_2 &= \operatorname{span}\{x_i \wedge x_j \wedge y_j \,|\, i \neq j\},\\
V_3 &= \operatorname{span}\{x_i \wedge x_j \wedge y_k \,|\, i<j, k \neq i,j\}, & V_4 &= \operatorname{span}\{x_i \wedge y_i \wedge y_j \,|\, i \neq j\},\\
V_5 &= \operatorname{span}\{x_i \wedge y_j \wedge y_k \,|\, j<k, i \neq j,k\}, & V_6 &= \operatorname{span}\{y_i \wedge y_j \wedge y_k \,|\, i<j<k\},
\end{aligned}$$

Let A denote the matrix of $\delta - I$ with respect to the basis above, and A_{ij} the block whose rows correspond to V_i and columns correspond to V_j.

Lemma 5.7. *The submatrices A_{ij} satisfy the following:*

(1) A_{13}*,* A_{24}*, and* A_{35} *are nonsingular,*
(2) $\operatorname{coker}(A_{24}) \cong \operatorname{coker}(Q)^{g-1}$*,*
(3) $\operatorname{coker}(A_{35}) \cong \prod_{i<j<k} (\mathbb{Z}/(q_i))^2 \times \mathbb{Z}/(2q_j q_k/q_i)$*,*
(4) $\operatorname{coker}(A_{56}) \cong \prod_{i=1}^{g-2} (\mathbb{Z}/(q_i))^{\binom{g-i}{2}}$
(5) $A_{12}, A_{14}, A_{15}, A_{16}, A_{23}, A_{25}, A_{26}, A_{34}, A_{36}, A_{45}, A_{46}$*, and* A_{ij} *for* $i \geq j$ *are all* 0*.*

Proof. The matrix A_{13} is nonsingular because its rows each have exactly one nonzero entry. Indeed, the row corresponding to $x_i \wedge x_j \wedge x_k$ has q_k in the column corresponding to $x_i \wedge x_j \wedge y_k$, and 0 for the remaining entries. Next, A_{24} is a square $g(g-1)$ matrix and $(\delta - 1)(x_i \wedge x_j \wedge y_j) = q_i \cdot y_i \wedge x_j \wedge y_j \mod F_3 L$. Therefore A_{24} is nonsingular and its cokernel is of the desired form. Now consider A_{35}. Given $i<j<k$, set

$$\begin{aligned}
V_{ijk} &= \operatorname{span}\{x_i \wedge x_j \wedge y_k,\ x_j \wedge x_k \wedge y_i,\ x_k \wedge x_i \wedge y_j\}\\
W_{ijk} &= \operatorname{span}\{x_k \wedge y_i \wedge y_j,\ x_i \wedge y_j \wedge y_k,\ x_j \wedge y_k \wedge y_i\}.
\end{aligned}$$

The matrix A_{35} may be arranged into block-diagonal form, where each block has rows indexed by W_{ijk} and columns by V_{ijk}. With respect to the bases above, this block is

$$\begin{pmatrix} 0 & q_j & q_i \\ q_j & 0 & q_k \\ q_i & q_k & 0 \end{pmatrix}$$

whose invariant factors are $q_i, q_i, 2q_j q_k/q_i$. Therefore, $\operatorname{coker} A_{35}$ is isomorphic to the product of the $(\mathbb{Z}/(q_i))^2 \times \mathbb{Z}/(q_j q_k/q_i)$. Because each block matrix is nonsingular, A_{35} is also nonsingular. This completes the proof of (1), (2), and (3).

Now consider (4). Because $(\delta - I)(x_i \wedge y_j \wedge y_k) = q_i \cdot y_i \wedge y_j \wedge y_k$, each column of A_{56} has exactly one nonzero entry. By only performing column operations, we may form a diagonal matrix B_{56} from A_{56} so that each diagonal entry is the gcd of the integers in its corresponding row. Because the nonzero entries of the row in A_{56} corresponding to

$y_i \wedge y_j \wedge y_k$ are q_i, q_j, q_k, the corresponding diagonal entry in B_{56} is $\gcd(q_i, q_j, q_k) = q_i$. From this, we see that $\operatorname{coker}(A_{56})$ has the desired form.

Finally consider (5). The matrices A_{ij} for $i \geq j$ are all 0 because the matrix of $\delta - I$, with respect to the above decomposition, is strictly lower-triangular. The remaining listed matrices are 0 because $\delta - I : X \to Y$ is diagonal with respect to the given bases. $\square$

Proposition 5.8. *The maps $\delta - I : \operatorname{gr}^F_{q-1} L_{\mathbb{Q}} \to \operatorname{gr}^F_q L_{\mathbb{Q}}$ are surjective when $q = 2, 3$ and injective when $q = 1, 2$. In particular,*

$$A(\delta) \cong \frac{F_2 L}{(\delta - I) F_1 L}.$$

Proof. The surjectivity claim follows from Lemma 5.2. The matrix A_{13} is nonsingular by Lemma 5.7(1), therefore $\delta - I : \operatorname{gr}^F_0 L_{\mathbb{Q}} \to \operatorname{gr}^F_1 L_{\mathbb{Q}}$ is injective. Because $\delta - I : \operatorname{gr}^F_1 L_{\mathbb{Q}} \to \operatorname{gr}^F_2 L_{\mathbb{Q}}$ is surjective and both spaces have the same dimension (equal to $g\binom{g}{2}$), it is also injective. The last statement follows from Proposition 5.4 and Proposition 5.5. $\square$

Proposition 5.9. *We have an isomorphism*

$$B(\delta) \cong \operatorname{coker}(Q)^{g-1} \times \prod_{i<j<k} (\mathbb{Z}/(q_i))^2 \times \mathbb{Z}/(2q_j q_k / q_i).$$

Proof. In terms of the decomposition above, $\delta - I : \operatorname{gr}^1_F L \to \operatorname{gr}^2_F L$ is given by the block matrix

$$A = \begin{pmatrix} A_{24} & A_{25} \\ A_{34} & A_{35} \end{pmatrix}.$$

The proposition now follows from Lemma 5.7. $\square$

Proposition 5.10. *We have an isomorphism*

$$A(\delta) \cong B(\delta) \times \prod_{i=1}^{g-2} (\mathbb{Z}/(q_i))^{\binom{g-i}{2}}$$

Proof. Under the identifications $A(\delta) \cong F_2 L/(\delta - I)F_1 L$ and $B(\delta) \cong F_2 L/((\delta - I)F_1 L + F_3 L)$ from Propositions 5.8 and 5.4, the map $A(\delta) \to B(\delta)$ given by the projection

$$\frac{F_2 L}{(\delta - I)F_1 L} \to \frac{F_2 L}{(\delta - I)F_1 L + F_3 L}$$

is surjective. Its kernel is isomorphic to $F_3 L/(F_3 L \cap (\delta - I)F_1 L)$. The map $\delta - I : \operatorname{gr}^F_1 L \to \operatorname{gr}^F_2 L$ is injective by Proposition 5.8, so $F_3 L \cap (\delta - I)F_1 L = (\delta - I)F_2 L$ by Proposition 5.5. Therefore, we have an exact sequence

$$0 \to \frac{F_3 L}{(\delta - I)F_2 L} \to A(\delta) \to B(\delta) \to 0$$

We claim that this exact sequence splits. The decomposition $F_2 L \cong \operatorname{gr}^F_2 L \times F_3 L$ yields a projection $\pi : F_2 L \to F_3 L/(\delta - I)F_2 L$. Given $a \in (\delta - I)F_1$, express it as $a = a_1 + a_2$ with $a_1 \in \operatorname{gr}^F_2 L$ and $a_2 \in F_3$. In fact, $a_2 \in (\delta - I)F_2$ since $\delta - I : \operatorname{gr}^F_1 L \to \operatorname{gr}^F_2 L$ is injective. Therefore, $a \in \ker \pi$, and hence π induces a splitting of $F_3 L/(\delta - I)F_2 L \to A(\delta)$. Finally, $F_3 L/(\delta - I)F_2 L \cong \prod_{i=1}^{g-2} (\mathbb{Z}/(q_i))^{\binom{g-i}{2}}$ by Lemma 5.7(4) and (5). $\square$

Corollary 5.11. *When Y has maximal rank,*

$$|A(\delta)| = 2^{\binom{g}{3}} \det(Q)^{\binom{g}{2}} \prod_{i=1}^{g-2} q_i^{\binom{g-i}{2}} \quad \text{and} \quad |B(\delta)| = 2^{\binom{g}{3}} \det(Q)^{\binom{g}{2}}.$$

Proof. Observe that $\prod_{i<j<k} q_i q_j q_k = \det(Q)^{\binom{g-1}{2}}$ because each invariant factor occurs exactly $\binom{g-1}{2}$ times. The formulas for $|B(\delta)|$ and $|A(\delta)|$ now follow from Propositions 5.9 and 5.10, respectively. □

5.3. **The symplectic case.** Finally, we consider the case where H is equipped with a symplectic form $\omega \in \wedge^2 H$, and δ is an element of $\mathrm{Sp}(H)$ such that $(\delta - I)^2 = 0$. We embed H into L via $h \mapsto \omega \wedge h$. Because δ preserves the form ω, it acts on L/H. We define

$$F_q(L/H) = (F_q L + H)/H.$$

Recall that Y is the saturation of $\mathrm{im}(\delta - I)$, which is isotropic since δ is symplectic, and X a subgroup such that $H = X \oplus Y$. Because $H \subset F_1 L$, $F_2 L + H = F_2 L \oplus X$, and $F_3 L \cap H = 0$, each $F_q(L/H)$ is saturated in L/H. In particular, the graded pieces $\mathrm{gr}^F_q(L/H)$ are free. By Lemma 5.1, $(\delta - I)$ takes $F_q(L/H)$ to $F_{q+1}(L/H)$, hence induces a map $\mathrm{gr}^F_q(L/H) \to \mathrm{gr}^F_{q+1}(L/H)$. We denote by $\overline{A(\delta)}$ and $\overline{B(\delta)}$ the groups

$$\overline{A(\delta)} = \mathrm{im}(H^1(\langle\delta\rangle, F_2(L/H)) \to H^1(\langle\delta\rangle, L/H)),$$
$$\overline{B(\delta)} = \mathrm{coker}(\delta - I : \mathrm{gr}^F_1(L/H) \to \mathrm{gr}^F_2(L/H)).$$

As we will show in the next section, the Ceresa class $\nu(T_\Lambda)$ of a multitwist T_Λ on a closed surface, with symplectic representation δ, lives in $\overline{A(\delta)}$, provided Y has maximal rank. In general, only a multiple of $\nu(T_\Lambda)$ will live in $\overline{A(\delta)}$. In this subsection, we will show that $\overline{A(\delta)}$ is finite, from which we conclude that the Ceresa class is torsion. When Y has maximal rank, $\overline{A(\delta)}$ naturally surjects onto $\overline{B(\delta)}$. It is much easier to compute the image of $\nu(T_\Lambda)$ which is an element of $\overline{B(\delta)}$, see Corollary 6.8. We will use this in §7 to determine non-triviality of $\nu(T_\Lambda)$ in some examples.

The following three propositions, and their proofs, are similar to Propositions 5.4, 5.8, and 5.10.

Proposition 5.12. *We have isomorphisms*

$$\overline{A(\delta)} \cong \frac{F_2 L + H}{((\delta - I)L + H) \cap (F_2 L + H)}, \quad \overline{B(\delta)} \cong \frac{F_2 L + H}{(\delta - I)(F_1 L) + F_3 L + H}.$$

In particular, $\overline{A(\delta)}$ and $\overline{B(\delta)}$ are finite.

Proposition 5.13. *The map $(\delta - I)_{\mathbb{Q}} : \mathrm{gr}^F_{q-1}(L/H)_{\mathbb{Q}} \to \mathrm{gr}^F_q(L/H)_{\mathbb{Q}}$ is surjective when $q = 2, 3$. When Y has maximal rank, this map is injective for $q = 1, 2$ and*

$$\overline{A(\delta)} \cong \frac{F_2 L + H}{(\delta - I) F_1 L + H}.$$

Proposition 5.14. *If Y has maximal rank, then*

$$\overline{A(\delta)} \cong \overline{B(\delta)} \times \prod_{i=1}^{g-2} (\mathbb{Z}/(q_i))^{\binom{g-i}{2}}.$$

The next two propositions compare $A(\delta)$ and $B(\delta)$ from the previous subsection to their counterparts $\overline{A(\delta)}$ and $\overline{B(\delta)}$.

Proposition 5.15. *If Y has maximal rank, then we have an exact sequence*

$$0 \to \operatorname{coker}(Q) \to B(\delta) \to \overline{B(\delta)} \to 0.$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{\wedge\omega} & F_1L & \longrightarrow & F_1(L/H) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta-I} & & \downarrow{\scriptstyle \delta-I} & & \downarrow{\scriptstyle \delta-I} & & \\ 0 & \longrightarrow & Y & \xrightarrow{\wedge\omega} & F_2L & \longrightarrow & F_2(L/H) & \longrightarrow & 0 \end{array}$$

whose rows are exact. The map $\delta - I : F_1(L/H) \to F_2(L/H)$ is injective because it becomes an isomorphism after tensoring with $\mathbb{Q}$ by Proposition 5.13. We now get the desired exact sequence by the snake lemma. □

Proposition 5.16. *If Y has maximal rank, then we have an exact sequence*

$$0 \to \operatorname{coker}(Q) \to A(\delta) \to \overline{A(\delta)} \to 0.$$

Proof. First, observe that $A(\delta) \to \overline{A(\delta)}$ induced by

$$\frac{F_2L}{(\delta - I)F_1L} \to \frac{F_2L + H}{(\delta - I)F_1L + H}$$

is surjective. Let K denote its kernel. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & A(\delta) & \longrightarrow & \overline{A(\delta)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{coker}(Q) & \longrightarrow & B(\delta) & \longrightarrow & \overline{B(\delta)} & \longrightarrow & 0 \end{array}$$

whose rows are exact. The vertical map on the left is an isomorphism by Propositions 5.10, 5.14, and the snake lemma. □

Corollary 5.17. *When Y has maximal rank,*

$$|\overline{A(\delta)}| = 2^{\binom{g}{3}} \det(Q)^{\binom{g}{2}-1} \prod_{i=1}^{g-2} q_i^{\binom{g-i}{2}} \quad \textit{and} \quad |\overline{B(\delta)}| = 2^{\binom{g}{3}} \det(Q)^{\binom{g}{2}-1}.$$

6. The Ceresa class of a multitwist

6.1. **The Dehn Twists and Multitwists.** In this subsection, we recall some basic facts about Dehn twists. We refer the reader to [9] for a more detailed treatment.

Lemma 6.1. *Let $f \in \Gamma_g$ (resp. Γ_g^1) and a the isotopy class of a simple closed curve. We have*

(1) $T_{f(a)} = fT_af^{-1}$, in particular, $[T_a, f] = T_aT_{f(a)}^{-1}$, and

(2) f commutes with T_a if and only if $f(a) = a$; in particular $T_aT_b = T_bT_a$ if and only if the geometric intersection number $i(a, b) = 0$.

Proof. See [9, Facts 3.7, 3.8]. □

As before, set $H = H_1(\Sigma_g, \mathbb{Z}) = H_1(\Sigma_g^1, \mathbb{Z})$. We write $[a]$ for the homology class of an isotopy class a of a simple closed curve. The induced map $(T_a)_* \in \mathrm{Sp}(H)$ only depends on the homology class $[a]$, and

$$(T_a)_*([b]) = [b] + \hat{\imath}([a],[b])[a], \tag{6.1}$$

see [9, Proposition 6.3].

Let $\Lambda = \{\ell_1, \ldots, \ell_N\}$ be a collection of isotopy classes of pairwise non-intersecting essential simple closed curves in Σ_g (resp. Σ_g^1). By Lemma 6.1(2), the subgroup $\mathcal{B} = \langle T_{\ell_1}, \ldots T_{\ell_N} \rangle$ of Γ_g (resp. Γ_g^1) is abelian. Given positive integers $c_1, \ldots, c_N$, set

$$T_\Lambda = \prod T_{\ell_i}^{c_i} \in \mathcal{B}.$$

Such an element is called a *positive multitwist*. We write $\delta \in \mathrm{Sp}(H)$ for the image of T_Λ under the symplectic representation and $Y < H$ for the saturation of $\mathrm{im}(\delta - I)$. We say that Λ is *Lagrangian* if the rank of Y is half the rank of H, the largest possible rank of Y.

Lemma 6.2. *With $\mathcal{B}$, δ and Y as above, and $f \in \mathcal{B}$, we have*

(1) $Y = \langle [\ell_1], \ldots, [\ell_N] \rangle$,
(2) $(f_* - I)(H) \leq Y$,
(3) $(f_* - I)(Y) = 0$, *and*
(4) $(f_* - I)^2 = 0$ *as an endomorphism of* H.

Proof. We temporarily denote by Y' the subgroup $\langle [\ell_1], \ldots, [\ell_N] \rangle$ of H. Applying Equation (6.1) to $f = \prod T_{\ell_i}^{b_i}$ yields

$$(f_* - I)(h) = \sum_{j=1}^{N} b_j \, \hat{\imath}([\ell_j], h)[\ell_j], \tag{6.2}$$

and therefore $(f_* - I)(H) \leq Y'$. In particular, $Y \leq Y'$. Because of this, and the fact that Y and Y' are saturated subgroups of H, the equality $Y_{\mathbb{Q}} = Y'_{\mathbb{Q}}$ will imply $Y = Y'$. Denote by $(Y'_{\mathbb{Q}})^\omega \leq H_{\mathbb{Q}}$ the symplectic complement to $Y'_{\mathbb{Q}}$, i.e.,

$$(Y'_{\mathbb{Q}})^\omega = \{h \in H_{\mathbb{Q}} \mid \hat{\imath}(y, h) = 0 \text{ for all } y \in Y'_{\mathbb{Q}}\}$$

and set $W_{\mathbb{Q}} = H_{\mathbb{Q}}/(Y'_{\mathbb{Q}})^\omega$. Because $Y'_{\mathbb{Q}}$ is an isotropic subspace of $H_{\mathbb{Q}}$, we have $\dim W_{\mathbb{Q}} = \dim Y'_{\mathbb{Q}}$. Since $(Y'_{\mathbb{Q}})^\omega$ lies in the kernel of $(\delta - I)_{\mathbb{Q}}$ by Equation (6.2), the following

$$(\alpha, \alpha') = \hat{\imath}(\alpha, (\delta - I)\alpha')$$

defines a bilinear form on $W_{\mathbb{Q}}$. We claim that this is actually an inner product. Indeed, we have

$$\hat{\imath}(\alpha, (\delta - I)\alpha') = \sum_{j=1}^{N} c_j \, \hat{\imath}([\ell_j], \alpha)\hat{\imath}([\ell_j], \alpha'),$$

and hence this bilinear form is symmetric. If α is a nonzero element of $W_{\mathbb{Q}}$, then there is some ℓ_i such that $\hat{\imath}([\ell_i], \alpha) \neq 0$. By the above equation, we see that $(\alpha, \alpha) > 0$, which establishes the positive definite property. We conclude that $(\delta - I)_{\mathbb{Q}} : W_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is an injective linear map of vector spaces that have the same dimension, and hence also surjective, This proves $Y_{\mathbb{Q}} = Y'_{\mathbb{Q}}$, and (1) now follows. Because $(f_* - I)(H) \leq Y'$, we also have (2).

Statement (3) follows from Equation (6.2) and the fact that the $\ell_1, \dots, \ell_N$ are pairwise non-intersecting. Statement (4) follows from (1) and (3). $\square$

6.2. **The Ceresa Class of a Multitwist.** In this subsection, we prove Theorem 6.7, which asserts that for any positive multitwist $T_\Lambda \in \Gamma_g$ (resp. $T_{\Lambda'} \in \Gamma_g^1$), its Ceresa class $\nu(T_\Lambda) \in H^1(\langle\delta\rangle, L/H)$ (resp. $\mu(T_{\Lambda'}) \in H^1(\langle\delta\rangle, L)$) is torsion. We start by working with the once punctured surface Σ_g^1, and retain the setup from the previous subsection. Lemma 6.2(4) implies that $(\delta - I)^2 = 0$, and so we may apply the results from §5.1. Let F_*L be the filtration on L associated to δ from Equation (5.1). By Proposition 5.4, $A(\delta)$ is a finite subgroup of $H^1(\langle\delta\rangle, L)$. Thus, to prove Theorem 6.7, it suffices to show that a multiple of $J([T_{\Lambda'}, \tau])$ lies in $F_2L + (\delta - I)L$, for some hyperelliptic quasi-involution τ. In the process, we will also show that $\mu(T_{\Lambda'}) \in A(\delta)$ (resp. $\nu(T_\Lambda) \in \overline{A(\delta)}$) when Λ is Lagrangian.

Lemma 6.3. *For any $f \in \mathcal{B}$ we have:*

(1) $f_*(F_iL) \leq F_iL$,
(2) $f_*((\delta - I)L) \leq (\delta - I)L$.

Proof. Statement (1) directly follows from Lemma 6.2 (3). Because $\mathcal{B}$ is abelian, δ and f_* commutes with each other. Thus for any $y \in L$ with $y = (\delta - I)(x)$, we have $f_*(y) = (\delta - I)(f_*(x))$, and hence statement (2) holds. $\square$

Lemma 6.4. *Take elements $f_1, \dots, f_M \in \mathcal{B}$, let f denote their product, and τ a hyperelliptic quasi-involution. Then*

$$\mu_\tau(f) = J([f, \tau]) = \sum_{i=1}^{M} (g_i)_* \cdot J([f_i, \tau])$$

for some $g_1, \dots, g_M \in \mathcal{B}$.

Proof. We proceed by induction on M. When $M = 1$ the Lemma is clear. Suppose that the Lemma is true for $M - 1$. Then by Proposition 2.1 we have

$$J([f, \tau]) = J([f_1, \tau]) + (f_1)_* \cdot J([f_2 \cdots f_M, \tau]) = J([f_1, \tau]) + (f_1)_* \sum_{i=2}^{M} (g_i)_* \cdot J([f_i, \tau]).$$

Since each f_i lies in $\mathcal{B}$, we have placed $J([f, \tau])$ in the desired form. $\square$

By Lemmas 6.3 and 6.4, to show that a multiple of $J([T_{\Lambda'}, \tau])$ lies in $F_2L + (\delta - I)L$, it suffices to show that a multiple of $J([T_{\ell_i}, \tau])$ lies in $F_2L + (\delta - I)L$ for each $i = 1, \dots, N$. By Lemma 6.1, we have $[T_{\ell_i}, \tau] = T_{\ell_i} T_{\tau(\ell_i)}^{-1}$. The homology classes of ℓ_i and $\tau(\ell_i)$ agree, which conforms with the fact that $T_{\ell_i} T_{\tau(\ell_i)}^{-1}$ lies in the Torelli group.

If ℓ_i is a separating curve, then so is $\tau(\ell_i)$, and $J(T_{\ell_i}) = J(T_{\tau(\ell_i)}) = 0$, see §2.1. Thus we need only consider the case where ℓ_i is a non-separating curve. We begin by assuming ℓ_i and $\tau(\ell_i)$ are disjoint, in which case $T_{\ell_i} T_{\tau(\ell_i)}^{-1}$ is a bounding pair map.

Lemma 6.5. *If γ, γ' are disjoint non-separating curves on Σ_g^1 such that $[\gamma] = [\gamma']$ and both classes are in Y, then some multiple of $J(T_\gamma T_{\gamma'}^{-1})$ lies in $F_2L + (\delta - I)L$. If Y has maximal rank ($\operatorname{rank} Y = g$), then $J(T_\gamma T_{\gamma'}^{-1})$ lies in F_2L.*

Proof. Let S, S' be the subsurfaces of Σ_g^1 with boundary components γ_1 and γ_2 in which S does not contain the puncture of Σ_g^1. By Equation (2.4), we have

$$J(T_\gamma T_{\gamma'}^{-1}) = \omega_W \wedge [\gamma]$$

where ω_W is the restriction of the symplectic form and W is the image of $H_1(S, \mathbb{Z}) \to H$. Write Y_W for the intersection of $Y_\mathbb{Q}$ with W. Then Y_W is an isotropic subspace of W. By Witt's theorem, we can choose a standard symplectic basis $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ for W such that $\omega_W = \sum_j \alpha_j \wedge \beta_j$ and Y_W is spanned by $\beta_1, \ldots, \beta_{\dim Y_W}$. We note that, for $j > \dim Y_W$, both α_j and β_j are orthogonal to every element of Y_W, and whence are orthogonal to every element of $Y_\mathbb{Q}$, since $Y_\mathbb{Q}$ is spanned by Y_W, $[\gamma]$, and loops on S', which are thus disjoint from α_j and β_j. It now follows from Equation (6.2) that $(\delta - I)(\alpha_j) = (\delta - I)(\beta_j) = 0$, so α_j and β_j both lie in $Z = \ker(\delta - I)$. Thus

$$\omega_W \wedge [\gamma] = \sum_{i \le \dim Y_W} \alpha_i \wedge \beta_i \wedge [\gamma] + \sum_{i > \dim Y_W} \alpha_j \wedge \beta_j \wedge [\gamma]. \tag{6.3}$$

The first sum lies in $F_2 L$, whereas the second sum lies in $Z \wedge Z \wedge Y$. By Proposition 5.6, this means that a multiple of $J(T_{\gamma_1} T_{\gamma_2}^{-1})$ lies in $F_2 L + (\delta - I)L$, as required.

If Y has maximal rank, then $\dim Y_W = (\dim W)/2$. In particular, the second summand in Equation (6.3) is vacuous, so $J(T_\gamma T_{\gamma'}^{-1})$ lies in $F_2 L$. □

We now show that the condition of disjointness in Lemma 6.5 can be discarded.

Proposition 6.6. *If γ, γ' are non-separating curves on Σ_g^1 ($g \geq 3$) such that $[\gamma] = [\gamma']$ and both classes are in Y, then some multiple of $J(T_\gamma T_\gamma^{-1})$ lies in $F_2 L + (\delta - I)L$. If Y is Lagrangian ($\operatorname{rank} Y = g$), then $J(T_\gamma T_{\gamma'}^{-1})$ lies in $F_2 L$.*

Proof. By a theorem of Putman [17, Theorem 1.9], there are curves $\gamma_0 = \gamma, \gamma_1, \ldots, \gamma_N = \gamma'$ such that $\gamma_j \cap \gamma_{j+1} = \emptyset$, and $[\gamma_j] = [\gamma]$ for all j. In Putman's setup, the surface is not punctured, but this doesn't matter for us; we can embed $\Sigma_g^1 \hookrightarrow \Sigma_g$, choose the curves on the unpunctured Σ_g and then perturb them to avoid the image of the puncture without changing their homology class.

By Lemma 6.5, each $J(T_{\gamma_{j-1}} T_{\gamma_j}^{-1})$ has a multiple lying in $F_2 L + (\delta - I)L$. Since

$$J(T_\gamma T_{\gamma'}^{-1}) = \sum_{j=1}^{N} J(T_{\gamma_{j-1}} T_{\gamma_j}^{-1}),$$

we conclude that a multiple of $J(T_\gamma T_{\gamma'}^{-1})$ also lies in $F_2 L + (\delta - I)L$. If Y is Lagrangian, then each $J(T_{\gamma_{j-1}} T_{\gamma_j}^{-1})$ lies in $F_2 L$, and hence so does $J(T_\gamma T_{\gamma'}^{-1})$. □

With Proposition 6.6 in hand, we are now ready to prove the following main theorem.

Theorem 6.7. *The Ceresa class $\nu(T_\Lambda) \in H^1(\mathbb{Z}, L/H)$ (resp. $\mu(T_{\Lambda'}) \in H^1(\mathbb{Z}, L)$) of a multitwist $T_\Lambda \in \Gamma_g$ (resp. $T_{\Lambda'} \in \Gamma_g^1$) is torsion. If Λ is Lagrangian, then $\mu(T_\Lambda) \in \overline{A(\delta)}$ (resp. $\mu(T_{\Lambda'}) \in A(\delta)$).*

Proof. Let $\Lambda = \{\ell_1, \dots, \ell_N\}$ be a collection of nontrivial isotopy classes of curves and $T_\Lambda = \prod_{i=1}^N T_{\ell_i}^{c_i}$ where $c_1, \dots, c_N \in \mathbb{Z}_{>0}$. Let $\iota : \Sigma_g^1 \hookrightarrow \Sigma_g$ be an inclusion such that the puncture is disjoint from the supporting simple closed curves $\{\ell_1, \dots, \ell_N\}$. Consider the Dehn twists associated to the preimages of these curves denoted as $\Lambda' = \{\ell_i', \dots, \ell_N'\}$ in Σ_g^1 and The multitwist $T_{\Lambda'} = \prod_{i=1}^N T_{\ell_i'}^{c_i} \in \Gamma_g^1$ maps to T_Λ via tha natural map $\Gamma_g^1 \to \Gamma_g$. Denote by $\delta \in \mathrm{Sp}(H)$ the image of T_Λ under the symplectic representation. The class $\nu(T_\Lambda)$ agrees with the image of $\mu(T_{\Lambda'})$ under the natural map $H^1(\langle\delta\rangle, L) \to H^1(\langle\delta\rangle, L/H)$. So the torsionness of $\nu(T_\Lambda)$ will follow from that of $\mu(T_{\Lambda'})$. Moreover, when Λ is Lagrangian, $\mu(T_{\Lambda'}) \in A(\delta)$ will imply $\nu(T_\Lambda) \in \overline{A(\delta)}$.

Next, we will show that for each Λ' there is a τ such that some multiple of $J([T_{\Lambda'}, \tau])$ lies in $F_2L + (\delta - I)H$. Suppose that $g \geq 3$ and let $\tau \in \Gamma_g^1$ be any hyperelliptic quasi-involution. For each non-separating Dehn twist $T_{\ell_i'}$, some multiple of $J([T_{\ell_i'}, \tau])$ lies in $F_2L + (\delta - 1)L$ by Proposition 6.6 applied to the loops ℓ_i' and $\tau(\ell_i')$. If ℓ_i' is separating, then $J(T_{\ell_i'}) = J(T_{\tau(\ell_i')}) = 0$. It then follows from Lemma 6.4 that some multiple of $J([T_{\Lambda'}, \tau])$ lies in $F_2L + (\delta - 1)L$. Moreover, if Λ' is Lagrangian, then $J([T_{\Lambda'}, \tau])$ lies in F_2L.

The case $g = 2$ must be handled separately since we cannot apply Proposition 6.6. By Lemmas 6.4, 6.5, and the fact that $J([T_\gamma, \tau]) = 0$ when γ is separating, to show that a multiple of $J([T_{\Lambda'}, \tau])$ lies in $F_2L + (\delta - I)L$, resp. $J([T_{\Lambda'}, \tau]) \in F_2L$ when Λ' is Lagrangian, we just need to show that there exists a hyperelliptic quasi-involution τ such that ℓ_i and $\tau(\ell_i)$ are disjoint for each non-separating curve $\ell_i \in \Lambda'$. In fact, it suffices to show this for the maximal collections. For $g = 2$ with one puncture, there are 3 such configurations, as displayed in Figure 6.1. The requisite hyperelliptic quasi-involution τ is rotation by 180° horizontal to the page.

By definition, $\mu(T_{\Lambda'})$ is the class in $H^1(\langle\delta\rangle, L)$ sending δ to $J([T_{\Lambda'}, \tau])$. To say that a multiple of $J([T_{\Lambda'}, \tau])$ lies in $F_2L + (\delta - 1)L$ is exactly to say that a multiple of $\mu(T_{\Lambda'})$ lies in the image of $H^1(\langle\delta\rangle, F_2L)$ in $H^1(\langle\delta\rangle, L)$, which is the group $A(\delta)$ defined §5.1. We conclude that $\mu(T_{\Lambda'})$ is torsion from the finiteness of $A(\delta)$, see Proposition 5.4. Similarly, if Λ' is Lagrangian, then to say that $J([T_{\Lambda'}, \tau])$ lies in F_2L means that $\mu(T_{\Lambda'})$ lies in $A(\delta)$. ☐

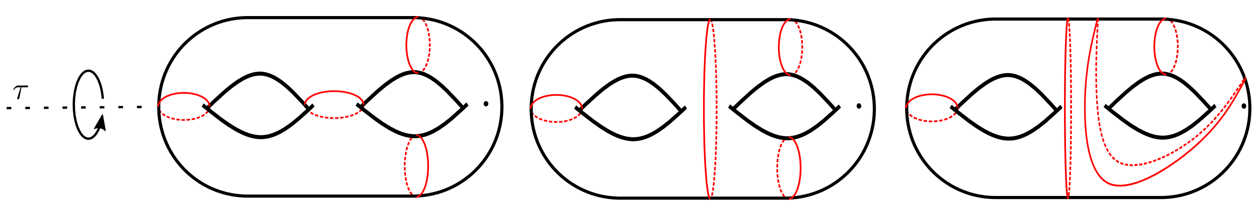

FIGURE 6.1.

Recall from Propositions 5.10 and 5.14 that, in the maximal rank case, there are natural surjective maps $A(\delta) \to B(\delta)$ and $\overline{A(\delta)} \to \overline{B(\delta)}$. The following Corollary gives a simple way to compute the projection of $\nu(T_\Lambda)$ (resp. $\mu(T_{\Lambda'})$) to $\overline{B(\delta)}$ (resp. $B(\delta)$).

Corollary 6.8. *Let Λ (resp. Λ') be a Lagrangian arrangement of curves on Σ_g (resp. Σ_g^1), and v_Λ (resp. $w_{\Lambda'}$) be the image of $\nu(T_\Lambda)$ in $\overline{B(\delta)}$ (resp. $\mu(T_{\Lambda'})$ in $B(\delta)$). Then*

$$v_\Lambda = \sum_{i=1}^{N} c_i J([T_{\ell_i}, \tau]), \quad \textit{resp.} \quad w_{\Lambda'} = \sum_{i=1}^{N} c_i J([T_{\ell_i'}, \tau]),$$

Proof. We will prove the formula for $\nu(T_\Lambda)$ as the pointed case is similar. Because Λ is Lagrangian, $\nu(T_\Lambda)$ lies in $\overline{A(\delta)}$ by Theorem 6.7, so we may view it as an element of $\overline{B(\delta)}$ via the surjection $\overline{A(\delta)} \to \overline{B(\delta)}$ from Proposition 5.14. Next, by Lemma 6.4,

$$J([T_\Lambda, \tau]) = \sum_{i=1}^{N} g_i \cdot J([T_{\ell_i}, \tau]) \quad \text{in } \overline{B(\delta)}$$

where each $g_i \in \mathcal{B}$. The corollary now follows from the fact that $\mathcal{B}$ acts trivially on $\overline{B(\delta)}$. $\square$

7. Examples

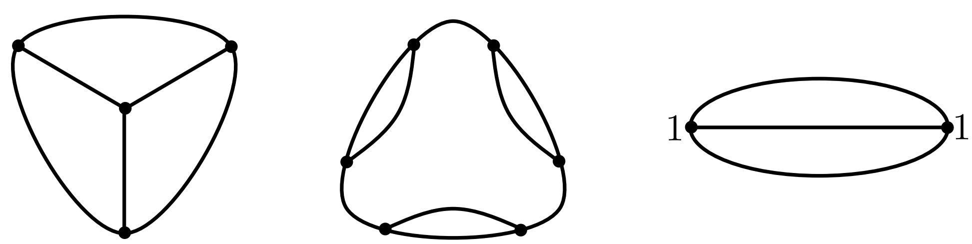

FIGURE 7.1. From left to right, these are the graphs K_4, TL_3. The vertices of the theta graph each have weight 1

In this section, we will compute the Ceresa class for the multitwist T_Γ for tropical curves whose underlying vertex-weighted graph is displayed in Figure 7.1. In the first two examples, we will consider tropical curves Γ whose vertex-weights are all 0 (see the left and middle graphs in Figure 7.1). Let us describe our strategy for determining Ceresa nontriviality in this case.

Let Γ be a genus $g \geq 3$ tropical curve with integral edge-lengths $c_1, \dots, c_N$ and whose vertex weights are all 0. Let $\ell_1, \dots, \ell_N$ be the collection of loops on Σ_g corresponding to Γ as in §4.1. By the second part of Theorem 6.7, the Ceresa class $\nu(T_\Gamma)$ lies in $\overline{A(\delta_\Gamma)}$. Denote by v_Γ the projection of $\nu(T_\Gamma)$ to $\overline{B(\delta_\Gamma)}$. To compute v_Γ, it suffices to compute each $J([T_{\ell_i}, \tau])$ for some hyperelliptic involution τ. While the choice of τ does not matter, it is essential that we use the same τ to compute each $J([T_{\ell_i}, \tau])$. Now, observe that $[T_{\ell_i}, \tau] = T_{\ell_i} T_{\tau(\ell_i)}^{-1}$ and ℓ_i is homologous to $\tau(\ell_i)$. Three things may happen.

- If ℓ_i is a separating curve, then so is $\tau(\ell_i)$, and hence $J(T_{\ell_i} T_{\tau(\ell_i)}^{-1}) = 0$.
- If ℓ_i is a non-separating curve that does not intersect $\tau(\ell_i)$, then we may use Formula (2.4) to compute $J([T_{\ell_i}, \tau])$.
- If ℓ_i is a non-separating curve that intersects $\tau(\ell_i)$, we may find a sequence of loops $\ell_i = \gamma_0, \gamma_1, \dots, \gamma_k = \tau(\ell_i)$ so that γ_j does not intersect γ_{j+1}, and compute

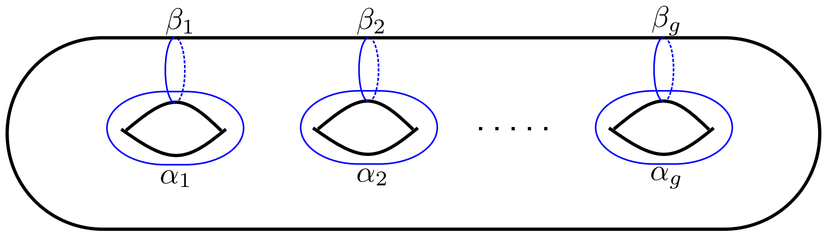

FIGURE 7.2. Symplectic basis for $H_1(\Sigma_g, \mathbb{Z})$

each $J(T_{\gamma_j} T_{\gamma_{j+1}}^{-1})$ using Formula (2.4). As we saw in the proof of Proposition 6.6, such a sequence must exist because $g \geq 3$.

With an explicit formula for v_Γ in hand, let us show how to determine if it represents the trivial element of $\overline{B(\delta_\Gamma)}$. Recall that $\overline{B(\delta_\Gamma)}$ is the cokernel of the map

$$\delta_\Gamma - I : \mathrm{gr}_1^F(L/H) \to \mathrm{gr}_2^F(L/H)$$

Because the vertex weights of Γ are all 0, the polarization Q_Γ, viewed as a map $H/Y \to Y$, is nonsingular, and the map $\delta_\Gamma - I$ above is invertible after tensoring with $\mathbb{Q}$. An explicit inverse is

$$(\delta_\Gamma - I)_{\mathbb{Q}}^{-1}(y \wedge y' \wedge h) = \frac{1}{2}\left(Q_\Gamma^{-1} y \wedge y' \wedge h + y \wedge Q_\Gamma^{-1} y' \wedge h - Q_\Gamma^{-1} y \wedge Q_\Gamma^{-1} y' \wedge Q_\Gamma h\right) \tag{7.1}$$

for $y, y' \in Y$ and $h \in H$. Thus, the class v_Γ is trivial if and only if $u_\Gamma = (\delta_\Gamma - I)_{\mathbb{Q}}^{-1}(v_\Gamma)$ is integral, i.e., lies in $\mathrm{gr}_1^F(L/H)$. Thus, determining triviality of v_Γ is almost as simple as computing the coordinates of u_Γ in $\mathrm{gr}_2^F(L)_{\mathbb{Q}}$ using a basis of H and seeing if they are integral, yet we still need to account for taking the quotient by H. Recall from Equation (2.3) that $\omega = \alpha_1 \wedge \beta_1 + \cdots + \alpha_g \wedge \beta_g$ for any symplectic basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of H. Therefore, any two representatives in L of an element in L/H differ only in coordinates of the form $\alpha_i \wedge \alpha_j \wedge \beta_i$ or $\alpha_i \wedge \beta_i \wedge \beta_j$. In conclusion, we have the following way to determine nontriviality of the Ceresa class.

Proposition 7.1. *Suppose that the vertex weights of Γ are all 0. The Ceresa class $\nu(T_\Gamma)$ is nontrivial if u_Γ has a coordinate of the form $\alpha_i \wedge \alpha_j \wedge \beta_k$, where i, j, k are distinct, whose coefficient is not integral.*

We will now illustrate this analysis in some examples.

Example 7.2. *Suppose Γ is a tropical curve whose underlying vertex-weighted graph is K_4 (the left graph in Figure 7.1). Then Γ is Ceresa nontrivial.*

By Proposition 4.3, it suffices to show that $\nu(T_\Gamma) \neq 0$ whenever Γ has integral edge lengths. Let $\ell_1, \ldots, \ell_6$ be configuration of essential closed curves illustrated in Figure 7.3. Because the vertex weights of Γ are all 0, this arrangement of curves is Lagrangian, and hence $\nu(T_\Gamma) \in \overline{A(\delta_\Gamma)}$. Let v_Γ denote the projection of $\nu(T_\Gamma)$ to $\overline{B(\delta_\Gamma)}$. Let τ be the hyperelliptic involution given by a rotation of 180° through the axis horizontal to the page. Observe that $[T_{\ell_i}, \tau] = 1$ for $i = 1, 3, 4, 6$. The remaining two commutators $[T_{\ell_2}, \tau] = T_{\ell_2} T_{\tau(\ell_2)}^{-1}$ and $[T_{\ell_5}, \tau] = T_{\ell_5} T_{\tau(\ell_5)}^{-1}$ are bounding pair maps, so we may compute their images under J using Equation (2.4). With respect to the basis of $H = H_1(\Sigma_3, \mathbb{Z})$ in Figure 7.2, these are

$$\begin{aligned} J([T_{\ell_2}, \tau]) &= \alpha_1 \wedge \beta_1 \wedge \beta_2 \\ J([T_{\ell_5}, \tau]) &= -\alpha_2 \wedge \beta_1 \wedge \beta_2 - \alpha_2 \wedge \beta_2 \wedge \beta_3 + \alpha_2 \wedge \beta_1 \wedge \beta_3. \end{aligned}$$

By Corollary 6.8, the class v_Γ is

$$v_\Gamma = c_2 \cdot \alpha_1 \wedge \beta_1 \wedge \beta_2 + c_5 \cdot (-\alpha_2 \wedge \beta_1 \wedge \beta_2 - \alpha_2 \wedge \beta_2 \wedge \beta_3 + \alpha_2 \wedge \beta_1 \wedge \beta_3). \tag{7.2}$$

Next we compute $u_\Gamma = (\delta_\Gamma - I)^{-1}_{\mathbb{Q}}(v_\Gamma)$ using Equation 7.1:

$$\begin{aligned} \det(Q_\Gamma)\, u_\Gamma = \\ &- (c_1c_2c_3 + c_1c_2c_4 + c_1c_2c_5 + c_2c_3c_5 + c_2c_4c_5 + c_2c_3c_6 + c_2c_4c_6 + c_2c_5c_6 \\ &\quad + c_3c_5c_6) \cdot \alpha_1 \wedge \alpha_2 \wedge \beta_1 \\ &- c_2c_4(c_1 + c_5 + c_6) \cdot \alpha_1 \wedge \alpha_3 \wedge \beta_1 + c_1c_5c_6 \cdot \alpha_2 \wedge \alpha_3 \wedge \beta_1 \\ &+ (c_2c_3c_5 + c_2c_4c_5 + c_3c_4c_5 + c_2c_3c_6 + c_2c_4c_6 + c_2c_5c_6 + c_3c_5c_6) \cdot \alpha_1 \wedge \alpha_2 \wedge \beta_2 \\ &+ c_2c_4c_6 \cdot \alpha_1 \wedge \alpha_3 \wedge \beta_2 + c_1c_5(c_2 + c_4 + c_6) \cdot \alpha_2 \wedge \alpha_3 \wedge \beta_2 \\ &- c_3c_4c_5 \cdot \alpha_1 \wedge \alpha_2 \wedge \beta_3 + c_2c_4c_5 \cdot \alpha_1 \wedge \alpha_3 \wedge \beta_1 - c_1c_4c_5 \cdot \alpha_2 \wedge \alpha_3 \wedge \beta_3, \end{aligned}$$

where

$$Q_\Gamma = \begin{pmatrix} c_1 + c_5 + c_6 & -c_6 & -c_5 \\ -c_6 & c_2 + c_4 + c_6 & -c_4 \\ -c_5 & -c_4 & c_3 + c_4 + c_5 \end{pmatrix}. \tag{7.3}$$

By Equation (4.1), $\det(Q_\Gamma)$ is the first Symanzik polynomial of G. Observe that the absolute value of the coordinates of $\det(Q_\Gamma)u_\Gamma$ consist of a sum of monomials of the form c_T for spanning trees T, each appearing with coefficient 1. Thus for any positive value of the c_i's, each coordinate of u_Γ has absolute value strictly between 0 and 1. By Proposition 7.1, the Ceresa class $\nu(T_\Gamma)$ is nontrivial.

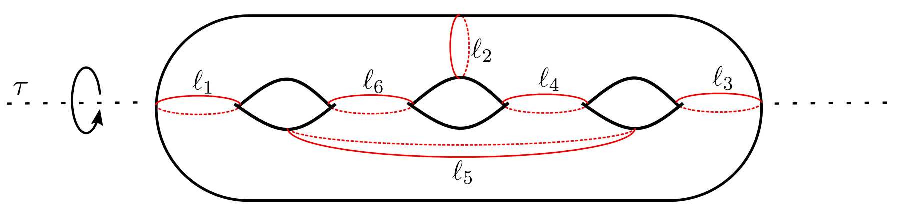

FIGURE 7.3. An arrangement of simple closed curves on Σ_3 dual to K_4

Remark 7.3. This is an opportune moment to remark on the relation between the definition of tropical Ceresa class in the present paper and the definition given by Zharkov in [21]. Consider the element $w_\Gamma = (\delta_\Gamma - 1)v_\Gamma$, which lies in $\mathrm{gr}^F_3(L/H)$. If v_Γ lies in $(\delta_\Gamma - 1)\,\mathrm{gr}^F_1(L/H)$, then certainly w_Γ lies in $(\delta_\Gamma - 1)^2\,\mathrm{gr}^F_1(L/H)$. So the Ceresa class maps to

$$\mathrm{gr}^F_3(L/H)/(\delta_\Gamma - 1)^2\,\mathrm{gr}^F_1(L/H).$$

Using the expression for v_Γ in Equation (7.2), we see that

$$w_\Gamma = -2c_2c_5 \cdot \beta_1 \wedge \beta_2 \wedge \beta_3.$$

The group $(\delta_\Gamma - I)^2 \operatorname{gr}_1^F(L/H)$ is generated by $(\delta_\Gamma - I)^2(\alpha_i \wedge \alpha_j \wedge \beta_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $i < j$. Because

$$(\delta_\Gamma - I)^2(\alpha_i \wedge \alpha_j \wedge \beta_k) = 2\, Q\alpha_i \wedge Q\alpha_j \wedge \beta_k = 2 \begin{vmatrix} q_{si} & q_{sj} \\ q_{ti} & q_{tj} \end{vmatrix} \beta_1 \wedge \beta_2 \wedge \beta_3$$

where (s, t, k) is an even permutation of $(1, 2, 3)$, we see that $(\delta_\Gamma - I)^2 \operatorname{gr}_1^F(L/H)$ is generated by 2 times the 2×2 minors of the symmetric matrix Q_Γ from Equation (7.3). Thus, this subgroup is generated by

$$\begin{array}{cc} 2(c_1c_4 - c_2c_5), & 2(c_1c_4 - c_3c_6), \\ 2(c_2c_5 + c_4c_5 + c_4c_6 + c_5c_6), & 2(c_2c_5 + c_1c_2 + c_1c_6 + c_2c_6), \\ 2(c_2c_5 + c_1c_3 + c_1c_5 + c_3c_5), & 2(c_2c_5 + c_2c_3 + c_2c_4 + c_3c_4). \end{array}$$

So the Ceresa class of T_Γ is nontrivial whenever $-2c_2c_5$ does not lie in the subgroup of $\mathbb{Z}$ generated by the six integers above. This is precisely the condition Zharkov computes in [21, § 3.2] for the algebraic nontriviality of his Ceresa cycle for a tropical curve with underlying graph K_4. It remains to be understood whether this relation between our tropical Ceresa class and Zharkov's holds in general.

Remark 7.4. We observe that the element $u_\Gamma = (\delta_\Gamma - I)_{\mathbb{Q}}^{-1}(v_\Gamma)$ is a point in a 6-dimensional torus (identified by our choice of basis here with $(\mathbb{R}/\mathbb{Z})^6$) which is zero if and only if the Ceresa class vanishes. What's more, u_Γ does not change when the edge lengths are scaled simultaneously, since both $\delta_\Gamma - 1$ and v_Γ are homogeneous of degree 1 in the edge lengths. The space $M(K_4)$ of tropical curves with underlying graph K_4 is a positive orthant in $\mathbb{R}^6$, or more precisely the quotient of this orthant by the action of the automorphism group S_4 (see [7] for a full description) and the class u_Γ can be thought of as a map from the projectivization of this orthant to the 6-torus. The content of Example 7.2 is then that the image of this map does not include 0. It would be interesting to understand whether this map can be naturally extended to the whole tropical moduli space of genus 3 curves.

Remark 7.5. Consider the case $c_1 = \cdots = c_6 = 1$. The invariant factors of Q_Γ are $q_1 = 1$, $q_2 = 4$, and $q_3 = 4$, and hence the projection $\overline{A(\delta_\Gamma)} \to \overline{B(\delta_\Gamma)}$ is an isomorphism. We compute v_Γ and u_Γ to be

$$\begin{aligned} v_\Gamma =& \alpha_1 \wedge \beta_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1 \wedge \beta_2 - \alpha_2 \wedge \beta_2 \wedge \beta_3 + \alpha_2 \wedge \beta_1 \wedge \beta_3, \\ u_\Gamma =& \frac{1}{16}(-9\, \alpha_1 \wedge \alpha_2 \wedge \beta_1 - 3\, \alpha_1 \wedge \alpha_3 \wedge \beta_1 + \alpha_2 \wedge \alpha_3 \wedge \beta_1 + 7 \cdot \alpha_1 \wedge \alpha_2 \wedge \beta_2 + \alpha_1 \wedge \alpha_3 \wedge \beta_2 \\ &+ 3\, \alpha_2 \wedge \alpha_3 \wedge \beta_2 - \alpha_1 \wedge \alpha_2 \wedge \beta_3 + \alpha_1 \wedge \alpha_3 \wedge \beta_1 - \alpha_2 \wedge \alpha_3 \wedge \beta_3). \end{aligned}$$

From this, we see that v_Γ has order 16 in $\overline{B(\delta_\Gamma)}$, and therefore the Ceresa class $\nu(T_\Gamma)$ also has order 16 in $\overline{A(\delta_\Gamma)}$.

There were two key features in this example: each $[T_{\ell_i}, \tau]$ was either trivial or a bounding pair map, and each coordinate of $(\delta_\Gamma - I)_{\mathbb{Q}}^{-1}(\nu(T_\Gamma))$ had absolute value strictly between 0 and 1. In the next example, neither of these properties will hold.

Example 7.6. *Suppose Γ is a tropical curve whose underlying vertex-weighted graph is TL_3 (the middle graph in Figure 7.1). Then Γ is Ceresa nontrivial.*

As in the previous example, it suffices to show that $\nu(T_\Gamma)$ is nontrivial whenever Γ has integral edge lengths. Let v_Γ denote the image of $\nu(T_\Gamma)$ in $\overline{B(\delta_\Gamma)}$. Let $\ell_1, \ldots, \ell_9$ be the configuration of essential closed curves in Figure 7.4, and choose τ to be rotation by 180° through the axis horizontal to the page. To compute each $J([T_{\ell_i}, \tau])$, we will use the symplectic basis of $H = H_1(\Sigma_4, \mathbb{Z})$ displayed on the right in Figure 7.4; this yields a much nicer expression for $u_\Gamma = (\delta_\Gamma - I)^{-1}_{\mathbb{Q}}(v_\Gamma)$. Clearly, $[T_{\ell_i}, \tau] = 1$ for $i = 1,2,3,4$. Next, $[T_{\ell_i}, \tau] = T_{\ell_i} T^{-1}_{\tau(\ell_i)}$ are bounding pair maps when $i = 5,6,7,9$, hence $J([T_{\ell_i}, \tau])$ may be computed using Formula (2.4). The only remaining loop is ℓ_8, which intersects $\tau(\ell_8)$. However, $T_{\ell_8} T^{-1}_{\tau(\ell_8)} = (T_{\ell_8} T^{-1}_{\ell_9})(T_{\ell_9} T^{-1}_{\tau(\ell_8)})$ expresses $T_{\ell_8} T^{-1}_{\tau(\ell_8)}$ as a product of bounding pair maps, which we may use to compute $J([T_{\ell_8}, \tau])$. Then

$$\begin{aligned} v_\Gamma = &- c_6 \cdot \alpha_2 \wedge \beta_1 \wedge \beta_2 + -c_5 \cdot \alpha_1 \wedge \beta_1 \wedge \beta_2 \\ &+ (-c_5 - c_7 + c_8 - c_9)(\alpha_1 \wedge \beta_1 \wedge \beta_3 + \alpha_1 \wedge \beta_1 \wedge \beta_4) \\ &+ (c_6 - c_7 + c_8 + c_9)(\alpha_2 \wedge \beta_2 \wedge \beta_3 + \alpha_2 \wedge \beta_2 \wedge \beta_4). \end{aligned}$$

Next we compute u_Γ using Equation 7.1:

$$\begin{aligned}
&\det(Q_\Gamma)\, u_\Gamma = \\
&c_5(c_1c_3c_4 + c_1c_3c_6 + c_1c_4c_6 + c_3c_4c_6 + 2c_1c_3c_8 + 2c_1c_4c_8 + 2c_3c_6c_8 + 2c_4c_6c_8) \cdot \alpha_1 \wedge \alpha_2 \wedge \beta_1 \\
&c_4(c_1 + c_6)(c_2c_5 - c_2c_8 - c_5c_8 + c_2c_9 + c_5c_9 + c_2c_7 + c_5c_7) \cdot \alpha_1 \wedge \alpha_3 \wedge \beta_1 \\
&c_3(c_1 + c_6)(c_2c_5 - c_2c_8 - c_5c_8 + c_2c_9 + c_5c_9 + c_2c_7 + c_5c_7) \cdot \alpha_1 \wedge \alpha_4 \wedge \beta_1 \\
&c_6(c_2c_3c_4 + c_2c_3c_5 + c_2c_4c_5 + c_3c_4c_5 + 2c_2c_3c_7 + 2c_2c_4c_7 + 2c_3c_5c_7 + 2c_4c_5c_7) \cdot \alpha_1 \wedge \alpha_2 \wedge \beta_2 \\
&- c_4(c_2 + c_5)(c_1c_6 + c_1c_8 + c_6c_8 + c_1c_9 + c_6c_9 - c_1c_7 - c_6c_7) \cdot \alpha_2 \wedge \alpha_3 \wedge \beta_2 \\
&- c_3(c_2 + c_5)(c_1c_6 + c_1c_8 + c_6c_8 + c_1c_9 + c_6c_9 - c_1c_7 - c_6c_7) \cdot \alpha_2 \wedge \alpha_4 \wedge \beta_2 \\
&c_5c_6(c_3c_4 + c_3c_8 + c_4c_8 - c_3c_9 - c_4c_9 + c_3c_7 + c_4c_7) \cdot \alpha_1 \wedge \alpha_2 \wedge \beta_3 \\
&c_4c_6(c_2c_5 - c_2c_8 - c_5c_8 + c_2c_9 + c_5c_9 + c_2c_7 + c_5c_7) \cdot \alpha_1 \wedge \alpha_3 \wedge \beta_3 \\
&c_3c_6(c_2c_5 - c_2c_8 - c_5c_8 + c_2c_9 + c_5c_9 + c_2c_7 + c_5c_7) \cdot \alpha_1 \wedge \alpha_4 \wedge \beta_3 \\
&- c_4c_5(c_1c_6 + c_1c_8 + c_6c_8 + c_1c_9 + c_6c_9 - c_1c_7 - c_6c_7) \cdot \alpha_2 \wedge \alpha_3 \wedge \beta_3 \\
&- c_3c_5(c_1c_6 + c_1c_8 + c_6c_8 + c_1c_9 + c_6c_9 - c_1c_7 - c_6c_7) \cdot \alpha_2 \wedge \alpha_4 \wedge \beta_3 \\
&c_5c_6(c_3c_4 + c_3c_8 + c_4c_8 - c_3c_9 - c_4c_9 + c_3c_7 + c_4c_7) \cdot \alpha_1 \wedge \alpha_2 \wedge \beta_4 \\
&c_4c_6(c_2c_5 - c_2c_8 - c_5c_8 + c_2c_9 + c_5c_9 + c_2c_7 + c_5c_7) \cdot \alpha_1 \wedge \alpha_3 \wedge \beta_4 \\
&c_3c_6(c_2c_5 - c_2c_8 - c_5c_8 + c_2c_9 + c_5c_9 + c_2c_7 + c_5c_7) \cdot \alpha_1 \wedge \alpha_4 \wedge \beta_4 \\
&- c_4c_5(c_1c_6 + c_1c_8 + c_6c_8 + c_1c_9 + c_6c_9 - c_1c_7 - c_6c_7) \cdot \alpha_2 \wedge \alpha_3 \wedge \beta_4 \\
&- c_3c_5(c_1c_6 + c_1c_8 + c_6c_8 + c_1c_9 + c_6c_9 - c_1c_7 - c_6c_7) \cdot \alpha_2 \wedge \alpha_4 \wedge \beta_4,
\end{aligned}$$

where

$$Q_\Gamma = \begin{pmatrix} c_1 + c_6 & 0 & c_6 & c_6 \\ 0 & c_2 + c_5 & c_5 & c_5 \\ c_6 & c_5 & c_3 + c_5 + c_6 + c_7 + c_8 + c_9 & c_5 + c_6 + c_7 + c_8 + c_9 \\ c_6 & c_5 & c_5 + c_6 + c_7 + c_8 + c_9 & c_4 + c_5 + c_6 + c_7 + c_8 + c_9 \end{pmatrix}.$$

By Equation (4.1), $\det(Q_\Gamma)$ is the first Symanzik polynomial of G. The coordinates of $\det(Q_\Gamma) u_\Gamma$ of the form $\alpha_i \wedge \alpha_j \wedge \beta_k$, for i, j, k distinct, are a sum of monomials c_T for spanning trees T, each appearing with coefficient ± 1. Thus for any positive value of the c_i's,

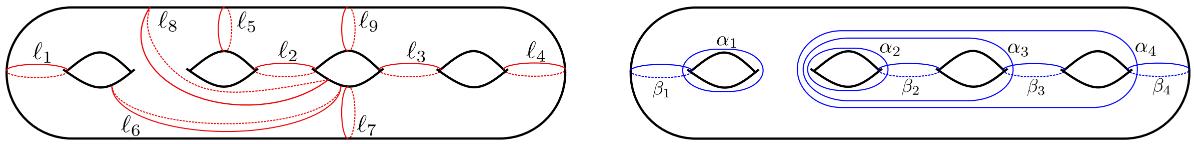

FIGURE 7.4. Left: an arrangement of simple closed curves on Σ_4 dual to TL_3. Right: the basis we use to compute $\nu(T_\Gamma)$, oriented so that $\hat{i}(\alpha_i, \beta_i) = 1$

each coordinate of these coordinates is strictly between -1 and 1. If they are all equal to 0, then

$$\begin{aligned}(c_1+c_6)c_7 &= c_1c_6 + c_1c_8 + c_6c_8 + c_1c_9 + c_6c_9,\\ (c_2+c_5)c_8 &= c_2c_5 + c_2c_9 + c_5c_9 + c_2c_7 + c_5c_7,\\ (c_3+c_4)c_9 &= c_3c_4 + c_3c_8 + c_4c_8 + c_3c_7 + c_4c_7.\end{aligned}$$

Solving for c_7 in the first equation and substituting this expression in the second equation yields

$$c_1c_2c_5 + c_1c_2c_6 + c_1c_5c_6 + c_2c_5c_6 + 2c_1c_2c_9 + 2c_1c_5c_9 + 2c_2c_6c_9 + 2c_5c_6c_9 = 0$$

which cannot happen if every c_i is positive. Therefore, the Ceresa class $\nu(T_\Gamma)$ is nontrivial by Proposition 7.1.

In the previous two examples, each collection of curves is Lagrangian, and hence $\nu(T_\Gamma)$ lies in $\overline{A(\delta_\Gamma)}$. However, the Ceresa class $\nu(T_\Lambda)$ may not live in $\overline{A(\delta_\Gamma)}$, as we shall see in the following example.

Example 7.7. *Let Γ be a tropical curve whose underlying graph is a theta graph. Suppose the two vertices each have weight 1, and each edge has length 1, see the right graph in Figure 7.1. Then $\nu(T_\Gamma) \notin \overline{A(\delta_\Gamma)}$, but $3\,\nu(T_\Gamma) \in \overline{A(\delta_\Gamma)}$. In particular Γ is Ceresa nontrivial.*

Let ℓ_1, ℓ_2, ℓ_3 be the configuration of essential closed curves illustrated in Figure 7.5. Consider the following basis of $H_1(\Sigma_4, \mathbb{Z})$, written in terms of the basis in Figure 7.2:

$$\alpha_2' = -\alpha_2,\ \alpha_3' = 2\alpha_2 + \alpha_3,\ \beta_2' = \beta_3 - 2\beta_2,\ \beta_3' = \beta_2$$

and $\alpha_i' = \alpha_i$, $\beta_i' = \beta_i$ for $i = 1, 4$. On this basis, $(\delta_\Gamma - I)(\alpha_2') = \beta_2'$, $(\delta_\Gamma - I)(\alpha_3') = 3\beta_3'$, and $(\delta_\Gamma - I)(\alpha_i')$, $(\delta_\Gamma - I)(\beta_i') = 0$ otherwise. So $Y' = \mathrm{span}_{\mathbb{Z}}\{\beta_2', 3\beta_3'\}$, and $Y = \mathrm{span}_{\mathbb{Z}}\{\beta_2', \beta_3'\}$. Next, we compute

$$\begin{aligned}J([T_\Gamma, \tau]) &= J([T_{\ell_1}, \tau]) + T_{\ell_1} \cdot J([T_{\ell_2}, \tau]) + T_{\ell_1}T_{\ell_2} \cdot J([T_{\ell_3}, \tau])\\ &= \alpha_1' \wedge \beta_1' \wedge \beta_2' + \alpha_1' \wedge \beta_1' \wedge \beta_3' + \alpha_2' \wedge \beta_2' \wedge \beta_3'.\end{aligned}$$

Clearly, $\alpha_2' \wedge \beta_2' \wedge \beta_3' \in F_2L$, and $\alpha_1' \wedge \beta_1' \wedge \beta_2' \in (\delta_\Gamma - I)(L)$ by Proposition 5.6. However, the simple wedge $\alpha_1' \wedge \beta_1' \wedge \beta_3'$ is not contained in $F_2L + (\delta_\Gamma - I)L + H$, so $\nu(T_\Lambda) \notin \overline{A(\delta_\Gamma)}$. Nevertheless, the class $3\,\nu(T_\Gamma)$ lies in $\overline{A(\delta_\Gamma)}$ because $Y' \leq 3\,Y$.

Remark 7.8. The ℓ-adic Ceresa class of a hyperelliptic algebraic curve is trivial, which we may interpret as a property of hyperelliptic Jacobians via the Torelli theorem. Nevertheless, the tropical analog of this does not hold because the property of being hyperelliptic cannot be determined by the Jacobian alone. A tropical curve whose Jacobian is isomorphic to the Jacobian of a hyperelliptic tropical curve, as polarized tropical abelian

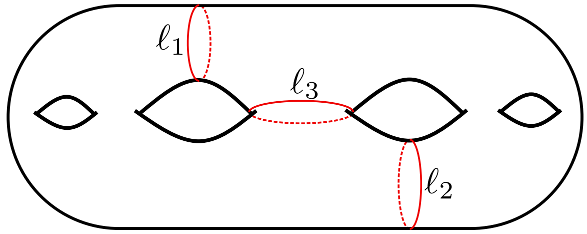

FIGURE 7.5. An arrangement of simple closed curves on Σ_4 dual to a theta graph with a weight on each vertex

varieties, is said to be of *hyperelliptic type*. By [8, Theorem 1.1], the tropical curve from Example 7.7 is of hyperelliptic type, yet it is Ceresa nontrivial. Thus, the Ceresa class for a tropical curve is not an invariant of its Jacobian and can be used to distinguish hyperelliptic tropical curve from tropical curves of hyperelliptic type. One can ask whether the tropical Ceresa class is trivial exactly for hyperelliptic tropical curves. In topological terms, we could ask: if γ is a multitwist and τ an involution in the mapping class group whose commutator lies in the Johnson kernel, is there an involution τ' which commutes with γ?

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