KAC-WAKIMOTO CONJECTURE FOR THE PERIPLECTIC LIE SUPERALGEBRA

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To Nicolas Andruskiewitsch for his 60th birthday

Abstract. We prove an analogue of the Kac-Wakimoto conjecture for the periplectic Lie superalgebra $\mathfrak{p}(n)$, stating that any simple module lying in a block of non-maximal atypicality has superdimension zero.

1. Introduction

1.1. Consider a complex vector superspace V, and let $\mathbb{C}^{0|1}$ be the odd one-dimensional vectors superspace.

The (complex) periplectic Lie superalgebra $\mathfrak{p}(V)$ is the Lie superalgebra of endomorphisms of a complex vector superspace V equipped with a non-degenerate symmetric form $\omega : S^2V \to \mathbb{C}^{0|1}$ (this form is also referred to as an "odd form"). An example of such superalgebra is $\mathfrak{p}(n) = \mathfrak{p}(\mathbb{C}^{n|n})$ for $V = \mathbb{C}^{n|n}$ with $\omega_n : \mathbb{C}^{n|n} \otimes \mathbb{C}^{n|n} \to \mathbb{C}^{0|1}$ pairing the even and odd parts of the vector superspace $\mathbb{C}^{n|n}$.

The periplectic Lie superalgebras has an interesting non-semisimple representation theory; some results on the category $\mathcal{F}_n$ of finite-dimensional integrable representations of $\mathfrak{p}(n)$ can be found in [BDE+16, Che15, Cou16, DLZ15, Gor01, Moo03, Ser02].

In [BDE+16], the blocks of the category $\mathcal{F}_n$ were classified: it was shown that (up to change of parity) the blocks can be enumerated by integers $-n, -n+2, -n+4, \ldots, n-4, n-2, n$. We say that a block has maximal atypicality if it contains one-dimensional even or odd representation of $\mathfrak{p}(n)$. Such block has number 0 for even n and ± 1 for odd n.

In this paper we prove the following version of the Kac-Wakimoto conjecture:

Theorem 1. *Every object of a block which is not of maximal atypicality has zero superdimension, where the superdimension* $\mathrm{sdim} M$ *is defined as* $\dim M_{\bar{0}} - \dim M_{\bar{1}}$.

The main ingredients in the proof of this theorem, are the translation functors acting on $\mathcal{F}_n$, and the Duflo-Serganova functor $DS : \mathcal{F}_n \to \mathcal{F}_{n-2}$. The translation functors are direct summands of the functor $-\otimes V$, whose action on the blocks $\mathcal{F}_n$ was described in [BDE+16]; the functor $DS : \mathcal{F}_n \to \mathcal{F}_{n-2}$ is a symmetric monoidal (SM) functor which preserves superdimension. Using this functor we reduce the problem of computing superdimensions in $\mathcal{F}_n$ to a similar problem in $\mathcal{F}_{n-2}$.

We also prove the following statement:

Theorem 2. *Let* $M \in \mathcal{F}_n$ *be an object lying in a certain block as described in Section 2.3.5 and* [BDE+16].

1. *The object* M^* *also lies in the same block of* $\mathcal{F}_n$.
2. *We have a natural isomorphism*

$$\Theta_k(M^*) \cong \Pi\left(\Theta_{-k}(M)\right)^*$$

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where Θ_k is the k-th translation functor on $\mathcal{F}_n$ (see Definition 2.3.5).

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2. Preliminaries

2.1. **General.** Throughout this paper, we will work over the base field $\mathbb{C}$, and all the categories considered are $\mathbb{C}$-linear.

A *vector superspace* will be defined as a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The *parity* of a homogeneous vector $v \in V$ will be denoted by $p(v) \in \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ (whenever the notation $p(v)$ appears in formulas, we always assume that v is homogeneous).

By Π we denote the functor $- \otimes \mathbb{C}^{0|1}$ on the category of vector superspaces.

2.2. **Tensor categories.** In the context of symmetric monoidal (SM) categories, we will denote by $\mathbb{1}$ the unit object, and by σ the symmetry morphisms.

A functor between symmetric monoidal categories will be called a *SM functor* if it respects the SM structure.

Given an object V in a SM category, we will denote by

$$coev : \mathbb{1} \to V \otimes V^*, \quad ev : V^* \otimes V \to \mathbb{1}$$

the coevaluation and evaluation maps for V. We will also denote by $\mathfrak{gl}(V) := V \otimes V^*$ the internal endomorphism space with the obvious Lie algebra structure on it. The object V is then a module over the Lie algebra $\mathfrak{gl}(V)$; we denote the action by $act : \mathfrak{gl}(V) \otimes V \to V$. For two functors F, G we write $F \vdash G$ if F is left adjoint of G.

2.3. **The periplectic Lie superalgebra.**

2.3.1. *Definition of the periplectic Lie superalgebra.* Let $n \in \mathbb{Z}_{>0}$, and let V_n be an $(n|n)$-dimensional vector superspace equipped with a non-degenerate odd symmetric form

(1) $\quad \beta : V_n \otimes V_n \to \mathbb{C}, \quad \beta(v, w) = \beta(w, v), \quad \text{and} \quad \beta(v, w) = 0 \text{ if } p(v) = p(w).$

Then $\mathrm{End}_{\mathbb{C}}(V_n)$ inherits the structure of a vector superspace from V_n. We denote by $\mathfrak{p}(n)$ the Lie superalgebra of all $X \in \mathrm{End}_{\mathbb{C}}(V_n)$ preserving β, i.e. satisfying

$$\beta(Xv, w) + (-1)^{p(X)p(v)}\beta(v, Xw) = 0.$$

Remark 2.3.1. Choosing dual bases $v_1, v_2, \ldots, v_n$ in $V_{\bar{0},n}$ and $v_{1'}, v_{2'}, \ldots v_{n'}$ in $V_{\bar{1},n}$, we can write the matrix of $X \in \mathfrak{p}(n)$ as $\left(\begin{smallmatrix} A & B \\ C & -A^t \end{smallmatrix}\right)$ where A, B, C are $n \times n$ matrices such that $B^t = B$, $C^t = -C$.

There is a grading $\mathfrak{p}(n) \cong \mathfrak{p}(n)_{-1} \oplus \mathfrak{p}(n)_0 \oplus \mathfrak{p}(n)_1$ where

$$\mathfrak{p}(n)_0 \cong \mathfrak{gl}(n), \quad \mathfrak{p}(n)_{-1} \cong \Pi \wedge^2 (\mathbb{C}^n)^*, \quad \mathfrak{p}(n)_1 \cong \Pi S^2\mathbb{C}^n.$$

Note that the action of $\mathfrak{p}(n)_{\pm 1}$ on any $\mathfrak{p}(n)$-module is $\mathfrak{p}(n)_0$-equivariant.

2.3.2. *Weights for the periplectic superalgebra.* We choose a Cartan subalgebra of $\mathfrak{p}(n)$ equal to the subalgebra of diagonal matrices in $\mathfrak{p}(n)_0 = \mathfrak{gl}(n)$ and fix the standard basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ in the dual space. The lattice of integral weights is by definition $span_{\mathbb{Z}}\{\varepsilon_i\}_{i=1}^n$.

- $\star$ We fix a set of simple roots $\varepsilon_2 - \varepsilon_1, \ldots, \varepsilon_n - \varepsilon_{n-1}, -\varepsilon_{n-1} - \varepsilon_n$, the last root is odd and all others are even.

  With respect to this choice dominant integral weights are of the form $\lambda = \sum_i \lambda_i \varepsilon_i$ with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

- ⋆ We fix an order on the weights of $\mathfrak{p}(n)$: for weights μ, λ, we say that $\mu \geq \lambda$ if $\mu_i \leq \lambda_i$ for each i.

  Remark 2.3.2. It was shown in [BDE+16, Section 3.3] that if $\leq$ corresponds to a highest-weight structure on the category of finite-dimensional representations of $\mathfrak{p}(n)$. Note that in the cited paper we use a different set of simple roots $-\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n$.

- ⋆ The simple finite-dimensional representation of $\mathfrak{p}(n)$ with highest weight λ and *even* highest weight vector will be denoted by $L_n(\lambda)$.

  Example 2.3.3. Let $n \geq 2$. The natural representation V_n of $\mathfrak{p}(n)$ has highest weight $-\varepsilon_1$, with odd highest weight vector; hence $V_n \cong \Pi L_n(-\varepsilon_1)$. The representation $\bigwedge^2 V_n$ has highest weight $-2\varepsilon_1$, and the representation $S^2 V_n$ has highest weight $-\varepsilon_1 - \varepsilon_2$; both have even highest weight vectors, so

$$\wedge^2 V_n \twoheadrightarrow L_n(-2\varepsilon_1), \quad L_n(-\varepsilon_1 - \varepsilon_2) \hookrightarrow S^2 V_n.$$

- ⋆ Set $\rho^{(n)} := \sum_{i=1}^n (i-1)\varepsilon_i$, and for any weight λ, let

$$\bar{\lambda} := \lambda + \rho^{(n)}.$$

- ⋆ We will associate to λ a weight diagram d_λ, defined as a labeling of the integer line by symbols $\bullet$ ("black ball") and $\circ$ ("empty") such that j has label $\bullet$ if $j \in \{\bar{\lambda}_i \mid i = 1, 2, \ldots\}$, and label $\circ$ otherwise.
- ⋆ We use the notations $|\lambda| := -\sum_i \lambda_i$ and $\kappa(\lambda) := \sum_i (-1)^{\bar{\lambda}_i}$.

2.3.3. *Representations of* $\mathfrak{p}(n)$. Let $\mathcal{F}_n$ be the category of finite-dimensional representations of $\mathfrak{p}(n)$ whose restriction to $\mathfrak{p}(n)_{\bar{0}} \cong \mathfrak{gl}(n)$ integrates to an action of $GL(n)$.

By definition, the morphisms in $\mathcal{F}_n$ will be *grading-preserving* $\mathfrak{p}(n)$-morphisms, i.e., $\mathrm{Hom}_{\mathcal{F}_n}(X, Y)$ is a vector space and not a vector superspace. This is important in order to ensure that the category $\mathcal{F}_n$ be abelian.

The category $\mathcal{F}_n$ is not semisimple. It is a highest-weight category, having simple, standard, costandard, and projective modules (the latter are also injective and tilting, per [BKN10]). Given a simple module $L_n(\lambda)$ in $\mathcal{F}_n$, we denote the corresponding standard, costandard, and projective modules by $\Delta_n(\lambda)$, $\nabla_n(\lambda)$, $P_n(\lambda)$ respectively.

2.3.4. *Tensor Casimir and translation functors.* In [BDE+16] we construct the following natural endomorphism $\Omega^{(n)}$ of the endofunctor $(-) \otimes V_n$ on $\mathcal{F}_n$.

Consider the involutive automorphism τ of $\mathfrak{gl}(V_n)$ defined by

$$\tau \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -D^t & B^t \\ -C^t & -A^t \end{pmatrix}.$$

Then $\mathfrak{p}(n)$ is the subalgebra of fixed points of τ and we have a $\mathfrak{p}(n)$-invariant decomposition

$$\mathfrak{gl}(V_n) \cong \mathfrak{p}(n) \oplus \mathfrak{p}(n)^*$$

where $\mathfrak{p}(n)^*$ is the eigenspace of τ with eigenvalue -1. Both $\mathfrak{p}_n$ and $\mathfrak{p}_n^*$ are maximal isotropic subspaces of $\mathfrak{gl}(V_n)$ with respect to the invariant symmetric form

$$(X, Y) = \operatorname{str} XY.$$

Definition 2.3.4 (Tensor Casimir)**.** For any $M \in \mathcal{F}_n$, let $\Omega^{(n)}_M$ be the composition

$$V_n \otimes M \xrightarrow{\mathrm{Id} \otimes coev \otimes \mathrm{Id}} V_n \otimes \mathfrak{p}(n)^* \otimes \mathfrak{p}(n) \otimes M \xrightarrow{i_* \otimes \mathrm{Id}} V_n \otimes \mathfrak{gl}(V_n) \otimes \mathfrak{p}(n) \otimes M \xrightarrow{act \otimes act} V_n \otimes M$$

where $i_* : \mathfrak{p}(n)^* \to \mathfrak{gl}(V_n)$ is the $\mathfrak{p}(n)$-equivariant embedding defined above.

Definition 2.3.5 (Translation functors)**.** For $k \in \mathbb{C}$, we define a functor ${\Theta'}_k^{(n)} : \mathcal{F}_n \to \mathcal{F}_n$ as the functor $\Theta^{(n)} = (-) \otimes V_n$ followed by the projection onto the generalized k-eigenspace for $\Omega^{(n)}$, i.e.

$$\text{(2)} \qquad {\Theta'}_k^{(n)}(M) := \bigcup_{m>0} \operatorname{Ker}(\Omega_M^{(n)} - k \operatorname{Id})^m_{|M \otimes V_n}$$

and set $\Theta_k^{(n)} := \Pi^k {\Theta'}_k^{(n)}$ in case $k \in \mathbb{Z}$ (it was proved in [BDE$^+$16] that $\forall k \notin \mathbb{Z},\ \Theta_k^{(n)} \cong 0$).

We use the following results from [BDE$^+$16] throughout the paper:

Theorem 2.3.6 (See [BDE$^+$16].)**.** *The relations on the translation functors $\Theta_j^{(n)}$, $j \in \mathbb{Z}$ induce a representation of the infinite Temperley-Lieb algebra $TL_\infty(q = i)$ on the Grothendieck ring on $\mathcal{F}_n$. Furthermore, for any $k \in \mathbb{Z}$, $\Theta_k^{(n)} \vdash \Theta_{k-1}^{(n)}$, i.e., $\Theta_k^{(n)}$ is left adjoint to $\Theta_{k-1}^{(n)}$ and right adjoint to $\Theta_{k+1}^{(n)}$.*

The functors $\Theta_k^{(n)}$ are exact, since $- \otimes V_n$ is an exact functor.

Theorem 2.3.7 (See [BDE$^+$16].)**.** *Let P be an indecomposable projective module in $\mathcal{F}_n$. Then for any i, $\Theta_i^{(n)} P$ is indecomposable projective or zero.*

For more details on the structure of $\mathcal{F}_n$ we refer the reader to [BDE$^+$16].

2.3.5. *Blocks.* Let $\mathcal{F}_n^k$ be the full subcategory of $\mathcal{F}_n$ consisting of modules whose composition functors are isomorphic to $L_n(\lambda)$ or $\Pi L_n(\lambda)$ with $\kappa(\lambda) = k$. Observe that $\kappa(\lambda)$ takes value in the set $\{-n, -n+2, \ldots, n-2, n\}$. It is proven in [BDE$^+$16] that

$$\mathcal{F}_n = \bigoplus_{k \in \{-n, -n+2, \ldots, n-2, n\}} \mathcal{F}_n^k.$$

Furthermore

$$\mathcal{F}_n^k = (\mathcal{F}_n^k)^+ \oplus (\mathcal{F}_n^k)^-$$

with the functor Π establishing equivalence between $(\mathcal{F}_n^k)^+$ and $(\mathcal{F}_n^k)^-$. The subcategories $(\mathcal{F}_n^k)^\pm$ are blocks of $\mathcal{F}_n$. Since parity of a module is not important for this paper, by abuse of terminology, we will just call $\mathcal{F}_n^k$ "blocks".

Remark 2.3.8. We call blocks $\mathcal{F}_n^{\pm n}$ typical. If $\kappa(\lambda) = \pm n$ then $L_n(\lambda) = \nabla_n(\lambda)$, see [BDE$^+$16, Remark 9.1.3]. On the other hand, $\dim L_n(\lambda) = (1|0)$ implies $\lambda = a(\sum_{i+1}^n \varepsilon_i)$ hence $\kappa(\lambda) = \pm 1$ for odd n and 0 for even n. We call the corresponding blocks maximally atypical.

Theorem 2.3.9 (See [BDE$^+$16].)**.** *Let $i \in \mathbb{Z}$, $k \in \{-n, -n+2, \ldots, n-2, n\}$. Then we have*

$$\Theta_i^{(n)} \mathcal{F}_n^k \subset \begin{cases} \mathcal{F}_n^{k+2} & \text{if } i \text{ is odd} \\ \mathcal{F}_n^{k-2} & \text{if } i \text{ is even} \end{cases}$$

Finally, we introduce some notation:

Notation 2.3.10. Let $I = (i_1, i_2, \ldots, i_k)$ be a sequence with $i_1, i_2, \ldots, i_k \in \mathbb{Z}$.

(1) We denote by

$$\Theta_I^{(n)} := \Theta_{i_1}^{(n)} \circ \Theta_{i_2}^{(n)} \circ \ldots \circ \Theta_{i_k}^{(n)}$$

the composition of the corresponding translation functors.

(2) We set
$$t(I) := \sum_{s=0}^{k} (-1)^{i_s+1}.$$

The following is an immediate corollary of Theorem 2.3.9:

Corollary 2.3.11. *For any $n \geq 1$, $l \in \{-n, -(n-2), \ldots, n-2, n\}$ and any integer sequence $I = (i_1, i_2, \ldots, i_k)$ we have:*
$$M \in \mathcal{F}^n_l \implies \Theta^{(n)}_I M \in \mathcal{F}^n_{l+2t(I)}.$$

2.4. **The Duflo-Serganova functor.**

2.4.1. *Definition and basic properties.* Let $n \geq 3$, and let $x \in \mathfrak{p}(n)$ be an odd element such that $[x, x] = 0$.

Definition 2.4.1 (See [DS05])**.** Let $M \in \mathcal{F}_n$. We define
$$DS_x(M) = {Ker(x|_M)} \Big/ {Im(x|_M)}.$$

The vector superspace $\mathfrak{p}_x := DS_x\mathfrak{p}(n)$ is naturally equipped with a Lie superalgebra structure. One can check by direct computations that $\mathfrak{p}_x$ is isomorphic to $\mathfrak{p}(n-s)$ where s is the rank of x. The above correspondence defines an SM-functor $DS_x : \mathcal{F}_n \to \mathcal{F}_{n-s}$, called the *Duflo-Serganova functor*. Such functors were introduced in [DS05].

3. The Duflo-Serganova functor and the tensor Casimir

In this section we recall the definition of the Duflo-Serganova functor and prove that it commutes with translation functors.

Let $n \geq 3$, and let $x \in \mathfrak{p}(n)_{\bar{1}}$ be such that $[x, x] = 0$. Let $s := \mathrm{rk}(x)$.

Definition 2.4.1 then gives us a functor $DS_x : \mathcal{F}_n \to \mathcal{F}_{n-s}$.

Lemma 3.0.1. *We have: $DS_x(\Omega^{(n)}) = \Omega^{(n-s)} DS_n$, where $\Omega^{(n)}$ is the tensor Casimir for $\mathfrak{p}(n)$, and $\Omega^{(n-s)}$ is the tensor Casimir for $\mathfrak{p}(n-s)$.*

That is, for any $M \in \mathcal{F}_n$, $DS_x(\Omega^{(n)}_M) = \Omega^{(n-s)}_{DS_x(M)}$ as endomorphisms of $V_{n-s} \otimes DS_x(M)$.

Proof. This follows directly from the definition of the tensor Casimir (Definition 2.3.4), as well as the fact that DS_x is a symmetric monoidal functor. □

Corollary 3.0.2. *The functor DS_x commutes with translation functors, that is we have a natural isomorphism of functors*
$$DS_x \Theta^{(n)}_k \xrightarrow{\sim} \Theta^{(n-s)}_k DS_x$$
for any $k \in \mathbb{Z}$.

Proof. Recall that DS_x is a SM functor and $DS_x(V_n) \cong V_{n-s}$. Hence we have a natural isomorphism $\eta : DS_x\Theta^{(n)} \longrightarrow \Theta^{(n-s)} DS_x$, where $\Theta^{(n)} = (-) \otimes V_n$ is as in Definition 2.3.5.

Now, consider $DS_x\left(\Omega^{(n)}\right)$ (the tensor Casimir). By Lemma 3.0.1, the diagram below commutes:
$$\begin{tikzcd} DS_x\Theta^{(n)} \arrow[r, "\eta"] \arrow[d, "DS_x\left(\Omega^{(n)}\right)"'] & \Theta^{(n-s)} DS_x \arrow[d, "\Omega^{n-s} DS_x"] \\ DS_x\Theta^{(n)} \arrow[r, "\eta"] & \Theta^{(n-s)} DS_x \end{tikzcd}.$$

Hence η induces an isomorphism $DS_x\Theta^{(n)}_k \cong \Theta^{(n-s)}_k DS_x$ for any $k \in \mathbb{Z}$, as required. □

4. Main result

Throughout this section, we will work with functors $DS_x : \mathcal{F}_n \to \mathcal{F}_{n-2}$ where $x \in \mathfrak{p}(n)_{-1}$ has rank 2 and satisfies $[x, x] = 0$. Here $\mathfrak{p}(n)_{-1}$ is as in Section 2.3.1.

Theorem 4.0.1. *If $M \in \mathcal{F}_n^k$ and $k \neq 0, \pm 1$ then* $\mathrm{sdim} M = 0$.

Proof. Let $x \in \mathfrak{p}(n)_{-1}$ such that $[x, x] = 0$, $\mathrm{rk}(x) = 2$.

We prove the statement by induction on n.

The base case $n = 1$ is tautological (in that case, there are no non-zero blocks except $\mathcal{F}_1^{\pm 1}$); in the base case $n = 2$, it is enough to check for a simple module $M \in \mathcal{F}_2^{\pm 2}$. The blocks $\mathcal{F}_2^{\pm 2}$ are typical, see Remark 2.3.8, hence any simple object in $\mathcal{F}_2^{\pm 2}$ is isomorphic to $\nabla_n(\mu)$, and hence has superdimension zero (cf. [BDE+16]).

Next, for the inductive step, let $n \geq 3$, and assume that our statement holds for $n - 2$.

Let $M \in \mathcal{F}_n^k$, $k \notin \{0, \pm 1\}$. We use Proposition 4.0.2 below, which states that DS_x preserves blocks, to show that $DS_x(M)$ lies in the corresponding block $\mathcal{F}_{n-2}^k$ (if $n = k$, then $DS_x(M) = 0$).

The fact that the functor $DS_x : \mathcal{F}_n \to \mathcal{F}_{n-2}$ is SM, and hence preserves superdimension, allows us to use the inductive assumption to show that $\mathrm{sdim} DS_x(M) = 0$, and hence $\mathrm{sdim} M = 0$, as required.

$\square$

Proposition 4.0.2. *Let $x \in \mathfrak{p}(n)_{-1}$ such that $[x, x] = 0$, $\mathrm{rk}(x) = 2$. Let $M \in \mathcal{F}_n^k$. Then $DS_x(M) \in \mathcal{F}_{n-2}^k$ (if $k = \pm n$, then $DS_x(M) = 0$).*

Proof. The proposition is proved in several steps.

1. We prove that DS_x commutes with translation functors. This is done in Lemma 3.0.2.
2. We prove that it is possible to translate any simple module into a typical block; that is, for any simple $L \in \mathcal{F}_n$,
   - There exists an integer sequence $I = (i_1, i_2, \ldots, i_s)$ such that
   $$\Theta_I^{(n)} L \in \mathcal{F}_n^n \text{ and } \Theta_I^{(n)} L \neq 0.$$
   - There exists an integer sequence $I = (i_1, i_2, \ldots, i_s)$ such that
   $$\Theta_I^{(n)} L \in \mathcal{F}_{-n}^n \text{ and } \Theta_I^{(n)} L \neq 0.$$

   This is proved in Lemma 4.0.3.
3. We prove the statement in the case $k = \pm n$. In that case, we need to show that $DS_x(M) = 0$ for any $M \in \mathcal{F}_n^{\pm n}$. Any simple module in such a block is also costandard, see Remark 2.3.8. Hence it is a free $U(\mathfrak{p}(n)_{-1})$-module. Any $\mathfrak{p}(n)$-module in $\mathcal{F}_n^{\pm n}$ has a finite filtration with simple subquotients, hence is a free $U(\mathfrak{p}(n)_{-1})$-module. This implies that $DS_x(M) = 0$ for any $M \in \mathcal{F}_n^{\pm n}$.
4. Consider a simple module $L \in \mathcal{F}_n^k$, and a simple subquotient L' of $DS_x(L)$. Let l be such that $L' \in \mathcal{F}_{n-2}^l$.

   We will show that $k = l$.

   Assume $l < k$. Recall that $k \equiv l \pmod 2$, as explained in Section 2.3.5.

   By (2), there exists an integer sequence $I := (i_1, i_2, \ldots, i_s)$ such that the translation functor
   $$\Theta_I^{(n-2)} := \Theta_{i_1}^{(n-2)} \circ \Theta_{i_2}^{(n-2)} \circ \ldots \circ \Theta_{i_s}^{(n-2)}$$
   on $\mathcal{F}_{n-2}$ satisfies:
   $$\Theta_I^{(n-2)} DS_x L' \neq 0 \text{ and } DS_x L' \in \mathcal{F}_{n-2}^{n-2}.$$

Furthermore, by Corollary 2.3.11, we have: $t(I) = (n-2-l)/2$.

Let
$$\Theta_I^{(n)} := \Theta_{i_1}^{(n)} \circ \Theta_{i_2}^{(n)} \circ \ldots \circ \Theta_{i_s}^{(n)}$$
be the corresponding translation functor on $\mathcal{F}_n$.

By (1), we have an isomorphism
$$\Theta_I^{(n-2)} DS_x L \cong DS_x \Theta_I^{(n)} L.$$

By our construction, this object has a non-zero direct summand in the typical block $\mathcal{F}_{n-2}^{n-2}$.

Let us show that $\Theta_I^{(n)} L \in \mathcal{F}_n^n$. Indeed, we may apply Corollary 2.3.9 to get:
$$\Theta_I^{(n)} L = \Theta_{i_1}^{(n)} \circ \Theta_{i_2}^{(n)} \circ \ldots \circ \Theta_{i_s}^{(n)} L \in \mathcal{F}_n^{k+2t(I)}$$
We already computed that $t(I) = (n-2-l)/2$ hence $\mathcal{F}_n^{k+2t(I)} = \mathcal{F}_n^{n+(k-l)-2}$. Recall that $k > l$ and they have the same parity, so $n + (k-l) - 2 \geq n$. Now, if $n + (k-l) - 2 > n$, then $\mathcal{F}_n^{n+(k-l)-2} = 0$, so we can just say that $\Theta_I^{(n)} L \in \mathcal{F}_n^n$.

This implies that $DS_x \Theta_I^{(n)} L = 0$ (by (3)), and hence cannot have a non-zero direct summand in the typical block $\mathcal{F}_{n-2}^{n-2}$. Thus we obtained a contradiction.

A similar proof shows that we cannot have $l > k$: in that case, we translate to typical block $\mathcal{F}_{n-2}^{-(n-2)}$.

$\square$

Lemma 4.0.3. *For any simple module $L \in \mathcal{F}_n$,*

(1) There exists a composition of translation functors $\Theta_I^{(n)}$ where $I = (i_1, i_2, \ldots, i_s)$ is an integer sequence, such that $\Theta_I^{(n)} L \neq 0$ sits in the typical block $\mathcal{F}_n^n$.

(2) There exists a composition of translation functors $\Theta_I^{(n)}$ where $I = (i_1, i_2, \ldots, i_s)$ is an integer sequence, such that $\Theta_I^{(n)} L \neq 0$ sits in the typical block $\mathcal{F}_n^{-n}$.

Proof. We use the results of [BDE$^+$16] on the action of translation functors. In particular, we use the description of the action of translation functors on projective modules given in [BDE$^+$16, Section 7.2] as well as the adjunction $\Theta_i^{(n)} \vdash \Theta_{i-1}^{(n)}$ for any $i \in \mathbb{Z}$, see Theorem 2.3.6.

We first prove (1).

Let λ be the highest weight of L (hence $L = L_n(\lambda)$), let $P = P_n(\lambda)$ be the projective cover of L.

Fix a typical weight μ of $\mathfrak{p}(n)$ with $\mu_i < \lambda_i$ and $\mu_i \in 2\mathbb{Z}$ for all i. Such a weight clearly exists: take for example $\mu_n \in \mathbb{Z}$ such that $\mu_n - n \in 2\mathbb{Z}$ and $\mu_n < \lambda_1$. Set $\mu_k := \mu_n - (n-k)$ for any $k = 1, 2, \ldots, n-1$. Then $\mu_k < \mu_n < \lambda_1 < \lambda_k$ for any $k = 1, \ldots, n$, and
$$\mu_k + k \equiv \mu_n - n \pmod 2$$
which implies $\mu_k + k \in 2\mathbb{Z}$. Hence μ is a typical weight, and $P' := P_n(\mu)$ sits in $\mathcal{F}_n^n$.

By [BDE$^+$16, Section 7.2], we have an integer sequence $J = (j_1, j_2, \ldots, j_s)$ such that $\Theta_J^{(n)} P' = P$.

Set $I = (j_s - 1, j_{s-1} - 1, \ldots, j_2 - 1, j_1 - 1)$. Then $\Theta_J^{(n)} \vdash \Theta_I^{(n)}$, and we have:
$$\mathrm{Hom}_{\mathfrak{p}(n)}(P', \Theta_I^{(n)} L) = \mathrm{Hom}_{\mathfrak{p}(n)}(\Theta_J^{(n)} P', L) = \mathrm{Hom}_{\mathfrak{p}(n)}(P, L) = \mathbb{C}.$$
Therefore, $\Theta_I^{(n)} L$ is a non-trivial quotient of P'. This proves (1).

Similarly, we prove (2). Fix a typical weight μ of $\mathfrak{p}(n)$ with $\mu_i < \lambda_i$ and $\mu_i \in 2\mathbb{Z}+1$ for all i. Again, such a weight can be constructed very explicitly. Then $P' := P_n(\mu)$ sits in $\mathcal{F}_n^{-n}$, and we can apply exact the same arguments as before.

□

5. Dual modules and blocks

In this section we prove Theorem 2.

Proposition 5.0.1. *Let $M \in \mathcal{F}_n^k$. Then M^* also lies in the block $\mathcal{F}_n^k$.*

Proof. First of all, notice that it is enough to prove the statement for a simple module $M = L(\lambda) \in \mathcal{F}_n^k$.

Consider the costandard module $\nabla_n(\lambda)$ having $L_n(\lambda)$ as its socle. This module is indecomposable, so $\nabla_n(\lambda) \in \mathcal{F}_n^k$. Consider the dual module $\nabla(\lambda)^*$. This is also an indecomposable costandard module, with cosocle $L(\lambda)^*$, so it is enough to check that $\nabla(\lambda)^* \in \mathcal{F}_n^k$ as well. Now, by [BDE+16, Lemma 3.6.1], $\nabla(\lambda)^* \cong \nabla(\mu)$, where $\mu + \rho = -w_0(\lambda + \rho)$, where w_0 is the longest element in the Weyl group. That is, d_μ is obtained from d_λ by reflecting the diagram with respect to zero.

Hence, $\kappa(\lambda) = \kappa(\mu)$, and so $L(\mu) \in \mathcal{F}_n^k$. □

Proposition 5.0.2. *There exists a natural isomorphism*

$$\Pi\Theta_{-k}^{(n)}(-)^* \xrightarrow{\sim} \left(\Theta_k^{(n)}(-)\right)^*.$$

Proof. Consider the functor $- \otimes V_n : \mathcal{F}_n \to \mathcal{F}_n$, $M \mapsto M \otimes V_n$. We have natural isomorphisms

$$M^* \otimes V_n \otimes \Pi\mathbb{C} \xrightarrow{\mathrm{Id}\otimes\eta^{-1}} M^* \otimes V_n^* \xrightarrow{\sigma_{M^*,V_n^*}} V_n^* \otimes M^* \cong (M \otimes V_n)^*$$

where $\eta : V_n^* \to V_n \otimes \Pi\mathbb{C}$ is the isomorphism defined by the odd bilinear form on V_n.

Consider the tensor Casimir $\Omega_M^{(n)} : M \otimes V_n \to M \otimes V_n$ as in Definition 2.3.4. Choosing dual bases $\{X_i\}, \{X^i\}$ in $\mathfrak{p}(n)$ and $\mathfrak{p}(n)^* \subset \mathfrak{gl}(n|n)$, we can write $\Omega_M^{(n)} = \sum_i X_i|_M \otimes X^i|_{V_n}$.

Denote by

$$(\Omega_M^{(n)})^* : V_n^* \otimes M^* \to V_n^* \otimes M^*$$

the dual map. Then for any homogeneous $u \in V_n^*$, $f \in M^*$, we have:

$$(\Omega_M^{(n)})^*(u \otimes f) = \sum_i (-1)^{p(X_i)p(X^i)}(-1)^{p(X^i)p(f)} \left(X^i|_{V_n}\right)^* (u) \otimes (X_i|_M)^* (f).$$

We now construct the commutative diagram

$$\begin{tikzcd}
V_n^* \otimes M^* \arrow[r, "(\Omega_M^{(n)})^*"] & V_n^* \otimes M^* \arrow[d, "\sigma^{-1}_{M^*,V_n^*}"] \\
M^* \otimes V_n^* \arrow[u, "\sigma_{M^*,V_n^*}"] \arrow[r, "\phi"] & M^* \otimes V_n^* \arrow[d, "\mathrm{Id}\otimes\eta"] \\
M^* \otimes V_n \otimes \Pi\mathbb{C} \arrow[u, "\mathrm{Id}\otimes\eta^{-1}"] \arrow[r, "\phi'"] & M^* \otimes V_n \otimes \Pi\mathbb{C}
\end{tikzcd}$$

and we compute the lower two horizontal arrows.

We begin with the horizontal arrow $\phi : M^* \otimes V_n^* \to M^* \otimes V_n^*$.

By definition, $\phi = \sigma^{-1}_{M^*,V_n^*} \circ (\Omega_M^{(n)})^* \circ \sigma_{M^*,V_n^*}$.

For any homogeneous $f \in M^*, u \in V_n^*$, applying the map $(\Omega_M^{(n)})^* \circ \sigma_{M^*, V_n^*}$ to the element $f \otimes u$ we get:

$$(\Omega_M^{(n)})^* \left((-1)^{p(f)p(u)} u \otimes f\right) =$$
$$= \sum_i (-1)^{p(f)p(u)} (-1)^{p(X_i)p(X^i)} (-1)^{p(X^i)p(u)+1} (-1)^{p(X^i)p(f)} X^i.u \otimes (-1)^{p(X^i)p(f)+1} X_i.f =$$
$$= \sum_i (-1)^{p(X_i)p(X^i)+p(f)p(u)+p(X^i)p(u)} X^i.u \otimes X_i.f$$

Hence,

$$\phi(f \otimes u) = \sum_i (-1)^{p(X_i)p(X^i)+p(f)p(u)+p(X_i)p(u)} (-1)^{(p(X_i)+p(f))(p(X^i)+p(u))} X_i.f \otimes X^i.u =$$
$$= \sum_i (-1)^{p(X^i)p(f)} X_i.f \otimes X^i.u$$

Next, we compute the horizontal arrow $\phi' : M^* \otimes V_n \otimes \Pi\mathbb{C} \to M^* \otimes V_n \otimes \Pi\mathbb{C}$.

The elements $X^i \in \mathfrak{gl}(n|n)$ satisfy the following property (cf. [BDE+16, Proof of Proposition 4.4.1]):

$$\begin{tikzcd}
V_n^* \arrow[r, "X^i"] \arrow[d, "\eta"] & V_n^* \arrow[d, "\eta"] \\
V_n \otimes \Pi\mathbb{C} \arrow[r, "-X^i"'] & V_n \otimes \Pi\mathbb{C}
\end{tikzcd}$$

Given homogeneous $f \in M^*, v \in V_n \otimes \Pi\mathbb{C}$, we have

$$\phi'(f \otimes v) = -(-1)^{p(X^i)p(f)} X_i.f \otimes X^i.v$$

Hence, $\phi' = -\Pi\Omega_{M^*}^{(n)}$. Thus the natural isomorphism $\sigma_{M^*, V_n^*} \circ (\mathrm{Id} \otimes \eta^{-1})$ establishes a natural isomorphism between the eigenspace of $(\Omega_M^{(n)})^*$ corresponding to eigenvalue k and the eigenspace (shifted by Π) of $\Omega_{M^*}^{(n)} = -\Pi\phi'$ corresponding to eigenvalue $(-k)$. This implies the statement of the proposition. $\square$

Example 5.0.3. Let $n \geq 2$ and set $M = V_n$. Then $V_n^* \cong \Pi V_n$ and $\Theta_{-1}^{(n)} V_n \cong \wedge^2 V_n$ and hence

$$\left(\Theta_{-1}^{(n)} V_n\right)^* \cong S^2 V_n \cong \Pi\Theta_1^{(n)}(\Pi V_n) \cong \Pi\Theta_1^{(n)} V_n^*.$$

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