

Dimension of the Exceptional Set in the Aronszajn–Donoghue Theorem for Finite Rank Perturbations

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The classical Aronszajn–Donoghue theorem states that for a rank-one perturbation of a self-adjoint operator (by a cyclic vector) the singular parts of the spectral measures of the original and perturbed operators are mutually singular. As simple direct sum type examples show, this result does not hold for finite rank perturbations. However, the set of exceptional perturbations is pretty small. Namely, for a family of rank d perturbations $A_\alpha := A + \mathbf{B}\alpha\mathbf{B}^*$, $\mathbf{B} : \mathbb{C}^d \rightarrow \mathcal{H}$, with $\text{Ran}\mathbf{B}$ being cyclic for A , parametrized by $d \times d$ Hermitian matrices α , the singular parts of the spectral measures of A and A_α are mutually singular for all α except for a small exceptional set E . It was shown earlier by the 1st two authors, see [4], that E is a subset of measure zero of the space $\mathbf{H}(d)$ of $d \times d$ Hermitian matrices. In this paper, we show that the set E has small Hausdorff dimension, $\dim E \leq \dim \mathbf{H}(d) - 1 = d^2 - 1$.

1 Introduction

Consider a family of finite rank (self-adjoint) perturbations of a self-adjoint operator A (possibly unbounded),

$$A_\alpha := A + \mathbf{B}\alpha\mathbf{B}^*, \quad (1.1)$$

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parametrized by self-adjoint operators (Hermitian matrices) $\alpha : \mathbb{C}^d \rightarrow \mathbb{C}^d$. Here we assume that $\mathbf{B} : \mathbb{C}^d \rightarrow \mathcal{H}$ is an injective operator. We do not need to assume that \mathbf{B} is bounded, it is sufficient to assume that the operator $(\mathbf{I} + |A|)^{-1/2}\mathbf{B}$ is bounded. In this case, we are dealing with the so-called “form bounded” perturbations; the theory of such perturbations is well developed and does not differ much from the case of bounded perturbations (see e.g., [1]).

Isolating the interesting from the perturbation theory point of view case, we always assume that $\text{Ran}\mathbf{B}$ is cyclic for A . In the case of rank-one perturbations ($d = 1$), the classical Aronszajn–Donoghue theorem states that the singular parts of the spectral measures of A and A_α are always mutually singular.

As simple direct sum examples show, this is not the case for $d > 1$. So, the singular parts of the spectral measures of the original and perturbations and the singular parts of the *scalar* spectral measures of A and A_α are not always mutually singular. However, it was proved in [4] that they are mutually singular for almost all perturbations.

Moreover, it was proved in [4] that if $\alpha_1 > \mathbf{0}$ (i.e., positive definite) and $\alpha(t) = \alpha_0 + t\alpha_1$, then given a singular measure ν the spectral measures μ^t (equivalently their singular parts μ_s^t) of the operators $A_{\alpha(t)}$ are mutually singular with ν for all $t \in \mathbb{R}$ except, maybe, countably many.

This leads one to suspect that in fact one can say more about the exceptional set, i.e., the set of all Hermitian $d \times d$ matrices α for which the singular parts of the scalar spectral measures (by a scalar spectral measure, we always mean a scalar spectral measure of maximal spectral type) of A and of A_α are *not mutually singular*. It looks like a reasonable conjecture that the exceptional set is not just a set of measure 0, but it in fact has dimension strictly less than the full dimension d^2 of the set $\mathbf{H}(d)$ of all $d \times d$ Hermitian matrices. (It is not hard to see that the Hausdorff dimension of the set $\mathbf{H}(d)$ of all $d \times d$ Hermitian matrices is exactly d^2 .)

This turns out to be the case; the main result of this note is the following theorem:

Theorem 1.1. Let operators A_α be given by (1), and let $\text{Ran}\mathbf{B}$ be cyclic for A . Given a singular measure ν the scalar spectral measures μ^α of the operators A_α are mutually singular with ν for all $\alpha \in \mathbf{H}(d) \setminus E$, where the exceptional set E has Hausdorff dimension at most $\dim \mathbf{H}(d) - 1 = d^2 - 1$.

2 An Application of the Marstrand–Mattila Theorem

The tool to show that the dimension of the exceptional set is at most $\dim \mathbf{H}(d) - 1 = d^2 - 1$ is already available. Namely, the following result was proved in [6, Lemma 6.4]. Below, \mathcal{H}^s denotes the s -dimensional Hausdorff measure, and $G(m, n)$ denotes the set of all m -dimensional subspaces of \mathbb{R}^n .

Lemma 1. Let E be an \mathcal{H}^s measurable subset of \mathbb{R}^n with $0 < \mathcal{H}^s(E) < \infty$. Then

$$\dim(E \cap (V + x)) \geq s + m - n$$

for almost all $(x, V) \in E \times G(m, n)$.

In this lemma, \mathcal{H}^s measurable means *Carathéodory measurable* with respect to the outer measure \mathcal{H}^s .

This result was proved by Marstrand [5] for $n = 2$ and by Mattila [6] for general $n \in \mathbb{N}$.

A formal application of this result would immediately give us the desired estimate on the dimension (see the reasoning at the end of this section). However, we do not know anything about the exceptional set E (see the definition below in Section 3); we do not know whether it is \mathcal{H}^s measurable for $s > d^2 - 1$. But what is more important, we cannot say that $\mathcal{H}^s(E) < \infty$ for $s > d^2 - 1$.

However, as it was discussed in [6], if one assumes that E is an analytic (a.k.a. a Suslin) set, one can reduce the assumption to $\mathcal{H}^s(E) > 0$. The reason for this reduction is that, by the theorem of Davies [3], given an analytic set $E \subset \mathbb{R}^n$ with $\mathcal{H}^s(E) > 0$ one can find a compact $K \subset E$ with $0 < \mathcal{H}^s(E) < \infty$.

So, although it was not stated explicitly, the following statement was proved in [6].

Lemma 2. Let E be an analytic (Suslin) subset of \mathbb{R}^n such that $\mathcal{H}^s(E) > 0$. Then

$$\dim(E \cap (V + x)) \geq s + m - n$$

for almost all $(x, V) \in E \times G(m, n)$.

We will not be giving the definition of analytic (Suslin) sets; for our purposes, it is sufficient to know that every Borel set in \mathbb{R}^n is analytic, cf. [2, Section 6.6].

We do not know if the exceptional set E is analytic. However, we will be able to prove the following statement, which essentially is the main result of this paper.

Theorem 2.1. There exists a Borel set $\tilde{E} \subset \mathbf{H}(d)$ such that $E \subset \tilde{E}$ and such that the intersection of \tilde{E} with any line with the direction from the open cone of positive definite Hermitian matrices is at most countable.

Proof. Proof of Theorem 1.1 using Theorem 2.1 For a moment let us assume that $\dim \tilde{E} > \dim \mathbf{H}(d) - 1 = d^2 - 1$. Then $\mathcal{H}_s(\tilde{E}) > 0$ for some $s > d^2 - 1$. Applying Lemma 2 with $n = \dim \mathbf{H}(d) = d^2$ and $m = 1$ we see that for almost all lines in $\mathbf{H}(d)$ of the form $\alpha_0 + t\alpha$, $\alpha_0 \in \tilde{E}$, $\alpha \in \mathbf{H}(d)$, their intersection with the extended exceptional set \tilde{E} should have positive Hausdorff dimension (at least $s + 1 - d^2 > 0$). But as we just discussed above, for all such lines L with the directions α in the open cone of positive definite matrices α (i.e., for a set of non-zero measure), we have at most countable intersection, so the dimension of the intersection is 0. This gives a contradiction, and therefore $\dim \tilde{E} \leq \dim \mathbf{H}(d) - 1 = d^2 - 1$. \blacksquare

3 Preliminaries: Spectral Measures and the Exceptional Set

For the operator A with cyclic set, $\text{Ran } B$ define its matrix-valued spectral measure \mathbf{M} with values in $\mathbf{H}(d)$ as the unique measure satisfying

$$\mathbf{B}^*(A - z\mathbf{I})^{-1}\mathbf{B} = \int_{\mathbb{R}} (s - z)^{-1} d\mathbf{M}(s) =: \mathcal{CM}(z);$$

the spectral measures \mathbf{M}_α , $\alpha \in \mathbf{H}(d)$ are defined the same way with A replaced by A_α .

It follows from the standard resolvent identities that the Cauchy transforms \mathcal{CM} and \mathcal{CM}_α are related by the following well known formula

$$\mathcal{CM}_\alpha = \mathcal{CM}(\mathbf{I} + \alpha(\mathcal{CM}))^{-1} = (\mathbf{I} + (\mathcal{CM})\alpha)^{-1}\mathcal{CM}. \quad (3.1)$$

For a proof of these relations, see for example, [4, Lemma 3.1].

Define the scalar measure $\mu^\alpha := \text{tr } \mathbf{M}_\alpha$. Clearly \mathbf{M}_α is absolutely continuous with respect to μ^α ,

$$d\mathbf{M}_\alpha = W_\alpha d\mu^\alpha, \quad \|W(s)\| \leq 1 \quad \mu^\alpha\text{-a.e.}$$

It is not hard to see that μ^α is a scalar spectral measure of the operator A_α (recall that by a scalar spectral measure we always mean a scalar spectral measure of maximal type).

Recall that a scalar spectral measure is not unique, it is defined up to a multiplication by a non-vanishing weight. So the *spectral type* $[\mu^\alpha]$ of μ^α , that is the equivalence class of all mutually absolutely continuous with μ^α measures, gives us all possible scalar spectral measures of A_α . For our purposes, it does not matter which representative we choose, and μ^α is a convenient choice.

For a fixed finite singular measure ν on \mathbb{R} , define the exceptional set $E = E(\nu)$ to be the set of all $\alpha \in \mathbf{H}(d)$ for which the measures μ^α and ν are not mutually singular. Note that in the definition of E we can replace μ^α by its singular part, and the resulting set will be exactly the same.

If ν is the singular part of the spectral measure μ^{α_0} , then the exceptional set E is exactly the set of all $\alpha \in \mathbf{H}(d)$ such that the singular parts of μ^{α_0} and μ^α are *not mutually singular* for some α .

Note that the set E is explicitly defined, not just up to a set of measure zero. In other words, for each $\alpha \in \mathbf{H}(d)$ one can always say if $\alpha \in E$ or not.

However, we do not know whether the set E is Borel, or even a Suslin (analytic) set. In particular, we cannot directly approach the set E using measure theoretic tools. We bypass this problem by constructing a bigger set \tilde{E} , which is Borel, see the above Theorem 2.1.

Recall that for a (say finite) Borel measure μ on \mathbb{R} , its Poisson extension to the upper half-plane \mathbb{C}_+ (which we slightly abusing notation, denote as $\mu(z)$) is given as

$$\mu(z) = \pi^{-1} \operatorname{Im} \mathcal{C}\mu(z), \quad z \in \mathbb{C}_+,$$

where $\mathcal{C}\mu$ is the Cauchy transform of the measure μ ,

$$\mathcal{C}\mu(z) = \int_{\mathbb{R}} (s - z)^{-1} d\mu(s), \quad z \in \mathbb{C}_+.$$

Similarly, for a (say again finite) matrix measure \mathbf{M} , its Poisson extension $\mathbf{M}(z)$ to the point $z \in \mathbb{C}_+$ is given by

$$\mathbf{M}(z) = \pi^{-1} \operatorname{Im} \mathcal{C}\mathbf{M}(z), \quad \operatorname{Im} \mathcal{C}\mathbf{M}(z) = (2i)^{-1} (\mathcal{C}\mathbf{M}(z) - \mathcal{C}\mathbf{M}(z)^*).$$

This formula, together with (1), implies that the functions $(\alpha, z) \mapsto \mathbf{M}_\alpha(z)$ and $(\alpha, z) \mapsto \mu^\alpha(z) = \operatorname{tr} \mathbf{M}_\alpha(z)$ are continuous functions of the arguments $\alpha \in \mathbf{H}(d)$ and $z \in \mathbb{C}_+$.

4 Some Measure Theory

It is well known (see e.g., [4, Part (ii) of Theorem 3.4]) that for a finite non-negative measure μ on \mathbb{R} its singular part μ_s is *carried* by the set S_μ of all $x \in \mathbb{R}$ for which

$$\lim_{z \rightarrow x \leftarrow} \mu(z) = +\infty;$$

here by limit we understand the non-tangential limit, and $\mu(z)$ is the Poisson extension of the measure μ at the point $z \in \mathbb{C}_+$. The term *carried* here means that $\mu_s(\mathbb{R} \setminus S_\mu) = 0$.

Note that if we pick a *reasonable* sequence $y_n \downarrow 0$, for example, $y_n = 2^{-n}$ (or $y_n = 1/n$), then it follows easily from Harnack's inequality that for any aperture of the approach region

$$\lim_{z \rightarrow x \leftarrow} \mu(z) = +\infty \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} \mu(x + iy_n) = +\infty;$$

this equivalence holds for *all* $x \in \mathbb{R}$. In particular, this means that the set S_μ is always a Borel set.

Let us fix such a "reasonable" sequence y_n . Define the set $F^1 \subset \mathbf{H}(d) \times \mathbb{R}$, consisting of all pairs (α, x) such that

$$\lim_{n \rightarrow \infty} \mu^\alpha(x + iy_n) = +\infty.$$

As we discussed at the end of Section 3, $(\alpha, z) \mapsto \mu^\alpha(z)$ is a continuous (and so Borel measurable) function of the variables $\alpha \in \mathbf{H}(d)$, $z \in \mathbb{C}_+$. Therefore, the functions $(\alpha, x) \mapsto \mu^\alpha(x + iy_n)$ are Borel measurable functions of the variables $(\alpha, x) \in \mathbf{H}(d) \times \mathbb{R}$, so the set F^1 is a Borel subset of $\mathbf{H}(d) \times \mathbb{R}$.

It is well known that if $d\nu = w d\mu$, then

$$\lim_{z \rightarrow x \leftarrow} \nu(z)/\mu(z) = w(x) \quad \mu\text{-a.e.}$$

This implies that for the "reasonable" sequence of y_n we picked above, we have

$$\lim_{n \rightarrow \infty} \nu(x + iy_n)/\mu(x + iy_n) = w(x) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}.$$

As we discussed above, $d\mathbf{M}_\alpha = W_\alpha d\mu^\alpha$, μ^α -a.e., so for any fixed $\alpha \in \mathbf{H}(d)$

$$\lim_{n \rightarrow \infty} \mathbf{M}_\alpha(x + iy_n)/\mu^\alpha(x + iy_n) = W_\alpha(x) \quad \mu^\alpha\text{-a.e.} \quad (4.1)$$

Define the set $F^2 \subset \mathbf{H}(d) \times \mathbb{R}$ to be the set of all pairs (α, x) such that the limit $\lim_{n \rightarrow \infty} \mathbf{M}_\alpha(x + iy_n)/\mu^\alpha(x + iy_n)$ exists and is non-zero. Again, this set is clearly Borel.

Moreover, if for a set $F \subset \mathbf{H}(d) \times \mathbb{R}$, we denote by F_α its section,

$$F_\alpha := \{x \in \mathbb{R} : (\alpha, x) \in F\},$$

then for any $\alpha \in \mathbf{H}(d)$ we have

$$d\mathbf{M}_\alpha = \mathbf{1}_{F_\alpha^2} W_\alpha d\mu^\alpha \quad (4.2)$$

(i.e., defining the density W_α , we can ignore the set where the limit (1) does not exists or equals $\mathbf{0}$).

Noticing that for any Borel $E \subset \mathbb{R}$ the measure $\mu^\alpha(E) > 0$ if and only if $\mathbf{M}_\alpha(E) \neq \mathbf{0}$, we get the following statement.

Let $F = F^1 \cap F^2$, so according to our notation $F_\alpha = F_\alpha^1 \cap F_\alpha^2$.

Lemma 3. For any $\alpha \in \mathbf{H}(d)$, the singular part of μ^α is carried by F_α , meaning that $\mu_s^\alpha(\mathbb{R} \setminus F_\alpha) = 0$. This implies, in particular, that given a singular measure ν the measures ν and μ^α (equivalently ν and the singular part of μ^α) are not mutually singular only if $\nu(F_\alpha) > 0$.

Proof. As we discussed above, the singular part of μ^α is supported on the set F_α^1 . Formula (2) implies that $\mathbf{M}_\alpha(F_\alpha^1 \setminus F_\alpha^2) = \mathbf{0}$. Since $\mu^\alpha = \text{tr}\mathbf{M}_\alpha$, we conclude that $\mu^\alpha(F_\alpha^1 \setminus F_\alpha^2) = 0$. So the singular part of μ^α is indeed supported on F_α .

The 2nd statement follows trivially. ■

Lemma 4. Let $\tilde{E} = \tilde{E}(\nu)$ be the set of all $\alpha \in \mathbf{H}(d)$ such that $\nu(F_\alpha) > 0$. Then \tilde{E} contains the exceptional set E and \tilde{E} is Borel.

Proof. The containment $E \subset \tilde{E}$ follows from Lemma 3. The Borel measurability of \tilde{E} is an immediate corollary of the Tonelli theorem. ■

5 Directional Support of the Singular Part

Lemma 5. Suppose that for $\alpha \in \mathbf{H}(d)$ and for $x \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} \mu(x + iy_n) = +\infty, \quad \lim_{n \rightarrow \infty} \mu^\alpha(x + iy_n) = +\infty, \quad (5.1)$$

and that the limits

$$\lim_{n \rightarrow \infty} \mathbf{M}(x + iy_n)/\mu(x + iy_n) =: W(x), \quad (5.2)$$

$$\lim_{n \rightarrow \infty} \mathbf{M}_\alpha(x + iy_n)/\mu^\alpha(x + iy_n) =: W_\alpha(x)$$

exist and are non-zero.

Then

$$\text{Ran}W(x) \perp \alpha \text{Ran}W_\alpha(x) \quad \text{or, equivalently,} \quad \alpha \text{Ran}W(x) \perp \text{Ran}W_\alpha(x).$$

Remark. We do not need to assume that the limits are non-zero: if one of the limits is zero, the statement is trivial.

Remark 1. The above Lemma 5 looks very much like [4, Theorem 6.2], and the proof below is essentially the proof from [4]. But the important difference is that in [4] the orthogonality condition was satisfied $\mu_s + \mu_s^\alpha$ a.e., but in Lemma 5 we need it to hold for *all* x satisfying (1) and (2). Since the devil is often in details, we present a complete proof below.

Proof of Lemma 5 By [4, Theorem 6.7], the matrix measures \mathbf{M} and $\tilde{\mathbf{M}}_\alpha := \alpha \mathbf{M}_\alpha \alpha$ satisfy the joint *two weight matrix A_2 condition*, that is,

$$\left\| \mathbf{M}(z)^{1/2} \tilde{\mathbf{M}}_\alpha(z)^{1/2} \right\| \leq C < \infty \quad \forall z \in \mathbb{C}_+.$$

We can rewrite this inequality as

$$(\mu(z)\mu^\alpha(z))^{1/2} \left\| \left(\mathbf{M}(z)/\mu(z) \right)^{1/2} \left(\tilde{\mathbf{M}}_\alpha(z)/\mu^\alpha(z) \right)^{1/2} \right\| \leq C < \infty \quad \forall z \in \mathbb{C}_+. \quad (5.3)$$

Let us substitute $z = z_n = x + iy_n$ from the statement of the lemma into (3) and take the limit as $n \rightarrow \infty$. Taking (1) into account, we can conclude from (3) that

$$\lim_{n \rightarrow \infty} \left\| \left(\mathbf{M}(z_n)/\mu(z_n) \right)^{1/2} \left(\tilde{\mathbf{M}}_\alpha(z_n)/\mu^\alpha(z_n) \right)^{1/2} \right\| = 0. \quad (5.4)$$

It follows from the identities (2) that

$$\lim_{n \rightarrow \infty} \mathbf{M}(z_n)/\mu(z_n) = W(x), \quad \lim_{n \rightarrow \infty} \tilde{\mathbf{M}}_\alpha(z_n)/\mu^\alpha(z_n) = \alpha W_\alpha(x) \alpha,$$

so the limit in (4) is exactly

$$\left\| (W(x))^{1/2} (\alpha W_\alpha(x) \alpha)^{1/2} \right\| = 0.$$

But the last identity could happen only if the ranges of (self-adjoint) matrices $W(x)$ and $\alpha W_\alpha(x) \alpha$ are orthogonal. ■

6 Countable Intersections with Extended Exceptional Set

In the theorem below, F_α is defined as in Section 4.

Theorem 6.1. Let $\alpha, \alpha_0 \in H(d)$ with $\alpha > 0$, and let $\alpha(t) = \alpha_0 + t\alpha$. Then, given a finite singular Borel measure ν on \mathbb{R} , for all t except maybe countably many

$$\nu(F_{\alpha(t)}) = 0.$$

We need the following lemma, the trivial proof of which we chose to omit.

Lemma 6. Let A be a positive definite matrix. There exists a constant $c = c(A)$ depending on A such that the condition $(Ax, y) = 0$ implies

$$\|x - y\|^2 \geq c(A)(\|x\|^2 + \|y\|^2).$$

Proof. Proof of Theorem 6.1 using Lemma 6. Pick $t, t' \in \mathbb{R}$ such that $\nu(F_{\alpha(t)}) \neq 0$ and $\nu(F_{\alpha(t')}) \neq 0$. Recall that for all $(\alpha, x) \in F$ (where F is as defined in Section 4), the limit

$$\lim_{n \rightarrow \infty} \mathbf{M}_\alpha(x + iy_n) / \mu^\alpha(x + iy_n) =: W_\alpha(x)$$

exists and is non-zero.

We had shown in Section 4 that the set F is Borel measurable and that the function $(\alpha, x) \mapsto W_\alpha(x)$ is a measurable function on F (as a limit of a sequence of continuous functions). Extending it by 0 outside of F , we will get a measurable function defined on the whole $H(d)$.

It is then an easy exercise to show that we can find a vector-valued Borel measurable function $(\alpha, x) \mapsto \Phi(\alpha, x) \in \text{Ran } W_\alpha(x) \subset \mathbb{C}^d$ such that $\Phi(\alpha, x) \neq 0$ if and only if $(\alpha, x) \in F$. Multiplying this function by an appropriate measurable function depending on α only we can assume that without loss of generality

$$\int_{\mathbb{R}} \|\Phi(\alpha, x)\|_{\mathbb{C}^d}^2 d\nu(x) = 1$$

whenever $\nu(F_\alpha) \neq 0$; if $\nu(F_\alpha) = 0$ the integral is trivially 0.

Define $f(x) := \Phi(\alpha(t), x)$, $g(x) := \Phi(\alpha(t'), x)$. Note that

$$\|f\|_{L^2(\nu)} = \|g\|_{L^2(\nu)} = 1.$$

Since $\alpha(t') = \alpha(t) + (t' - t)\alpha$, Lemma 5 (applied to $W_{\alpha(t)}$ instead of W and $W_{\alpha(t')}$ instead of W_α) implies that $\alpha \text{Ran} W_{\alpha(t)} \perp \text{Ran} W_{\alpha(t')}$, so

$$(\alpha f(x), g(x))_{\mathbb{C}^d} = 0 \quad \forall x \in \mathbb{R};$$

if both points $(\alpha(t), x)$ and $(\alpha(t'), x)$ are in F , this follows from Lemma 5; if not, this is trivial because one of the vectors $f(x), g(x)$ is zero.

Applying Lemma 6, we get that for all $x \in \mathbb{R}$

$$\|f(x) - g(x)\|_{\mathbb{C}^d}^2 \geq c \cdot \left(\|f(x)\|_{\mathbb{C}^d}^2 + \|g(x)\|_{\mathbb{C}^d}^2 \right).$$

Integrating this inequality, we see that

$$\|f - g\|_{L^2(\nu)}^2 \geq c \cdot \left(\|f(x)\|_{L^2(\nu)}^2 + \|g(x)\|_{L^2(\nu)}^2 \right) = c.$$

So, for all $t \in \mathbb{R}$ such that $\nu(F_{\alpha(t)}) \neq 0$, we constructed unit vectors $f_t = \Phi(\alpha(t), \cdot) \in L^2(\nu) = L^2(\nu; \mathbb{C}^d)$ such that the distance between any two such vectors is at least some fixed $c > 0$. Since the space $L^2(\nu; \mathbb{C}^d)$ is separable, there can be at most countably many such $t \in \mathbb{R}$. ■

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