

CLASSICAL THETA LIFTS FOR HIGHER METAPLECTIC COVERING GROUPS

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ABSTRACT. The classical theta correspondence establishes a relationship between automorphic representations on special orthogonal groups and automorphic representations on symplectic groups or their double covers. This correspondence is achieved by using as integral kernel a theta series on the metaplectic double cover of a symplectic group that is constructed from the Weil representation. There is also an analogous local correspondence. In this work we present an extension of the classical theta correspondence to higher degree metaplectic covers of symplectic and special orthogonal groups. The key issue here is that for higher degree covers there is no analogue of the Weil representation, and additional ingredients are needed. Our work reflects a broader paradigm: constructions in automorphic forms that work for algebraic groups or their double covers should often extend to higher degree metaplectic covers.

1. INTRODUCTION

Theta series provide a way to construct correspondences between spaces of automorphic forms. For example, suppose that $G_1 = SO(V_1)$ and $G_2 = Sp(V_2)$ are orthogonal and symplectic groups, resp. Then these groups embed in a symplectic group $G := Sp(V_1 \otimes V_2)$. Let F be a number field and $\mathbb{A}$ its ring of adeles. There is a family of theta functions θ^ϕ on the adelic metaplectic double cover $Mp(V_1 \otimes V_2)(\mathbb{A})$ (depending on some additional data ϕ), and these functions may be used to create a correspondence between automorphic forms f_1 on $G_1(\mathbb{A})$ and automorphic forms f_2 on $G_2(\mathbb{A})$ or its double cover:

$$f_2(g_2, \phi) = \int \theta^\phi(g_1 \cdot g_2) f_1(g_1) dg_1,$$

where the integral is over the adelic quotient $G_1(F) \backslash G_1(\mathbb{A})$. This correspondence, which in a low-rank case can be used to recreate the Shimura correspondence, has been studied by many authors over the past half century (see for example Rallis [Ra2] and the references therein). There is also a local correspondence of irreducible smooth representations, the Howe correspondence, that is obtained by restricting the Weil representation to the image of $G_1 \cdot G_2$ in G (see Howe [Ho]), that has likewise received a great deal of attention. The goal of this paper is to extend these constructions to higher metaplectic covers of the groups G_1, G_2 (beyond the double cover Mp) and to initiate an analysis of the resulting map of representations.

The double cover of the symplectic group, Mp , was introduced by Weil [We] in his treatment of theta series. Over local or global fields F with enough roots of unity, there are higher

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degree covers of classical groups as well, related to the work of Bass-Milnor-Serre on the congruence subgroup problem. These were first treated for simply connected algebraic groups over F whose F -points are simple, simply connected, split over F and of rank at least two by Matsumoto [Mat], and in the context of number theory and automorphic forms for GL_n by Kubota [Ku] (when $n = 2$) and Kazhdan-Patterson [K-P] (for general n). The construction of covering groups was extended to a wide class of groups K -theoretically by Brylinski and Deligne [B-D]. Fan Gao [Gao1] gave a thorough treatment of metaplectic covers over local and global fields using the Brylinski-Deligne construction, and generalized Langlands's work on the constant term of Eisenstein series to Eisenstein series on covering groups. Though for more general groups there are sometimes a number of inequivalent covers of a given degree, for the groups we treat here the covers are essentially unique, and locally may be regarded as restrictions of the covers of Matsumoto. We will describe the covers in detail below.

In this paper, we construct a global theta lifting taking automorphic representations on covers of a symplectic group to automorphic forms on covers of an orthogonal group. Here the degrees of the covers must be compatible. The construction is via a theta kernel. The difficulty in this construction is that there is no known analogue of the Weil representation for general degree covers. The classical theta functions may also be obtained as residues of Eisenstein series on $Mp(\mathbb{A})$, as shown by Ikeda [Ik1] (this may be regarded as an instance of the Siegel-Weil formula; see Ikeda [Ik2]), and so a first attempt would be to mimic the construction above:

$$\tilde{f}_2(g_2) = \int \tilde{\theta}(g_1 \cdot g_2) \tilde{f}_1(g_1) dg_1,$$

where now $\tilde{\theta}$ is an automorphic function realized as a residue of an Eisenstein series. This is indeed useful in many cases, including the lift of Bump and the authors [B-F-G2] for the double cover of an orthogonal group and the recent work of Leslie [L], which constructs CAP representations of the four-fold cover of a symplectic group using such a theta kernel. However, it is not sufficient to produce the lifting here. The reason is that this residue is too large a representation, that is, it is not attached to the minimal nontrivial coadjoint orbit or some other orbit of small Gelfand-Kirillov dimension. Thus it is too big to give a correspondence by mimicking the standard procedure (i.e., restricting to the tensor product embedding).

Instead we consider the product of two different theta functions—one coming from a residue of an Eisenstein series, and the other coming from the Weil representation—and use a Fourier coefficient of this product as integral kernel (given precisely in (3.3) below). In the case of the trivial cover of the orthogonal group, one of the theta functions and the Fourier coefficient integral become trivial, and our construction reduces to the classical theta kernel construction. Hence the construction presented here may be viewed as an extension of the classical theta integral construction. There is also a local analogue, and we show that for principal series induced from unramified quasicharacters in general position the lifting in the equal rank case gives a correspondence that is indeed compatible with the expected map of L -groups (these are discussed below). For the classical theta correspondence, local functoriality was established (in greater generality) by Rallis [Ra1], Section 6.

To explain the L -group formalism that is related to our construction, let G denote one of the groups Sp_{2l} or SO_k . Let r denote a positive integer, and $G^{(r)}(\mathbb{A})$ the r -fold metaplectic covering group of $G(\mathbb{A})$. For this group to be defined we need to assume that the field F

contains all r -th roots of unity; for convenience we shall assume that F contains the $2r$ -th roots of unity. The notion of an L-group for metaplectic covering groups was developed in general by Weissman [W], following a description in the split case by McNamara [McN]. The assumption that F contains the $2r$ -th roots avoids certain complications in Weissman's construction (and is required by McNamara). In particular, though the Langlands dual group for a cover of an even orthogonal group goes as usual: $(SO_{2a}^{(r)})^\vee = SO_{2a}(\mathbb{C})$, the Langlands dual group for covers of odd orthogonal and symplectic groups depends on the parity of r :

$$(Sp_{2l}^{(r)})^\vee = \begin{cases} SO_{2l+1}(\mathbb{C}) & \text{if } r \text{ is odd} \\ Sp_{2l}(\mathbb{C}) & \text{if } r \text{ is even} \end{cases} \quad (SO_{2a+1}^{(r)})^\vee = \begin{cases} Sp_{2a}(\mathbb{C}) & \text{if } r \text{ is odd} \\ SO_{2a+1}(\mathbb{C}) & \text{if } r \text{ is even.} \end{cases}$$

As in the classical case, one then expects functorial liftings of automorphic representations between the metaplectic covering groups of symplectic and orthogonal groups. However, as one can see from these dual groups, the groups involved in such a theta correspondence should sometimes be different than in the classical case. Specifically, if r is odd, then for suitable a, l one expects a correspondence between automorphic representations of $Sp_{2l}^{(r)}(\mathbb{A})$ and $SO_{2a}^{(r)}(\mathbb{A})$, and between automorphic representations of $Sp_{2l}^{(2r)}(\mathbb{A})$ and $SO_{2a+1}^{(r)}(\mathbb{A})$. For $r = 1$, these maps are the classical theta correspondence. In this paper we introduce a construction which gives them for any odd r .

Residues of Eisenstein series, when square-integrable, contribute to the residual spectrum, and these residues are frequently not generic. Indeed, the residue we consider generates an automorphic representation that should have many vanishing classes of Fourier coefficients, and this property is key to establishing our results. However, the question of determining the maximal unipotent orbit that supports a non-zero Fourier coefficient, or locally a nonzero twisted Jacquet module, is rather delicate, and has only recently been addressed for the general linear group (Cai [C1]; see also Leslie [L]). We make a conjecture about this exact orbit (Conjecture 1), establish the vanishing of Fourier coefficients that it would imply, and establish a number of partial results towards the full conjecture, including proving it in certain cases. Though we do not resolve it in full, these are sufficient for the results presented here. See Section 4 below.

The construction we present here can most likely be taken farther. For example, one can consider the inverse correspondence from covers of orthogonal groups to covers of symplectic groups obtained by the same integral kernel, and in the local case one may formulate analogues of the Howe Conjecture. One can also ask for the first non-zero occurrence of the lift (for the classical theta lift, see Roberts [Ro]), and this will be treated in a subsequent paper [F-G5]. In particular, we will show that for given r , if Conjecture 1 below holds, then the integrals presented here do not vanish on a full theta tower, and also that a generic cuspidal genuine automorphic representation of $Sp_{2n}^{(r)}(\mathbb{A})$ always lifts nontrivially to a generic genuine representation on $SO_{2n+r+1}^{(r)}(\mathbb{A})$. Since Conjecture 1 is established here for $r = 3, 5$, this implies these results unconditionally for these two cases. One can also consider the case of r even, but this will require additional information about the unipotent orbits of theta residues.

Our work suggests a broader paradigm: *constructions* in the theory of automorphic forms should generalize to covers. We note three examples. First, as mentioned above, Fan Gao [Gao1] has extended Langlands's work on the constant term of Eisenstein series to covers. Second, the doubling integrals of the authors, Cai and Kaplan [CFGK1] may be extended

to covers, as loosely sketched in our announcement [CFGK2] and worked out in detail by Kaplan in [Kap]. See also Cai [C2]. Third, the work here indicates that the classical theta correspondence also generalizes. As noted above, Weissman [W] has defined a metaplectic L -group. Our suggestion is that not only the formalism of functoriality but also the integrals that give L -functions or correspondences should often generalize. Of course, doing so must involve new ideas, as is the case here. As another example, it is not straightforward to extend Langlands-Shahidi theory to covers as this theory uses the uniqueness of the Whittaker model, and such uniqueness does not hold, even locally, for most covering groups. However, it should be possible to generalize this theory by replacing the Whittaker model for a cuspidal automorphic representation τ on a metaplectic covering group with the Whittaker model for the Speh representation attached to τ . Indeed, this Speh representation, a residue of an Eisenstein series (and defined only conjecturally at the moment), is expected to have a unique Whittaker model by a generalization of Suzuki's conjecture [Su1], [Su2]. (Suzuki's conjecture concerns covers of GL_n and we expect a similar phenomenon in general.)

To conclude this introduction we describe the algebraic and representation theoretic structures that are behind our construction. A classical reductive dual pair is a pair of subgroups (G_1, G_2) inside a symplectic group $Sp(W)$ which are mutual centralizers and which act reductively on W . Our algebraic structure is this:

- (1) Groups G_1, G_2 , and two symplectic vector spaces W_1, W_2 , with monomorphisms

$$\iota_1 : G_1 \times G_2 \rightarrow Sp(W_1), \quad \iota_2 : G_1 \times G_2 \rightarrow Sp(W_2),$$

such that

- (a) via the map ι_1 , (G_1, G_2) is a reductive dual pair in $Sp(W_1)$
- (b) the images under ι_2 of $G_1 \times 1$ and $1 \times G_2$ in $Sp(W_2)$ commute, though they need not be mutual centralizers;
- (2) A unipotent group $U \subset Sp(W_2)$ which is normalized by $\iota_2(G_1, G_2)$;
- (3) A character $\psi_U : U \rightarrow \mathbb{C}$ such that $\iota_2(G_1, G_2)$, acting by conjugation, stabilizes ψ_U , and the index of $\iota_2(G_1, G_2)$ in the full stabilizer of ψ_U is finite;
- (4) A homomorphism $l : U \rightarrow \mathcal{H}(W_1)$, where $\mathcal{H}(W_1)$ denotes the Heisenberg group attached to W_1 .

In this paper, we take $(G_1, G_2) = (SO_k, Sp_{2n})$. The map ι_1 is the tensor product embedding into Sp_{2nk} , just as in the classical theta correspondence, while remaining ingredients are given in Sections 2 and 3 below. We actually consider covers, and extend the maps ι_1, ι_2 to covers there.

To explain the representation theory that is employed, let ω_ψ denote the Weil representation of $\mathcal{H}(W_1) \rtimes Mp(W_1)$ with additive character ψ , over either $\mathbb{A}$ or a completion of F . As noted above, the global theta correspondence (resp. locally, the Howe Correspondence) is based on restricting theta series constructed from ω_ψ (resp. the representation ω_ψ) to the image of a reductive dual pair $\iota_1(G_1, G_2)$. In our structure, we have two symplectic groups, and we restrict the tensor product of two small representations. Specifically, we form the representation $\omega_\psi \otimes \Theta_2^{(r)}$, where $\Theta_2^{(r)}$ is a theta representation of $Sp^{(r)}(W_2)$. We then restrict this to $Mp(W_1) \times Sp^{(r)}(W_2)$ where the groups act on the two representations via ι_1 and ι_2 , resp. Globally, in making the integral kernel we also take a Fourier coefficient with respect to U and ψ_U , where the action of U on the first factor is via ℓ ; see (3.3) below. The local map, conjecturally a correspondence, makes use of a twisted Jacquet module with respect

to U , where in defining this Jacquet module, the action of U on the first factor is again via ℓ . That is, if we write (V_1, ω_ψ) and $(V_2, \Theta_2^{(r)})$ as the representations over a local field, then $J_{U, \psi_U}(\omega_\psi \otimes \Theta_2^{(r)}) = (V_1 \otimes V_2)/V_3$ with

$$V_3 = \langle \omega_\psi(\ell(u))v_1 \otimes \Theta_2^{(r)}(u)v_2 - v_1 \otimes \psi_U(u)v_2 \mid v_1 \in V_1, v_2 \in V_2, u \in U \rangle.$$

It is this twisted Jacquet module that is restricted to $G_1 \times G_2$ (with suitable covers, and once again using the embeddings ι_1, ι_2).

Section 2 introduces the notation to be used in the sequel and treats the necessary foundations concerning metaplectic groups in some detail. In particular, we shall have occasion to use isomorphic but different covers of groups (these arise by modifying a 2-cocycle by a coboundary), and these are described precisely. Then Section 3 presents the integral (3.3) that describes the theta lifting. As indicated above, this integral requires a theta function on a higher odd degree cover of a symplectic group Sp_{2l} and also a second theta function on the double cover of a symplectic group, which is constructed using the Weil representation. The associated higher theta representation $\Theta_{2l}^{(r)}$ is analyzed in Section 4. In Conjecture 1 we describe the expected unipotent orbit behavior of this representation, and in Theorem 1 we give a proof of the Conjecture in certain cases and a partial proof in others. Section 5 concerns cuspidality. In Theorem 2, we show that an analogue of the Rallis theta tower is valid for this new class of theta lifts. The proof uses in an essential way both the properties of the higher theta representation $\Theta_{2l}^{(r)}$ and the description of the classical theta function on the double cover. Section 6, the final section of this work, concerns the unramified lift. In Theorem 3, this is shown to be functorial for characters in general position in the equal rank case.

2. GROUPS AND COVERING GROUPS

Let F be a number field with ring of adeles $\mathbb{A}$. If H is any algebraic group defined over F , we write $[H]$ for the quotient $H(F) \backslash H(\mathbb{A})$. In particular, $[\mathbb{G}_a] = F \backslash \mathbb{A}$.

We will work with the symplectic and special orthogonal groups defined over F . We realize them as follows. For $m \geq 1$ let J_m be the $m \times m$ matrix with 1 on the anti-diagonal and zero elsewhere. Let Sp_{2m} denote the symplectic group

$$Sp_{2m} = \left\{ g \in GL_{2m} \mid {}^t g \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix} \right\}$$

and for $k \geq 2$ let SO_k denote the split special orthogonal group

$$SO_k = \{ g \in GL_k \mid {}^t g J_k g = J_k \}.$$

(The construction presented here can be extended to the non-split case but we shall not do so here.)

Throughout the paper, we make use of two embeddings. First, let $n \geq 1$, $k \geq 2$, and let $\iota_1 : SO_k \times Sp_{2n} \rightarrow Sp_{2nk}$ denote the usual tensor product embedding. In matrices, if $h \in SO_k(\mathbb{A})$ and $g \in Sp_{2n}(\mathbb{A})$, then $\iota_1((1, g)) = \text{diag}(g, \dots, g)$ where g appears k times, and $\iota_1((h, 1)) = (h_{ij} I_{2n})$. Second, let $r \geq 1$ be odd and let $\iota_2 : SO_k \times Sp_{2n} \rightarrow Sp_{2n+k(r-1)}$ be the embedding given by $\iota_2(h, g) = \text{diag}(h, \dots, h, g, h^*, \dots, h^*)$, where h, h^* each appear $(r-1)/2$ times and h^* is determined so that the matrix is symplectic.

Let $\text{Mat}_{a \times b}$ denote the algebraic group of all matrices of size $a \times b$, and when $a = b$, write $\text{Mat}_a = \text{Mat}_{a \times a}$. For $l \geq 1$, let $\mathcal{H}_{2l+1}$ denote the Heisenberg group in $2l + 1$ variables. This is the group with elements (X, Y, z) where $X, Y \in \text{Mat}_{1 \times l}$ and $z \in \text{Mat}_1$, and multiplication given by

$$(X_1, Y_1, z_1)(X_2, Y_2, z_2) = (X_1 + X_2, Y_1 + Y_2, z_1 + z_2 + \frac{1}{2}(X_1 J_l {}^t Y_2 - Y_1 J_l {}^t X_2)).$$

The Heisenberg group $\mathcal{H}_{2l+1}$ is isomorphic to a subgroup of Sp_{2l+2} by the map

$$\tau(X, Y, z) = \begin{pmatrix} 1 & X & \frac{1}{2}Y & z \\ & I_l & & Y^* \\ & & I_l & X^* \\ & & & 1 \end{pmatrix},$$

where the starred entries are uniquely determined by the requirement that the matrix be symplectic ($X^* = -J_l {}^t X$, $Y^* = \frac{1}{2}J_l {}^t Y$). Also, let $\text{Mat}_a^0 = \{Z \in \text{Mat}_a \mid {}^t Z J_a = J_a Z\}$, and let $\text{Mat}_a^{00} = \{Z \in \text{Mat}_a \mid {}^t Z J_a = -J_a Z\}$.

Next we define some parabolic and unipotent groups that will be of use. All parabolic subgroups here are standard parabolic subgroups whose unipotent radical consists of upper triangular unipotent matrices, and all of the unipotent groups we introduce are groups of upper triangular unipotent matrices.

For non-negative integers a, b and c , let $P_{a,b,c}$ denote the parabolic subgroup of $Sp_{2(ab+c)}$ whose Levi part is $GL_a \times \dots \times GL_a \times Sp_{2c}$ where GL_a appears b times. Let $U_{a,b,c}$ denote the unipotent radical of the group $P_{a,b,c}$. Let $1 \leq i \leq b-1$, $\alpha = a(i-1)$, $\beta = 2(ab+c) - 2a(i+1)$, and set

$$(2.1) \quad u_{a,b,c}^i(X) = \begin{pmatrix} I_\alpha & & & & & \\ & I_a & X & & & \\ & & I_a & & & \\ & & & I_\beta & & \\ & & & & I_a & X^* \\ & & & & & I_a \\ & & & & & & I_\alpha \end{pmatrix}, \quad X \in \text{Mat}_a.$$

The set of all matrices $\{u_{a,b,c}^i(X) \mid X \in \text{Mat}_a\}$ is a subgroup of $U_{a,b,c}$ which we denote by $U_{a,b,c}^i$. Given $u \in U_{a,b,c}$, one may write it uniquely as $u = u_{a,b,c}^i(X_i)u'$ where $X_i \in \text{Mat}_a$ and $u' \in U_{a,b,c}$ is such that all its (p, j) entries are zero when $\alpha + 1 \leq p \leq \alpha + a$ and $\alpha + a + 1 \leq j \leq \alpha + 2a$. We call $u_{a,b,c}^i(X_i)$ the i -th coordinate of u .

Another subgroup of $U_{a,b,c}$ is the group $U'_{a,c}$ that consists of all matrices

$$(2.2) \quad u'_{a,c}(Y, Z) = \begin{pmatrix} I_{a(b-1)} & & & & \\ & I_a & Y & Z & \\ & & I_{2c} & Y^* & \\ & & & I_a & \\ & & & & I_{a(b-1)} \end{pmatrix}, \quad Y \in \text{Mat}_{a \times 2c}, Z \in \text{Mat}_a^0.$$

Similarly, we define the $u'_{a,c}(Y, Z)$ coordinate of u . Then every $u \in U_{a,b,c}$ has a factorization

$$(2.3) \quad u = u'_{a,c}(Y, Z) \prod_{i=1}^{b-1} u_{a,b,c}^i(X_i) u_1,$$

where $u_1 \in U_{a,b,c}$ has entries zero in the first b $a \times a$ blocks directly above the main diagonal and in positions Y and Z in (2.2) above (and so also zero in the corresponding last b superdiagonal blocks and in Y^*). Let $U_{a,b,c}^1$ denote the subgroup of $U_{a,b,c}$ consisting of all matrices (2.3) such that $Y = 0$. Also, let $U_{a,b,c}^0$ denote the subgroup of $U_{a,b,c}$ consisting of all matrices (2.3) such that $Y = Z = 0$. Note that $U_{a,b,c}^0$ is a subgroup of $U_{a,b,c}^1$.

The group $U'_{a,c}$ has a structure of a generalized Heisenberg group. Define a homomorphism $l : U_{a,b,c} \rightarrow \mathcal{H}_{2ac+1}$ as follows. Let $y_i \in \text{Mat}_{1 \times 2c}$ denote the i -th row of Y , and define

$$(2.4) \quad l(u'_{a,c}(Y, Z)) = (y_a, y_{a-1}, \dots, y_1, \frac{1}{2} \text{tr}(T_k Z)),$$

where T_k is the identity matrix I_k if k is even, and $T_k = \text{diag}(I_{[k/2]}, 2, I_{[k/2]})$ if k is odd. (Note that the center of $U_{a,c}$ consists of all matrices (2.2) such that $Y = 0$, and this is mapped to the center of $\mathcal{H}_{2ac+1}$.) Then extend l trivially from $U'_{a,c}$ to $U_{a,b,c}$.

Also, when we work with Weyl groups, we shall always work with representatives which have one non-zero entry in each row and column whose value is ± 1 ; we call such matrices Weyl group elements. For the symplectic group Sp_{2a} , such a matrix is determined uniquely by its entries in the first a rows and we sometimes specify it in this way (e.g. the proof of Proposition 5).

We now set the notation for metaplectic groups. Fix an integer $m \geq 1$ such that F contains a full set of m -th roots of unity μ_m . (Below, we will take m to be an odd integer r or to be $2r$.) If G is either the F_ν points (where ν is a place of F) or the $\mathbb{A}$ points of a linear algebraic group (that will be SO or Sp), then an m -fold covering group $\tilde{G}$ of G is a topological central extension of G by μ_m . The group consists of pairs (g, ϵ) , $g \in G$, $\epsilon \in \mu_m$, with multiplication described in terms a (Borel measurable) 2-cocycle $\sigma \in Z^2(G, \mu_m)$:

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \sigma(g_1, g_2)).$$

After fixing an embedding of μ_m into $\mathbb{C}^\times$, such groups naturally give rise to central extensions of G by $\mathbb{C}^\times$ by the same multiplication rule. If $\beta : G \rightarrow \mathbb{C}^\times$ is a (Borel measurable) map with $\beta(e) = 1$, then multiplying σ by the 2-coboundary $(g_1, g_2) \mapsto \beta(g_1)\beta(g_2)/\beta(g_1 g_2)$ gives an isomorphic group, and we also consider such groups. One can construct a global 2-cocycle from local cocycles at each place by taking the product, but only if almost all factors are 1 on the adelic points of G .

We begin with the local constructions that we will use. Fix $r > 1$ odd, let K be a local field containing a full set of r -th roots of unity and let $(\ , \)_r \in \mu_r$ be the local Hilbert symbol. If $Sp_{2m}(K)$ is any symplectic group, let $Sp_{2m}^{(r)}(K)$ (or simply $Sp_{2m}^{(r)}$) denote the r -fold covering of $Sp_{2m}(K)$ constructed by Matsumoto [Mat] using the Steinberg symbol corresponding to $(\ , \)_r^{-1}$. Since Sp_{2m} is simple and simply connected, this is the unique topological r -fold covering group of $Sp_{2m}(K)$; see for example Moore [Mo]. Hence, the cocycle is unique, but only up to a coboundary. For any integer m , let $\sigma_m \in Z^2(SL_m(K), \mu_r)$ denote the 2-cocycle on $SL_m(K)$ constructed by Banks, Levy and Sepanski [B-L-S], Section 2 (when we need to indicate the cover, we write this $\sigma_m^{(r)}$). This cocycle enjoys a block compatibility property ([B-L-S], Theorem 1) that will be of use. We shall realize $Sp_{2m}^{(r)}$ by restricting the cocycle $\sigma_{2m} \in Z^2(SL_{2m}(K), \mu_r)$ to $Sp_{2m}(K) \times Sp_{2m}(K)$, as in Kaplan [Kap], Section 1.4. Doing so, the block compatibility implies that if $g_1, g_2 \in SO_k(K)$, $h_1, h_2 \in Sp_{2n}(K)$, then

$$(2.5) \quad \sigma_{2n+k(r-1)}(\iota_2(h_1, g_1), \iota_2(h_2, g_2)) = \sigma_k(h_1, h_2)^{r-1} \sigma_{2n}(g_1, g_2).$$

Since σ_k take values in μ_r , we have $\sigma_k(h_1, h_2)^{r-1} = \sigma_k(h_1, h_2)^{-1}$. We realize the r -fold cover $SO_k^{(r)}(K)$ of the orthogonal group $SO_k(K)$ by restricting the cocycle σ_k^{-1} to $SO_k(K) \times SO_k(K)$. Then (2.5) implies that the map ι_2 extends to a homomorphism of the covering groups $\iota_2^{(r)} : SO_k^{(r)}(K) \times Sp_{2n}^{(r)}(K) \rightarrow Sp_{2n+k(r-1)}^{(r)}(K)$, given by

$$(2.6) \quad \iota_2^{(r)}((h, \epsilon_1), (g, \epsilon_2)) = (\iota_2(h, g), \epsilon_1 \epsilon_2).$$

We will also make use of the double cover $Sp_{2nk}^{(2)}(K)$ (this cover is unique up to isomorphism), and its pullback via the tensor product embedding ι_1 . Recall that when k is even (resp. k is odd), the pair (SO_k, Sp_{2n}) (resp. $(SO_k, Sp_{2n}^{(2)})$) forms a reductive dual pair inside $Sp_{2nk}^{(2)}$. We recall the local theory and give a reference for the adelization; this construction underlies the classical theta correspondence.

For each $m \geq 1$, we realize the group $Sp_{2m}^{(2)}(K)$ by a Leray cocycle $\tilde{\sigma}_{2m}$, which takes values in (generally) the eighth roots of unity. (We will be specific in the next paragraph.) This cocycle may be regarded as arising from the classical Weil representation attached to a fixed nontrivial additive character and realized in the Schrödinger model; see Kudla [Ku2], Theorem 3.1. The cocycle is cohomologous to a ‘‘Rao cocycle’’ with values in $\mu_2 = \{\pm 1\}$ (see Rao [Rao]).

Information about the behavior of the metaplectic double cover with respect to the tensor product embedding may be found in Kudla [Ku1], [Ku2]. We use a cocycle given in those works, adjusting the notation to take into account some different normalizations. Kudla considers vector spaces V, W over the local field K of dimensions k and $2n$, resp., equipped with nondegenerate bilinear (resp. symplectic) forms, and defines $\mathbb{W} = V \otimes_K W$. The tensor product of the forms gives a symplectic form on $\mathbb{W}$. He then defines j_W, j_V by using the tensor product embedding $SO(V) \times Sp(W) \rightarrow Sp(\mathbb{W})$ and restricting to the first and second factors, resp. Let λ be the natural isomorphism from $Sp_{2nk}(K)$ (defined above) to the group $Sp(\mathbb{W})$ in Kudla’s work (i.e. moving one symplectic form to another), such that $\lambda(\iota_1(h, 1)) = j_W(h)$ and $\lambda(\iota_1(1, g)) = j_V(g)$ for all $h \in SO_k(K)$, $g \in Sp_{2n}(K)$. Let $c_{\mathbb{Y}}$ be the Leray cocycle on $Sp_{2nk}(K)^2$ in Kudla’s notation. Then we realize $Sp_{2nk}^{(2)}(K)$ by means of the cocycle $\tilde{\sigma}_{2nk}(g_1, g_2) = c_{\mathbb{Y}}(\lambda(g_1), \lambda(g_2))$ (for $g_1, g_2 \in Sp_{2nk}(K)$). For $k = 1$ this defines $Sp_{2n}^{(2)}(K)$, with Y the maximal isotropic subspace of W given on p. 18 of [Ku2], which corresponds to $2n$ -vectors with first n entries zero in our normalization. Then it is shown in [Ku2], Ch. II, Proposition 3.2, that there is a map $\beta_k : Sp_{2n}(K) \rightarrow S^1 \subset \mathbb{C}^\times$ such that the map $\iota_1^{(2)}$ given by

$$\begin{aligned} \iota_1^{(2)} : SO_k(K) \times Sp_{2n}(K) &\rightarrow Sp_{2nk}^{(2)}(K) & \iota_1^{(2)}(h, g) &= (\iota_1(h, g), \beta_k(g)^{-1}) & \text{if } k \text{ is even} \\ \iota_1^{(2)} : SO_k(K) \times Sp_{2n}^{(2)}(K) &\rightarrow Sp_{2nk}^{(2)}(K) & \iota_1^{(2)}(h, (g, \epsilon)) &= (\iota_1(h, g), \beta_k(g)^{-1} \epsilon) & \text{if } k \text{ is odd} \end{aligned}$$

is an isomorphism.

We now turn to the global construction. Let $m \geq 1$ be fixed and write $M = Sp_{2m}$. We will construct the r -fold cover of $M(\mathbb{A})$. (We will then choose $m = 2n + k(r - 1)$.) Let F be a number field, and for each place ν let σ_ν be the cocycle $\tilde{\sigma}_{2m}^{(r)}$ described above for $M(F_\nu)$. Then at almost all places, the local cocycle σ_ν may be adjusted by a coboundary η_ν so that the new cocycle ρ_ν satisfies $\rho_\nu(\kappa, \kappa') = 1$ if κ, κ' are in the hyperspecial maximal compact group $M(\mathcal{O}_\nu)$, where $\mathcal{O}_\nu$ is the ring of integers of F_ν . (This may not be true, and there

is no adjustment, at a finite number of places, namely the ramified places of F , the places dividing $2r$ and the archimedean places.)

More precisely, for unramified places, let $\eta_\nu : M(F_\nu) \rightarrow S^1$ be a continuous map that satisfies

$$(2.7) \quad \sigma_\nu(m, m') = \eta_\nu(mm')\eta_\nu(m)^{-1}\eta_\nu(m')^{-1} \quad \text{for all } m, m' \in M(O_\nu).$$

This uniquely determines η_ν on $M(O_\nu)$, and it is then extended to $M(F_\nu)$. See [Kap], Section 1.5. We will choose η_ν so that for all $h \in SO_k(F_\nu)$, $g \in Sp_{2n}(F_\nu)$,

$$(2.8) \quad \eta_\nu(\iota_2(h, 1))\eta_\nu(\iota_2(1, g)) = \eta_\nu(\iota_2(h, g)).$$

Indeed, if $h \in SO_k(O_\nu)$, $g \in Sp_{2n}(O_\nu)$, then (2.8) holds, since $\sigma_\nu(\iota_2(h, 1), \iota_2(1, g)) = 1$ by (2.5), and then (2.8) follows from (2.7). We may then choose the extension of η_ν to $M(F_\nu)$ so that (2.8) holds in general. Then introduce the cocycle

$$\rho_\nu(g, g') = \frac{\eta_\nu(g)\eta_\nu(g')}{\eta_\nu(gg')}\sigma_\nu(g, g') \quad g, g' \in M(F_\nu),$$

and the 2-cocycle $\rho = \prod_v \rho_v$ of $M(\mathbb{A})$. This is well-defined since if $g, g' \in M(\mathbb{A})$, then at almost all places $g, g' \in M(O_\nu)$ and $\rho_\nu(g, g') = 1$. We shall realize the global r -fold metaplectic cover $M^{(r)}(\mathbb{A})$ of $M(\mathbb{A})$ as the central extension of $M(\mathbb{A})$ by μ_r with respect to this cocycle. For $g \in M(\mathbb{A})$ such that $\eta(g_v) = 1$ for almost all v , let $\eta(g) = \prod_v \eta_v(g_v)$; in particular, this function is defined for $g \in M(F)$ (see Takeda [Tak], Proposition 1.8 and Kaplan [Kap], Section 1.5). Then (as a consequence of Hilbert reciprocity) $M(F)$ embeds in $M^{(r)}(\mathbb{A})$ by the homomorphism $m \mapsto (m, \eta^{-1}(m))$. Since it will always be clear from context whether we are working in a matrix group or a covering group, we shall not introduce a separate notation for the image of $M(F)$; instead, we regard $M(F)$ as a subgroup of $M^{(r)}(\mathbb{A})$ via this map. We may then form the automorphic quotient $M(F) \backslash M^{(r)}(\mathbb{A})$ and consider genuine automorphic representations on this quotient.

For our construction below, we will proceed as follows. Given n, k , let $m = 2n + k(r - 1)$ and construct the 2-cocycle ρ on $M(\mathbb{A})$. We then define 2-cocycles on $SO_k(\mathbb{A})$ and $Sp_{2n}(\mathbb{A})$ by pulling back ρ via the maps $\iota_2(\star, 1)$ and $\iota_2(1, \star)$, resp. In view of (2.5), this gives r -fold covers of these groups, which we write $SO_k^{(r)}(\mathbb{A})$, $Sp_{2n}^{(r)}(\mathbb{A})$, whose local components at ν are isomorphic to the local metaplectic r -fold covers constructed above (with $K = F_\nu$). Also, using (2.8) it is not difficult to check that the map $\iota_2^{(r)} : SO_k^{(r)}(\mathbb{A}) \times Sp_{2n}^{(r)}(\mathbb{A}) \rightarrow Sp_{2n+k(r-1)}^{(r)}(\mathbb{A})$ given by (2.6) is a homomorphism.

Similarly one can define a global two-fold cover $Sp_{2nk}^{(2)}(\mathbb{A})$ whose multiplication is given by a cocycle $\sigma_{2nk}^{(2)}$ which is the product of the cocycle $\tilde{\sigma}_{2nk, \nu}$ above adjusted by a coboundary at each place ν . This cocycle restricts to the trivial cocycle on $\iota_1(SO_k(\mathbb{A}), 1)^2$ and to a cocycle which is either cohomologically trivial if k is even or nontrivial if k is odd on $\iota_1(1, Sp_{2n}(\mathbb{A}))^2$. See for example Sweet [Sw], Sections 1.8 and 2.4. Thus there are homomorphisms

$$\begin{aligned} \iota_1^{(2)} : SO_k(\mathbb{A}) \times Sp_{2n}(\mathbb{A}) &\rightarrow Sp_{2nk}^{(2)}(\mathbb{A}) & \iota_1^{(2)}(h, g) &= (\iota_1(h, g), \delta_k(g)) & \text{if } k \text{ is even} \\ \iota_1^{(2)} : SO_k(\mathbb{A}) \times Sp_{2n}^{(2)}(\mathbb{A}) &\rightarrow Sp_{2nk}^{(2)}(\mathbb{A}) & \iota_1^{(2)}(h, (g, \epsilon)) &= (\iota_1(h, g), \delta_k(g)\epsilon) & \text{if } k \text{ is odd,} \end{aligned}$$

where $\delta_k : Sp_{2n}(\mathbb{A}) \rightarrow S^1$ is a Borel measurable map. We also introduce the r -fold cover $\widetilde{Sp}_{2n}^{(r)}(\mathbb{A})$, that is obtained by multiplying the cocycle ρ by the coboundary $\delta_k(g_1, g_2) :=$

$\delta_k(g_1 g_2) \delta_k^{-1}(g_1) \delta_k^{-1}(g_2)$. The map $i(g, \epsilon) = (g, \delta_k(g) \epsilon)$ is an isomorphism from $Sp_{2n}^{(r)}(\mathbb{A})$ to $\widetilde{Sp}_{2n}^{(r)}(\mathbb{A})$.

We realize the $2r$ -fold cover $Sp_{2n}^{(2r)}(\mathbb{A})$ by using the 2-cocycle which is the product of the 2-cocycles ρ and $\sigma_{2n}^{(2)}$. For any a there is a canonical projection $p^{(1)} : Sp_{2n}^{(a)}(\mathbb{A}) \rightarrow Sp_{2n}(\mathbb{A})$ given by projection onto the first factor. Then the $2r$ -fold cover $Sp_{2n}^{(2r)}(\mathbb{A})$ is isomorphic to the fibre product of $Sp_{2n}^{(r)}(\mathbb{A})$ and $Sp_{2n}^{(2)}(\mathbb{A})$ over $Sp_{2n}(\mathbb{A})$ with respect to these projections. Fixing such an isomorphism (equivalently, an isomorphism $\mu_{2r} \cong \mu_r \times \mu_2$), the group $Sp_{2n}^{(2r)}(\mathbb{A})$ thus comes equipped with projections $p^{(r)} : Sp_{2n}^{(2r)}(\mathbb{A}) \rightarrow Sp_{2n}^{(r)}(\mathbb{A})$, $p^{(2)} : Sp_{2n}^{(2r)}(\mathbb{A}) \rightarrow Sp_{2n}^{(2)}(\mathbb{A})$ that are homomorphisms. We also use $p^{(1)}$ for the projection from $SO_k^{(r)}(\mathbb{A}) \rightarrow SO_k(\mathbb{A})$. Last, for fixed k odd we introduce the group $\widetilde{Sp}_{2n}^{(2r)}(\mathbb{A})$ by using the 2-cocycle $\rho \sigma_{2n}^{(2)} \delta_k$. Once again the map $i(g, \epsilon) = (g, \delta_k(g) \epsilon)$ is an isomorphism from $Sp_{2n}^{(2r)}(\mathbb{A})$ to $\widetilde{Sp}_{2n}^{(2r)}(\mathbb{A})$.

As noted above, the group $Sp_{2n}(F)$ embeds in $Sp_{2n}^{(\kappa r)}(\mathbb{A})$ for $\kappa = 1, 2$. Though $Sp_{2n}^{(\kappa r)}$ is not an algebraic group, we abuse the notation slightly and write $[Sp_{2n}^{(\kappa r)}]$ for the automorphic quotient $Sp_{2n}(F) \backslash Sp_{2n}^{(\kappa r)}(\mathbb{A})$. Also, $\delta_k(Sp_{2n}(F)) = 1$ (see Sweet [Sw], Proposition 2.4.2), so i induces a bijection of the automorphic quotients of $Sp_{2n}^{(r)}(\mathbb{A})$ and $\widetilde{Sp}_{2n}^{(r)}(\mathbb{A})$.

We conclude this section by mentioning several additional groups of matrices that embed in the covering groups. First, for any local field F_ν and the cover $SL_d^{(r)}(F_\nu)$ described above, any upper unipotent subgroup $N(F_\nu)$ of $SL_d(F_\nu)$ is canonically split by the trivial section $u \mapsto (u, 1)$. Passing to the adelic group requires changing each local cocycle by a coboundary, so the group $N(\mathbb{A})$ splits in $SL_d^{(r)}(\mathbb{A})$ by means of a section of $N(\mathbb{A})$, $n \mapsto (n, \eta(n)^{-1})$. Moreover, this section is canonical (Mœglin and Waldspurger [M-W], Appendix 1). Any lower unipotent group is also split by a canonical section $n_- \mapsto (n_-, \eta(n_-))$ and these splittings are compatible with the action of the Weyl group (which embeds in the covering group) by conjugation. That is, if n_- is lower triangular and ${}^w n_- := w n_- w^{-1}$ is in $N(\mathbb{A})$, then ${}^w(n_-, \eta(n_-)) = ({}^w n_-, \eta({}^w n_-))$. (See for example, [Kap], equation (1.11).) From now on, we consider all unipotent groups as embedded in the relevant covering groups by such sections (and once again do not introduce a separate notation). With this convention, all notions related to unipotent orbits extend to covering groups without change.

Similarly, working over F_ν or $\mathbb{A}$, for $a \neq 0$, let $t(a) = \text{diag}(a, a^{-1}, \dots, a, a^{-1}) \in Sp_{2l}$ ($l \geq 1$). If $a_1, a_2 \in F_\nu^*$ are r -th powers, then $\sigma_{2l, \nu}^{(r)}(t(a_{1, \nu}), t(a_{2, \nu})) = 1$ (this follows from the well-known formula for the cocycle $\sigma_{2l, \nu}^{(r)}$ on diagonal matrices; see for example [B-L-S], Section 3). This equality implies that the map η_ν restricted to the group $\{t(a) \mid a \in O_\nu^{*, r}\}$ is a homomorphism. Using this observation and the explicit description of η_ν on $SL_2(O_\nu)$ (due to Kubota), it follows that $\eta_\nu(t(a_\nu)) = 1$ for each place ν such that $a_\nu \in O_\nu^{*, r}$. Accordingly, the map $t(a) \mapsto \widetilde{t}(a) := (t(a), \eta^{-1}(t(a)))$ is a well-defined embedding of $\{t(a) \mid \mathbb{A}^{*, r}\}$ into $Sp_{2l}^{(r)}(\mathbb{A})$. This embedding will appear in the proof of Proposition 6 below.

3. DESCRIPTION OF THE INTEGRAL KERNEL

As mentioned in the introduction, the integral kernel we work with here requires two different theta functions. The first is a function on the metaplectic double cover $Sp_{2l}^{(2)}(\mathbb{A})$ of $Sp_{2l}(\mathbb{A})$. Let ψ denote a nontrivial character of $F \backslash \mathbb{A}$. Let $\Theta_{2l}^{(2)}$ denote the theta representation

defined on the group $Sp_{2l}^{(2)}(\mathbb{A})$ formed using ψ . We refer to [G-R-S1], Section 1, part 6, for the definitions and the action of the Weil representation. This representation extends to the group $\mathcal{H}_{2l+1}(\mathbb{A}) \cdot Sp_{2l}^{(2)}(\mathbb{A})$. The theta kernel will involve a function $\theta_{2l}^{(2),\psi}$ in $\Theta_{2l}^{(2)}$.

To give the second, let $r > 1$ be a fixed odd integer, and suppose as above that the number field F contains a full set of r -th roots of unity μ_r . Fix an embedding $\epsilon : \mu_r \rightarrow \mathbb{C}^\times$. Let $\Theta_{2l}^{(r)}$ be the theta representation on the group $Sp_{2l}^{(r)}(\mathbb{A})$ that is genuine with respect to ϵ ; that is, the functions $\theta^{(r)}$ in $\Theta_{2l}^{(r)}$ transform by central character ϵ , i.e., $\theta^{(r)}((I_{2l}, \mu)g) = \epsilon(\mu)\theta^{(r)}(g)$. This representation is defined via residues of the minimal parabolic Eisenstein series on $Sp_{2l}^{(r)}(\mathbb{A})$, similarly to the construction for GL_r in [K-P]. For its definition and basic properties, see [F-G2, Section 2]; see also Gao [Gao2]. We will discuss the Fourier coefficients attached to different unipotent orbits for this representation in Section 4 below.

To describe the global construction, we work with the groups $U_{a,b,c}$ defined just before (2.1) above. Define a character $\psi_{U_{a,b,c}}$ of the group $[U_{a,b,c}]$ as follows. Given $u \in U_{a,b,c}$, write $u = u'_{a,c}(Y, Z) \prod_i u^i_{a,b,c}(X_i)u_1$ as in (2.3). Define

$$(3.1) \quad \psi_{U_{a,b,c}}(u) = \psi(\text{tr}(X_1 + \cdots + X_{b-1})).$$

By restriction, this character is also a character of $[U_{a,b,c}^0]$. We will be concerned with a specific choice of the numbers a, b and c .

Fix two integers k and n , which will index the sizes of the orthogonal and symplectic groups, resp., as in Section 2. Let $r_1 = (r-1)/2$, and set $a = k$, $b = r_1$ and $c = n$. In our main construction, we will take a Fourier coefficient corresponding to the unipotent orbit $((r-1)^k 1^{2n})$ of the group $Sp_{2n+k(r-1)}$ (see, for example, [G1]). This is an integral over $[U_{k,r_1,n}]$ against the character $\psi_{U_{k,r_1,n}}$. It follows from [C-M] that the image of $SO_k \times Sp_{2n}$ under the embedding ι_2 is in the stabilizer of this orbit and this Fourier coefficient, so the Fourier coefficient gives rise to a function on this product. In view of our work with cocycles above, this extends to covering groups.

We are now ready to discuss our global integrals. Fix k and let $\kappa = 1$ if k is even, and $\kappa = 2$ if k is odd. Let $g \in Sp_{2n}^{(\kappa r)}(\mathbb{A})$ and $h \in SO_k^{(r)}(\mathbb{A})$. Then the Fourier coefficient of interest to us, corresponding to the unipotent orbit $((r-1)^k 1^{2n})$, is

$$(3.2) \quad \int_{[U_{k,r_1,n}]} \theta_{2nk}^{(2),\psi}(l(u)\iota_1^{(2)}(p^{(1)}(h), p^{(\kappa)}(g))) \theta_{2n+k(r-1)}^{(r)}(u\iota_2^{(r)}(h, p^{(r)}(g))) \psi_{U_{k,r_1,n}}(u) du.$$

Here $\theta_{2n+k(r-1)}^{(r)}$ is a vector in the space of the representation $\Theta_{2n+k(r-1)}^{(r)}$, $\theta_{2nk}^{(2),\psi}$ is a vector in the space of the representation $\Theta_{2nk}^{(2)}$ (which depends on ψ), and the map l is as in (2.4) with $a = k$, $b = r_1$ and $c = n$. Note that both of these theta functions are genuine functions. We shall use this Fourier coefficient as our integral kernel.

To formulate the theta lift, let $\pi^{(\kappa r)}$ denote a genuine irreducible cuspidal automorphic representation of $Sp_{2n}^{(\kappa r)}(\mathbb{A})$ where the metaplectic groups are constructed in the prior section. For fixed k , by means of the isomorphism i we may realize $\pi^{(\kappa r)}$ by means of genuine automorphic functions on the group $\widetilde{Sp}_{2n}^{(\kappa r)}(\mathbb{A})$, and we do so henceforth. Here genuine means that the functions in $\pi^{(\kappa r)}$ transform under the center of $\widetilde{Sp}_{2n}^{(\kappa r)}(\mathbb{A})$, $\{(1, \mu) \mid \mu \in \mu_{\kappa r}\}$, by a fixed embedding $\epsilon' : \mu_{\kappa r} \rightarrow \mathbb{C}^\times$ which is compatible with ϵ and, if k is odd, with the isomorphism of μ_{2r} with $\mu_r \times \mu_2$ selected above. Specifically, let p_2 denote the projection

from any covering group to its second factor (a root of unity of F). Then we require that $\epsilon'(\mu) = \epsilon(p_2(p^{(r)}(1, \mu))) p_2(p^{(\kappa)}(1, \mu))$ for all $\mu \in \mu_{\kappa r}$. Let $\varphi^{(\kappa r)}$ denote a vector in the space of $\pi^{(\kappa r)}$.

Definition 1. *The theta lift of $\pi^{(\kappa r)}$ is the representation $\sigma_{n,k}^{(r)}$ of $SO_k^{(r)}(\mathbb{A})$ generated by the functions $f(h)$ defined by*

$$(3.3) \quad f(h) = \int_{[Sp_{2n}^{(\kappa r)}]} \int_{[U_{k,r_1,n}]} \overline{\varphi^{(\kappa r)}(i(g))} \theta_{2nk}^{(2),\psi}(l(u) \iota_1^{(2)}(p^{(1)}(h), p^{(\kappa)}(g))) \theta_{2n+k(r-1)}^{(r)}(u \iota_2^{(r)}(h, p^{(r)}(g))) \psi_{U_{k,r_1,n}}(u) du dg.$$

as each of $\varphi^{(\kappa r)}$, $\theta_{2nk}^{(2),\psi}$, $\theta_{2n+k(r-1)}^{(r)}$ varies over the functions in its representation space.

For convenience we sometimes shorten the notation in working with (3.3) below, writing $\iota_1(h, g)$ in place of $\iota_1^{(2)}(p^{(1)}(h), p^{(\kappa)}(g))$ and $\iota_2(h, g)$ in place of $\iota_2^{(r)}(h, p^{(r)}(g))$.

In (3.3) the covers in g are compatible (otherwise this integral would vanish identically for trivial reasons). Also, the integral converges absolutely. This follows from the cuspidality of the representation $\pi^{(\kappa r)}$. Each function f is a genuine function on $SO_k^{(r)}(\mathbb{A})$. The construction given by the space generated by the integrals (3.3) defines a mapping from the set of irreducible cuspidal representations of $Sp_{2n}^{(\kappa r)}(\mathbb{A})$ to the set of representations of the quotient $SO_k(F) \backslash SO_k^{(r)}(\mathbb{A})$. In the following Sections we will study the properties of this mapping.

As mentioned in the introduction, this construction may be viewed as an extension of the well-known theta lift associated to the reductive dual pair (SO_k, Sp_{2n}) if k is even, and to the reductive dual pair $(SO_k, Sp_{2n}^{(2)})$ if k is odd. Indeed, in integral (3.3) we assumed that $r \geq 3$ is odd. If we formally set $r = 1$, let $\Theta_{2n+k(r-1)}^{(r)}$ be the trivial representation, and let $U_{k,0,n}$ be the trivial group, then we get the theta lift. Just as the theta lift can be used to go from either group in the reductive dual pair to the other, we could also use the same integral kernel to construct a mapping from the set of irreducible cuspidal automorphic representations of $SO_k^{(r)}(\mathbb{A})$ to automorphic representations of $Sp_{2n}^{(\kappa r)}(\mathbb{A})$.

There is also a local analogue of this integral given by a Hom space, which naturally gives rise to a map of representations over local fields that formally generalizes the Howe correspondence. This will be treated for unramified principal series in Section 6 below.

To conclude this section, let us mention that the theta kernel above is consistent with the Dimension Equation of [G2], [F-G4]. As explained in [F-G4], Section 6, the Dimension Equation states that when a theta kernel gives a correspondence, there is an equality relating the dimensions of the groups in the integral and the dimensions of the automorphic representations appearing in and arising from the construction. Here the dimension of an automorphic representation means its Gelfand-Kirillov dimension in the sense of [G1], Definition 5.15. In our case, this is the equality

$$\dim(\pi) + \dim(\Theta_{2nk}^{(2)}) + \dim(\Theta_{2n+k(r-1)}^{(r)}) = \dim(Sp_{2n}) + \dim(U_{k,r_1,n}) + \dim(\sigma),$$

where π , σ are as above. In the case that π and σ are both generic, and when $n = [k/2]$, we expect a functorial correspondence between $SO_k^{(r)}$ and $Sp_{2n}^{(\kappa r)}$. And indeed, it may be checked (using formulas (2), (3) in [F-G4]) that in this situation the above equation holds, assuming

Conjecture 1 below. This equality is another motivation for choosing the orbit $((r-1)^k 1^{2n})$ for the unipotent integration in (3.2).

4. THE UNIPOTENT ORBIT OF THE THETA REPRESENTATION

In this section we discuss the Fourier coefficients of the automorphic theta representation $\Theta_{2l}^{(r)}$. These coefficients, obtained by integrating functions in the representation space against certain characters of unipotent groups, are indexed by unipotent orbits, which may be described by means of certain partitions of $2l$; see, for example, [G1], Section 2, for more information. The set of unipotent orbits of Sp_{2l} is a partially ordered set. Let $\mathcal{O}(\Theta_{2l}^{(r)})$ denote the set of unipotent orbits $\mathcal{O}$ which are maximal with respect to the property that $\Theta_{2l}^{(r)}$ has a non-zero Fourier coefficient with respect to $\mathcal{O}$. We expect that this set is a singleton. More precisely, we make the following conjecture.

Conjecture 1. *Let $r > 1$ be an odd integer, and write $2l = \alpha r + \beta$ where $0 \leq \beta < r$. Let $\mathcal{O}_c(\Theta_{2l}^{(r)})$ denote the unipotent orbit*

$$\mathcal{O}_c(\Theta_{2l}^{(r)}) = \begin{cases} (r^\alpha \beta) & \text{if } \alpha \text{ is even} \\ (r^{\alpha-1}(r-1)(\beta+1)) & \text{if } \alpha \text{ is odd.} \end{cases}$$

Then $\mathcal{O}(\Theta_{2l}^{(r)}) = \{\mathcal{O}_c(\Theta_{2l}^{(r)})\}$.

This conjecture is also presented in [F-G2]. Gao and Tsai [G-T] have recently generalized Conjecture 1 to other groups, and also given an archimedean analogue. The compatibility of their conjecture with Conjecture 1 is established in Section 4 of their paper.

Although we are not able to prove Conjecture 1 in general, we can prove it in some cases, as follows.

Theorem 1. (1) *For all positive integers l , if $\mathcal{O} \in \mathcal{O}(\Theta_{2l}^{(r)})$, then $\mathcal{O} \leq \mathcal{O}_c(\Theta_{2l}^{(r)})$.*
(2) *Assume that $l = 0, 1, 2, r-3, r-2, r-1$. Let n denote a non-negative integer, and assume that if $l = 0$, then $n \geq 1$. Then Conjecture 1 holds for the group $Sp_{2(l+nr)}^{(r)}(\mathbb{A})$. In particular, when $r = 3, 5$ the conjecture holds for all l and n .*

The first case for which we cannot prove Conjecture 1 is the group $Sp_6^{(7)}(\mathbb{A})$, where the conjecture states that $\Theta_6^{(7)}$ is generic. The unramified constituents of $\Theta_6^{(7)}$ are generic by [Gao2], but we do not know of a global result.

We start with the vanishing property of the representations $\Theta_{2l}^{(r)}$. The proof requires some local preparations. If U is any unipotent subgroup of a p -adic group or a metaplectic cover G , λ is a character of U , and (π, V) is a representation of G , then the Jacquet module $J_{U,\lambda}(\pi)$ is the quotient of V by the subspace spanned by vectors of the form $\pi(u)v - \lambda(u)v$ with $u \in U$, $v \in V$. When λ is nontrivial, sometimes we refer to this as the twisted Jacquet module. When λ is trivial we omit it from the notation, and denote this quotient as $J_U(\pi)$, the untwisted Jacquet module. The map $V \mapsto J_{U,\lambda}(\pi)$ is the Jacquet functor.

Let ν be a finite place of F , $\Theta_{GL_m, \nu}^{(r)}$ denote the local theta representation on the r -fold cover of $GL_m(F_\nu)$ treated by Kazhdan and Patterson [K-P], Section I, with $c = 0$ (or the corresponding character if $m = 1$), and $\Theta_{2l, \nu}^{(r)}$ be the local theta representation for $Sp_{2l}^{(r)}(F_\nu)$. Then we have the following proposition.

Proposition 1. *Let P be the unipotent radical of Sp_{2l} whose Levi factor is $GL_m \times Sp_{2l-2m}$, $1 \leq m \leq l$, and let U be its unipotent radical. Then*

$$J_U \left(\Theta_{2n,\nu}^{(r)} \right) = \Theta_{GL_m,\nu}^{(r)} \otimes \Theta_{2l-2m,\nu}^{(r)}.$$

Here the factor $\Theta_{2l-2m,\nu}^{(r)}$ is omitted if $m = l$.

The proof of this result follows exactly the proof of [B-F-G1], Theorem 2.3, which gives the same statement in a slightly different situation, and so is omitted here. There is also a global analogue; see for example [F-G2], Proposition 1.

The proof of vanishing requires two lemmas that are cases of more general results. The first, due to Gao [Gao2], gives information about when an unramified local constituent of $\Theta_{2l}^{(r)}$ fails to be generic. (Similar information for covers of the general linear group is due to Kazhdan and Patterson [K-P]; see Theorem I.3.5 there.)

Lemma 1 (Gao). *Suppose that $2l > r$. Then the local theta representation $\Theta_{2l,\nu}^{(r)}$ at an unramified place ν is not generic.*

This follows directly from [Gao2] by combining (the more general) Theorem 1.1 with Proposition 5.1 there.

The second is the following result, which is a special case of Leslie [L], Proposition 6.5.

Lemma 2 (Leslie). *Let π be a non-zero smooth admissible genuine representation of $Sp_{2l}^{(2r)}(F_\nu)$ that is unramified. Then π is not supercuspidal.*

We now give our result on the vanishing of Fourier coefficients for $\Theta_{2l}^{(r)}$.

Proposition 2. *Suppose that $\mathcal{O}$ is a unipotent orbit which is greater than $\mathcal{O}_c(\Theta_{2l}^{(r)})$ or that is not related to $\mathcal{O}_c(\Theta_{2l}^{(r)})$. Then the representation $\Theta_{2l}^{(r)}$ has no non-zero Fourier coefficient corresponding to $\mathcal{O}$.*

Note that Proposition 2 is equivalent to the first part of Theorem 1.

Proof. Recall that we write $2l = \alpha r + \beta$ with $0 \leq \beta < r$. If $\alpha = 0$ then $\mathcal{O}_c(\Theta_{2l}^{(r)}) = (\beta)$, so $\mathcal{O} \leq \mathcal{O}_c(\Theta_{2l}^{(r)})$ for all orbits $\mathcal{O}$, and there is nothing to prove. So suppose that $\alpha > 0$. Let $\mathcal{O}$ be a unipotent orbit satisfying the conditions of Proposition 2, let $U_{\mathcal{O}}$ be the corresponding upper triangular unipotent subgroup of Sp_{2l} , and let $\psi_{\mathcal{O}}$ denote any character of $U_{\mathcal{O}}$ such that the Fourier coefficient

$$(4.1) \quad \int_{[U_{\mathcal{O}}]} \phi(ug) \psi_{\mathcal{O}}(u) du$$

corresponds to the unipotent orbit $\mathcal{O}$ (see for example [B-F-G1], Section 4). (This is a slight abuse of notation as $\psi_{\mathcal{O}}$ is not uniquely determined by $\mathcal{O}$.) We need to prove that integral (4.1) is zero for all vectors ϕ in the space of $\Theta_{2l}^{(r)}$ and all $g \in Sp_{2l}(\mathbb{A})$.

First we show that it is enough to prove the vanishing of the coefficients for $\mathcal{O}_k := ((2k)1^{2l-2k})$ with $2k > r$. Indeed, by [G-R-S3], Lemma 2.6, this implies the vanishing for all orbits of the form $\mathcal{O} = ((2k)p_1^{e_1} \dots p_d^{e_d})$ with all $p_i \leq 2k$, $e_i \geq 0$. This in turn implies the vanishing for orbits of the form $((2k+1)^2 p_1^{e_1} \dots p_d^{e_d})$ with all $p_i \leq 2k+1$, $e_i \geq 0$ by [G-R-S3], Lemma 2.4. These two Lemmas are formulated for the symplectic groups but their proofs

go over to their metaplectic covers without change. Also, the arguments can be made local without difficulty, i.e. the corresponding $\psi_{\mathcal{O}}$ -twisted Jacquet module at a good unramified place is zero.

We are reduced to showing the vanishing for $\mathcal{O}_k = ((2k)1^{2l-2k})$ with $2k > r$. It is enough to prove the corresponding local statement. That is, we will show that at a good unramified finite place ν (i.e. $\text{ord}_{\nu}(2r) = 0$), the $\psi_{\mathcal{O}_k}$ -twisted Jacquet module $J_{V_2(\mathcal{O}_k), \psi_{\mathcal{O}_k}}(\pi_{2l})$ is zero, where π_{2l} is the local constituent of $\Theta_{2l}^{(r)}$ at ν . (We use π_{2l} in place of $\Theta_{2l, \nu}^{(r)}$ to simplify the notation.) In particular π_{2l} is a genuine representation of the covering group $Sp_{2l}^{(r)}(F_{\nu})$ over the nonarchimedean local field F_{ν} . To establish the vanishing, we follow Leslie [L], Section 8, who investigated a similar problem for the 4-fold cover. Let m be the integer such that the $\psi_{\mathcal{O}}$ -twisted Jacquet module of π_{2l} for $\mathcal{O} = ((2p)1^{2l-2p})$ is zero for $p > m$ but such that the module for $((2m)1^{2l-2m})$ is nonzero. Observe that $m < l$. Indeed, if $m = l$ this would imply that π_{2l} is generic. Since $2l > r$, this contradicts Lemma 1.

We also suppose by induction on l that the twisted Jacquet modules for π_{2j} , $1 \leq j < l$, vanish on the orbits $((2q)1^{2j-2q})$ when $2q > r$. Note that the base of the induction, that is the case $l = 2$, is clear. We remark that even the case $l = 2$ requires $2k > r$. Indeed, if $r = 3$ then the theta representation for $Sp_4^{(3)}$ has a nonzero Fourier coefficient corresponding to (21^2) , but not corresponding to (4) (see [F-G3], Lemma 2 and Proposition 3).

The idea of the proof is to establish that the twisted Jacquet modules $J_{V_2(\mathcal{O}_k), \psi_{\mathcal{O}_k}}(\pi_{2l})$ vanish by studying their descent properties. To prove that these modules vanish, it is sufficient to show that the Fourier Jacobi modules $FJ_{m, \alpha}(\pi_{2l})$ defined in [L], Section 6 (here $\alpha \in F_{\nu}^{\times}$), all vanish ([L], Corollary 6.4; as noted there this is the local version of [G-R-S3], Lemma 1.1). To do so, we will show that each Fourier Jacobi module $FJ_{m, \alpha}(\pi_{2l})$ is a supercuspidal representation of $Sp_{2l-2m}^{(2r)}(F_{\nu})$. Since $l > m$, if it were nonzero this would contradict Lemma 2. Note that in taking the Fourier Jacobi module the cover is doubled since r is odd and the Fourier Jacobi module uses a twist by the Weil representation which lives on the double cover, so $FJ_{m, \alpha}(\pi_{2l})$ is a genuine unramified representation of the $2r$ -fold cover $Sp_{2l-2m}^{(2r)}(F_{\nu})$. Aside from this the proof that the vanishing of all $FJ_{m, \alpha}(\pi_{2l})$ implies the vanishing of $J_{V_2(\mathcal{O}_k), \psi_{\mathcal{O}_k}}(\pi_{2l})$ is the same as that in [L]. (Fourier Jacobi modules and descents are also discussed in [G-R-S4], Section 3.8 and following.)

To prove that the representation $FJ_{m, \alpha}(\pi_{2l})$ is supercuspidal we must check that its untwisted Jacquet functors are all zero. For $1 \leq a \leq l - m$, let J_a be the (untwisted) Jacquet functor on admissible representations of $Sp_{2l-2m}^{(2r)}(F_{\nu})$ corresponding to the unipotent radical of the parabolic subgroup of Sp_{2l-2m} with Levi component $GL_a \times Sp_{2l-2m-2a}$. Then to prove supercuspidality, we must establish the vanishing of $J_a(FJ_{m, \alpha}(\pi_{2l}))$, $1 \leq a \leq l - m$. After conjugating by a suitable Weyl element, this Jacquet module is isomorphic to the ψ_a -twisted Jacquet module of π_{2l} with respect to the subgroup

$$U_a(F_{\nu}) := \left\{ \begin{pmatrix} I_a & X & Y \\ & v & X^* \\ & & I_a \end{pmatrix} \right\}$$

where $X \in \text{Mat}_{a \times (2l-2a)}(F_{\nu})$ has zero entries in the first column and v is upper triangular unipotent with center block $I_{2(l-a)}$, and where the character ψ_a of $U_a(F_{\nu})$ is given by

$$\psi_a(u) = \psi(u_{a+1, a+2} + \cdots + u_{a+m, a+m+1} + \alpha u_{a+m, 2l-a-m+1}).$$

Let $H \subset Sp_{2l}$ be the subgroup of unipotent matrices which are zero above the main diagonal except on column $a + 1$ and row $2l - a$. Using the Geometrical Lemma of [B-Z], p. 448, if the module $J_{U_a, \psi_a}(\pi_{2l})$ is nonzero, then $J_{H, \psi_H}(J_{U_a, \psi_a}(\pi_{2l}))$ is nonzero for some character ψ_H of H . There are two orbits under the action of GL_a . Using Proposition 1, the module with $\psi_H = 1$ factors through a coefficient of π_{2l-2a} with respect to the orbit $((2m)1^{2l-2m-2a})$, and this vanishes by induction (note that since $2l > r$, also $2m > r$). The non-trivial orbit contributes a similar Jacquet module with a replaced by $a - 1$. Under the action of GL_{a-1} there are again two orbits. As above the contribution from the trivial orbit is zero by the induction hypothesis. Thus, we are again left with the non-trivial orbit. Repeating, we arrive at the nonvanishing of a twisted Jacquet module of π_{2l} with respect to the orbit $((2m + 2a)1^{2l-2m-2a})$. However, m was chosen to be maximal such that the module for $((2m)1^{2l-2m})$ is nonzero, and $a \geq 1$. We conclude that $J_{U_a, \psi_a}(\pi_{2l}) = 0$, and hence that all the Jacquet modules $J_a(FJ_{m, \alpha}(\pi_{2l}))$ indeed vanish. Thus the representation $FJ_{m, \alpha}(\pi_{2l})$ is supercuspidal, as claimed. This completes the proof of the vanishing. $\square$

The local information developed above will also be used in treating the unramified correspondence in Section 6. We record the local vanishing in Proposition 7 below (there we use $\Theta_{2l}^{(r)}$ for the representation denoted $\Theta_{2l, \nu}^{(r)}$ here).

Our next proposition requires a result of Jiang and Liu ([J-L], Proposition 3.3) about the Fourier coefficients of automorphic representations on symplectic groups and composite partitions which we recall here for completeness. The notation is as in [J-L]. In fact the result of Jiang and Liu is bit sharper since they give information about the characters supporting the Fourier coefficients but we will not need this here. Their result is not stated for covering groups but the proof is identically the same.

Proposition 3 (Jiang and Liu). *Let π be an irreducible automorphic representation of $Sp_{2n}(\mathbb{A})$ or a metaplectic cover that is realized in the space of automorphic forms, and $\underline{p} = [(2k+1)^2 p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}]$ be a symplectic partition of $2n$ with $2k+1 \geq p_1 \geq p_2 \geq \cdots \geq p_r$; $e_i = 1$ if p_i is even; and $e_i = 2$ if p_i is odd. Then π has a non-vanishing Fourier coefficient attached to $\underline{p}$ if and only if it has a non-vanishing Fourier coefficient attached to the composite partition $[(2k+1)^2 1^{2n-4k-2}] \circ [p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}]$.*

Next we study the nonvanishing of the Fourier coefficients of theta functions. We start with a proposition.

Proposition 4. *Suppose Conjecture 1 holds for a given r and l . Then it holds for $r+l$ and l as well.*

Proof. We know from Proposition 2 that if $\mathcal{O} \in \mathcal{O}(\Theta_{2(l+r)}^{(r)})$ then $\mathcal{O} \leq \mathcal{O}_c(\Theta_{2(l+r)}^{(r)})$. Thus, we only need to prove that the representation $\Theta_{2(l+r)}^{(r)}$ has a non-zero Fourier coefficient corresponding to the unipotent orbit $\mathcal{O}_c(\Theta_{2(l+r)}^{(r)})$.

Write $2l = \alpha r + \beta$ with $0 \leq \beta < r$. Then $2(l+r) = (\alpha+2)r + \beta$. By assumption, there is a unipotent subgroup $U(\mathcal{O}_c(\Theta_{2l}^{(r)}))$ of Sp_{2l} , and a character ψ_U of the quotient $[U(\mathcal{O}_c(\Theta_{2l}^{(r)}))]$ such that the Fourier coefficient

$$(4.2) \quad \int_{[U(\mathcal{O}_c(\Theta_{2l}^{(r)}))]} \varphi_{2l}^{(r)}(u) \psi_U(u) du$$

corresponds to the unipotent orbit $\mathcal{O}_c(\Theta_{2l}^{(r)})$ and is not zero for some $\varphi_{2l}^{(r)}$ in the space of $\Theta_{2l}^{(r)}$. It follows from [F-G2], Proposition 2.1, or as in Offen and Sayag [O-S], p. 10, that the constant terms of theta functions with respect to the unipotent radical of a Levi in fact realize products of theta functions on the Levi. (In [O-S], this is formulated in terms of the surjectivity of an intertwining operator.) This is the global analogue of Proposition 1 above. We conclude that there is a function $\varphi_{2(l+r)}^{(r)}$ in the space of $\Theta_{2(l+r)}^{(r)}$ such that the integral

$$(4.3) \quad \int_{[U(\mathcal{O}_c(\Theta_{2l}^{(r)}))]} \int_{[V_2]} \int_{[V_1]} \varphi_{2(l+r)}^{(r)} \left(v_1 \begin{pmatrix} v_2 & & \\ & u & \\ & & v_2^* \end{pmatrix} \right) \psi_{V_2}(v_2) \psi_U(u) dv_1 dv_2 du$$

is not zero. Here V_1 is the unipotent radical of the maximal parabolic subgroup of $Sp_{2(l+r)}$ whose Levi part is $GL_r \times Sp_{2l}$, V_2 is the maximal unipotent subgroup of GL_r consisting of upper unipotent matrices, and the character ψ_{V_2} is the Whittaker character defined on the group V_2 .

To relate the Fourier coefficient in equation (4.3), to the Fourier coefficient corresponding to the unipotent orbit $\mathcal{O}_c(\Theta_{2(l+r)}^{(r)})$, we first use the result of Jiang and Liu recalled in Proposition 3. We deduce from this that the Fourier coefficient corresponding to the unipotent orbit $\mathcal{O}_c(\Theta_{2(l+r)}^{(r)})$ is not zero for some choice of data, if and only if the Fourier coefficient corresponding to the orbit $(r^2 1^{2l}) \circ \mathcal{O}_c(\Theta_{2l}^{(r)})$ is not zero for some choice of data. The Fourier coefficient corresponding to the last orbit can be written as follows

$$(4.4) \quad \int_{[U_{2,r_1,l+1}]} \int_{[U(\mathcal{O}_c(\Theta_{2l}^{(r)}))]} \varphi_{2(l+r)}^{(r)} \left(v \begin{pmatrix} I_{2r} & & \\ & u & \\ & & I_{2r} \end{pmatrix} \right) \psi'_{U_{2,r_1,l+1}}(v) \psi_U(u) du dv.$$

We recall that the group $U_{2,r_1,l+1}$ was defined right before equation (2.1). Also, the character ψ_U was defined in equation (4.2). The character $\psi'_{U_{2,r_1,l+1}}$ is defined as follows. For $v \in U_{2,r_1,l+1}$ consider its factorization given by equation (2.3). Then we define $\psi'_{U_{2,r_1,l+1}}(v) = \psi_{U_{2,r_1,l+1}}(v) \psi'_V(v)$ where $\psi_{U_{2,r_1,l+1}}$ is defined in equation (3.1) and $\psi'_V(v)$ is defined as follows. Let $Y = (y_{i,j}) \in \text{Mat}_{2 \times 2(l+1)}$. Define $\psi'(Y) = \psi(y_{1,1} + y_{2,2(l+1)})$. Then, for v as in equation (2.3) we define $\psi'_V(v) = \psi'(Y)$.

Using Proposition 2 we claim that integral (4.3) is not zero for some choice of data if and only if integral (4.4) is not zero for some choice of data. Since integral (4.3) is not zero for some choice of data, this claim will imply the Proposition.

The proof of the claim is standard, and follows in a similar way as the proof of Proposition 3.3 in [J-L]. We give some details. Let w denote the monomial matrix in the Weyl group of $Sp_{2(n+l)}$ defined as

$$w = \begin{pmatrix} \epsilon_1 & & \epsilon_2 \\ & I_{2l} & \\ \epsilon_3 & & \epsilon_4 \end{pmatrix} \quad \epsilon_i \in \text{Mat}_r$$

Here $\epsilon_1(i, 2i-1) = 1$ for all $1 \leq i \leq (r+1)/2$, and $\epsilon_2(i + (r+3)/2, 2i+2) = 1$ for all $0 \leq i \leq (r-3)/2$. This determines w uniquely. We have $\varphi_{2(l+r)}^{(r)}(h) = \varphi_{2(l+r)}^{(r)}(wh)$. Hence, after conjugating w to the right, integral (4.3) is not zero for some choice of data if and only

if the integral

$$\int_{[U(\mathcal{O}_c(\Theta_{2l}^{(r)}))]} \int_{[V_2]} \varphi_{2(l+r)}^{(r)} \left(\begin{pmatrix} I_r & A & B \\ & I_{2l} & A^* \\ & & I_r \end{pmatrix} \begin{pmatrix} v_2 & & \\ & u & \\ & & v_2^* \end{pmatrix} \begin{pmatrix} I_r & & \\ C & I_{2l} & \\ D & C^* & I_r \end{pmatrix} \right) \psi_{V_2}(v_2) \psi_U(u) du dv$$

is not zero for some choice of data. Here, the matrices A , B , C and D are suitable matrices which are obtained in a similar way as in the proof of Proposition 3.3 in [J-L]. We omit the details. Performing a root exchange as in [G-R-S4], Section 7.1, and using Proposition 2, we deduce that the above integral is not zero for some choice of data if and only if integral (4.4) is not zero for some choice of data. $\square$

It follows from Proposition 4 that given an odd number r , to prove Conjecture 1 for all symplectic groups $Sp_{2l}^{(r)}$ it is enough to prove the result for the symplectic groups $Sp_{2l}^{(r)}$ with $1 \leq l \leq r-1$.

Note that when $2 \leq 2l < r$, Conjecture 1 states that $\mathcal{O}(\Theta_{2l}^{(r)}) = \{(2l)\}$, that is, that the theta representation is generic. When $r < 2l < 2r-1$, the conjecture is that $\mathcal{O}(\Theta_{2l}^{(r)}) = \{((r-1)(2l-r+1))\}$. The next Proposition gives a lower bound for some element of the set $\mathcal{O}(\Theta_{2l}^{(r)})$ (or for the associated orbit if this set is, as expected, a singleton).

Proposition 5. *Assume that $1 \leq l \leq r$. Then the representation $\Theta_{2l}^{(r)}$ has a non-zero Fourier coefficient corresponding to the unipotent orbit (l^2) .*

Proof. Let V_l denote the unipotent radical of the parabolic subgroup of Sp_{2l} whose Levi part is isomorphic to $GL_2^{l/2}$ if l is even and to $GL_2^{(l+1)/2}$ if l is odd. Let ψ_{V_l} denote the character of $[V_l]$ given by $\psi_{V_l}(v) = \psi(\sum_{i=1}^{l-1} v_{i,i+2})$. To prove the proposition we will assume that the integral

$$(4.5) \quad \int_{[V_l]} \varphi^{(r)}(vg) \psi_{V_l}(v) dv$$

is zero for all functions $\varphi^{(r)}$ in $\Theta_{2l}^{(r)}$ and $g \in Sp_{2l}^{(r)}(\mathbb{A})$ and derive a contradiction. We may take $g = e$. Let $w \in Sp_{2l}(F)$ be the monomial matrix with non-zero entries ± 1 and such that $w_{i,2i-1} = 1$ for all $1 \leq i \leq l$. Since $w \in Sp_{2l}(F)$, we have $\varphi^{(r)}(v) = \varphi^{(r)}(wv)$. Conjugating w across v , we deduce that the integral

$$(4.6) \quad \int \varphi^{(r)} \left(\begin{pmatrix} I_l & A \\ & I_l \end{pmatrix} \begin{pmatrix} B & \\ & B^* \end{pmatrix} \begin{pmatrix} I_l & \\ C & I_l \end{pmatrix} \right) \psi_0(B) dA dB dC$$

is zero for all choices of data. Here A and C are integrated over $[\text{Mat}_{l,1}^0]$ where $\text{Mat}_{l,1}^0$ is the subgroup of Mat_l^0 consisting of all matrices $X = (X_{i,j}) \in \text{Mat}_l^0$ such that $X_{i,j} = 0$ for all $i \geq j$ (the group Mat_l^0 was defined at the beginning of Section 2). The variable B is integrated over $[L_l]$ where L_l is the unipotent subgroup of GL_l consisting of all upper triangular matrices, and ψ_0 is the Whittaker character defined on L_l .

Integral (4.6) is a special case of the situation dealt with in [J-L] Propositions 3.2 and 3.3. Performing root exchange as in [J-L] (see also [G-R-S4], Section 7.1), and using the fact that if an automorphic function is zero then all its Fourier coefficients are zero, we deduce that

the integral

$$\int \varphi^{(r)} \left(\begin{pmatrix} I_l & A \\ & I_l \end{pmatrix} \begin{pmatrix} B & \\ & B^* \end{pmatrix} \right) \psi_0(B) dA dB$$

is zero for all choices of data. Here A is integrated over the quotient $[\text{Mat}_l^0]$, and B is integrated as in (4.6). Applying [F-G2], Proposition 2.1 (or, again, as in [O-S] p. 10), we deduce that the Whittaker coefficient of the theta representation of the group $GL_l^{(r)}(\mathbb{A})$ is zero for all choices of data. However, since $1 \leq l \leq r$ it follows from [K-P] that this theta representation on $GL_l^{(r)}(\mathbb{A})$ is generic. Thus we have derived a contradiction. $\square$

Denote by $e_{i,j}$ the square matrix whose (i, j) entry is one and with zeros elsewhere. Given $l \geq 1$ and $1 \leq i, j \leq 2l$, let $e'_{i,j} = e_{i,j} \pm e_{2l-j+1, 2l-i+1}$, with the sign determined so that the matrix $I_{2l} + e'_{i,j}$ is in Sp_{2l} . Also, let

$$E_{i, 2l+1-i} = \{I_{2l} + te_{i, 2l+1-i}\}, \quad E'_{i,j} = \{I_{2l} + te'_{i,j}\} \quad (i + j \neq 2l + 1)$$

be the associated one parameter unipotent subgroups, corresponding to the long (resp. short) roots of Sp_{2l} .

Proposition 6. *Suppose that $1 \leq l \leq r - 1$.*

- (1) *If l is odd, then the set $\mathcal{O}(\Theta_{2l}^{(r)})$ contains an orbit which is greater than or equal to the unipotent orbit $((l+1)(l-1))$.*
- (2) *Suppose l is even, $l \neq r - 1$ and that $\mathcal{O} \geq (l^2)$ for all $\mathcal{O} \in \mathcal{O}(\Theta_{2l}^{(r)})$. Then $\mathcal{O}(\Theta_{2l}^{(r)})$ is a singleton and $\mathcal{O}(\Theta_{2l}^{(r)}) \geq ((l+2)(l-2))$.*

Proof. Consider first the case that l is odd. By Proposition 5, the Fourier coefficient (4.5) is nonzero. However, the stabilizer of the unipotent orbit (l^2) in Sp_{2l} is SL_2 , which embeds in the parabolic subgroup with Levi isomorphic to $GL_2^{(l+1)/2}$ by the diagonal embedding. Note that since $1 \leq l \leq r - 1$ and l is odd, in fact $1 \leq l \leq r - 2$. Thus the cover $Sp_{2l}^{(r)}(\mathbb{A})$ restricts via this embedding to a group isomorphic to $SL_2^{(r)}(\mathbb{A})$. The integral (4.5) then gives an automorphic function of $g \in SL_2^{(r)}(\mathbb{A})$ which is genuine. Therefore, it can not be constant, and so this function has a nontrivial Whittaker coefficient.

For $y \in \mathbb{A}$, let $x(y) \in Sp_{2l}(\mathbb{A})$ be given by $x(y) = I_{2l} + \sum_{i=1}^{(l-1)/2} ye'_{2i-1, 2i} + ye_{l, l+1}$. We conclude that there a choice of data such that the integral

$$(4.7) \quad \int_{[V_l] [\mathbb{G}_a]} \int \varphi^{(r)}(vx(y)g) \psi_{V_l}(v) \psi(\alpha y) dy dv$$

is not zero. Here $\alpha \in F^*$. Then is not hard to check that, after root exchange, the integral (4.7) has as inner integration a Fourier coefficient corresponding to the unipotent orbit $((l+1)(l-1))$. We refer to [F-G1], following (9) there, for a similar computation. Hence the first part is proved.

Next we consider the case when l is even. The argument is different since we do not have unipotent elements inside the stabilizer of this unipotent orbit. However, the stabilizer contains the subgroup generated by the matrices $t(a) = \text{diag}(a, a^{-1}, \dots, a, a^{-1})$ and $w_0 = \text{diag}(J_2, \dots, J_2)$, with J_2 defined in Section 2. Recall that the elements $\tilde{t}(a) = (t(a), \eta^{-1}(t(a))) \in Sp_{2l}^{(r)}(\mathbb{A})$ were introduced at the end of Section 2.

Let $a \in \mathbb{A}^*$ be an r -th power, and define

$$(4.8) \quad \Lambda(a) = \int_{[V_l]} \varphi^{(r)}(v\tilde{t}(a)w_0) \psi_{V_l}(v) dv.$$

Denote also by $\Lambda_0(e)$ integral (4.5) with $g = e$. Since w_0 stabilizes the character ψ_{V_l} we have $\Lambda(e) = \Lambda_0(e)$.

Performing Fourier expansions and root exchanges similar to those in the proof of Proposition 5, we obtain that $\Lambda(a)$ is equal to

$$(4.9) \quad \int \varphi^{(r)} \left(\begin{pmatrix} I_l & A \\ & I_l \end{pmatrix} \begin{pmatrix} B & \\ & B^* \end{pmatrix} \begin{pmatrix} I_l & \\ C & I_l \end{pmatrix} \tilde{h}(a)ww_0 \right) \psi_0(B) dA dB dC.$$

Here, A is integrated over $[\text{Mat}_l^0]$, B is integrated as in (4.6), and C is integrated over $\text{Mat}_{l,1}^0(\mathbb{A})$ (defined after (4.6)). The Weyl element w was defined following (4.5), and $\tilde{h}(a) = w\tilde{t}(a)w^{-1} = (h(a), \eta(t(a))^{-1})$, where $h(a) = \text{diag}(aI_l, a^{-1}I_l)$. We want to emphasize the difference between the computation performed here, and the one performed in Proposition 5. In that Proposition we needed to show that a certain integral vanished for all choices of data. In this Proposition we need a precise identity. This is why we made the assumption that $\mathcal{O} \geq (l^2)$ for all $\mathcal{O} \in \mathcal{O}(\Theta_{2l}^{(r)})$. Indeed, this assumption implies that the set $\mathcal{O}(\Theta_{2l}^{(r)})$ is a singleton, and that the representation $\Theta_{2l}^{(r)}$ has no non-zero Fourier coefficient corresponding to unipotent orbits which are not related to (l^2) , for example the orbit $((l+2)1^{l-2})$. To prove that $\Lambda(a)$ is equal to integral (4.9), we need to make use of this.

Recall that a is an r -th power. In (4.9) conjugate the matrix $\tilde{h}(a)$ to the left. First, we get the factor $|h|^{-l^2/2}$ from the change in variables in C . The torus $h(a)$ commutes with the matrix $\text{diag}(B, B^*)$. We are left with the computation of the constant term consisting of all matrices $\begin{pmatrix} I & A \\ & I \end{pmatrix}$ where $A \in \text{Mat}_l^0$. It follows from [F-G2] Proposition 2.1 that we obtain the factor of $\chi_{S_{p_{2l}^{(r)}}, \Theta}(\tilde{h}(a))$. By the formula in [F-G2], top of p. 93, this last term is equal to $|a|^{l(l+1)(r-1)/2r}$. Putting this together, we have proved that

$$(4.10) \quad \Lambda(a) = |a|^{\frac{l(l+1)(r-1)}{2r} - \frac{l^2}{2}} \int \varphi^{(r)} \left(\begin{pmatrix} I_l & A \\ & I_l \end{pmatrix} \begin{pmatrix} B & \\ & B^* \end{pmatrix} \begin{pmatrix} I_l & \\ C & I_l \end{pmatrix} ww_0 \right) \psi_0(B) dA dB dC.$$

Hence, $\Lambda(a) = |a|^{\frac{l(l+1)(r-1)}{2r} - \frac{l^2}{2}} \Lambda(e) = |a|^{\frac{l(l+1)(r-1)}{2r} - \frac{l^2}{2}} \Lambda_0(e)$.

Next we compute $\Lambda(a)$ in a different way. Going back to the definition in (4.8), we first conjugate w_0 to the left. We obtain

$$\Lambda(a) = \int_{[V_l]} \varphi^{(r)}(v\tilde{t}(a^{-1})) \psi_{V_l}(v) dv.$$

Repeating the same computations we performed in equations (4.9) and (4.10), we obtain $\Lambda(a) = |a|^{-\frac{l(l+1)(r-1)}{2r} + \frac{l^2}{2}} \Lambda_0(e)$. By Proposition 5, there is a choice of data such that $\Lambda_0(e)$ is not zero. Hence, we must have $\frac{l(l+1)(r-1)}{2r} = \frac{l^2}{2}$. This is equivalent to $l = r - 1$, so the second part follows. $\square$

The first part of Theorem 1 is in Proposition 2. We now give the proof of second part of the Theorem.

Proof. Consider first the case $l = 0$. Then applying Proposition 4, it is enough to prove the theorem for the group $Sp_{2r}^{(r)}(\mathbb{A})$. According to Conjecture 1, we need to prove that $\Theta_{2r}^{(r)}$ has a non-zero Fourier coefficient corresponding to the unipotent orbit (r^2) . This was proved in Proposition 5.

For all other cases stated in Theorem 1, part 2, using Proposition 4, it is enough to prove the result for the group $Sp_{2l}^{(r)}(\mathbb{A})$. The case $l = 1$ is clear. Next consider $l = 2$. If $r = 3$, it follows from Proposition 2 that $\Theta_4^{(3)}$ is not generic. From Proposition 4, $\Theta_4^{(3)}$ has a nonzero Fourier coefficient corresponding to the orbit (2^2) . If $r \geq 5$, it follows from Proposition 6 that $\mathcal{O}(\Theta_4^{(r)})$ is greater than (2^2) . Hence $\Theta_4^{(r)}$ is generic and we are done.

When $l = r - 2$, the Conjecture states that $\mathcal{O}(\Theta_{2(r-2)}^{(r)}) = \{((r-1)(r-3))\}$. This follows from Proposition 6. When $l = r - 1$, the Conjecture states that $\mathcal{O}(\Theta_{2(r-1)}^{(r)}) = \{((r-1)^2)\}$. This follows from Proposition 5. Thus the second part of the Theorem is proved. $\square$

5. CUSPIDALITY OF THE LIFT

In this Section we discuss the cuspidality of the representation $\sigma_{n,k}^{(r)}$. The main result is that the first non-zero occurrence of the generalized theta lift is automatically cuspidal. This generalizes Rallis's tower property, and is found in Theorem 2 below. The proof requires showing the vanishing of constant terms. We will establish this by considering various Fourier expansions, using the process of root exchange (see [G-R-S4, Section 7.1]), and making use of two key ingredients: the smallness of the representation $\Theta_{2n+k(r-1)}^{(r)}$ established in Proposition 2, and the cuspidality of the representation $\pi^{(\kappa r)}$.

We begin with several Lemmas. Let U_m be the unipotent subgroup $U_m = U_{1,m+1,n+kr_1-m-1}^0$, and let ψ_{U_m} denote the character $\psi_{U_m} = \psi_{U_{1,m+1,n+kr_1-m-1}}$ of this group (see (3.1)). The first Lemma is closely related to Lemma 2.2 in [G-R-S3].

Lemma 3. *Suppose that $r \leq m \leq n + kr_1 - 1$. Then the integral*

$$(5.1) \quad \int_{[U_m]} \theta_{2n+k(r-1)}^{(r)}(u) \psi_{U_m}(u) du$$

is zero for all choices of data.

Proof. Let $x(p) = I + pe_{m+1,2n+k(r-1)-m}$, with I the identity matrix of size $2n + k(r-1)$. Expand (5.1) along the abelian group $\{x(p)\}$. This is a sum of integrals against characters $\psi(\alpha p)$, $\alpha \in F$. The nontrivial terms contribute zero. Indeed, the Fourier coefficient we obtain from a nontrivial term corresponds to the unipotent orbit $((2m+2)1^{2+k(r-1)-2m-2})$. Since $m \geq r$, this last unipotent orbit is not comparable with the unipotent orbit $\mathcal{O}_c(\Theta_{2n+k(r-1)}^{(r)})$ defined in Conjecture 1 above. By Proposition 2, these Fourier coefficients vanish.

We are left with the contribution from $\alpha = 0$. That is, integral (5.1) is equal to

$$(5.2) \quad \int_{[U_m]} \int_{[\mathbb{G}_a]} \theta_{2n+k(r-1)}^{(r)}(ux(p)) \psi_{U_m}(u) dp du.$$

The quotient group $\{x(p)\}U_m \backslash U_{m+1}$ can be identified with a row vector of size $2(n-m-1) + k(r-1)$. Expand integral (5.2) along this quotient. There are two orbits under the action of the group $Sp_{2(n-m-1)+k(r-1)}(F)$. The trivial orbit contributes zero to integral (5.2).

Indeed, to prove this, we use Proposition 1 in [F-G2], which identifies the constant term of a theta function with lower rank theta functions, with a there equal to $m + 1$. This implies that as an inner integration we obtain the Whittaker coefficient of the theta function defined on an r -fold cover of $GL_{m+1}(\mathbb{A})$. Since $m + 1 > r$, it follows from [K-P] that this Whittaker coefficient is zero. We conclude that the integral (5.1) is a sum of integrals of the form

$$\int_{[U_{m+1}]} \theta_{2n+k(r-1)}^{(r)}(u) \psi_{U_{m+1}}(u) du.$$

Applying induction, these integrals are all zero. Hence integral (5.1) is zero for all choices of data. $\square$

In the next Lemma we establish another vanishing result. Assume again that $r \leq m \leq n + kr_1 - 1$. Let V_m denote the maximal unipotent subgroup of GL_{m+1} consisting of upper triangular unipotent matrices. Let Mat_{m+1}^{00} denote the subgroup of Mat_{m+1}^0 of all matrices $x \in \text{Mat}_{m+1}^0$ such that $x_{i,j} = 0$ for all $i \geq j$. Let U_m^0 denote the subgroup of U_m which consists of all matrices

$$t(v, x) = \begin{pmatrix} v & & x \\ & I_{2(n+kr_1-m-1)} & \\ & & v^* \end{pmatrix} \quad v \in V_m, \quad x \in \text{Mat}_{m+1}^{00}.$$

By restriction, the character ψ_{U_m} is a character of U_m^0 .

Lemma 4. *Assume that $r \leq m \leq n + kr_1 - 1$. Then the integral*

$$(5.3) \quad \int_{[U_m^0]} \theta_{2n+k(r-1)}^{(r)}(u) \psi_{U_m}(u) du$$

is zero for all choices of data.

Proof. We start by defining some unipotent subgroups of $Sp_{2n+k(r-1)}$. Let $a = 2(n + kr_1 - m - 1)$. Let $b(m) = c(m) = m/2$ if m is even, and $b(m) = (m + 1)/2$, $c(m) = (m - 1)/2$ if m is odd. For $1 \leq j \leq b(m)$ let Y_j be the upper triangular unipotent subgroup

$$Y_j = \{y_j(p_1, \dots, p_{a+j}) = I_{2(n+kr_1)} + \sum_{i=1}^{a+j} p_i e'_{j,m+i+1}\},$$

and for $1 \leq j \leq c(m)$ let Y'_j be the upper triangular unipotent subgroup

$$Y'_j = \{y'_j(p_1, \dots, p_{a+c(m)-j+1}, q) = I_{2(n+kr_1)} + \sum_{i=1}^{a+c(m)-j+1} p_i e'_{b(m)+j,m+i+1} + q e_{b(m)+j,a+c(m)+m-j+3}\}.$$

We also define corresponding lower unipotent groups. For $1 \leq j \leq c(m)$ let

$$Z_j = \{z_j(p_1, \dots, p_{a+j}) = I_{2(n+kr_1)} + \sum_{i=m+2}^{a+j+m+1} p_i e'_{j,j+1}\},$$

and for $1 \leq j \leq b(m)$ let

$$\begin{aligned} Z'_j &= \{z'_j(p_1, \dots, p_{a+b(m)-j+1}) = \\ &= I_{2(n+kr_1)} + \sum_{i=1}^{a+b(m)-j} p_i e'_{i,c(m)+j+1} + p_{a+b(m)-j+1} e_{a+b(m)-j+1,c(m)+j+1}\}. \end{aligned}$$

To prove the Lemma, we expand (5.3) along the quotient $[Y_1]$. Doing so, we see that the integral (5.3) is equal to

$$(5.4) \quad \sum_{\xi_i \in F} \int_{[Y_1]} \int_{[U_m^0]} \theta_{2n+k(r-1)}^{(r)}(y_1(p_1, \dots, p_{a+1})u) \psi_{U_m}(u) \psi\left(\sum \xi_i p_i\right) dy_1 du.$$

Since the function $\theta_{2n+k(r-1)}^{(r)}$ is automorphic, for $\xi_j \in F$ we have

$$\theta_{2n+k(r-1)}^{(r)}(z_1(-\xi_1, \dots, -\xi_{a+1})h) = \theta_{2n+k(r-1)}^{(r)}(h).$$

Using this in the integral (5.4) and then conjugating the matrix $z_1(-\xi_1, \dots, -\xi_{a+1})$ to the right, we obtain (after a change of variables in u),

$$(5.5) \quad \sum_{\xi_i \in F} \int_{[Y_1]} \int_{[U_m^0]} \theta_{2n+k(r-1)}^{(r)}(y_1(p_1, \dots, p_{a+1})u z_1(-\xi_1, \dots, -\xi_{a+1})) \psi_{U_m}(u) dy_1 du.$$

Thus if we prove that the inner integration in (5.5) is zero for all choices of data, this will imply that the integral (5.3) is zero for all choices of data.

Now we repeat this process with the inner integration of (5.5), this time using the groups Y_2 and Z_2 . The process is visibly inductive, and we repeat it $b(m)$ times. More precisely, for all $1 \leq j \leq b(m)$, we expand the corresponding integral along the group Y_j , and use the group Z_j as above unless $j = b(m)$, m odd, in which case we use the group Z'_1 . Let U_m^{00} denote the unipotent group generated by U_m^0 and by all Y_j for $1 \leq j \leq b(m)$. Then this shows that if the integral

$$(5.6) \quad \int_{[U_m^{00}]} \theta_{2n+k(r-1)}^{(r)}(u) \psi_{U_m}(u) du$$

is zero for all choices of data, then the integral (5.3) is zero for all choices of data. Here ψ_{U_m} is a character of U_m^{00} obtained from U_m^0 by extending it trivially.

Next, expand the integral (5.6) along the unipotent abelian group

$$\{x'(p) = I_{2(n+kr_1)} + p e_{b(m)+1,a+c(m)+m+2}\}.$$

We claim that the contribution to the expansion from the non-constant terms is zero. Indeed, for these terms we obtain a Fourier coefficient which corresponds to the unipotent orbit $((2b(m) + 2)1^{2(n+kr_1-b(m)-1)})$. Since $r \leq m$, it follows that this unipotent orbit is not related to the orbit $\mathcal{O}_c(\Theta_{2n+k(r-1)}^{(r)})$. Hence by Proposition 2 these coefficients vanish. Thus the Lemma will follow once we prove that the integral

$$(5.7) \quad \int_{[\mathbb{G}_a]} \int_{[U_m^{00}]} \theta_{2n+k(r-1)}^{(r)}(x'(p)u) \psi_{U_m}(u) dp du$$

is zero for all choices of data. To show this, observe that the group $\{x'(p)\}$ is the center of the group Y'_1 . Thus we can expand the integral (5.7) along the quotient $Y'_1/\{x'(p)\}$. Depending on the parity of m , we use Z'_1 or Z'_2 as above. Once again the argument is inductive, and after we carry it out with the groups Y'_j for $1 \leq j \leq c(m)$, we obtain the integral (5.1). By Lemma 3, since $r \leq m$, this integral is zero for all choices of data. Lemma 4 follows. $\square$

For the next Lemma, let α and l be two positive integers which satisfy $2 \leq 2\alpha \leq k$, and $(r-1)/2 < l \leq r-1$. We work with the unipotent group $U_{a,b,c}$ with $a = \alpha, b = l$ and $c = n + kr_1 - l\alpha$. Consider the Fourier coefficient

$$(5.8) \quad f(g) = \int_{[U_{\alpha,l,n+kr_1-l\alpha}^0]} \theta_{2n+k(r-1)}^{(r)}(ug) \psi_{U_{\alpha,l,n+kr_1-l\alpha}}(u) du, \quad g \in Sp_{2n+k(r-1)}^{(r)}(\mathbb{A}).$$

For short we shall write ψ_0 for $\psi_{U_{\alpha,l,n+kr_1-l\alpha}}$. Because of the factorization in equation (2.3), we may view (5.8) as a function of $u'_{\alpha,n+kr_1-l\alpha}(0, Z)$ where $Z \in \text{Mat}_{\alpha}^0(\mathbb{A})$. We have

Lemma 5. *For α and l in the range specified above, the Fourier coefficient (5.8) is invariant under the adelic points of the group $\{u'_{\alpha,n+kr_1-l\alpha}(0, Z)\}$. That is, for all $Z \in \text{Mat}_{\alpha}^0(\mathbb{A})$ we have $f(u'_{\alpha,n+kr_1-l\alpha}(0, Z)g) = f(g)$.*

Proof. Since the group of all matrices $u'_{\alpha,n+kr_1-l\alpha}(0, Z)$ is an abelian subgroup of $Sp_{2n+k(r-1)}$, we can expand the function $f(g)$ along it. We obtain

$$(5.9) \quad f(g) = \sum_{\gamma \in \text{Mat}_{\alpha}^0(F)} \int_{[\text{Mat}_{\alpha}^0]} f(u'_{\alpha,n+kr_1-l\alpha}(0, Z)g) \psi_{\gamma}(Z) dZ,$$

where $\gamma \rightarrow \psi_{\gamma}$ is an isomorphism of the abelian group $\text{Mat}_{\alpha}^0(F)$ with its dual. We need to prove that the nontrivial characters contribute zero to the expansion. The group $GL_{\alpha}(F)$, embedded in $Sp_{2n+k(r-1)}(F)$ by the map $\delta \mapsto \text{diag}(\delta, \delta, \dots, \delta, I, \delta^*, \dots, \delta^*)$, acts on the set $\{\gamma\}$. Here $\delta \in GL_{\alpha}(F)$ appears l times. It is enough to consider representatives under this action. If γ is not zero then there are two cases to consider. The first case is when $\psi_{\gamma}(x(p))$ is not zero for $x(p) = I_{2n+k(r-1)} + pe_{j_1, j_2}$ where $j_1 = (l-1)\alpha + 1$ and $j_2 = 2n+k(r-1) - (l-1)\alpha - 1$. The second case is when $\psi_{\gamma}(x(p))$ is not zero for $x(p) = I_{2n+k(r-1)} + pe'_{j_1, j_2}$ where $j_1 = (l-1)\alpha + 1$ and $j_2 = 2n+k(r-1) - l\alpha + 1$.

We start with the first case. Let w_0 be the Weyl group element $w_0 = \text{diag}(w, I_{2(n+kr_1-l\alpha)}, w^*)$ in $Sp_{2n+k(r-1)}(F)$. Here w in $GL_{l\alpha}(F)$ is the matrix whose only nonzero entries are 1 in positions $(j, (j-1)\alpha + 1)$ for $1 \leq j \leq l$ and positions (j_1, j_2) for $j_1 = l + (j-1)\alpha + a - j + 1$ and $j_2 = (j-1)\alpha + a + 1$ with $1 \leq a \leq \alpha - 1$ and $1 \leq j \leq l$.

Conjugating the integral in equation (5.9) by w_0 , we obtain the integral

$$(5.10) \quad \int_{[X]} \int_{[U'_l]} \int_{[\mathbb{G}_a]} \theta_{2n+k(r-1)}^{(r)}(uu'_{1,n+kr_1-l}(0, m)x) \psi_{U_l}(u) \psi(\beta m) dm du dx$$

as an inner integration to the integral (5.9). Here $\beta \in F^*$, and U'_l is the subgroup of $U_{1,l+1,n+kr_1-l-1}^0$ defined as follows. An element $u = (u_{i,j}) \in U'_l$ if $u_{i,j} = 0$ for all $1 \leq i \leq l-1$

and $l + 1 \leq j \leq l + i(\alpha - 1)$. As for the group X , it consists of all matrices of the form

$$(5.11) \quad x = \begin{pmatrix} I_l & & & & \\ y & I_{l(\alpha-1)} & & & \\ & & I & & \\ & & & I_{l(\alpha-1)} & \\ & & & y^* & I_l \end{pmatrix}$$

where $y \in \text{Mat}_{l(\alpha-1) \times l}$ satisfies the conditions $y_{i,j} = 0$ for all (i, j) such that $1 \leq j \leq l$ and $(j - 1)(\alpha - 1) + 1 \leq i \leq l(\alpha - 1)$. Thus, to prove that the contribution from this case to the sum in equation (5.9) is zero, it is enough to prove that the integral (5.10) is zero for all choices of data.

We claim that, by means of root exchange, the vanishing of the integral (5.10) follows from the vanishing of

$$(5.12) \quad \int_{[U_l]} \int_{[\mathbb{G}_a]} \theta_{2n+k(r-1)}^{(r)}(uu'_{1,n+kr_1-l}(0, m)) \psi_{U_l}(u) \psi(\beta m) dm du dx$$

for all choices of data. To prove this claim, for $1 \leq j \leq l - 1$ we consider the two families of unipotent subgroups. Let V_j denote the unipotent subgroup of U_l defined by

$$V_j = \{x_i(p_i) = I_{2n+k(r-1)} + p_i e'_{j,l+i}, \quad : 1 \leq i \leq j(\alpha - 1)\},$$

and let X_j denote the unipotent subgroup of X defined by matrices of the form (5.11) with all entries of y equal to zero outside the $(j + 1)$ -st column. We proceed inductively, starting with $j = 1$. We expand the integral (5.5) along the group $[V_j]$, and then we perform root exchange with the group X_j . After the root exchange corresponding to $j = l - 1$, we obtain the integral (5.12) as an inner integration to integral (5.10).

However, the integral (5.12) corresponds to the unipotent orbit $((2l+2)1^{2(n+kr_1-l-1)})$. Since $(r - 1)/2 < l$, this unipotent orbit is not comparable with the unipotent orbit $\mathcal{O}_c(\Theta_{2(n+kr_1)}^{(r)})$ in Conjecture 1. Hence Proposition 2 implies that (5.12) is zero for all choices of data. This completes the first case of representative in (5.9).

Next we consider the second case. For this case we use a different Weyl group element in $Sp_{2(n+kr_1)}(F)$, which we denote by w . To define w , we set $w_{i,(i-1)\alpha+1} = w_{l+i,2n+k(r-1)-(l-i+1)\alpha} = 1$ for $1 \leq i \leq l$. Then we extend it in an arbitrary way to a Weyl group element of $Sp_{2(n+kr_1)}(F)$. Conjugating the corresponding integral in the expansion (5.9), we obtain integral (5.3) with $m = 2l - 2$ as an inner integration. Applying Lemma 4, the result follows. $\square$

With this preparation, we now establish the tower property for this theta lift. Fix $k \geq 3$.

Theorem 2. *Suppose that the representation $\sigma_{n,m}^{(r)}$ is zero (i.e. every function in this space is identically zero) for all m , $1 \leq m < k$, $m \equiv k \pmod{2}$. Then $\sigma_{n,k}^{(r)}$ is a cuspidal representation.*

Proof. To prove the cuspidality of the lift, we need to show that the constant terms of the representation $\sigma_{n,k}^{(r)}$ along any unipotent radical of a standard maximal parabolic subgroup of SO_k are zero for all choices of data. These are the subgroups N_α , $1 \leq \alpha \leq [k/2]$, consisting

of all matrices in SO_k of the form

$$\begin{pmatrix} I_\alpha & * & * \\ & I_\beta & * \\ & & I_\alpha \end{pmatrix}, \quad \beta = k - 2\alpha.$$

Thus, with $f(h)$ given by (3.3), we need to prove that the integral

$$(5.13) \quad \int_{[N_\alpha]} f(n_\alpha h) dn_\alpha$$

is zero for all choices of data.

We start by unfolding the theta series $\theta_{2nk}^{(2),\psi}$, which is expressed as a sum over $\xi \in F^{nk}$. We choose the polarization $\xi = (\xi_1, \xi_2)$, where $\xi_1 \in F^{2n\alpha}$ and $\xi_2 \in F^{n\beta}$. Write $\xi_1 = (\xi_{1,1}, \xi_{1,2}, \dots, \xi_{1,\alpha})$ where $\xi_{1,i} \in F^{2n}$. The action of Sp_{2n} is given by multiplication on the right on each $\xi_{1,i}$. Now the projection $l(u)$ that appears in (3.3) depends only on the values of $l(u'_{k,n}(Y, Z))$ (for the notation, see (2.3)). Write

$$(5.14) \quad Y = \begin{pmatrix} y_\alpha \\ y_\beta \\ y'_\alpha \end{pmatrix}, \quad Z = \begin{pmatrix} z_\alpha & * & * \\ z_1 & z_\beta & * \\ z_2 & * & * \end{pmatrix} \in \text{Mat}_k^0.$$

Here $y_\alpha, y'_\alpha \in \text{Mat}_{\alpha \times 2n}$ and $y_\beta \in \text{Mat}_{\beta \times 2n}$. Also, $z_\alpha \in \text{Mat}_\alpha$, $z_\beta \in \text{Mat}_\beta^0$, $z_1 \in \text{Mat}_{\beta \times \alpha}$ and $z_2 \in \text{Mat}_\alpha^0$. Using the formulas for the Weil representation ω_ψ (see for example [G-R-S1, Section 1, part 3]), we obtain

$$(5.15) \quad \theta_{2nk}^{(2),\psi}(l(u'_{k,n}(Y, Z))\iota_1(n_\alpha, g)) = \sum_{\xi_1, \xi_2} \omega_\psi((\xi_1, 0)l(u'_{k,n}(Y, Z))\iota_1(n_\alpha, g))\phi(0, \xi_2)$$

Here ϕ is a Schwartz function of $\mathbb{A}^{nk}$.

From the definition of the homomorphism l , we have $l(u'_{k,n}(Y_{\xi_1}, 0)) = (\xi_1, 0)$ where $Y_{\xi_1} = \begin{pmatrix} 0 \\ \xi_1 \end{pmatrix}$. In the integral (5.13) the variables y'_α , defined in (5.14), are integrated over the quotient $\text{Mat}_{\alpha \times 2n}(F) \backslash \text{Mat}_{\alpha \times 2n}(\mathbb{A})$. Hence, after conjugating the element $u'_{k,n}(Y_{\xi_1}, 0)$ to the right we may combine summation with integration. It follows that to prove the vanishing of the integral (5.13) for all choices of data, it suffices to prove the vanishing of the integral

$$(5.16) \quad \int_{[Sp_{2n}]} \int \overline{\varphi^{(\kappa r)}(i(g))} \theta_{2n(k-2\alpha)}^{(2),\psi}(l_0(u'_{k,n}(Y, Z))\iota_1(1, g))$$

$$\theta_{2n+k(r-1)}^{(r)}(u'_{k,n}(Y, Z)u\iota_2(n_\alpha, g)) \psi_{U_{k,r_1,n}}(u) \psi_\alpha(Z) dY dZ du dn_\alpha dg$$

for all choices of data, where the notation is as follows.

In the coordinates of (5.14), l_0 is defined by $l_0(u'_{k,n}(Y, Z)) = (y_\beta, \text{tr}(z_\beta T_{k-2\alpha})) \in \mathcal{H}_{2n\beta+1}$ (where the rows of y_β are listed with the bottom row first, similarly to the definition of the map l in Section 2). The variable Y is integrated over $[Y_0]$, where Y_0 is the subgroup of $\text{Mat}_{k \times 2n}$ given by all matrices Y as in (5.14) with $y'_\alpha = 0$, and Z is integrated over $[\text{Mat}_{k \times k}^0]$. Also, u is integrated over $[U_{k,r_1,n}^0]$, and n_α is integrated over $[N_\alpha]$. Using the coordinates of equation (5.14), the character ψ_α is defined as $\psi_\alpha(Z) = \psi(\text{tr}(z_\alpha))$. Also, notice that (when $\beta > 0$) the theta series appearing in (5.16) is defined on the double cover of $Sp_{2n(k-2\alpha)}(\mathbb{A})$. This follows since after the above collapsing of summation and integration we are left with

the summation over $\xi_2 \in F^{2n(k-2\alpha)}$. The rightmost argument of the theta series is $\iota_1(1, g)$ since $\omega_\psi(\iota_1(n_\alpha, g))\phi(0, \xi_2) = \omega_\psi((\iota_1(1, g))\phi(0, \xi_2))$. If $k_0 = \beta = 0$ and $k = 2\alpha$ then the theta function $\theta_{2n(k-2\alpha)}^{(2), \psi}$ in (5.16) is omitted.

The next step is to define a certain Weyl group element $w \in Sp_{2n+k(r-1)}(F)$, which we will then use to conjugate the various groups. This Weyl element is of the form

$$(5.17) \quad w = \begin{pmatrix} w_1 & & w_2 \\ & I_{2n} & \\ w_3 & & w_4 \end{pmatrix}$$

where $w_i \in \text{Mat}_{kr_1}$. To specify it, it is enough to specify the matrices w_1 and w_2 . These matrices will be block matrices whose only nonzero entries are the identity matrices I_α and I_β . To describe the location of each such identity block, it is enough to specify the location in each w_j , $j = 1, 2$, of its first 1 on the diagonal. The matrix w_1 has an identity matrix I_α whose first 1 is at position $(\alpha(i-1) + 1, k(i-1) + 1)$ for $1 \leq i \leq r_1$. Since w has only one non-zero entry in each row, note that this implies that the first αr_1 rows of w_2 are all zeros. Next, the matrix w_2 has an identity matrix I_α whose first 1 is at position $(\alpha r_1 + \alpha(i-1) + 1, k(i-1) + 1)$ for $1 \leq i \leq r_1$. This then implies that the corresponding rows in the matrix w_1 are all zero. Finally, in w_1 there is a block of I_β whose first 1 is at position $(\alpha(r-1) + \beta(i-1) + 1, k(i-1) + \alpha + 1)$ for $1 \leq i \leq r_1$.

For example, when $r = 7$, we have

$$w_1 = \begin{pmatrix} I_\alpha & 0_\beta & 0_\alpha & 0_\alpha & 0_\beta & 0_\alpha & 0_\alpha & 0_\beta & 0_\alpha \\ 0_\alpha & & & I_\alpha & & & & & \\ 0_\alpha & & & & & & I_\alpha & & \\ 0_\alpha & & & & & & & & \\ 0_\alpha & & & & & & & & \\ 0_\alpha & & & & & & & & \\ 0_\beta & I_\beta & & & & & & & \\ 0_\beta & & & I_\beta & & & & & \\ 0_\beta & & & & & & & I_\beta & \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0_\alpha & 0_\beta & 0_\alpha & 0_\alpha & 0_\beta & 0_\alpha & 0_\alpha & 0_\beta & 0_\alpha \\ 0_\alpha & & & & & & & & \\ 0_\alpha & & & & & & & & \\ I_\alpha & & & & & & & & \\ 0_\alpha & & & I_\alpha & & & & & \\ 0_\alpha & & & & & & I_\alpha & & \\ 0_\beta & & & & & & & I_\alpha & \\ 0_\beta & & & & & & & & \\ 0_\beta & & & & & & & & \end{pmatrix}$$

where all the blank entries are zero.

Before conjugating by w , we perform a certain root exchange. To do that, let $L_{\alpha, \beta}$ denote the unipotent subgroup of GL_k consisting of all matrices of the form

$$(5.18) \quad l = \begin{pmatrix} I_\alpha & a & b \\ & I_\beta & c \\ & & I_\alpha \end{pmatrix}$$

and let L_0 denote any unipotent subgroup of $L_{\alpha, \beta}$ such $L_{\alpha, \beta} = L_0 N_\alpha$. For example, one may choose the group of all matrices l as above such that $c = 0$ and such that bJ_α is strictly upper triangular. Consider the direct sum $L_{\alpha, \beta} \oplus \dots \oplus L_{\alpha, \beta} \oplus L_0$ where $L_{\alpha, \beta}$ appears $r_1 - 1$ times. We embed this group inside $Sp_{2n+k(r-1)}$ as

$$(5.19) \quad \text{diag}(l_1, l_2, \dots, l_{r_1-1}, l_0, I_{2n}, l_0^*, l_{r_1-1}^*, \dots, l_1^*).$$

We will also need to consider the subgroup of Mat_k which consists of all matrices of the form

$$l^- = \begin{pmatrix} 0_\alpha & & \\ a_1 & 0_\beta & \\ b_1 & c_1 & 0_\alpha \end{pmatrix}$$

We denote this group by L^- .

Now we carry out root exchange with the embedded copies of the groups $L_{\alpha,\beta}$ and L^- . We begin with $L_{\alpha,\beta}$ is embedded in the first component of $L_{\alpha,\beta} \oplus \dots \oplus L_{\alpha,\beta} \oplus L_0$ and then inside $Sp_{2n+k(r-1)}$ as in (5.19). Since $L_{\alpha,\beta}$ is not abelian, this root exchange needs to be carried out in stages, as follows. First, we exchange the unipotent elements which are in the center of $L_{\alpha,\beta}$, i.e. the (abelian) group of all matrices l in (5.18) such that $a = c = 0$, with the group of all matrices $u_{k,r_1,n}^1(l^-)$ (see (2.1)) such that $l^- \in L^-$ with $a_1 = c_1 = 0$. After doing this, we proceed with the abelian group consisting of all matrices $l \in L_{\alpha,\beta}$ with $b = c = 0$, and then with the abelian group of all matrices with $a = b = 0$. Next, we exchange $L_{\alpha,\beta}$ embedded in the second component of $L_{\alpha,\beta} \oplus \dots \oplus L_{\alpha,\beta} \oplus L_0$ and then inside $Sp_{2n+k(r-1)}$ as in (5.19). For this group we use the copy of L^- embedded in $Sp_{2n+k(r-1)}$ as $l^- \mapsto u_{k,r_1,n}^2(l^-)$. We continue this process for all i with $1 \leq i \leq r_1 - 1$, exchanging the i -th copy of $L_{\alpha,\beta}$ inside $L_{\alpha,\beta} \oplus \dots \oplus L_{\alpha,\beta} \oplus L_0$ with a subgroup of $u_{k,r_1,n}^i(L^-)$. Then, we perform root exchange corresponding to $l^- \in L^-$, embedded in $U_{k,r_1,n}^0$ as all matrices $l^- \mapsto u_{k,r_1,n}^1(l^-)$. We exchange this group with the group of all matrices $u'_{k,n}(0, Z)$ where

$$(5.20) \quad Z = \begin{pmatrix} z_1 & \\ z_2 & z_1^* \end{pmatrix}.$$

After performing these root exchanges, we conjugate by the Weyl element w defined in (5.17). Thus, to prove that integral (5.16) is zero for all choice of data, we conclude that it is enough to prove that the integral

$$(5.21) \quad \int \overline{\varphi^{(\kappa r)}(i(g))} \theta_{2n(k-2\alpha)}^{(2),\psi}(l_0(u)\iota_1(1, g)) \psi_{U_{\beta,r_1,n}}(u) \psi_{V_{\alpha,r-1}}(v) \\ \theta_{2n+k(r-1)}^{(r)} \left(\begin{pmatrix} I_{\alpha(r-1)} & C & D \\ & I & C^* \\ & & I_{\alpha(r-1)} \end{pmatrix} \begin{pmatrix} v \\ u \\ v^* \end{pmatrix} \begin{pmatrix} I_{\alpha(r-1)} & & \\ A & I & \\ B & A^* & I_{\alpha(r-1)} \end{pmatrix} \iota_2(1, g) \right) d(\dots)$$

is zero for all choices of data. Here I denotes the identity matrix of size $2n + (k - 2\alpha)(r - 1)$. The description of the variables in this integral and the domain of integration is elaborate, and we give it now.

First, g is integrated as in the integral (5.16). The variable u is integrated over the quotient $[U_{\beta,r_1,n}]$, where $U_{\beta,r_1,n} \subseteq Sp_{2n+\beta(r-1)}$ is embedded inside $Sp_{2n+k(r-1)}$ by the map $u \rightarrow \text{diag}(I_{\alpha(r-1)}, u, I_{\alpha(r-1)})$. Also, $V_{\alpha,r-1}$ denotes the subgroup of $U_{\alpha,r-1,n+\beta r_1}$ generated by all matrices of the form $\prod_{i=1}^{r-2} u_{\alpha,r-1,n+\beta r_1}^i(X_i)$ with $X_i \in \text{Mat}_\alpha$; $V_{\alpha,r-1}$ is isomorphic to a unipotent subgroup of $GL_{\alpha(r-1)}$. The character $\psi_{V_{\alpha,r-1}}$ is the restriction of $\psi_{U_{\alpha,r-1,n+\beta r_1}}$ to $V_{\alpha,r-1}$. The variable v is integrated over $[V_{\alpha,r-1}]$.

Next we define the regions over which the variables A, B, C and D (each matrices of a certain size) are integrated. These are given by considering them as block matrices and imposing various conditions. We start with the variable D . Consider the subgroup $D_0 \subset$

$\text{Mat}_{\alpha(r-1)}^0$ consisting of block matrices with blocks of size α such that each (i, j) block with $j < i$ is the zero matrix 0_α . Thus, for example if $r - 1 = 4$, then D_0 consists of all matrices of the form

$$D = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ 0_\alpha & X_5 & X_6 & X_3^* \\ 0_\alpha & 0_\alpha & X_5^* & X_2^* \\ 0_\alpha & 0_\alpha & 0_\alpha & X_1^* \end{pmatrix} \quad X_1, X_2, X_3, X_5 \in \text{Mat}_\alpha; \quad X_4, X_6 \in \text{Mat}_\alpha^0.$$

With these notations D is integrated over $[D_0]$. Similarly, let $B_0 \subset \text{Mat}_{\alpha(r-1)}^0$ consist of block matrices with blocks of size α such that each (i, j) block with $j \leq i + 1$ is the zero matrix 0_α . For example if $r - 1 = 5$, then B_0 consists of all matrices of the form

$$B = \begin{pmatrix} 0_\alpha & 0_\alpha & X_1 & X_2 & X_3 \\ 0_\alpha & 0_\alpha & 0_\alpha & X_4 & X_2^* \\ 0_\alpha & 0_\alpha & 0_\alpha & 0_\alpha & X_1^* \\ 0_\alpha & 0_\alpha & 0_\alpha & 0_\alpha & 0_\alpha \\ 0_\alpha & 0_\alpha & 0_\alpha & 0_\alpha & 0_\alpha \end{pmatrix} \quad X_1, X_2 \in \text{Mat}_\alpha; \quad X_3, X_4 \in \text{Mat}_\alpha^0.$$

Then B is integrated over $[B_0]$.

As for C , define a subgroup $C_0 \subset \text{Mat}_{\alpha(r-1) \times (2n+2\beta r_1)}$ as follows. Write

$$C = \begin{pmatrix} C_1 & C_2 & C_3 \\ 0 & 0 & C_4 \end{pmatrix},$$

with $C_2 \in \text{Mat}_{\alpha r_1 \times 2n}$ and $C_1, C_3, C_4 \in \text{Mat}_{\alpha r_1 \times \beta r_1}$. Then C_0 is the subgroup of such matrices such that C_1 and C_4 may be written as block matrices, with blocks of size $\alpha \times \beta$, such that each (i, j) block with $j < i$ is $0_{\alpha \times \beta}$. For example, when $r_1 = 3$, then both C_1 and C_4 are matrices of the form

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ 0 & X_4 & X_5 \\ 0 & 0 & X_6 \end{pmatrix} \quad X_i \in \text{Mat}_{\alpha \times \beta},$$

where the zeroes indicate the zero matrices of the corresponding sizes. The variable C is integrated over $[C_0]$.

Finally, let A_0 denote the subgroup of $\text{Mat}_{(2n+\beta(r-1)) \times \alpha(r-1)}$ consisting of matrices of the form

$$A = \begin{pmatrix} 0 & A_1 & A_2 \\ 0 & 0 & A_3 \\ 0 & 0 & A_4 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} A_2 \\ A_3 \end{pmatrix} \in \text{Mat}_{(2n+\beta r_1) \times \alpha(r_1-1)},$$

where the first column has width 2α , the last row has height β , and $A_1, A_4 \in \text{Mat}_{\beta(r_1-1) \times \alpha(r_1-1)}$ satisfy the following conditions. First viewing A_1 as a block matrix with blocks of size $\alpha \times \beta$, all (i, j) blocks with $j < i$ are zero matrices. Second, viewing A_4 similarly, all (i, j) blocks with $j \leq i$ are the zero matrix. For example, for $r_1 - 1 = 4$ we have

$$(5.22) \quad A_1 = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ 0 & X_5 & X_6 & X_7 \\ 0 & 0 & X_8 & X_9 \\ 0 & 0 & 0 & X_{10} \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & Y_1 & Y_2 & Y_3 \\ 0 & 0 & Y_4 & Y_5 \\ 0 & 0 & 0 & Y_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad X_i, Y_j \in \text{Mat}_{\beta \times \alpha}.$$

Then A is integrated over $[A_0]$.

Our goal is to prove that the integral (5.21) is zero for all choices of data. To do that we start with a sequence of root exchanges, and a repeated application of Lemma 5. Recall that in terms of blocks of height α , the matrices C and D each have $r-1$ rows. The integral runs over the first row of blocks of each, while for the second row, we have variables of integration everywhere except for the blocks in position $(2, 1)$, a block which is $0_{\alpha \times \beta}$ for C and 0_α for D , and so on for the remaining rows.

We first perform the root exchange using $(1, 3)$ block in B and then using the $(1, 3)$ block of A . Note that there is a compatibility in the block sizes with the $(2, 1)$ blocks of the matrices D and C , resp. Indeed, the $(1, 3)$ block of B is of size $\alpha \times \alpha$ which is exactly the size of the block in the $(2, 1)$ position of D . Similarly, the $(1, 3)$ block of A is of size $\beta \times \alpha$ which is exactly what is needed for root exchange with the block matrix at the $(2, 1)$ position of the C , which is of size $\alpha \times \beta$. We continue this process with the third row of the block matrices C and D . For these two matrices the $(3, 1)$ and $(3, 2)$ blocks are each zero. We integrate over the $(1, 4)$ and $(2, 4)$ block matrices in A and B , which allows us to perform root exchange. Repeating this process with all rows up to and including the r_1 -th row, we obtain the integral (5.8) with $l = (r+1)/2$ as inner integration. Applying Lemma 5, we get the invariance of this integral along the adelic points of the unipotent subgroup $u'_{\alpha, n+kr_1-l\alpha}(0, Z)$ with $l = (r+1)/2$. Notice that this subgroup is realized as the block matrices of size α which are in position $((r+1)/2, (r-1)/2)$ in D . With this invariance, we can then proceed with root exchange of the blocks of C and D with the corresponding blocks of A and B .

We conclude from this root exchange that the integral (5.21) is zero for all choices of data if the integral

$$(5.23) \quad \int \overline{\varphi^{(\kappa r)}(i(g))} \theta_{2n(k-2\alpha)}^{(2), \psi}(l_0(u) \iota_1(1, g)) \psi_{U_{\beta, r_1, n}}(u) \psi_{U_{\alpha, r-1, n+\beta r_1}}(u_1) \\ \theta_{2n+k(r-1)}^{(r)}(u'_{\alpha, n+\beta r_1}(Y_{3, r_1}, Z) u_1 \begin{pmatrix} I_{\alpha(r-1)} & & \\ & u & \\ & & I_{\alpha(r-1)} \end{pmatrix} \iota_2(1, g)) du_1 du dY_{3, r_1} dZ dg$$

is zero for all choices of data. Here u and g are integrated as in (5.21), u_1 is integrated over the quotient $[U_{\alpha, r-1, n+\beta r_1}]$, and Z is integrated over $[\text{Mat}_\alpha^0]$. Also, Y_{3, r_1} is integrated over $[\text{Mat}_{\alpha \times \beta}]$, where the notation Y_{3, r_1} is explained as follows. Recall that $u'_{\alpha, n+\beta r_1}(Y, Z)$ was defined in (2.2). Here we have $Y \in \text{Mat}_{\alpha \times 2(n+\beta r_1)}$. Write $Y = (Y_1 \ Y_2 \ Y_3)$, where $Y_1, Y_3 \in \text{Mat}_{\alpha \times \beta r_1}$ and $Y_2 \in \text{Mat}_{\alpha \times 2n}$, and let

$$Y_3 = (Y_{3,1} \ Y_{3,2} \ \dots \ Y_{3,r_1}) \quad Y_{3,i} \in \text{Mat}_{\alpha \times \beta}; \ 1 \leq i \leq r_1$$

Then in (5.23) instead of writing $Y = (0_{\alpha \times 2(n+\beta(r_1-1))} \ Y_{3, r_1})$ we write Y_{3, r_1} for short.

Next we expand the integrand in (5.23) along the group $Y_3 = (0 \ 0 \ \dots \ 0 \ Y_{3, r_1-1} \ 0)$ where $Y_{3, r_1-1} \in [\text{Mat}_{\alpha \times \beta}]$. The nontrivial characters in this expansion correspond to matrices of size $\beta \times \alpha$ of fixed positive rank. We will prove that all nontrivial terms are zero. Consider the group $GL_\alpha(F) \times GL_\beta(F)$ embedded in $Sp_{2n+k(r-1)}(F)$ by

$$(h_1, h_2) \rightarrow \text{diag}(h_1, \dots, h_1, h_2, \dots, h_2, I_{2n}, h_2^*, \dots, h_2^*, h_1^*, \dots, h_1^*),$$

where $h_1 \in GL_\alpha(F)$ appears $r-1$ times and $h_2 \in GL_\beta(F)$ appears r_1 times. This group may be used to collect terms of this expansion in the usual way. As representatives with

respect to it, we choose the characters

$$\psi_A(Y_{3,r_1-1}) = \psi(\text{tr}(Y_{3,r_1-1}A)) \quad A = \begin{pmatrix} I_m & \\ & 0 \end{pmatrix} \in \text{Mat}_{\beta \times \alpha}.$$

The contribution to the integral (5.23) from a nontrivial orbit is

$$(5.24) \quad \int \overline{\varphi^{(\kappa r)}(i(g))} \theta_{2n(k-2\alpha)}^{(2),\psi}(l_0(u)\iota_1(1,g)) \psi_{U_{\beta,r_1,n}}(u) \psi_{U_{\alpha,r-1,n+\beta r_1}}(u_1) \psi_A(Y_{3,r_1-1}) \\ \theta_{2n+k(r-1)}^{(r)}(u'_{\alpha,n+\beta r_1}(Y_{3,r_1-1}, Y_{3,r_1}, Z)u_1 \begin{pmatrix} I_{\alpha(r-1)} & & \\ & u & \\ & & I_{\alpha(r-1)} \end{pmatrix} \iota_2(1,g)) du_1 du dY_{3,r_1-1} dZ dg.$$

To prove that this integral is zero, let w be a Weyl element of $Sp_{2n+k(r-1)}(F)$ which has entry 1 at positions $(i, \alpha(i-1)+1)$, $1 \leq i \leq r-1$, and at positions $(r, (\alpha+\beta)(r-1)+2(n-\beta)+1)$ and $(r+1, (\alpha+\beta)(r-1)+2n-\beta+1)$. (We do not specify it in rows $r+2$ to $n+kr_1$.) Using the automorphicity of $\theta_{2n+k(r-1)}^{(r)}$ we can conjugate the argument of this function by w . After doing so, we obtain the integral (5.3) with $m=r$ as inner integration, and then from Lemma 4 it follows that (5.24) is zero for all choices of data. We conclude that the only nonzero contribution to the integral (5.23) from the expansion along Y_{3,r_1-1} is from the constant term.

Continuing this process, next with Y_{3,r_1-2} , we obtain by induction that (5.23) is equal to the integral

$$(5.25) \quad \int \overline{\varphi^{(\kappa r)}(i(g))} \theta_{2n(k-2\alpha)}^{(2),\psi}(l_0(u)\iota_1(1,g)) \psi_{U_{\beta,r_1,n}}(u) \psi_{U_{\alpha,r-1,n+\beta r_1}}(u_1) \\ \theta_{2n+k(r-1)}^{(r)}(u'_{\alpha,n+\beta r_1}(Y_3, Z)u_1 \begin{pmatrix} I_{\alpha(r-1)} & & \\ & u & \\ & & I_{\alpha(r-1)} \end{pmatrix} \iota_2(1,g)) du_1 du dY_3 dZ dg,$$

where Y_3 is integrated over $[\text{Mat}_{\alpha \times \beta r_1}]$.

Next, we expand (5.25) along Y_2 (as defined above, following (5.23)) over the quotient $[\text{Mat}_{\alpha \times 2n}]$. We will show that all nontrivial characters contribute zero to this expansion. Note that the group $GL_{\alpha}(F) \times Sp_{2n}(F)$, embedded in $Sp_{2n+k(r-1)}(F)$ by the map

$$(h, g) \rightarrow \text{diag}(h, \dots, h, I_{\beta r_1}, g, I_{\beta r_1}, h^*, \dots, h^*) \quad h \in GL_{\alpha}(F); \quad g \in Sp_{2n}(F),$$

acts, so we may consider the characters modulo this action. There are two types of representatives for the nontrivial orbits, as follows.

The first is given by characters of the form

$$\psi_A(Y_2) = \psi(\text{tr}(Y_2A)) \quad A = \begin{pmatrix} 1 & 0 & * \\ 0_{2(n-1) \times 1} & 0_{2(n-1) \times 1} & * \\ 0 & 1 & * \end{pmatrix} \in \text{Mat}_{2n \times \alpha}(F).$$

These appear in the expansion only if $\alpha \geq 2$. Let w denote a Weyl element of $Sp_{2n+k(r-1)}(F)$ with entry 1 in positions $(i, \alpha(i-1)+1)$ and $(r+i, (\alpha+\beta)(r-1)+2n+i\alpha-1)$ for $1 \leq i \leq r-1$, and in position $(r, \alpha(r-1)+\beta r_1+1)$. Then, arguing as above with (5.24), by conjugating by this w , we obtain the integral (5.3) with $m=2r-2$ as inner integration.

Notice that since $\alpha \geq 2$, then $k \geq 4$, and we do have $r \leq 2r - 2 \leq n + kr_1 - 1$. Applying Lemma 4, we see that the contribution to the expansion from these representatives is zero.

The second type of representative is given by the characters

$$(5.26) \quad \psi_A(Y_2) = \psi(\text{tr}(Y_2 A)) \quad A = \begin{pmatrix} I_a & & \\ & & \end{pmatrix} \in \text{Mat}_{2n \times \alpha}(F); \quad 1 \leq a \leq n, \alpha.$$

(If $a > n$ then the orbit is represented by a character already considered above.) The stabilizer inside Sp_{2n} contains the unipotent radical of the maximal parabolic subgroup of Sp_{2n} whose Levi part is $GL_a \times Sp_{2(n-a)}$. We denote this unipotent group by R_a . It is embedded inside $Sp_{2n+k(r-1)}$ as all matrices of the form

$$(5.27) \quad \begin{pmatrix} I_{kr_1} & & & & \\ & I_a & B & C & \\ & & I_{2(n-a)} & B^* & \\ & & & I_a & \\ & & & & I_{kr_1} \end{pmatrix} \quad B \in \text{Mat}_{a \times 2(n-a)}; \quad C \in \text{Mat}_{a \times a}^0.$$

The claim is that, as a function of $g \in Sp_{2n}(\mathbb{A})$, the integral

$$(5.28) \quad \int \theta_{2n(k-2\alpha)}^{(2),\psi}(l_0(u)\iota_1(1,g)) \psi_{U_{\beta,r_1,n}}(u) \psi_{U_{\alpha,r-1,n+\beta r_1}}(u_1) \psi_A(Y) \\ \theta_{2n+k(r-1)}^{(r)}(u'_{\alpha,n+\beta r_1}(Y,Z)u_1 \begin{pmatrix} I_{\alpha(r-1)} & & \\ & u & \\ & & I_{\alpha(r-1)} \end{pmatrix} \iota_2(1,g)) du_1 du dY dZ$$

is left invariant under $R_a(\mathbb{A})$. Here Y runs over all matrices of the form $\begin{pmatrix} 0 & Y_2 & Y_3 \end{pmatrix}$ with Y_2 integrated over $[\text{Mat}_{\alpha \times 2n}]$ and Y_3 integrated as in (5.25). The character $\psi_A(Y)$ is the trivial extension of $\psi_A(Y_2)$ defined above.

To prove the claim we first unfold the theta series $\theta_{2n(k-2\alpha)}^{(2),\psi}$, similarly to (5.15). After collapsing summation and integration, we obtain

$$(5.29) \quad \int \sum_{\xi \in F^m} \omega_\psi(l_0(u)\iota_1(1,g))\phi(0,\xi) \psi_{U_{\beta,r_1,n}}(u) \psi_{U_{\alpha,r-1,n+\beta r_1}}(u_1) \psi_A(Y) \\ \theta_{2n+k(r-1)}^{(r)}(u'_{\alpha,n+\beta r_1}(Y,Z)u_1 \begin{pmatrix} I_{\alpha(r-1)} & & \\ & u & \\ & & I_{\alpha(r-1)} \end{pmatrix} \iota_2(1,g)) du_1 du dY dZ.$$

Here $m = 0$ if k is even, and $m = n - a$ if k is odd. The integration in u is over the quotient $U'_{\beta,r_1,n}(F) \backslash U_{\beta,r_1,n}(\mathbb{A})$ where $U'_{\beta,r_1,n}$ is a certain subgroup of $U_{\beta,r_1,n}$ (whose definition we omit as it plays no role in the sequel). It follows from the action of the Weil representation that the function $\sum_{\xi \in F^m} \omega_\psi(\iota_1(1,g))\phi(0,\xi)$ is left invariant under $R_a(\mathbb{A})$. After moving the matrix $r_a \in R_a(\mathbb{A})$ to the right and changing variables, it is enough to prove that the function of g

$$(5.30) \quad \int \theta_{2n+k(r-1)}^{(r)}(u'_{\alpha,n+\beta r_1}(Y,Z)u_1 \iota_2(1,g)) \psi_{U_{\alpha,r-1,n+\beta r_1}}(u_1) \psi_A(Y) du_1 dY dZ$$

is left invariant under $r_a \in R_a(\mathbb{A})$. Here, the variables u_1, Y and Z are integrated as in (5.29).

To prove this invariance, we argue as in the proof of Lemma 5. First expand (5.30) along the abelian group $\text{Mat}_{a \times a}^0$, embedded in $Sp_{2n+k(r-1)}$ as the group of all matrices of the form (5.27) with $B = 0$. Next observe that only the trivial character contributes. Indeed, arguing similarly to the proof of Lemma 5, we obtain as inner integration either a Fourier coefficient which corresponds to a unipotent orbit which is not related to $\mathcal{O}(\Theta_{2n+k(r-1)}^{(r)})$, or an integral of the form of (5.3) with $r \leq m$. Then expand along the group $\text{Mat}_{a \times 2(n-a)}$ embedded inside $Sp_{2n+k(r-1)}$ as all matrices of the form (5.27) with $C = 0$. Similar arguments imply that only the constant term gives a non-zero contribution. From this it follows that integral (5.30), and hence integral (5.28), is left invariant under $r_a \in R_a(\mathbb{A})$. Using this invariance property in the expansion of (5.25) along Y_2 , when we consider the contribution from the characters (5.26), we obtain (after measure factorization) the integral

$$\int_{[R_a]} \overline{\varphi^{(\kappa r)}(i(r_a)i(g))} dr_a$$

as inner integration. By cuspidality this integral is zero for all choices of data.

We deduce that the integral (5.25) is equal to

$$(5.31) \quad \int \overline{\varphi^{(\kappa r)}(i(g))} \theta_{2n(k-2\alpha)}^{(2),\psi}(l_0(u)\iota_1(1,g)) \psi_{U_{\beta,r_1,n}}(u) \psi_{U_{\alpha,r-1,n+\beta r_1}}(u_1) \\ \theta_{2n+k(r-1)}^{(r)}(u'_{\alpha,n+\beta r_1}(Y_2, Y_3, Z)u_1 \begin{pmatrix} I_{\alpha(r-1)} & & \\ & u & \\ & & I_{\alpha(r-1)} \end{pmatrix} \iota_2(1,g)) du_1 du dY_2 dY_3 dZ dg.$$

The final expansion we need to consider is the expansion of (5.31) along the quotient $Y_1 \in [\text{Mat}_{\alpha \times \beta r_1}]$. Here Y_1 is embedded inside $Sp_{2n+k(r-1)}$ as the group of all matrices $u'_{\alpha,n+\beta r_1}(Y, 0)$ with $Y = \begin{pmatrix} Y_1 & 0 & 0 \end{pmatrix}$ (with Y as described following (5.23)). Similarly to the expansion of (5.23) along the subgroup Y_3 , we see that all nontrivial characters for Y_1 give zero. Thus (5.31) is equal to

$$(5.32) \quad \int \overline{\varphi^{(\kappa r)}(i(g))} \theta_{2n(k-2\alpha)}^{(2),\psi}(l_0(u)\iota_1(1,g)) \psi_{U_{\beta,r_1,n}}(u) \psi_{U_{\alpha,r-1,n+\beta r_1}}(u_1) \\ \theta_{2n+k(r-1)}^{(r)}(u'_{\alpha,n+\beta r_1}(Y, Z)u_1 \begin{pmatrix} I_{\alpha(r-1)} & & \\ & u & \\ & & I_{\alpha(r-1)} \end{pmatrix} \iota_2(1,g)) du_1 du dY dZ dg$$

where Y is now integrated over $[\text{Mat}_{\alpha \times 2n+\beta r_1}]$.

To complete the proof of the Theorem we need to prove that (5.32) is zero for all choices of data. Recall that u_1 is integrated over $[U_{\alpha,r-1,n+\beta r_1}^0]$. Combining this integration with the integration over $u'_{\alpha,n+\beta r_1}(Y, Z)$, we obtain as inner integration the constant term of the function $\theta_{2n+k(r-1)}^{(r)}$ along the unipotent radical of the maximal parabolic subgroup of $Sp_{2n+k(r-1)}$ whose Levi part is $GL_{\alpha(r-1)} \times Sp_{2n+\beta(r-1)}$. This means that we can use Proposition 1 in [F-G2] to obtain as inner integration a function which is realized in the space of the representation $\sigma_{n,\beta}^{(r)} = \sigma_{n,k-2\alpha}^{(r)}$. Indeed, this follows since u in (5.32) is integrated over $[U_{\beta,r_1,n}]$. By our assumption this representation is zero. This completes the proof of the Theorem. $\square$

In fact it follows from the above proof that we have the following Corollary.

Corollary 1. *Suppose that the representation $\sigma_{n,k}^{(r)}$ is zero. Then, for all $1 \leq m \leq [k/2]$, the representation $\sigma_{n,k-2m}^{(r)}$ is zero.*

Proof. Suppose that $\sigma_{n,k}^{(r)}$ is zero. Then for $1 \leq \alpha \leq k_1$ the constant term of this representation along the unipotent subgroup N_α of SO_k is zero for all choices of data. (The group N_α was defined before (5.13).) In Theorem 2 we proved that if the representation $\sigma_{n,k-2\alpha}^{(r)}$ is zero, then the constant term given by (5.13) is zero for all choices of data. In fact the converse is also true. The key point is that the process of root exchange shows that one expression vanishes for all choices of data if and only if another does (see [G-R-S4], Corollary 7.1). Accordingly, the steps of the above proof may be reversed. We omit the detail. $\square$

6. THE UNRAMIFIED CORRESPONDENCE

Our goal in this section is to establish that the theta correspondence developed here is functorial on unramified principal series. In this section we let F be a nonarchimedean local field containing μ_r , and to simplify the notation we write a group or covering group for its F -points (so we write Sp_{2n} instead of $Sp_{2n}(F)$, etc.). Recall that given an unramified character of the torus of a symplectic or orthogonal group, one may define its associated principal series by parabolic induction from the Borel subgroup. One may do this for covering groups as well, but this requires a two-step process. Let G be the $(F\text{-points})$ of one of the groups considered above and $B = TU$ its standard Borel subgroup, with T be its maximal split torus. Let $\tilde{G}$ be one of the covering groups under consideration here, and if H is any subgroup of G let $\tilde{H}$ denote its inverse image in $\tilde{G}$. Then $\tilde{T}$ is generally not abelian, but it is a two-step nilpotent group. If χ is a genuine character of the center $Z(\tilde{T})$ of $\tilde{T}$, then genuine principal series representations are constructed by first extending χ to a character of a maximal abelian group A of $\tilde{T}$ containing $Z(\tilde{T})$, next inducing this extension from A to $\tilde{T}$, then extending trivially on N to obtain a representation of $\tilde{B}$, and finally taking the normalized induction from $\tilde{B}$ to $\tilde{G}$. By an analogue of the Stone-Von Neumann Theorem, this representation is determined by its central character χ , and we write the representation $\pi(\chi)$. For more details see for example [McN] (analogously to the general linear group, in [K-P], Section I.1) or [Gao2]. For characters in general position these induced representations are irreducible.

Let χ be the unramified character of the maximal torus T of Sp_{2n} given by

$$\chi(\text{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1})) = \prod_{i=1}^n \chi_i(t_i),$$

where the χ_i are unramified quasicharacters of $F^\times$. This character determines a genuine character χ of $Z(\tilde{T})$, and we form the unramified principal series $\pi_{Sp_{2n}}^{(\kappa r)}(\chi)$ as outlined above. If $\kappa = 2$ then this requires a Weil factor, as in [B-F-H], equation (1.10). The functions in this space are genuine with respect to the character ϵ' specified in Section 3. Similarly, let ξ be an unramified character of the maximal torus of SO_k ,

$$\xi(\text{diag}(t_1, \dots, t_{k_1}, I_{k-2k_1}, t_{k_1}^{-1}, \dots, t_1^{-1})) = \prod_{i=1}^{k_1} \xi_i(t_i),$$

where $k_1 = \lfloor k/2 \rfloor$, the ξ_j are quasicharacters, and the maximal torus has a 1 in the middle entry if k is odd, and form the unramified principal series $\pi_{SO_k}^{(r)}(\xi)$ (for more details see [B-F-G1], Section 6). We consider the case that $k_1 = n$, so that each representation has the same number of Satake parameters, and study the theta correspondence between these representations. (One may describe matters slightly more generally using the notion of a Brylinski-Deligne extension, as in Gao [Gao1]; see for example Leslie [L], Section 3. This does not change the discussion below in any significant way.)

Let ψ be an additive character of F which is trivial on the ring of integers of F but on no larger fractional ideal, and let $(\omega_\psi, V_{\omega_\psi})$ be the Weil representation of $Sp_{2nk}^{(2)}$ with respect to ψ . Let $(\Theta^{(r)}, V_{\Theta^{(r)}})$ be the local theta representation of $Sp_{2n+k(r-1)}^{(r)}$. (We shall write $\Theta_{2n+k(r-1)}^{(r)}$ when we want to indicate the size of the symplectic group.) Form the vector space $V_{\omega_\psi} \otimes V_{\Theta^{(r)}}$. This space admits a representation $\omega_\psi \otimes \Theta^{(r)}$ of $(Sp_{2nk}^{(2)} \rtimes \mathcal{H}_{2kn+1}) \times Sp_{2n+k(r-1)}^{(r)}$. We restrict this to a representation of $SO_k^{(r)} \times Sp_{2n}^{(\kappa r)}$ using the same embeddings ι_1, ι_2 as in the global integral. More precisely, the action of $(h, g) \in SO_k^{(r)} \times Sp_{2n}^{(\kappa r)}$ on a vector $v_1 \otimes v_2$, $v_1 \in V_{\omega_\psi}$, $v_2 \in V_{\Theta^{(r)}}$ is given by

$$(\omega_\psi \otimes \Theta^{(r)})(h, g) \cdot (v_1 \otimes v_2) = \omega_\psi(\iota_1^{(2)}(p^{(1)}(h), p^{(\kappa)}(g)))v_1 \otimes \Theta^{(r)}(\iota_2^{(r)}(h, p^{(r)}(g)))v_2.$$

Note that the subgroup $\{((1, \zeta), (1, \zeta^{-1})) \mid \zeta \in \mu_r\}$ acts trivially. Let $J_{U_{k,r_1,n}, \psi_{U_{k,r_1,n}}}$ or, for notational convenience, simply $J_{U_{k,r_1,n}, \psi}$ be the twisted Jacquet functor with respect to the character $\psi_{U_{k,r_1,n}}$ of $U_{k,r_1,n} \subseteq Sp_{2n+k(r-1)}$, acting on the first factor in the tensor product by the map $l : U_{k,r_1,n} \rightarrow \mathcal{H}_{2kn+1}$ and acting on the second factor by the theta representation. (This functor is defined near the end of Section 1 above.) This is the local analogue of the integral (3.2). Similarly to the treatment of the global situation, the group $SO_k^{(r)} \times Sp_{2n}^{(\kappa r)}$ with the above action stabilizes the group $U_{k,r_1,n}$ and character, so the Jacquet module $J_{U_{k,r_1,n}, \psi}(\omega_\psi \otimes \Theta^{(r)})$ also affords a representation of $SO_k^{(r)} \times Sp_{2n}^{(\kappa r)}$. Let

$$\text{Hom}_{SO_k^{(r)} \times Sp_{2n}^{(\kappa r)}}(J_{U_{k,r_1,n}, \psi}(\omega_\psi \otimes \Theta^{(r)}), \pi_{SO_k}^{(r)}(\xi) \otimes \pi_{Sp_{2n}}^{(\kappa r)}(\chi))$$

denote the space of $(SO_k^{(r)} \times Sp_{2n}^{(\kappa r)})$ -equivariant maps from $J_{U_{k,r_1,n}, \psi}(\omega_\psi \otimes \Theta^{(r)})$ to $\pi_{SO_k}^{(r)}(\xi) \otimes \pi_{Sp_{2n}}^{(\kappa r)}(\chi)$. Then we shall show

Theorem 3. *Let $k = 2n$ or $2n + 1$ and suppose that the characters χ and ξ are in general position. If*

$$(6.1) \quad \text{Hom}_{SO_k^{(r)} \times Sp_{2n}^{(\kappa r)}}(J_{U_{k,r_1,n}, \psi}(\omega_\psi \otimes \Theta^{(r)}), \pi_{SO_k}^{(r)}(\xi) \otimes \pi_{Sp_{2n}}^{(\kappa r)}(\chi)) \neq 0$$

then after applying a Weyl group element, $\xi_i = \chi_i$ for each i .

Here we recall that the Weyl group of Sp_{2n} acts on χ by permuting the indices and by inverting each χ_i ; the Weyl group of SO_k acts on the ξ_i , by a similar action (if k is odd) and by permutations and by inversions of an even number of quasicharacters (if k is even). Hence the conclusion is equivalent to the assertion that the sets $\{\chi_i^{\pm 1}\}$ and $\{\xi_j^{\pm 1}\}$, which are each of cardinality $2n$ as the characters are in general position, are in bijection. Also, if k is even then we recall that the principal series representations of $SO_k^{(r)}$ attached to $(\xi_1, \dots, \xi_{n-1}, \xi_n)$ and $(\xi_1, \dots, \xi_{n-1}, \xi_n^{-1})$ are isomorphic under the outer automorphism of SO_k that is conjugation

by

$$\begin{pmatrix} I_{k/2-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{k/2-1} \end{pmatrix}.$$

The proof of Theorem 3 requires the local version of the smallness of the representation $\Theta_{2l}^{(r)}$. Recall that $\mathcal{O}_c(\Theta_{2l}^{(r)})$ is defined in Conjecture 1 above. As in Section 4, for a given unipotent orbit $\mathcal{O}$ let $U_{\mathcal{O}}$ denote the upper unipotent subgroup attached to this orbit. Then the proof of Proposition 2 gives the following result.

Proposition 7. *Suppose that $\mathcal{O}$ is a unipotent orbit which is greater than $\mathcal{O}_c(\Theta_{2l}^{(r)})$ or that is not related to $\mathcal{O}_c(\Theta_{2l}^{(r)})$. Then for any character $\psi_{\mathcal{O}}$ attached to $\mathcal{O}$, the twisted Jacquet module $J_{U_{\mathcal{O}}, \psi_{\mathcal{O}}}(\Theta_{2l}^{(r)}) = 0$.*

We turn to the proof of Theorem 3. We will follow, in a loose sense, the approaches of Bump and the authors [B-F-G2], Section 6, and Leslie [L], Section 10, in their proofs of unramified correspondences for two other theta maps that use a single theta function as an integral kernel. Below, we refer to these proofs for some details that are similar.

Proof of Theorem 3. We shall prove this by induction on n . More precisely, suppose that the nonvanishing (6.1) holds for given n , k , χ , and ξ with $k = 2n$ or $k = 2n + 1$ and the characters χ , ξ in general position. Then we show that up to permutation of the indices, we have $\chi_1 = \xi_1$ or $\chi_1 = \xi_1^{-1}$. We also show that a similar hypothesis holds true for (n, k) replaced by $(n - 1, k - 2)$ and (χ, ξ) replaced by (χ', ξ') where $\chi' = (\chi_2, \dots, \chi_n)$, $\xi' = (\xi_2, \dots, \xi_n)$, that is, we show that

$$\text{Hom}_{SO_{k-2}^{(r)} \times Sp_{2n-2}^{(\kappa r)}}(J_{U_{k-2, r_1, n-1, \psi}}(\omega_{\psi} \otimes \Theta^{(r)}), \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi')) \neq 0$$

Then the result follows by reduction to the case $n = 1$, which is straightforward to confirm by the same method used below.

Let $P'_{1, k-2}$ denote the unipotent radical of SO_k with Levi factor $M'_{1, k-2} := GL_1 \times SO_{k-2}$ and unipotent radical $U'_{1, 2k-2}$. Let $P_{1, 2n-2}$ denote the unipotent radical of Sp_{2n} with Levi factor $M_{1, 2n-2} := GL_1 \times Sp_{2n-2}$ and unipotent radical $U_{1, 2n-2}$. Denote the inverse images of these groups in their respective covering groups by a tilde. First, by transitivity of induction, we realize $\pi_{SO_k}^{(r)}(\xi)$ as parabolically induced from $\xi_1 \otimes \pi_{SO_k}^{(r)}(\xi')$ with $\xi' = (\xi_2, \dots, \xi_n)$. (Note that the inverse images of GL_1 and SO_{k-2} in the local covering group $SO_k^{(r)}$ commute due to block compatibility, so the tensor product construction is straightforward.) Similarly, $\pi_{Sp_{2n}}^{(\kappa r)}(\chi)$ is parabolically induced from the representation $\chi_1 \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi')$ where $\chi' = (\chi_2, \dots, \chi_n)$. Second, the Jacquet module $J_{U'_{1, k-2}}(\pi_{SO_k}^{(r)}(\xi))$ is isomorphic to the direct sum of $\xi_i^{\pm 1} \otimes \pi_{SO_{k-2}}^{(r)}(\xi'_{i, \pm})$ where $\xi'_{i, +}$ is obtained from ξ by removing ξ_i and $\xi'_{i, -}$ is obtained from ξ by removing ξ_i and inverting one of the remaining quasicharacters. (See, for example, Section 5.2 of [B-Z].) Note that the action of the Weyl group of SO_k on ξ permutes these factors. Similarly the Jacquet module $J_{U_{1, 2n-2}}(\pi_{Sp_{2n}}^{(\kappa r)}(\chi))$ is isomorphic to the direct sum of $\chi_j^{\pm 1} \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi'_j)$ where χ'_j is obtained from χ by removing χ_j . It follows that the tensor product $\pi_{Sp_{2n}}^{(\kappa r)}(\chi) \otimes \pi_{SO_k}^{(r)}(\xi)$

is a sum of the induced representations

$$\text{Ind}_{\tilde{P}'_{1,k-2} \times \tilde{P}'_{1,2n-2}}^{SO_k^{(r)} \times Sp_{2n}^{(\kappa r)}} (\xi_i^{\pm 1} \otimes \pi_{SO_{k-2}}^{(r)}(\xi'_{i,\pm}) \otimes \chi_j^{\pm 1} \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi'_j) \delta_{P'_{1,k-2}}^{1/2} \delta_{P_{1,2n-2}}^{1/2}).$$

(The weaker statement that this tensor product is glued from these representations (see [B-Z]) would be sufficient below.) Since the Weyl groups permutes these factors for different indices, we shall focus on the case $\xi_i^{\pm 1} = \xi_1$, $\chi_j^{\pm 1} = \chi_1$ without loss of generality, and in this case we write $\xi'_{1,+} = \xi'$, $\chi'_1 = \chi'$ (as in the prior paragraph).

Applying Frobenius reciprocity, the nonvanishing hypothesis (6.1) in Theorem 3 implies that, up to the action of the Weyl groups as just explained, there is a nonzero $\widetilde{M}'_{1,k-2} \times \widetilde{M}_{1,n-1}$ -equivariant map

$$J_{U'_{1,k-2}} J_{U_{1,2n-2}} (J_{U_{k,r_1,n},\psi}(\omega_\psi \otimes \Theta^{(r)})) \rightarrow \xi_1 \otimes \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \chi_1 \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi') \delta_{P'_{1,k-2}}^{1/2} \delta_{P_{1,2n-2}}^{1/2}.$$

To keep track of the characters that arise, let A, B denote the block matrices

$$(6.2) \quad A = \begin{pmatrix} a & & \\ & g' & \\ & & a^{-1} \end{pmatrix} \in Sp_{2n} \quad B = \begin{pmatrix} b & & \\ & h' & \\ & & b^{-1} \end{pmatrix} \in SO_k,$$

with $a, b \in GL_1$, $g' \in Sp_{2n-2}$, $h' \in SO_{k-2}$. Then $\delta_{P'_{1,k-2}}^{1/2}(B) = |b|^{k/2-1}$ and $\delta_{P_{1,2n-2}}^{1/2}(A) = |a|^n$.

Next we apply the argument in [G-R-S4], Proposition 6.6 (see p. 129). This implies that the Hom space of such maps is a quotient of the space

$$(6.3) \quad \text{Hom}_{\widetilde{M}'_{1,k-2} \times \widetilde{M}_{1,n-1}} (J_{V_{k,r_1,n},\psi}(\omega_\psi \otimes \Theta^{(r)}), \xi_1 \otimes \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \chi_1 \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi') \delta_{P'_{1,k-2}}^{1/2} \delta_{P_{1,2n-2}}^{1/2} \delta)$$

and hence this space is nonzero. Here $V_{k,r_1,n} \subseteq Sp_{2n+k(r-1)}$ is the upper unipotent subgroup $V_{k,r_1,n} = \iota_2(U'_{1,k-2}, U_{1,2n-2})U_{k,r_1,n}^\flat$ where the group $U_{k,r_1,n}^\flat$ is defined as follows. Let $\text{Mat}'_{k,2n}$ denote the subgroup of $\text{Mat}_{k,2n}$ consisting of all matrices $Y = (y_{i,j})$ whose entries $y_{j,1}$, $2 \leq j \leq k$ and $y_{k,i}$, $1 \leq i \leq 2n$ are each zero. Then $U_{k,r_1,n}^\flat$ is the subgroup of the group $U_{k,r_1,n}$ consisting of all matrices u whose factorization (2.3) (with $a = k$, $b = r_1$, $c = n$) has the property that Y is in $\text{Mat}'_{k,2n}$. The character ψ of $V_{k,r_1,n}$ is the character $\psi_{U_{k,r_1,n}}$ restricted to $U_{k,r_1,n}^\flat$ and then extended trivially to $\iota_2(U'_{1,k-2}, U_{1,2n-2})$. Also, the map l restricted to $V_{k,r_1,n}$ gives a homomorphism from $V_{k,r_1,n}$ (or $U_{k,r_1,n}^\flat$) onto the Heisenberg group $\mathcal{H}_{(n-1)(k-2)}$ and this defines the action of $V_{k,r_1,n}$ on ω_ψ . The character δ factors through the projection from $\widetilde{M}'_{1,k-2} \times \widetilde{M}_{1,n-1}$ to $M'_{1,k-2} \times M_{1,n-1}$, and evaluated on (6.2) has two factors, a contribution of $|a|^{k/2-1}|b|^n$ due to the action of the Weil representation and a factor from the modular function of the quotient $U_{k,r_1,n}/U_{k,r_1,n}^\flat$. This quotient may be identified with the group of all matrices $u'_{k,n}(Y_1, 0)$ (see (2.2)) such that Y_1 is an element in $\text{Mat}_{k,2n}^{00}$ which is the complement to $\text{Mat}'_{k,2n}$ in $\text{Mat}_{k,2n}$, so this factor is $|a|^{2-k}|b|^{-2n}$. Combining these, we see that the value of the character δ on (6.2) above is $|a|^{1-k/2}|b|^{-n}$, so $\delta_{P'_{1,k-2}}^{1/2} \delta_{P_{1,2n-2}}^{1/2} \delta$ takes the value $|ab^{-1}|^{1-k/2+n}$. We remark that this step is the local analogue of using the definition of the theta function as a sum and unfolding part of the sum, a process that is used in the proof of Theorem 2; see equation (5.15) and the discussion following, in the case $\alpha = 1$, $\beta = k - 2$.

Next, we make a series of root exchanges. We have already used root exchanges for global integrals. For root exchange in the context of representations of local fields, see [G-R-S2], Section 2.2. In the local case, they allow one to replace the Jacquet module in (6.3) by

another isomorphic Jacquet module and to conclude that the corresponding Hom space is nonzero. Globally or locally, the root exchanges require an additive character, and as in the global case, we use the additive character ψ of $V_{k,r_1,n}$ obtained from $\psi_{U_{k,r_1,n}}$ (see (3.1)) and also of the additive character that comes from the Weil representation ω_ψ .

The first set of exchanges are the same as the root exchanges used in the proof of Theorem 2 above, where we treated the groups $L_{\alpha,\beta}$ introduced in (5.18). We use these same exchanges in the case $\alpha = 1$, $\beta = k - 2$. The root exchanges described through (5.20) allow us to conclude that the space

$$\text{Hom}_{\widetilde{M}'_{1,k-2} \times \widetilde{M}_{1,n-1}}(J_{V''_{k,r_1,n},\psi}(\omega_\psi \otimes \Theta^{(r)}), \xi_1 \otimes \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \chi_1 \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi') \delta_1)$$

is nonzero. Here $V''_{k,r_1,n}$ is the unipotent subgroup of $Sp_{2n+k(r-1)}$ generated by the following two types of matrices. First, the matrices with factorization (2.3) with the X_i , $1 \leq i \leq r_1 - 1$, Y , and Z of the form

$$X_i = \begin{pmatrix} * & * & * \\ 0_{(k-2) \times 1} & * & * \\ 0 & 0_{1 \times (k-2)} & * \end{pmatrix} \in \text{Mat}_k, \quad Y = \begin{pmatrix} * & * & * \\ 0_{(k-2) \times 1} & * & * \\ 0 & 0_{1 \times (2n-2)} & 0 \end{pmatrix} \in \text{Mat}_{k \times 2n},$$

$$Z = \begin{pmatrix} * & * & * \\ 0_{(k-2) \times 1} & * & * \\ 0 & 0_{1 \times (k-2)} & * \end{pmatrix} \in \text{Mat}_k^0,$$

where the stars indicate arbitrary elements of F . Second, the block-diagonal matrices in the Levi of $P_{k,r_1,n}$ whose entries in the first r_1 diagonal $k \times k$ blocks are of the form (5.18) with $\alpha = 1$, $\beta = k - 2$. These root exchanges also change the character $\delta_{P'_{1,k-2}}^{1/2} \delta_{P_{1,2n-2}}^{1/2} \delta$ above to δ_1 whose value on (6.2) is $\delta_{P'_{1,k-2}}^{1/2} \delta_{P_{1,2n-2}}^{1/2} \delta |b|^{-k-(r_1-1)(2k-2)} = |ab^{-1}|^{1-k/2+n} |b|^{-k-(r_1-1)(2k-2)}$.

Next we conjugate by the Weyl group element $w \in Sp_{2n+k(r-1)}$ which is the shortest Weyl group element that conjugates the matrix

$$(6.4) \quad \text{diag}(B, \dots, B, A, B^*, \dots, B^*),$$

to the matrix

$$(6.5) \quad \text{diag}(b, \dots, b, a, h', \dots, h', g', (h')^*, \dots, (h')^*, a^{-1}, b^{-1}, \dots, b^{-1}).$$

In (6.4) above, A and B are as given in (6.2), the notation B^* is defined in Section 2, and each of B, B^* appears r_1 times; in (6.5), each of b, b^{-1} is repeated $r - 1$ times and each of $h', (h')^*$ is repeated r_1 times. After doing so, we conclude that the Hom space

$$\text{Hom}_{\widetilde{M}}(J_{V',\psi'}(\omega_\psi \otimes \Theta^{(r)}), \xi_1 \otimes \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \chi_1 \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi') \delta_1)$$

is nonzero, where $\widetilde{M}$ is isomorphic to $\widetilde{M}'_{1,k-2} \times \widetilde{M}_{1,n-1}$ and $\iota_2(\widetilde{M}')$ projects to the group of matrices of the form (6.5). The group $V' = wV''_{k,r_1,n}w^{-1}$ is closely related to the unipotent group that appeared in the integration in (5.21) above. More precisely, an element in V' may be written as a product of the form

$$(6.6) \quad \begin{pmatrix} I_r & C & D \\ & I & C^* \\ & & I_r \end{pmatrix} \begin{pmatrix} v & & \\ & u & \\ & & v^* \end{pmatrix} \begin{pmatrix} I_r & & \\ R & I & \\ S & R^* & I_r \end{pmatrix}$$

where $I = I_{2n+k(r-1)-2r}$, $u \in U_{k-2,r_1,n-1}$ and $v \in V_r^0$, which is defined as follows. Let V_r be the maximal upper unipotent subgroup of GL_r . Then V_r^0 is the subgroup of V_r consisting of all matrices $(v_{i,j})$ such that $v_{r-1,r} = 0$. The matrices C , D , R and S are defined similarly to the corresponding matrices C , D , A and B in integral (5.21). With these notations the character ψ' is described as follows. First

$$\psi'(\text{diag}(v, u, v^*)) = \psi_r^0(v) \psi_{U_{k-2,r_1,n-1}}(u)$$

where ψ_r^0 is the Whittaker character of V_r restricted to V_r^0 . Then ψ' is extended trivially to the rest of V' .

In view of this, the space we are studying is analogous to the integral (5.21), with the unipotent integration in this local context being the process of taking the twisted Jacquet module, except that in that integral one is integrating g over (a cover of) Sp_{2n} while here we have the Hom space with respect to the parabolic subgroup $\widetilde{M}_{1,n-1}$ which projects to Levi factor $GL_1 \times Sp_{2n-2}$. Note that in the proof of Theorem 2 we also conjugated by a Weyl group element, but not quite the same one we are using now, due to the difference between Sp_{2n} and $GL_1 \times Sp_{2n-2}$. We next use many of the same steps in the proof of Theorem 2 with minor modifications.

First, we carry out a series of root exchanges that are the same as the ones given following (5.22) with $\alpha = 1$, $\beta = k - 2$, but with an adjustment on the size of the blocks, replacing $r - 1$ by r . (For example, $I_{\alpha(r-1)}$ in (5.21) is now replaced by I_r , as in (6.6).) These are performed row by row. Specifically, we start with the second row of the matrix C as given in (6.6). Choosing the third column in the matrix R in (6.6), we perform root exchange; this is the same as the first in the sequence of root exchanges that was carried out following (5.22), and contributes the Jacobian $|b|^k$. We continue to make root exchanges, again following the procedure described following (5.22), until we have completed the first $r - 2$ rows. In the local case, these root exchanges imply the nonvanishing of the Hom space

$$(6.7) \quad \text{Hom}_{\widetilde{M}}(J_{V'_1, \psi'_1}(\omega_\psi \otimes \Theta^{(r)}), \xi_1 \otimes \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \chi_1 \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi') \delta_2).$$

Here V'_1 is the subgroup of $Sp_{2n+k(r-1)}$ of matrices which have a factorization (6.6) that satisfies the following conditions: for C : $C_{i,j} = 0$ if $i = r - 1$, $1 \leq j \leq 2n + (r - 1)(k - 2) - k$ or $i = r$, $1 \leq j \leq r_1(k - 2)$; for D , $D_{r-1,1} = D_{r-1,2} = D_{r,2} = 0$; for R , $R_{i,j} = 0$ if $1 \leq j \leq r - 1$ or if $j = r$, $i > (r_1 - 1)(k - 2)$; and $S = 0$. Also, ψ'_1 is the character of V'_1 that is ψ' extended trivially to V'_1 . (Note that studying this nonvanishing is the local analogue of studying the possible nonvanishing of (5.23) with $\alpha = 1$ and $r - 1$ replaced by r , and this same group and character, up to this modification, appear there.) The character δ_2 in (6.7) is computed as follows. In carrying out these root exchanges, the character δ_1 , evaluated on (6.2), is multiplied by a power of $|b|$ at each step of the root exchange process. For $1 \leq j \leq r_1$, to fill in the j -th row by root exchange the adjustment on the character is $|b|^{k(j-1)}$, and for row $r_1 + j$, $1 \leq j \leq r_1 - 1$, the adjustment is $|b|^{r_1 k + 2n - 2 + (j-1)(k-4)}$. The character δ_2 in (6.7) is obtained by combining these adjustments with the prior character δ_1 . (We will compute the final character later in the proof.)

We next treat the $(r - 1)$ -st row. To do so, we perform a local version of Fourier expansion. To describe the procedure, let L denote a unipotent abelian subgroup of $Sp_{2n+k(r-1)}$ such that LV'_1 is also a unipotent group, and such that ψ'_1 is a well-defined character of LV'_1 when extended trivially from V'_1 to LV'_1 . Also suppose that the group M normalizes this unipotent

group. Then it follows from the nonvanishing of the Hom space (6.7) that at least one of the spaces

$$(6.8) \quad \text{Hom}_{\widetilde{M}_{\psi_L}}(J_{L,\psi_L}(J_{V'_1,\psi'_1}(\omega_\psi \otimes \Theta^{(r)})), \xi_1 \otimes \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \chi_1 \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi') \delta_2).$$

is non-zero, as ψ_L runs over a set of representatives of the characters of the group L under the action of the group M . Here $\widetilde{M}_{\psi_L}$ is the stabilizer of ψ_L in $\widetilde{M}$. Indeed, this follows from the gluing described in Bernstein and Zelevinsky [B-Z], Theorem 5.2, applied as in the work of Bump and the authors [B-F-G2], pp. 392-3. As explained there, one obtains an exact sequence from the Geometrical Lemma of [B-Z], p. 448, and then applies a Jacquet functor which preserves the sequence (as it is an exact functor). One term involves a Jacquet functor applied to a compactly-induced representation, and to study the Hom space from this piece, one applies Mackey theory as extended in [B-Z].

We begin to make this expansion with the root supported on one-parameter subgroup $E_{r-1,2n+k(r-1)-r+2}$ (defined before Proposition 6). In fact, this is the same procedure used to analyze (5.23) and described in detail following (5.23). The Jacquet module with the character (the ‘non-constant term’ in the global computation) factors through the twisted Jacquet module corresponding to the orbit $((r+1)1^{2n+k(r-1)-r+1})$ for $\Theta^{(r)}$. However, this vanishes by the smallness of the representation $\Theta^{(r)}$, Proposition 7. Let $V'_2 = E_{r-1,2n+k(r-1)-r+2}V'_1$, and let ψ'_2 denote the character of V'_2 which is the trivial extension of ψ'_1 . Then we deduce that the space (6.7) is isomorphic to

$$(6.9) \quad \text{Hom}_{\widetilde{M}}(J_{V'_2,\psi'_2}(\omega_\psi \otimes \Theta^{(r)}), \xi_1 \otimes \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \chi_1 \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi') \delta_2),$$

hence nonzero.

We move to the next root, with root subgroup $E'_{r-1,2n+k(r-1)-r+1}$. For convenience denote this group L , and let

$$x(t) = I_{2n+k(r-1)} + te'_{r-1,2n+k(r-1)-r+1} \in L$$

($e'_{i,j}$ is defined before Proposition 6). The group M consisting of all matrices of the form (6.5) acts on L by conjugation with two orbits; indeed if m is given by (6.5), then $mx(t)m^{-1} = x(abt)$. Let $M_0 \subset M$ denote the stabilizer of this action; this is the subgroup consisting of all matrices (6.5) with $b = a^{-1}$. Let ψ_L denote a nontrivial character of L . Then, since (6.9) is not zero, we deduce that at least one of the two spaces

$$(6.10) \quad \text{Hom}_{\widetilde{M}_0}(J_{L,\psi_L}(J_{V'_2,\psi'_2}(\omega_\psi \otimes \Theta^{(r)})), \xi_1 \otimes \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \chi_1 \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi') \delta_2)$$

and

$$(6.11) \quad \text{Hom}_{\widetilde{M}}(J_{V'_3,\psi'_3}(\omega_\psi \otimes \Theta^{(r)}), \xi_1 \otimes \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \chi_1 \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi') \delta_2)$$

is also not zero. Here $V'_3 = LV'_2$ and the character ψ'_3 is obtained by extending ψ'_2 trivially to V'_3 .

We start with the case that the space (6.10) is not zero. (We will deal with the space (6.11) afterwards.) There the character ψ_L , which appears through the twisting, allows us to do root exchanges (similar to the treatment of (5.23) above), in order to fill in all the remaining positive root subgroups $E'_{r-1,j}$ in row $r-1$, as follows. First, we use ψ_L to move roots into row $r-1$ from positive roots $E'_{r,j}$ with $r+r_1(k-2) < j \leq 2n+k(r-1)-r$ and $E_{r,2n+k(r-1)-r+1}$ that appear in V'_2 in row r . We then use root exchange to exchange the $(r_1-1)(k-2)$ negative roots $E'_{j,r}$, $r < j \leq r+(r_1-1)(k-2)$ of V'_2 (these appear in

R and R^* of (6.6)) into row $r - 1$. Note that this root exchange is possible because there is an additive character appearing from the Weil representation ω_ψ . Doing these exchanges multiplies the character δ_2 by a factor of $|b|^{r_1 k + 2n - 2 + (r_1 - 1)(k - 4)} |a|^{(r_1 - 1)(k - 2)}$. Thus we obtain the nonvanishing of the Hom space

$$\text{Hom}_{\widetilde{M}_0}(J_{L', \psi_{L'}}(J_{V_2'', \psi_2'}(\omega_\psi \otimes \Theta^{(r)})), \xi_1 \otimes \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \chi_1 \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi') \delta_3).$$

Here L' is the (abelian) group generated by L and all the positive root subgroups of the form $E'_{r-1, j}$, $r \leq j \leq 2n + k(r - 1) - r$ (note that this group includes $E_{r-1, 2n+k(r-1)-r+2}$), $\psi_{L'}$ is ψ_L extended trivially from L to L' , V_2'' is the subgroup of V_2' consisting of upper unipotent matrices with 0 in position (r, j) , $r < j \leq 2n + k(r - 1) - r + 1$ and $\delta_3 = \delta_2 |b|^{r_1 k + 2n - 2 + (r_1 - 1)(k - 4)} |a|^{(r_1 - 1)(k - 2)}$.

At this point we conjugate by the shortest Weyl group element, call it w' , that interchanges a and a^{-1} in (6.5). We then fill in the remaining positive roots in row r . To do so we repeat the same process as above. We start with the group $E_{r, 2n+k(r-1)-r+1}$. Expanding, the twisted Jacquet module vanishes by the smallness of $\Theta^{(r)}$. Then we consider the group $L''/E_{r, 2n+k(r-1)-r+1}$ where L'' is the group generated by $E_{r, 2n+k(r-1)-r+1}$ and the $E'_{r, j}$ with $r + 1 \leq j \leq 2n + k(r - 1) - r$. The Hom spaces with nontrivial characters at these roots each are zero, since nonvanishing for a nontrivial character in row r would imply that the representation $\Theta^{(r)}$ supports a functional for a unipotent orbit that has at least $r + 1$ in its partition, and this would contradict Proposition 7.

Putting all this together, the nonvanishing of the Hom space (6.10) implies that

$$\text{Hom}_{\widetilde{M}_1}(J_{U_{k-2, r_1, n-1}, \psi}(\omega_\psi \otimes J_{U_{GL_r}, \psi_{Wh}}(J_{U_{r, 1, n+kr_1-r}}(\Theta^{(r)}))), \xi_1 \otimes \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \chi_1^{-1} \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi') \delta_3)$$

is nonzero. Here $\widetilde{M}_1 = w' \widetilde{M}_0 (w')^{-1}$. The projection of $\widetilde{M}_1$ to $Sp_{2n+k(r-1)}$ is

$$\text{diag}(a^{-1}, \dots, a^{-1}, h', \dots, h', g', (h')^*, \dots, (h')^*, a, \dots, a)$$

where a^{-1} and a are each repeated r times and h' , $(h')^*$ are each repeated r_1 times. The Jacquet functor applied to $\Theta^{(r)} := \Theta_{2n+k(r-1)}^{(r)}$ is the untwisted Jacquet functor with respect to $U_{r, 1, n+kr_1-r}$, the unipotent radical of the parabolic with Levi factor $GL_r \times Sp_{2n+k(r-1)-2r}$. By Proposition 1, this functor gives the module $\Theta_{GL_r}^{(r)} \otimes \Theta_{2n+2k(r-1)-2r}^{(r)}$. Here $\Theta_{GL_r}^{(r)}$ denotes the local theta representation for $GL_r^{(r)}(F)$. The Jacquet functor $J_{U_{GL_r}, \psi_{Wh}}$ is the twisted Jacquet functor with respect to the Whittaker character of GL_r , and is applied to the first component of this tensor product.

However, the theta representation on the r -fold cover of GL_r is generic by [K-P], and its central character is computed there. Thus the action of GL_1 (more precisely, the subgroup of central elements in the metaplectic group) is as follows. If B is the Borel of $Sp_{2n+k(r-1)}$ then we obtain a total contribution from this step of

$$\delta_B^{\frac{r-1}{2r}} \left(\begin{pmatrix} a^{-1} I_r & & \\ & I_{2n+k(r-1)-2r} & \\ & & a I_r \end{pmatrix} \right) = |a|^{-r_1(2n+(k-1)(r-1))}.$$

The character δ_3 is given on (6.2) with $b = a^{-1}$ by $|a|^{-1}$ raised to the power obtained by collecting all the terms arising from the unfolding and the root exchanges, each of which is

noted above. This power is

$$k + (r_1 - 1)(2k - 2) + \sum_{j=1}^{r_1-1} jk + r_1(r_1 k + 2n - 2) + \sum_{j=1}^{r_1-1} j(k - 4) - (r_1 - 1)(k - 2) \\ = r_1(2n + (k - 1)(r - 1)).$$

This power of $|a|^{-1}$ thus exactly matches the corresponding contribution from the Whittaker coefficient of $\Theta_{GL_r}^{(r)}$.

We conclude that if the Hom space (6.10) is nonzero, then $\chi_1 = \xi_1$ and moreover the space

$$\text{Hom}_{SO_{k-2}^{(r)} \times Sp_{2n-2}^{(\kappa r)}}(J_{U_{k-2, r_1, n-1, \psi}}(\omega_\psi \otimes \Theta_{2n-2+(k-2)(r-1)}^{(r)}), \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi'))$$

is nonzero. This is the same nonvanishing as in (6.1) but with (n, k) replaced by $(n-1, k-2)$ and (χ, ξ) replaced by (χ', ξ') . Thus in this case we are done by induction.

We must also analyze the situation that it is the Hom space (6.11) which is non-zero. In this case, we continue to expand along the roots of row $r-1$ using the gluing or local Fourier expansion procedure. We already have the roots in this row that sit over the last h' and the final r entries of the matrix (6.5). We first expand along the root spaces $E'_{r-1, j}$ with $r < j \leq 2n + k(r-1) - r - (k-2)$, taking into account the groups SO_{k-2} and Sp_{k-2} which act on the characters in the Fourier expansion by conjugation (see following (6.8)). Thus we expand using the abelian groups consisting of $k-2$ root spaces $E'_{r, j}$ at a time for the roots whose root spaces include an entry in the r -th row above each h' and h and $2n-2$ at a time for the roots above the middle g' . It follows by using the smallness of $\Theta^{(r)}$ or general position (when a copy of GL_1 , restricted to r -th powers, acts on the module under consideration by a fixed character) that the contributions coming from each nontrivial character in this expansion vanish. We conclude that the space

$$\text{Hom}_{\widetilde{M}}(J_{V'_4, \psi'_4}(\omega_\psi \otimes \Theta^{(r)}), \xi_1 \otimes \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \chi_1 \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi') \delta_2)$$

is non-zero, where V'_4 is the group generated by V'_3 and the root subgroups $E'_{r-1, j}$ with $r < j \leq 2n + k(r-1) - r + 1$, and the character ψ'_4 is ψ'_3 extended trivially to V'_4 .

We now do the expansion along the root subgroup $E'_{r-1, r}$. This is again glued from two terms, but the situation here is different. First, there is a constant term at this root. In this case, the Jacquet module arising from the expansion factors through $J_{U_{r-1, 1, 2n+(k-2)(r-1)}}(\Theta^{(r)})$. By Proposition 1, this gives the representation $\Theta_{GL_{r-1}}^{(r)} \otimes \Theta_{2n+(k-2)(r-1)}^{(r)}$. However, here the GL_1 in position b in (6.5) (restricted to r -th powers) acts by a fixed character. Since we are assuming that χ and ξ are in general position, the Hom space from this term is zero. We conclude that it is the non-constant piece that must be non-zero.

To analyze the non-constant piece, i.e. the contribution from a nontrivial character $\psi_{r-1, r}$ of the group $E'_{r-1, r}$, we must restrict to the stabilizer in $\widetilde{M}$ of this character. If $x_{r-1, r}(t) = I_{2n+k(r-1)} + te'_{r-1, r}$ and m is given by (6.5) then $mx_{r-1, r}(t)m^{-1} = x_{r-1, r}(ba^{-1}t)$. Thus this character is stabilized by all matrices (6.5) with $a = b$. One argues similarly to the above to see that this Hom space is nonzero only if $\chi_1 = \xi_1^{-1}$ and if

$$\text{Hom}_{SO_{k-2}^{(r)} \times Sp_{2n-2}^{(\kappa r)}}(J_{U_{k-2, r_1, n-1, \psi}}(\omega_\psi \otimes \Theta_{2n-2+(k-2)(r-1)}^{(r)}), \pi_{SO_{k-2}}^{(r)}(\xi') \otimes \pi_{Sp_{2n-2}}^{(\kappa r)}(\chi')) \neq (0).$$

Then we are done by induction.

This concludes the proof of Theorem 3. □

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