RANDOM VEERING TRIANGULATIONS ARE NOT GEOMETRIC

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Abstract. Every pseudo-Anosov mapping class φ defines an associated veering triangulation τ_φ of a punctured mapping torus. We show that generically, τ_φ is not geometric. Here, the word "generic" can be taken either with respect to random walks in mapping class groups or with respect to counting geodesics in moduli space. Tools in the proof include Teichmüller theory, the Ending Lamination Theorem, study of the Thurston norm, and rigorous computation.

1. Introduction

In 2011, Agol introduced the notion of a layered veering triangulation for certain hyperbolic mapping tori [1]. Given a hyperbolic surface S and a pseudo-Anosov homeomorphism $\varphi\colon S \to S$, the mapping torus M_φ with fiber S and monodromy φ is always hyperbolic. Drilling out the singularities of the φ–invariant foliations on S produces a punctured surface $\mathring{S}$ and a restricted pseudo-Anosov map $\mathring{\varphi} = \varphi|_{\mathring{S}}$, whose mapping torus $\mathring{M}_\varphi = M_{\mathring{\varphi}}$ is a surgery parent of M_φ. Agol's construction uses splitting sequences of train tracks to produce an ideal triangulation of $\mathring{M}_\varphi$ (that is, a decomposition of $\mathring{M}_\varphi$ into simplices whose vertices have been removed) called the **veering triangulation associated to** φ.

In Section 2.5, we give a detailed description of the veering triangulation $\tau = \tau_\varphi$ from an alternate point of view, introduced by Guéritaud [22]. For now, we mention that τ has very strong combinatorial and topological properties. The triangulation τ is layered, meaning that every edge is isotopic to an essential arc on the punctured fiber $\mathring{S}$. The triangulation τ contains a product region $\Sigma \times I$ for every large-distance subsurface $\Sigma \subset \mathring{S}$ [40]. Finally, τ_φ decorated with layering data is a complete invariant of the conjugacy class $[\varphi] \subset \mathrm{Mod}(S)$ [1, Corollary 4.3], which yields a fast practical solution to the conjugacy problem for pseudo-Anosovs [4, 34]. Given these combinatorial properties, it is natural to ask whether τ also has desirable geometric properties in the complete hyperbolic metric on $\mathring{M}_\varphi$.

Since every edge of τ is homotopically non-trivial, it is possible to homotope every ideal tetrahedron $\mathfrak{t} \subset \tau$ to a straight simplex $\mathfrak{t}'$, whose lift to the universal cover $\mathbb{H}^3$ is the convex hull of 4 points on $\partial\mathbb{H}^3$. This homotopy is natural, in the sense that it extends continuously to all of τ. The triangulation τ is called **geometric** if the straightening homotopy can be accomplished by isotopy. Equivalently, τ is called geometric if the complete hyperbolic structure on $\mathring{M}_\varphi$ can be obtained by taking positively oriented tetrahedra in $\mathbb{H}^3$ in bijection with the 3–simplices of τ, and gluing them by isometry in the combinatorial pattern of τ.

Agol asked whether veering triangulations are always geometric [1, Section 5]. Hodgson, Issa, and Segerman showed that the answer can be negative [25], by finding a veering triangulation with 13 tetrahedra, in which one tetrahedron is negatively oriented. (In the

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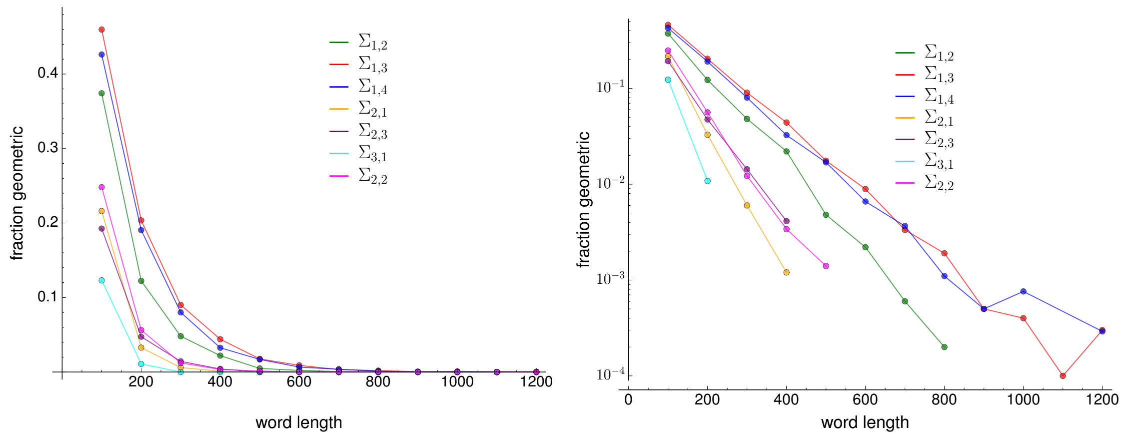

FIGURE 1. For a simple random walk in $\mathrm{Mod}(S)$, with generators shown in Figure 4, the probability that the veering triangulation is geometric decays exponentially with the length of the walk. Both graphs show the same data, with a linear plot on the left and a log-linear plot on the right. Each dot represents several thousand mapping classes. Figure from Worden [49].

straightening homotopy, two opposite edges of this tetrahedron must pass through each other before the tetrahedron can become straight.) In describing their example, they write,

> *It seems unlikely that a counterexample would have been found without a computer search, and it is still something of a mystery why veering triangulations are so frequently geometric.*

It is now clear that geometric veering triangulations are exceedingly rare. This was shown experimentally by Worden [49], who tested over 800,000 examples on a high-performance computing cluster. Given a hyperbolic surface S of complexity $\xi(S) \geqslant 2$, he found that for randomly sampled long words in $\mathrm{Mod}(S)$, the probability of the associated veering triangulation being geometric decays exponentially with the length of the word. See Figure 1.

The main result of this paper is a proof that the pattern of Figure 1 is indeed correct. We prove this in two separate probabilisitic regimes: first, with respect to random walks on $\mathrm{Mod}(S)$ (Theorem 1.1), and second, with respect to counting closed geodesics in moduli space (Theorem 1.2).

We use the symbol $\Sigma_{g,n}$ to denote the surface of genus g with n punctures. Every surface S mentioned below is presumed homeomorphic to some $\Sigma_{g,n}$; in particular, S is presumed connected and orientable. We define the **complexity** $\xi(\Sigma_{g,n}) = 3g - 3 + n$.

For a surface S as above, we show that with overwhelming probability, a random walk on $\mathrm{Mod}(S)$ produces a pseudo-Anosov mapping class with non-geometric veering triangulation.

Theorem 1.1. *Let S be a surface of complexity $\xi(S) \geqslant 2$, and consider a simple random walk on $\mathrm{Mod}(S)$ with respect to any finite generating set. Then, for almost every infinite sample path (φ_n), there is a positive integer n_0 such that for all $n \geqslant n_0$, the mapping class φ_n is pseudo-Anosov and the veering triangulation of $\mathring{M}_{\varphi_n}$ is non-geometric.*

In fact, the same result holds true for sample paths defined by a more general probability measure. See Corollary 1.5 for a precise statement.

We remark that every pseudo-Anosov on a surface satisfying $\xi(S) < 2$ has geometric veering triangulation. (See Theorem 1.3 and the ensuing discussion.) Thus Theorem 1.1 applies to the largest possible collection of surfaces.

We will prove Theorem 1.1 by combining two separate, logically independent ingredients. The first ingredient is Theorem 1.3: when $\xi(S) \geqslant 2$, there is at least one principal mapping class on S whose associated veering triangulation is non-geometric. A pseudo-Anosov mapping class $\varphi \in \mathrm{Mod}(S)$ is called **principal** if its invariant Teichmüller geodesic lies in the principal stratum. (See Section 2.1 for a discussion of strata and Teichmüller geodesics.) Equivalently, φ is principal if its stable foliation has 3–prong singularities at interior points of S and 1–prong singularities at punctures of S. By a theorem of Gadre and Maher [20], principal pseudo-Anosovs are generic from the point of view of random walks in $\mathrm{Mod}(S)$.

The second ingredient is a convergence result, Theorem 1.4, which shows that every principal mapping class occurs in a suitable sense as the limit of a random process, where the combinatorics of the triangulation and the geometry of the mapping torus both converge to the desired limit. This result works for any hyperbolic surface. In particular, given a principal mapping class φ with non-geometric veering triangulation, almost every sample path of a random walk also has non-geometric veering triangulation.

By replacing random walk techniques with work of Hamenstädt [24] and Eskin–Mirzakhani [13], we prove our second result concerning the scarcity of geometric veering triangulations. For $L > 0$, let $\mathcal{G}(L)$ be the finite set of conjugacy classes of pseudo-Anosov mapping classes in $\mathrm{Mod}(S)$ whose Teichmüller translation length is at most L. Equivalently, $\mathcal{G}(L)$ is the set of all conjugacy classes of pseudo-Anosovs whose dilatation is at most e^L. Recall that the veering triangulation τ_φ of the punctured mapping torus $\mathring{M}_\varphi$ only depends on the conjugacy class φ, i.e. on an element of $\mathcal{G}(L)$ for some L.

Theorem 1.2. *Let S be a surface with complexity $\xi(S) \geqslant 2$. Then*

$$\lim_{L\to\infty} \frac{1}{|\mathcal{G}(L)|} \left| \left\{ [\varphi] \in \mathcal{G}(L) : \textit{the veering triangulation of } \mathring{M}_\varphi \textit{ is not geometric} \right\} \right| = 1.$$

Just as with Theorem 1.1, the proof of Theorem 1.2 combines an existence statement with a convergence statement. The existence statement is again Theorem 1.3: there is a principal mapping class $\varphi \in \mathrm{Mod}(S)$ whose associated veering triangulation is non-geometric. The convergence statement roughly says that the axis of a typical element of $\mathcal{G}(L)$ fellow-travels the axis of φ for a very long distance. This statement, combined with ingredients from the proof of Theorem 1.4, implies the desired result. We refer to Section 7 for more details.

1.1. Existence of non-geometric triangulations.

As described above, we begin the proof of Theorems 1.1 and 1.2 by finding some pseudo-Anosov element of $\mathrm{Mod}(S)$ whose associated veering triangulation is non-geometric. In fact, we show the following.

Theorem 1.3. *Let $S \cong \Sigma_{g,n}$ be a hyperbolic surface. Then $\xi(S) \geqslant 2$ if and only if there exists a principal pseudo-Anosov $\varphi \in \mathrm{Mod}(S)$ such that the associated veering triangulation of the mapping torus $\mathring{M}_\varphi$ is non-geometric.*

The "if" direction of Theorem 1.3 is previously known. The only (connected, orientable) hyperbolic surfaces with $\xi(S) < 2$ are $\Sigma_{0,3}, \Sigma_{0,4}$, and $\Sigma_{1,1}$. Akiyoshi [2] and Lackenby [30] proved that all pseudo-Anosov mapping classes on $\Sigma_{1,1}$ and $\Sigma_{0,4}$ have geometric veering triangulations. Guéritaud gave a direct argument for the same conclusion [21]. Meanwhile, $\mathrm{Mod}(\Sigma_{0,3})$ is finite, hence $\Sigma_{0,3}$ has no pseudo-Anosov mapping classes at all. Thus the new content of Theorem 1.3 is the "only if" direction of the statement.

In the proof of Theorem 1.3, we characterize geometric triangulations using shape parameters. Given a tetrahedron $\mathfrak{t} \subset \mathring{M}$, endowed with an ordering of its ideal vertices, we lift $\mathfrak{t}$ to $\mathbb{H}^3$ and define the **shape parameter** $z_\mathfrak{t}$ to be the cross-ratio of the 4 vertices on the sphere at infinity. The cross-ratio $z_\mathfrak{t}$ determines the isometry type of the straightened tetrahedron $\mathfrak{t}'$ homotopic to $\mathfrak{t}$. In particular, $\mathfrak{t}'$ is positively oriented if and only if $\mathrm{Im}(z_\mathfrak{t}) > 0$. Shape parameters can be computed using `Snappy` [10], making it possible to test whether a particular triangulation τ is geometric.

To prove the "only if" direction of Theorem 1.3 for a finite list of fiber surfaces, we essentially follow the method of Hodgson–Issa–Segerman [25]. We find a suitable mapping class $\varphi \in \mathrm{Mod}(S)$ using a brute-force search, and use `flipper` [4] to certify that φ is a principal pseudo-Anosov. Then, we use rigorous interval arithmetic in `Snappy`, assisted by `HIKMOT` [27], to certify that the shape parameter of each $\mathfrak{t} \subset \tau_\varphi$ lies inside a small box in $\mathbb{C}$. One of these boxes has strictly negative imaginary part, implying that τ_φ is non-geometric. See Section 8 for details.

To extend our knowledge from finitely many surfaces to *all* the surfaces in Theorem 1.3, we exploit the fact that many fibered 3–manifolds fiber in infinitely many ways, organized via the Thurston norm. (See Section 9 for definitions and further details). All the fibers that appear in a single fibered cone of the Thurston norm ball have associated monodromies that induce the same veering triangulation of the same drilled manifold $\mathring{M}$. As a consequence, we can prove Theorem 1.3 for all $g \geqslant 1$, $n \geqslant 1$ (excluding $\Sigma_{1,1}$), using only two explicit examples. That is, we find two fibered manifolds with principal pseudo-Anosovs whose veering triangulations are non-geometric, and show that every such surface $\Sigma_{g,n}$, appears as (a cover of) a fiber for at least one of our two examples. Similar tricks handle the other surfaces with $\xi(S) \geqslant 2$.

1.2. Convergence to any principal pseudo-Anosov.

The following convergence theorem is the main technical result of this paper. Although the statement here is for the random walk model, it is derived from a more general result (Proposition 6.2) that also applies to counting geodesics in moduli space.

Consider a probability measure μ on $\mathrm{Mod}(S)$. We use the notation $\langle \mathrm{Supp}(\mu) \rangle_+$ to denote the semigroup generated by the support of μ. Say that $\langle \mathrm{Supp}(\mu) \rangle_+$ is **non-elementary** if it contains at least two pseudo-Anosov elements with distinct axes. In the setting of a simple random walk, μ is the uniform probability measure on a symmetric generating set, hence $\langle \mathrm{Supp}(\mu) \rangle_+ = \mathrm{Mod}(S)$ is non-elementary.

Theorem 1.4. *Let S be a hyperbolic surface, and fix a principal pseudo-Anosov $\varphi \in \mathrm{Mod}(S)$. Lift the veering triangulation τ_φ of the mapping torus $\mathring{M}_\varphi$ to a triangulation $\widetilde{\tau}$ of the infinite cyclic cover $\mathring{N}_\varphi$, corresponding to the fiber. Let $K \subset \widetilde{\tau}$ be any finite, connected sub-complex.*

Let μ be a probability distribution on $\mathrm{Mod}(S)$ with finite first moment, such that $\langle \mathrm{Supp}(\mu) \rangle_+$ is non-elementary and contains φ. Then, for almost every sample path $\omega = (\omega_n)$, there is a positive integer n_0 such that the following hold:

- *For all $n \geqslant n_0$, ω_n is a principal pseudo-Anosov.*
- *For all $n \geqslant n_0$, K embeds as a sub-complex of the veering triangulation τ_{ω_n} of the mapping torus $\mathring{M}_{\omega_n}$.*
- *For every tetrahedron $\mathfrak{t} \subset K$, the shape of $\mathfrak{t}$ in $\mathring{M}_{\omega_n}$ converges to the shape of $\mathfrak{t}$ in $\mathring{N}_\varphi$ as $n \to \infty$.*

The argument used to prove Theorem 1.4 can be summarized as follows. The first two conclusions are proven by combining a result about fellow-traveling of sample paths in Teichmüller space (Theorem 3.1, due to Gadre–Maher [20]), together with Corollary 5.6. Informally, Corollary 5.6 states that if appropriate quadratic differentials q_i converge to q, then their associated veering triangulations also converge, in the sense appearing in Theorem 1.4. A more precise formulation of this result requires Guéritaud's construction of the veering triangulation (given in Section 2.5), and is postponed until Section 5. The upshot is that if φ and ψ are pseudo-Anosov homeomorphisms whose axes in Techmüller space fellow travel for sufficiently long, then their associated veering triangulations of $\mathring{M}_\varphi$ and $\mathring{M}_\psi$ have large isomorphic subcomplexes.

The third conclusion of Theorem 1.4 follows by relating the convergence of the quadratic differentials referenced above to the convergence of the hyperbolic structures on the associated manifolds. The main tool for this is the Ending Lamination Theorem of Brock–Canary–Minsky [39, 8] together with a strengthening by Leininger–Schleimer [31]. In short, algebraic convergence of the surface group representations implies convergence in $\mathbb{H}^3$ of the ideal endpoints of the veering tetrahedra, which means that the shapes of these tetrahedra converge as desired. The details are given in Section 6.

One particular consequence of Theorem 1.4 is the following statement.

Corollary 1.5. *Let S be a hyperbolic surface. Let $\varphi \in \mathrm{Mod}(S)$ be a principal pseudo-Anosov $\varphi \in \mathrm{Mod}(S)$ whose veering triangulation τ_φ is non-geometric.*

Let μ be a probability distribution on $\mathrm{Mod}(S)$ with finite first moment, such that $\langle \mathrm{Supp}(\mu) \rangle_+$ is non-elementary and contains φ. Then, for almost every infinite sample path $\omega = (\omega_n)$, there is a positive integer n_0 such that for all $n \geqslant n_0$, the veering triangulation τ_{ω_n} is also non-geometric.

Proof. Let $K \subset \tilde{\tau}$ be (the lift to $\mathring{N}_\varphi$ of) a single tetrahedron $\mathfrak{t} \subset \tau_\varphi$ whose shape is negatively oriented. Theorem 1.4 says that $\mathfrak{t}$ also appears as a tetrahedron in τ_{ω_n} for $n \gg 0$. Furthermore, the shape of $\mathfrak{t}$ in $\mathring{M}_{\omega_n}$ converges to a negatively oriented limit as $n \to \infty$, hence $\mathfrak{t}$ has to be negatively oriented in $\mathring{M}_{\omega_n}$ for all $n \gg 0$. □

Now, observe that Theorem 1.1 follows immediately by combining Theorem 1.3 with Corollary 1.5. The proof of Theorem 1.2 follows a similar pattern, but replaces Corollary 1.5 with Corollary 7.2. See Section 7 for the full details.

1.3. **Organization.** Section 2 lays out definitions and background material from Teichmüller theory that will be needed in most of the subsequent arguments.

The proof of Theorem 1.4 spans Sections 3 to 6. We discus convergence of quadratic differentials in Sections 3 and 4, convergence of veering triangulations in Section 5, and finally convergence of geometric structures on 3–manifolds in Section 6. In Section 7, we combine these ingredients with measure-theoretic tools to prove Theorem 1.2.

Finally, Sections 8 and 9 contain the proof of Theorem 1.3.

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2. Background

The primary goal of this section is to survey some background material on Teichmüller theory, quadratic differentials, and measured foliations that will be heavily used in the following few sections. The reader is referred to [7, 14, 15, 44] for additional details. After this general background, we describe Guéritaud's construction of veering triangulations from quadratic differentials [22].

Throughout, we let $S = \Sigma_{g,n}$ be a surface of genus g with n punctures, and assume that $\xi(S) = 3g - 3 + n \geqslant 1$. This assumption implies that the Teichmüller space $\mathcal{T}(S)$, which is the space of complex structures on S up to isotopy, has real dimension $2\xi(S) \geqslant 2$.

2.1. Quadratic differentials and strata.

Let $X \in \mathcal{T}(S)$ be a complex structure on S. A **quadratic differential** q on X is a tensor locally defined in coordinates by $q = q(z)dz^2$ for some meromorphic function $q(z)$. The function $q(z)$ is required to be analytic inside S, but is allowed to have simple poles at the punctures of S. The poles and zeros of q are called **singularities**. By changing coordinates, we may assume that $q = dz^2$ in the neighborhood of a regular value of q, and $q = z^k dz^2$ in the neighborhood of a singularity ($k = -1$ for simple poles, and $k > 0$ for zeros). These are called natural coordinates and have the property that, away from singularities of q, the transition functions have the form $z \to \pm z + c$, for some complex number c. In particular, these transition functions preserve the standard Euclidean metric on $\mathbb{C}$.

A quadratic differential q determines a pair of transverse measured foliations $\mathcal{F}_q^-$ and $\mathcal{F}_q^+$, called the **horizontal** and **vertical** foliations. In the above natural coordinates $z = x + iy$ away from the singularities, these foliations are given by setting y and x (respectively) to be constant, with transverse measures $|dy|$ and $|dx|$. Near a zero of order k (where a pole corresponds to $k = -1$), each of the horizontal and vertical foliations has a $(k+2)$–pronged singularity.

Away from singularities, the transverse measures $|dx|$ and $|dy|$ induce a Euclidean metric $\sqrt{|dx|^2 + |dy|^2}$ on S. The completion of this metric on S is known as the **singular flat metric** corresponding to q. The area of S endowed with this metric is denoted $\|q\|$, and defines a norm on the space $\mathcal{QD}(S)$ of quadratic differentials on S. We denote by $\mathcal{QD}^1(S)$ the set of elements in $q \in \mathcal{QD}(S)$ with $\|q\| = 1$. The projection $\mathcal{QD}(S) \to \mathcal{T}(S)$ sending a quadratic differential to its underlying complex structure can be identified with the cotangent bundle of $\mathcal{T}(S)$.

The **principal stratum** of quadratic differentials $\mathcal{GQD}(S)$ is the subset of $\mathcal{QD}(S)$ that consists of all those quadratic differentials whose zeros are of order 1 (that is, 3–prong singularities), and whose punctures are all simple poles (that is, 1–prong singularities). In general, $\mathcal{QD}(S)$ decomposes into strata characterized by the order of the zeros and poles of $q(z)$. When $S \not\cong \Sigma_{1,1}$, the principal stratum is open and dense, while the other strata have positive codimension. (When $S \cong \Sigma_{1,1}$, the principal stratum as previously defined is empty. All nonzero quadratic differentials belong to a single stratum with a single 2–prong singularity at the puncture.)

2.2. Teichmüller geodesics and flows.

We recall the construction of the **Techmüller geodesic flow**, denoted $\Phi^t \colon \mathcal{QD}^1(S) \to \mathcal{QD}^1(S)$. Given a unit-area quadratic differential $q \in \mathcal{QD}^1(S)$, and a number $t \in \mathbb{R}$, the image $\Phi^t(q)$ is defined as follows. The underlying complex structure is $X_t = X_t(q)$, whose coordinate charts (away from singularities) are given by composing the natural coordinates for q with the affine map

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \tag{2.1}$$

Then, $\Phi^t(q) \in \mathcal{QD}^1(S)$ is the quadratic differential on X_t given by dz^2 in these coordinates. The flow Φ^t plays an important role in Section 7.

For a fixed $q \in \mathcal{QD}^1(S)$, the map $\mathbb{R} \to \mathcal{T}(S)$ defined by $t \mapsto X_t(q)$ is called a **Teichmüller geodesic**. Indeed, this line in $\mathcal{T}(S)$ is a parametrized geodesic for the **Teichmüller metric** $d_{\mathcal{T}}$. By Teichmüller's theorem, any pair of points X, Y in $\mathcal{T}(S)$ are joined by a unique segment of a Teichmüller geodesic, of length $d_{\mathcal{T}}(X, Y)$, which we often denote by $[X, Y]$. The map $X = X_0 \to X_t = Y$ defined by Equation (2.1) is called the **Teichmüller map**. If the quadratic differential q associated to a Teichmüller geodesic γ is in the principal stratum $\mathcal{GQD}(S)$, then we will say that γ is in the principal stratum of Teichmüller space.

Consider now a pseudo-Anosov mapping class $\varphi \in \mathrm{Mod}(S)$. Bers [5] showed that φ preserves a unique geodesic axis $\gamma_\varphi \subset \mathcal{T}(S)$ consisting of points $X \in \mathcal{T}(S)$ such that $d_{\mathcal{T}}(X, \varphi(X)) = \log \lambda_\varphi$, where $\lambda_\varphi > 1$ is the dilatation of φ. By Equation (2.1), the geodesic γ_φ corresponds to a one-parameter family q_t of quadratic differentials. The complex structure X_t underlying q_t is a point along γ_φ, and the projective classes of $\mathcal{F}^+(q_t)$ and $\mathcal{F}^-(q_t)$ are constant and equal to the invariant foliations of φ. If some (hence every) q_t lies in the principal stratum $\mathcal{GQD}(S)$, we say that φ is a **principal pseudo-Anosov**.

2.3. **Curves, foliations, and laminations.** One can study how conformal structures change along Teichmüller geodesics by understanding what happens to the lengths of curves and arcs. This perspective will be important in Section 3.

The **arc and curve graph** $\mathcal{AC}(S)$ is the graph whose vertices are isotopy classes of essential simple closed curves and simple proper arcs in S. Here, *essential* means that the curve or arc is not isotopic into a small neighborhood of a point or a puncture. Two vertices are joined by an edge in $\mathcal{AC}(S)$ if they have disjoint representatives. If we follow the same construction with vertices restricted to be closed curves on S, we obtain the **curve graph** $\mathcal{C}(S) \subset \mathcal{AC}(S)$, and similarly restricting to arcs yields the **arc graph** $\mathcal{A}(S) \subset \mathcal{AC}(S)$.

We have already encountered measured foliations as the vertical and horizontal foliations of a quadratic differential. A **singular measured foliation** $\mathcal{F}$ on S is a singular foliation endowed with a transverse measure (see [15] for a more thorough definition). A Whitehead move on a foliation $\mathcal{F}$ introduces or contracts a compact singular leaf on $\mathcal{F}$, by either splitting a singularity into a pair of singularities joined by a compact leaf, or by contracting such a leaf to collapse two singularities into one. In general, we let $\mathcal{MF}(S)$ denote Thurston's space of measured foliations of S, up to Whitehead equivalence. The space $\mathcal{PMF}(S)$ of **projective measured foliations** is obtained from $\mathcal{MF}(S)$ by identifying measures which differ by scaling. If the underlying topological singular foliation of $\mathcal{F}$ supports a unique projective measure class, then $\mathcal{F}$ is called **uniquely ergodic**. The subspace of uniquely ergodic foliations is denoted $\mathcal{UE}(S) \subset \mathcal{PMF}(S)$.

By the Uniformization theorem, every conformal structure X is realized by a unique hyperbolic metric. A **geodesic measured lamination** on a hyperbolic surface X is a non-empty collection of disjoint simple geodesics of X whose union is closed in X, along with a transverse measure that is invariant as we flow along the geodesics. There is an exact correspondence between measured laminations and measured foliations (up to Whitehead equivalence). For a precise treatment of this correspondence between foliations and laminations, see Levitt [32]. We denote the space of (geodesic) measured laminations on S by

$\mathcal{ML}(S)$. In analogy with $\mathcal{PMF}(S)$, we define the space $\mathcal{PML}(S)$ of projective measured laminations to be $\mathcal{ML}(S)$ modulo scaling of the measure. We will use the identifications $\mathcal{MF}(S) \cong \mathcal{ML}(S)$ and $\mathcal{PMF}(S) \cong \mathcal{PML}(S)$ without further comment.

Let $\mathcal{C}^0(S)$ be the vertex set of $\mathcal{C}(S)$. Endowing every curve with the counting measure embeds $\mathcal{C}^0(S)$ as a subset of $\mathcal{ML}(S)$. By the above correspondence, we also have a natural embedding $\mathcal{C}^0(S) \subset \mathcal{MF}(S)$. Thurston proved that the projectivization of $\mathcal{C}^0(S)$ is dense in both $\mathcal{PMF}(S)$ and $\mathcal{PML}(S)$. Furthermore, $\mathcal{PMF}(S)$ and $\mathcal{PML}(S)$ are compact [15].

A **filling lamination** is one that intersects every (essential) curve. The space of **ending laminations** of S, denoted $\mathcal{EL}(S)$, is obtained by restricting to the subset of $\mathcal{ML}(S)$ consisting of filling laminations, and quotienting by forgetting the measures. This space plays an important role in the theory of Kleinian groups; see Section 6.

2.4. **Intersection pairing.** Given two vertices $a, b \in \mathcal{AC}^0(S)$, the *geometric intersection number* of a and b is defined to be the minimal number of intersections between any pair of curves/arcs representing of a and b. In symbols,

$$i(a,b) = \min_{\alpha \in a,\, \beta \in b} |\alpha \cap \beta|.$$

Thurston showed that this function extends uniquely to a continuous, homogeneous function $i\colon \mathcal{MF}(S) \times \mathcal{MF}(S) \to \mathbb{R}$, also called the **geometric intersection number**. See [45, 7].

For a quadratic differential q, recall the vertical and horizontal measured foliations $\mathcal{F}_q^+$ and $\mathcal{F}_q^-$. For $a \in \mathcal{AC}(S)$, let $h_q(a)$ denote the (horizontal) length of a with respect to the transverse measure on $\mathcal{F}_q^+$. Similarly, $v_q(a)$ denotes the (vertical) length of a with respect to the transverse measure on $\mathcal{F}_q^-$. Then the ℓ^1 **length** of a with respect to the flat structure induced by q is $\ell_q^1(a) = h_q(a) + v_q(a)$. The intersection pairing $i(\cdot,\cdot)$ satisfies

$$h_q(a) = i(\mathcal{F}_q^+, a) \quad \text{and} \quad v_q(a) = i(\mathcal{F}_q^-, a).$$

Hence, $\ell_q^1(\cdot) = i(\mathcal{F}_q^+, \cdot) + i(\mathcal{F}_q^-, \cdot)$ extends to a continuous function on $\mathcal{MF}(S)$. In Section 5, we will need the stronger observation that the pairing

$$i^1\colon \mathcal{QD}(S) \times \mathcal{MF}(S) \to \mathbb{R}.$$

given by $(q, \mathcal{F}) \mapsto \ell_q^1(\mathcal{F})$ is continuous in both parameters. This follows from the continuity of the intersection pairing along with the fact that the assignment $q \mapsto (\mathcal{F}_q^+, \mathcal{F}_q^-)$ induces a homeomorphism $\mathcal{QD}(S) \to \mathrm{Fill}^2 \subset \mathcal{MF}(S) \times \mathcal{MF}(S)$ [28], where $\mathrm{Fill}^2 = \{(\mathcal{F}_1, \mathcal{F}_2) : i(\alpha, \mathcal{F}_1) + i(\alpha, \mathcal{F}_2) > 0 \text{ for all } \alpha \in \mathcal{C}(S)\}$.

2.5. **Veering triangulations.** We close this background section with a description of Guéritaud's construction of veering triangulations [22]. Before giving the details, we note that this was not the original construction. Agol's original definition used periodic train track splitting sequences associated to the invariant foliations of a pseudo-Anosov map [1]. A very quick combinatorial characterization of veering triangulations appears in [26]. See also [18, 40] for other perspectives.

Let S be a surface, and let $q \in \mathcal{QD}(S)$ be a quadratic differential. Let $\mathring{S}$ be the complement of the singularities of q. Then $\mathcal{F}^-$ and $\mathcal{F}^+$, the horizontal and vertical foliations of q, have singularities only at punctures of $\mathring{S}$. Recall that q defines a singular flat metric on S, which restricts to an incomplete metric on $\mathring{S}$. A **saddle connection** of q is a geodesic arc in the singular flat metric on S, with singularities at the endpoints but no singularities in its interior. Every saddle connection naturally yields in arc in $\mathring{S}$. For the following

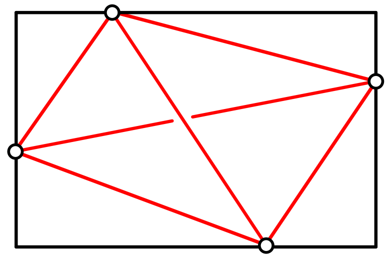

FIGURE 2. Guéritaud's construction: a maximal singularity-free rectangle $\mathcal{R}$ defines an oriented ideal tetrahedron in $\mathring{S} \times \mathbb{R}$ with a projection to $\mathcal{R}$.

construction, we will assume that q has *no horizontal or vertical saddle connections*; that is, no saddle connection is a leaf of $\mathcal{F}^{\pm}$.

Consider an immersed rectangle $\mathcal{R} \to S$, with horizontal boundary mapped to $\mathcal{F}^{-}$, vertical boundary mapped to $\mathcal{F}^{+}$, and interior mapped to $\mathring{S}$. It follows that the interior of $\mathcal{R}$ must miss all singularities of $\mathcal{F}^{\pm}$. We call $\mathcal{R}$ a **maximal (singularity-free) rectangle** of q if it is maximal with respect to inclusion. Since there are no horizontal or vertical saddle connections, every side of a maximal rectangle must meet exactly one puncture of $\mathring{S}$. Observe that the punctures of $\mathring{S}$ must lie at interior points of edges: if a puncture occurred at a corner, $\mathcal{R}$ could be extended, violating maximality. See Figure 2 and [40, Figure 2].

Every maximal singularity-free rectangle $\mathcal{R}$ defines an oriented tetrahedron $\mathfrak{t}$ with a map $\mathfrak{t} \to \mathcal{R}$, as follows. The vertices of $\mathfrak{t}$ map to the four preimages of punctures in $\partial\mathcal{R}$. The edges of $\mathfrak{t}$ map to the six saddle connections spanned by these four vertices. The orientation of $\mathfrak{t}$ is determined by the convention that the more-vertical edge (whose endpoints are on the horizontal edges of $\mathcal{R}$) lies above the more-horizontal edge. See Figure 2.

Performing this construction for all maximal rectangles gives a countable collection of tetrahedra whose vertices map to punctures of $\mathring{S}$. If tetrahedra $\mathfrak{t}$ and $\mathfrak{t}'$ contain the same triple of saddle connections (equivalently, if maximal rectangles $\mathcal{R}$ and $\mathcal{R}'$ intersect along a sub-rectangle that meets three punctures), we glue $\mathfrak{t}$ to $\mathfrak{t}'$ along their shared face. By a theorem of Guéritaud [22] (see also [40, Theorem 2.1]), the resulting 3–complex is an ideal triangulation τ_q of $\mathring{S} \times \mathbb{R}$:

Theorem 2.1 (Guéritaud)**.** *The complex of tetrahedra associated to maximal rectangles of q is an ideal triangulation τ_q of $\mathring{S} \times \mathbb{R}$. The maps of tetrahedra to their defining rectangles piece together to form a fibration $\pi \colon \mathring{S} \times \mathbb{R} \to \mathring{S}$.*

We call τ_q the **veering triangulation associated to** q. Observe that a saddle connection of q corresponds to an edge of τ_q if and only if it spans a singularity-free rectangle. This is because every singularity-free rectangle can be expanded to a maximal one.

Now, suppose that the quadratic differential q corresponds to a pseudo-Anosov homeomorphism $\varphi \colon S \to S$. Restricting φ to the punctured surface $\mathring{S}$ produces a pseudo-Anosov $\mathring{\varphi} \colon \mathring{S} \to \mathring{S}$. Then $\mathring{\varphi}$ permutes the (maximal) singularity-free rectangles of q, and therefore acts simplicially and π-equivariantly on the ideal triangulation τ_q of $\mathring{S} \times \mathbb{R}$. Consequently, τ_q projects to an ideal triangulation of the **punctured mapping torus**

$$\mathring{M}_\varphi = M_{\mathring{\varphi}} = \left. \mathring{S} \times [0,1] \middle/ (x,1) \sim (\mathring{\varphi}(x), 0) \right. .$$

The resulting **veering triangulation of** $M_{\mathring{\varphi}}$ is denoted τ_φ.

3. Convergence of quadratic differentials

In this section, we establish a statement about convergence of quadratic differentials that will form a key component for proving Theorem 1.4. This statement requires a handful of definitions.

For a pair of geodesics γ_1 and γ_2 in a metric space X, we say that γ_2 is a ρ–**fellow traveler** with γ_1 for distance D centered at $x = \gamma_1(t_0)$ if we have $d_X(\gamma_1(t), \gamma_2(t)) < \rho$ whenever $d_X(\gamma_1(t), x) \leqslant D/2$, for some unit speed parameterizations of γ_1 and γ_2.

For a given constant $\epsilon > 0$, let $\mathcal{T}_\epsilon(S)$ denote the set of all $X \in \mathcal{T}(S)$ such that the hyperbolic metric defined by X contains a closed geodesic shorter than ϵ. The complement $K_\epsilon := \mathcal{T}(S) \smallsetminus \mathcal{T}_\epsilon(S)$ is called the ϵ–**thick part** of Teichmüller space. Let γ_0 be a Teichmüller geodesic in a thick part of Teichmüller space, and let $X, Y \in \mathrm{Teich}(S)$ be two points on γ_0. By Masur's Criterion [36], the horizontal and vertical foliations of γ_0 are uniquely ergodic. Let $B_r(X)$ and $B_r(Y)$ be balls of radius r about X and Y, respectively. Define $\Gamma_r(X, Y)$ to be the set of all oriented geodesics γ passing first through $B_r(X)$, then through $B_r(Y)$, such that the vertical and horizontal foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ associated to γ are uniquely ergodic. By a result of Hubbard and Masur [28], any pair of uniquely ergodic foliations in $\mathcal{PMF}(S) \cong \partial\mathcal{T}(S)$ determine a unique Teichmüller geodesic, so we can also think of $\Gamma_r(X, Y)$ as a subset of $\mathcal{UE}(S) \times \mathcal{UE}(S)$.

Theorem 3.1 (Gadre–Maher)**.** *Let $g \in \mathrm{Mod}(S)$ be a principal pseudo-Anosov, with invariant geodesic γ_g. Let μ be a probability distribution on $\mathrm{Mod}(S)$ with finite first moment, such that $\langle \mathrm{Supp}(\mu) \rangle_+$ is non-elementary and contains g. Then there exists $\rho > 0$ such that, for any $D > 0$ sufficiently large and for almost every bi-infinite sample path $\omega = (\omega_n)$, there is a positive integer N such that for $n \geqslant N$, ω_n is a principal pseudo-Anosov whose Teichmüller geodesic γ_{ω_n} is a ρ–fellow traveler with $h_n\gamma_g$ for distance D, for some $h_n \in \mathrm{Mod}(S)$.*

In the above theorem, the statement that ω_n is principal appears in the statement of Gadre and Maher's [20, Theorem 1.1]. The statement that γ_{ω_n} fellow travels with a translate of γ_g forms a key ingredient in Gadre and Maher's proof that ω_n is principal. The claim that the fellow-traveling distance D can be taken arbitrarily large follows from examining their argument, but is not explicitly stated. Since our application (Corollary 3.4) requires fellow traveling for longer and longer distances, we write down a unified proof of Theorem 3.1 by reassembling many of the same tools used by Gadre and Maher. We remark that a similar theorem was obtained independently by Baik–Gekhtman–Hamenstädt [3, Theorem 6.8].

Proof of Theorem 3.1. Fix a basepoint X on the Teichmüller geodesic γ_g. By Kaimanovich–Masur [29], $\omega_n X$ and $\omega_{-n} X$ converge to distinct uniquely ergodic measured foliations $\mathcal{F}^+_\omega$ and $\mathcal{F}^-_\omega$ as $n \to \infty$. Let γ_ω be the unique Teichmüller geodesic determined by these foliations, parametrized so that $\gamma_\omega(0)$ is the closest point on γ_ω to the basepoint X.

In the following argument, we will first establish fellow traveling between γ_ω and some translate of γ_g. Then, we will establish fellow traveling between γ_ω and γ_{ω_n} for large n. This will imply fellow traveling between γ_{ω_n} and the translate of γ_g, which will also imply that ω_n is principal. As the proof involves many constants, we point the reader to Figure 3 for a sketch of how the ideas fit together.

Let $\epsilon > 0$ be small enough so that γ_g is in the ϵ–thick part K_ϵ. Let $l > 0$ be the drift of the random walk. Given this ϵ, let $F_0 > 0$ and $0 < e_0 < \frac{1}{2}$ be the constants from [20, Proposition 3.1]. For almost every ω, and n sufficiently large, that proposition guarantees the existence of points Y_0 and Y_1 on γ_{ω_n} and points $\gamma_\omega(T_0)$, $\gamma_\omega(T_1)$ on γ_ω such that

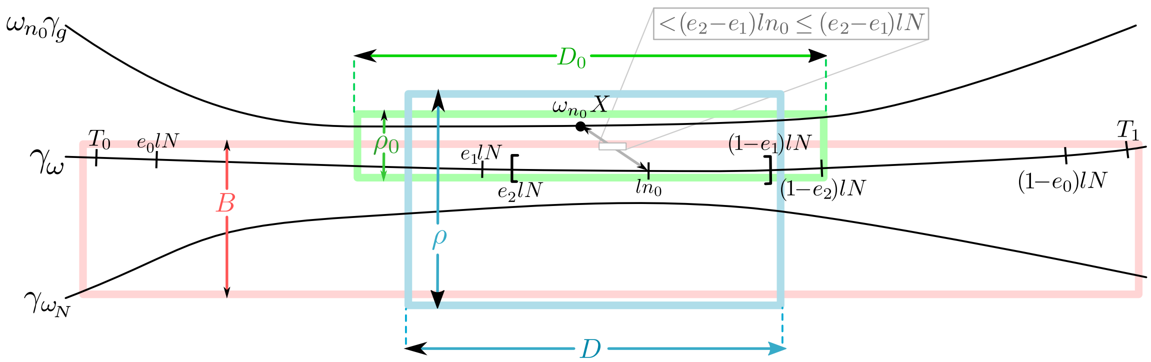

Figure 3. The logical structure of the proof of Theorem 3.1. First, establish ρ_0–fellow traveling of γ_ω and some translate of γ_g (green). Then, establish B–fellow traveling between γ_ω and γ_{ω_N} (red), on an interval containing the green interval. This implies ρ-fellow traveling between γ_{ω_N} and the translate of γ_g (blue). Note that the points marked on the geodesic γ_ω are really t-values: e.g. e_0lN should be $\gamma_\omega(e_0lN)$, and so on.

(1) $d_{\mathcal{T}}(\gamma_\omega(T_i), Y_i) \leqslant F_0$ for $i = 0, 1$.
(2) $T_0 \leqslant e_0ln \leqslant (1-e_0)ln \leqslant T_1 \leqslant ln$
(3) $\gamma_\omega(T_i)$ is in the thick part K_ϵ for $i = 0, 1$.

That such constants F_0, e_0 exist was originally shown in the proof of [11, Theorem 2.6].

Let $B = B(\epsilon, F_0)$ be the fellow traveling constant coming from [20, Theorem 2.3], originally due to Rafi [43, Theorem 7.1]. Since $d_{\mathcal{T}}(\gamma_\omega(T_i), Y_i) \leqslant F_0$ and $\gamma_\omega(T_i)$ is in the ϵ–thick part, this theorem guarantees B–fellow traveling between the segments $[Y_0, Y_1]$ and $[\gamma_\omega(T_0), \gamma_\omega(T_1)]$. By [20, Lemma 4.1], there is a constant $r > 0$ such that for every $Y, Z \in \gamma_g$, the probability $\mathbb{P}(\omega \in \Gamma_r(Y, Z))$ is strictly positive. Applying [20, Theorem 2.3] again, there is a constant $\rho_0 = \rho_0(\epsilon, r)$, such that any geodesic in $\Gamma_r(Y, Z)$ contains a sub-segment that ρ_0–fellow travels with γ_g on the entirety of $[Y, Z]$.

Define $\rho = B + \rho_0$, and let $D_1 > 0$ be the constant from [20, Proposition 4.3], which ensures that geodesics that ρ–fellow travel with γ_g for distance greater that D_1 lie in the principal stratum. Now fix any $D \geqslant D_1$.

Let $k > 0$ be the smallest positive integer such that

$$d_{\mathcal{T}}(g^{-k}X, g^kX) \geqslant D_0 := D + \rho.$$

By our choice of $\rho_0 = \rho_0(\epsilon, r)$, any geodesic in $\Gamma_r(g^{-k}X, g^kX)$ will ρ–fellow travel with γ_g on an interval of length D_0 centered at X. Let $\Omega \subset \mathrm{Mod}(S)^{\mathbb{Z}}$ consist of those sample paths ω such that the sequences $\omega_{-n}X$ and ω_nX converge to distinct uniquely ergodic foliations $(\mathcal{F}^-, \mathcal{F}^+) \in \Gamma_r(g^{-k}X, g^kX)$. By [20, Lemma 4.1], the subset Ω has positive probability P.

Let $\sigma\colon \mathrm{Mod}(S)^{\mathbb{Z}} \to \mathrm{Mod}(S)^{\mathbb{Z}}$ be the shift map. Ergodicity of σ implies that for almost every ω, there is some $n \geqslant 0$ such that $\sigma^n(\omega) \in \Omega$. For such n, we have that γ_ω is a ρ_0–fellow traveler with $\omega_n\gamma_g$ for distance D_0, centered at ω_nX.

For almost every ω, the probability that $n \in \{1, \ldots, N\}$ satisfies $\sigma^n(\omega) \in \Omega$, tends to P as $N \to \infty$. For any e_1 satisfying $e_0 < e_1 < \frac{1}{2}$, the probability that $e_1N \leqslant n \leqslant (1-e_1)N$ satisfies $\sigma^n(\omega) \in \Omega$ also tends to P as $N \to \infty$. This implies that given ω, there is an N_0 such that for all $N \geqslant N_0$, there exists an n such that $e_1N \leqslant n \leqslant (1-e_1)N$ and $\sigma^n(\omega) \in \Omega$.

By sublinear tracking in Teichmüller space, due to Tiozzo [47], almost every ω satisfies

$$\lim_{n\to\infty} \frac{1}{n} d_{\mathcal{T}}(\omega_n X, \gamma_\omega(ln)) = 0.$$

(Recall that we have parametrized γ_ω so that $\gamma_\omega(0)$ is the closest point to X.) Now, choose e_1 and e_2 so that $e_0 < e_1 < e_2 < \frac{1}{2}$ and $\frac{1}{2} - e_1 \ll e_1 - e_0$. Possibly replacing N_0 with a larger number, we can then assume that $d_{\mathcal{T}}(\omega_n X, \gamma_\omega(ln)) \leqslant (e_2 - e_1)ln$ for all $n \geqslant e_1 N_0$.

Combining the preceding two paragraphs, there exists n_0 such that

(1) $e_1 lN \leqslant ln_0 \leqslant (1 - e_1)lN$ and $\sigma^{n_0}(\omega) \in \Omega$.
(2) $d_{\mathcal{T}}(\omega_{n_0} X, \gamma_\omega(ln_0)) \leqslant (e_2 - e_1)ln_0$.

Since $\sigma^{n_0}(\omega) \in \Omega$, γ_ω and $\omega_{n_0}\gamma_g$ are ρ_0-fellow travelers for distance D_0 centered at the closest point on γ_ω to $\omega_{n_0} X$. Item (2) above guarantees that the point $\gamma_\omega(ln_0)$ is within $2(e_2 - e_1)lN$ of the point on γ_ω that is closest to $\omega_{n_0} X$. The key here is that $(e_2 - e_1)lN$ is small compared to $(e_1 - e_0)lN$, so we can guarantee that this ρ_0-fellow traveling all happens between $\gamma_\omega(e_0 lN)$ and $\gamma_\omega((1 - e_0)lN)$ (by making N larger if necessary). See Figure 3.

Now, recall that we used [20, Proposition 3.1] to find T_0, T_1 satisfying

$$T_0 \leqslant e_0 lN \leqslant (1 - e_0)lN \leqslant T_1 \leqslant lN$$

so that the interval $[\gamma_\omega(T_0), \gamma_\omega(T_1)]$ is a B–fellow traveler with γ_{ω_N}. It follows that γ_{ω_N} is a ρ–fellow traveler with a translate of γ_g for a distance $D_0 - (B + \rho_0) \geqslant D_0 - \rho = D$. Since $D \geqslant D_1$, [20, Proposition 4.3] implies that γ_{ω_N} lies in the principal stratum. □

We will use Theorem 3.1 in the form of Corollary 3.4 below. First, we need the following lemma which says that if a geodesic γ fellow travels the axis of a pseudo-Anosov for sufficiently long, γ gets arbitrarily close to the axis. By [20, Proposition 4.3], if this pseudo-Anosov axis is principal, then γ will be as well.

Lemma 3.2. *Let g be a pseudo-Anosov mapping class with axis γ_g. Fix $\rho > 0$, and suppose that γ_n is a sequence of Teichmüller geodesics such that $h_n\gamma_n$ is a ρ–fellow traveler with γ_g for distance D_n, where $h_n \in \mathrm{Mod}(S)$ and $D_n \to \infty$. Then there is a choice of quadratic differentials q_n associated to points along γ_n such that $h_n q_n$ converge to a quadratic differential q associated to γ_g.*

Proof. By replacing γ_n with $h_n\gamma_n$ and translating by a power of g, we may suppose that γ_n is a ρ–fellow traveler with γ_g for a length D_n subsegment of γ_g centered at some point $s \in \gamma_g$. Let q be the quadratic differential based at s associated to γ_g. We first make the following claim:

Claim 3.3. *There are $s_n \in \gamma_n$ such that $s_n \to s$.*

Proof of claim. If this were not the case, then after passing to a subsequence, γ_n converges to a Teichmüller geodesic γ with the properties that

(1) γ fellow travels γ_g, and
(2) s has distance at least $\delta > 0$ from γ.

Now let γ^+ be a positive ray in γ. Since γ^+ stays bounded distance from (the positive end of) γ_g, it follows that γ^+ accumulates in $\mathcal{PMF}(S)$ to foliations that are topologically equivalent to the stable foliation of g [37, Theorem 3.8]. Since this foliation is uniquely ergodic, we see that γ^+ in fact converges to the stable foliation of g. Similarly, γ^- converges to the unstable foliation of g. Hence, the vertical and horizontal foliations defining γ and γ_g agree, and so

γ and γ_g are equal, up to a reparametrization. This, however, contradicts (2), completing the proof of the claim. □

Returning to the proof of the lemma, let q_n be the quadratic differential associated to $s_n \in \gamma_n$. Now we claim that $q_n \to q$ in $\mathcal{QD}(S)$. This follows exactly as in the proof of the claim: if not, then after passing to a subsequence, $q_n \to q' \neq q$ based at s. But then γ_n would converge (uniformly on compact sets) to the Teichmüller geodesics determined by q'. Since we know that γ_n converges to γ, this gives a contradiction and completes the proof. □

Recall that $\mathcal{GQD}(S)$ denotes the principal stratum of quadratic differentials on S.

Corollary 3.4. *Let $g \in \mathrm{Mod}(S)$ be a principal pseudo-Anosov with Teichmüller axis γ_g. Let μ be a probability distribution on $\mathrm{Mod}(S)$ with finite first moment, such that $\langle \mathrm{Supp}(\mu) \rangle_+$ is non-elementary and contains g. Then for almost every sample path $\omega = (\omega_n)$ in $\mathrm{Mod}(S)$, there is a positive integer N such that for $n \geqslant N$, every ω_n is a principal pseudo-Anosov and $h_n q_{\omega_n} \to q_g$ in $\mathcal{GQD}(S)$, where $h_n \in \mathrm{Mod}(S)$ and q_g is some quadratic differential along the axis γ_g.*

Proof. Let $\rho > 0$ be the number guaranteed by Theorem 3.1. Let $PPA \subset \mathrm{Mod}(S)$ be the set of principal pseudo-Anosovs and define the set

$$\mathcal{G}_D = \{\omega : \exists N \geqslant 0 \text{ such that } \forall n \geqslant N,\, \omega_n \in PPA \text{ and } \gamma_{\omega_n} \text{ is a } \rho\text{–fellow traveler with a translate of } \gamma_g \text{ for distance } D\}.$$

By Theorem 3.1, $\mathbb{P}(\mathcal{G}_D) = 1$ for all sufficiently large D. Since $\mathcal{F}_{D'} \subset \mathcal{F}_D$ for $D \leqslant D'$, we set $\mathcal{G} = \bigcap \mathcal{G}_D$ and conclude that $\mathbb{P}(\mathcal{G}) = 1$.

Now for each $\omega \in \mathcal{G}$, there is a sequence of mapping classes h_n, such that $h_n \gamma_{\omega_n}$ is a ρ–fellow traveler with γ_g for distance D_n, where $D_n \to \infty$ as $n \to \infty$. Applying Lemma 3.2 and recalling that $\mathcal{GQD}(S)$ is open in $\mathcal{QD}(S)$ completes the proof. □

Remark 3.5. The only property of the principal stratum that we used in this section is that $\mathcal{GQD}(S)$ is open. As a consequence, all of the results in this section hold for $S \cong \Sigma_{1,1}$, with $\mathcal{GQD}(S)$ replaced by $\mathcal{QD}(S)$. In particular, Corollary 3.4 applies to every pseudo-Anosov in $\mathrm{Mod}(\Sigma_{1,1})$, with the word “principal” excised.

4. Transition to punctured surfaces

Recall from Section 2.5 that the construction of a veering triangulation starts with a quadratic differential $q \in \mathcal{QD}(S)$, punctures S along the singularities of q to obtain a surface $\mathring{S}$, and then builds an ideal triangulation of $\mathring{S} \times \mathbb{R}$. We need to analyze the veering triangulations not just for one $q \in \mathcal{QD}(S)$, but for an entire convergent sequence $q_n \to q$ that comes from Corollary 3.4. To do this, we need a coherent way to map the sequence $q_n \to q$ to a convergent sequence $\mathring{q}_n \to \mathring{q} \in \mathcal{QD}(\mathring{S})$.

Let $\mathcal{QD}_p(\mathring{S})$ denote the subspace of $\mathcal{QD}(\mathring{S})$ consisting of quadratic differentials whose singularities occur only at punctures. If $\mathring{q} \in \mathcal{QD}_p(\mathring{S})$ is a quadratic differential with at least 2 prongs at any puncture being filled in S, then $\mathring{q}$ defines a quadratic differential $q \in \mathcal{QD}(S)$. Implicit in this definition is the observation that markings on $\mathring{S}$ induce markings on S. The following is immediate:

Lemma 4.1. *Let $\mathring{q}_n \to \mathring{q}$ be a convergent sequence in a single stratum of $\mathcal{QD}(\mathring{S})$, with all singularities at the punctures. Let S be the result of filling some number of punctures of $\mathring{S}$, where these quadratic differentials have at least 2 prongs. Then the sequence $\mathring{q}_n \to \mathring{q}$ induces a convergent sequence $q_n \to q \in \mathcal{QD}(S)$.* □

We need to go in the opposite direction, puncturing S at singularities of $q \in \mathcal{QD}(S)$ to obtain $\mathring{S}$. This is is not as straightforward, because the surjection $\mathrm{Mod}(\mathring{S}) \to \mathrm{Mod}(S)$ has a large kernel, hence there is no consistent way to turn markings on S into markings on $\mathring{S}$. Nevertheless, this can be done locally in the principal stratum.

Lemma 4.2. *Let $q \in \mathcal{GQD}(S)$ be a quadratic differential in the principal stratum. Then there is an open neighborhood U of q, with an embedding $f\colon U \to \mathcal{QD}_p(\mathring{S})$ that forms a local inverse to the map of Lemma 4.1.*

Proof. Let $X(q) \in \mathcal{T}(S)$ be the marked conformal structure underlying q. Let $y_1^q, \ldots, y_k^q \in X(q)$ be the singularities of q. Let $\epsilon > 0$ be such that there are pairwise disjoint regular neighborhoods $N_\epsilon(y_1^q), \ldots, N_\epsilon(y_k^q)$.

Now, let $q' \in \mathcal{GQD}(S)$ be another quadratic differential in the principal stratum, with singularities $y_1^{q'}, \ldots, y_k^{q'} \in X(q)$. There is a unique Teichmüller map $h\colon X(q') \to X(q)$ which maps the singularities of q' to a k–tuple of points $h\big(y_1^{q'}\big), \ldots, h\big(y_k^{q'}\big) \in X(q)$. Because h is uniquely defined by the pair (q', q), these points of $X(q)$ are uniquely determined up to reordering. Thus there is an open neighborhood U of q such that for $q' \in U$, the singularities of q' can be ordered so that $h\big(y_i^{q'}\big) \in N_\epsilon(y_i^q)$, for a unique point y_i^q.

Let $\mathring{S} = X(q) \smallsetminus \{y_1^q, \ldots y_k^q\}$. For every $q' \in U$, we will define a marked conformal structure on $\mathring{S}$, as follows. Let $\mathring{X}(q') = X(q') \smallsetminus \{y_1^{q'}, \ldots, y_k^{q'}\}$. This conformal structure is marked by the composition map

$$\begin{aligned}\mathring{S} = X(q) \smallsetminus \{y_1^q, \ldots, y_k^q\} &\xrightarrow{\ r\ } X(q) \smallsetminus \{h\big(y_1^{q'}\big), \ldots, h\big(y_k^{q'}\big)\} \\ &\xrightarrow{h^{-1}} X(q') \smallsetminus \{y_1^{q'}, \ldots, y_k^{q'}\} \qquad = \mathring{X}(q').\end{aligned}$$

where r is the identity on the complement of $N_\epsilon(y_1^q) \cup \ldots \cup N_\epsilon(y_k^q)$. The composition $h^{-1} \circ r$ is well-defined up to isotopy because the mapping class group of a punctured disk is trivial.

Now, the quadratic differential q' on the marked Riemann surface $X(q')$ restricts to a quadratic differential $\mathring{q}'$ on the marked Riemann surface $\mathring{X}(q')$. By construction, all singularities of $\mathring{q}'$ are at the punctures, hence $\mathring{q}' \in \mathcal{QD}_p(\mathring{S})$. The map $f\colon U \to \mathcal{QD}_p(\mathring{S})$ defined via $q' \mapsto \mathring{q}'$ is continuous by construction. It is one-to-one because the map of Lemma 4.1 provides an inverse. □

For the next two sections, we will work primarily in the punctured surface $\mathring{S}$.

5. Convergence of veering triangulations

Let $\mathring{S}$ be a surface with at least one puncture. The main result of this section, Corollary 5.6, says that veering triangulations of $\mathring{S} \times \mathbb{R}$ depend continuously on their defining quadratic differentials. More precisely, we will show that given an appropriate convergent sequence $q_n \to q \in \mathcal{QD}(\mathring{S})$, the corresponding veering triangulations τ_{q_n} agree with τ_q on larger and larger finite sets of tetrahedra, limiting to the entire triangulation τ_q.

Recall from Section 4 that $\mathcal{QD}_p(\mathring{S})$ is the subspace of $\mathcal{QD}(\mathring{S})$ consisting of quadratic differentials whose singularities occur at punctures of $\mathring{S}$. We define $\mathcal{EQD}_p(\mathring{S}) \subset \mathcal{QD}_p(\mathring{S})$

to be the subspace of quadratic differentials without vertical or horizontal saddle connections. In Section 2.5, these are exactly the quadratic differentials on $\mathring{S}$ that define veering triangulations of $\mathring{S} \times \mathbb{R}$.

The following easy lemma will be useful in Section 6.

Lemma 5.1. *For every $q \in \mathcal{EQD}_p(\mathring{S})$, the foliations $\mathcal{F}_q^+$ and $\mathcal{F}_q^-$ are filling.*

Proof. Suppose for a contradiction that $\mathcal{F} = \mathcal{F}_q^+$ is not filling. Then there is some closed essential curve $\alpha \subset \mathring{S}$ with $i(\alpha, \mathcal{F}) = 0$. The q–geodesic representative α_q of α is a concatenation of saddle connections (see [42] or [12]) and since $i(\alpha, \mathcal{F}) = 0$, each of these saddle connections must be vertical, a contradiction. The proof for $\mathcal{F}_q^-$ is identical. □

As in Section 2.5, for each $q \in \mathcal{EQD}_p(\mathring{S})$ we have an associated veering triangulation $\tau = \tau_q$ of $\mathring{S} \times \mathbb{R}$. (Note that no further puncturing is necessary because all singularities are already at the punctures of $\mathring{S}$.) Let $\mathcal{A}(\tau) = \mathcal{A}(\tau_q)$ the subset of $\mathcal{A}(\mathring{S})$ consisting of arcs that correspond to edges of τ_q. As described immediately after Theorem 2.1, $\mathcal{A}(\tau_q)$ is precisely the set of saddle connections of q that span singularity free rectangles.

Let $a, a_1, \ldots, a_n \in \mathcal{A}(\mathring{S})$ be arcs. We call the collection $\{a_1, \ldots, a_n\}$ a **homotopical decomposition** of a, and write $a \sim \sum_i a_i$, if these arcs have lifts $\tilde{a}, \tilde{a}_1, \ldots, \tilde{a}_n$ to the universal cover of $\mathring{S}$ which bound an immersed ideal polygon. The decomposition is **nontrivial** if $n > 1$.

Recall from Section 2.4 that the horizontal and vertical lengths of a are denoted $h_q(a)$ and $v_q(a)$, whereas $\ell^1_q(a) = h_q(a) + v_q(a)$ is the total ℓ^1 length.

Lemma 5.2. *Let $q \in \mathcal{EQD}_p(\mathring{S})$ and $a \in \mathcal{A}(\mathring{S})$. Then $a \in \mathcal{A}(\tau_q)$ if and only if for any nontrivial homotopical decomposition $a \sim \sum a_i$ with $a_i \in \mathcal{A}(\mathring{S})$, we have*

$$\ell^1_q(a) < \sum \ell^1_q(a_i). \tag{5.1}$$

Proof. Suppose that a is homotopic to an edge σ of the veering triangulation and $a \sim \sum a_i$. Since σ spans a singularity free rectangle, the total horizontal or vertical length of the a_i must be strictly greater than that of σ. This is because, after lifting to the universal cover of $\mathring{S}$, all ℓ^1 geodesics between the endpoints of σ must lie in the rectangle spanned by σ. As we always have $h_q(a) \leqslant \sum h_q(a_i)$ and $v_q(a) \leqslant \sum v_q(a_i)$, the strict inequality (5.1) follows.

The converse direction follows from the work of Minsky and Taylor [40]. First recall that every arc a has a unique q–geodesic representative a_q. See [40, Proposition 2.2 and Figure 6]. This geodesic a_q follows a sequence of saddle connections, which we may call $a_1, \ldots, a_n$, such that $a \sim \sum a_i$. Since a_q is a geodesic, we have

$$\ell^1_q(a) = \ell^1_q(a_q) = \sum \ell^1_q(a_i).$$

Thus we have proved the negation of (5.1), unless a is itself homotopic to a saddle connection c, i.e., the sum $\sum a_i$ has only one term.

Now, suppose that $a = c$ is a saddle connection that is not an edge of τ_q. Then c does not span a singularity free rectangle of q. Hence, c does not span a singularity free right triangle to one of its sides. To this side, we apply the map $\mathbf{t}$ illustrated in [40, Figure 12]. The resulting object $\mathbf{t}(c)$ is a concatenation of (not necessarily disjoint) saddle connections c_j, forming a non-trivial decomposition $c \sim \sum_j c_j$. By [40, Lemma 4.2], these saddle connections have the property that, working in the universal cover of $\mathring{S}$, each leaf of the vertical/horizontal foliation of q meets the union $\bigcup_j c_j$ at most once. Hence,

$$\ell^1_q(a) = \ell^1_q(c) = \sum \ell^1_q(c_j)$$

and the sum is non-trivial, contradicting (5.1). □

Lemma 5.3. *Fix $q \in \mathcal{EQD}_p(\mathring{S})$. For any $L \geqslant 0$, there is a open neighborhood U of q in $\mathcal{QD}(\mathring{S})$ such that for any $q' \in U \cap \mathcal{EQD}_p(\mathring{S})$, every arc $\sigma \in \mathcal{A}(\tau_q)$ of length $\ell^1_q(\sigma) \leqslant L$ is also in $\mathcal{A}(\tau_{q'})$.*

Proof. For $q \in \mathcal{EQD}_p(\mathring{S})$ and $L \geqslant 0$, define $A_q(L) = \{a \in \mathcal{A}(\mathring{S}) : \ell^1_q(a) \leqslant L\}$. Note that $A_q(L)$ is always finite. Now fix $L \geqslant 0$ and let

$$U_1 = \{q' \in \mathcal{QD}(\mathring{S}) : \ell^1_{q'}(a) < L + 1 \text{ for all } a \in A_q(L)\}.$$

This is an open neighborhood of q in $\mathcal{QD}(\mathring{S})$. After making U_1 smaller if necessary, we can ensure that the closure $\overline{U}_1 \subset \mathcal{QD}(\mathring{S})$ is compact. This is done for the following claim:

Claim 5.4. *The set*

$$B = \left\{a \in \mathcal{A}(\mathring{S}) : \ell^1_{q'}(a) \leqslant L + 1 \textit{ for some } q' \in U_1\right\}$$

is finite.

Proof. For any arc $a \subset \mathring{S}$, there is an essential (multi-)curve c_a constructed as follows. Consider the punctures of $\mathring{S}$ to be marked points in a larger surface S; build a regular neighborhood P of a and the marked points that it meets; then, take the $\mathring{S}$–essential components ∂P. We remark that $P \cap \mathring{S}$ is a pair of pants containing a as its only $\mathring{S}$–essential arc, hence c_a determines a. For any q, we have $\ell^1_q(c_a) \leqslant 2 \cdot \ell^1_q(a)$ because a representative of c_a is given by traversing the q–geodesic representative for a at most twice.

Now suppose that the claim is false. Then there would be an infinite collection $a_i \in B$ and $q_i \in U_1$ with $\ell^1_{q_i}(a_i) < L+1$. Setting $c_i = c_{a_i}$, we obtain an infinite collection of distinct multi-curves c_i with $\ell^1_{q_i}(c_i) < 2(L+1)$. Since $\overline{U}_1$ is compact, we may pass to a subsequence such that $q_i \to q'$ for some $q' \in \overline{U}_1$. Passing to a further subsequence and using compactness of $\mathcal{PMF}(\mathring{S})$, there are constants $x_i \geqslant 0$ such that $x_i c_i$ converges in $\mathcal{MF}(\mathring{S})$ to $\alpha \neq 0$. It is also easy to see that $x_i \to 0$ as $i \to \infty$. Indeed, for an arbitrary (but fixed) hyperbolic metric ρ on S, $x_i c_i \to \alpha$ implies that $x_i \ell_\rho(c_i) \to \ell_\rho(\alpha) \in \mathbb{R}_+$. Since there are infinitely many distinct multi-curves c_i, we must have $\ell_\rho(c_i) \to \infty$, hence $x_i \to 0$.

Recall from Section 2.4 that the ℓ^1–length $\ell^1_q(\alpha)$ is continuous in both q and α. Thus

$$i(\mathcal{F}^+_{q'}, \alpha) + i(\mathcal{F}^-_{q'}, \alpha) \;=\; \ell^1_{q'}(\alpha) \;=\; \lim_{i\to\infty} \ell^1_{q_i}(x_i c_i) \;=\; \lim_{i\to\infty} x_i\, \ell^1_{q_i}(c_i) \;\leqslant\; 2(L+1) \lim x_i \;=\; 0.$$

However, a measured foliation α cannot have intersection number 0 with both $\mathcal{F}^+_{q'}$ and $\mathcal{F}^-_{q'}$, a filling pair of foliations. This contradiction completes the proof of the claim. □

We now return to the proof of the lemma. For each $a \in \mathcal{A}(\tau_q) \cap A_q(L)$, we define the function $f_a \colon U_1 \to \mathbb{R}$:

$$q' \in U_1 \;\longmapsto\; f_a(q') = \min\left\{\sum_i \ell^1_{q'}(a_i) - \ell^1_{q'}(a) \;:\; a \sim \sum_i a_i \text{ is nontrivial and } a_i \in B\right\},$$

Since B is finite, this is a minimum of finitely many continuous functions, hence f_a is continuous on U_1. Furthermore, since $a \in \mathcal{A}(\tau_q)$, Lemma 5.2 implies that $f_a(q) > 0$. Set $U_a = U_1 \cap \{q' : f_a(q') > 0\}$, which is open in $\mathcal{QD}(\mathring{S})$ and contains q.

Finally, define
$$U = \bigcap \{U_a : a \in \mathcal{A}(\tau_q) \cap A_q(L)\},$$
which by construction is an open neighborhood of q in $\mathcal{QD}(\mathring{S})$. To show that this neighborhood satisfies the conclusion of the lemma, let $q' \in U \cap \mathcal{EQD}_p(\mathring{S})$ and suppose that $\sigma \in \mathcal{A}(\tau_q)$ with $\ell^1_q(\sigma) \leqslant L$. If σ is *not* in $\tau_{q'}$, then by Lemma 5.2 there is a decomposition $\sigma \sim \sum \sigma_i$ with $\ell^1_{q'}(\sigma) = \sum \ell^1_{q'}(\sigma_i)$. Since $U \subset U_1$, we have that $\ell^1_{q'}(\sigma) < L+1$ and so similarly $\ell^1_{q'}(\sigma_i) < L+1$ for all i. Hence, by definition of B, we have $\sigma_i \in B$ for each i. Then the difference
$$\sum \ell^1_{q'}(\sigma_i) - \ell^1_{q'}(\sigma)$$
appears in the definition of f_σ. Since $f_\sigma(q') > 0$, the difference $\sum \ell^1_{q'}(\sigma_i) - \ell^1_{q'}(\sigma)$ is strictly positive, contradicting the choice of our decomposition of σ. This completes the proof. □

Remark 5.5. An alternate proof of Claim 5.4 relies on the fact that there is a constant K, depending only on the compact set $\overline{U}_1$, such that for any $q_1, q_2 \in \overline{U}_1$ the induced map $(\tilde{\mathring{S}}, \widetilde{q_1}) \to (\tilde{\mathring{S}}, \widetilde{q_2})$ is a K–quasi-isometry. Then $B \subset A_q(K(L+1)+K)$, and the claim follows.

Lemma 5.3 has the following useful corollary:

Corollary 5.6. *Let $q \in \mathcal{EQD}_p(\mathring{S})$, and let $K \subset \tau_q$ be any finite sub-complex. Then there is a neighborhood U of q in $\mathcal{QD}(\mathring{S})$ such that $K \subset \tau_{q'}$ for every $q' \in U \cap \mathcal{EQD}_p(\mathring{S})$.*

Proof. Define
$$L = \max\left\{\ell^1_q(a) : a \in K^{(1)}\right\}.$$
By Lemma 5.3, there is a neighborhood U such that every arc $\sigma \in \mathcal{A}(\tau_q)$ with $\ell^1_q(\sigma) \leqslant L$ also belongs to $\mathcal{A}(\tau_{q'})$ for $q' \in U \cap \mathcal{EQD}_p(\mathring{S})$. In particular, every arc in the 1–skeleton of K belongs to $\mathcal{A}(\tau_{q'})$. Since the edges of every tetrahedron $\mathfrak{t} \subset K$ belong to $\mathcal{A}(\tau_{q'})$, we have $\mathfrak{t} \subset \tau_{q'}$. □

6. Convergence of tetrahedron shapes

In this section, we prove Theorem 1.4. To do so, we establish a technical result (Proposition 6.2) which states, roughly, that as quadratic differentials converge, so do the hyperbolic shapes of the associated veering tetrahedra. This result will also be used in the proof of Theorem 1.2 in Section 7. We begin by reviewing some needed background about doubly degenerate representations of surface groups.

For a surface S, let $\mathrm{AH}(S)$ denote the space of conjugacy classes of discrete and faithful representations $\rho \colon \pi_1(S, x_0) \to \mathrm{PSL}_2(\mathbb{C})$ such that peripheral elements map to parabolic isometries. In the algebraic topology on $AH(S)$, conjugacy classes $[\rho_n]$ converge to $[\rho]$ if and only if there are conjugacy representatives $\rho_n \colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ such that for every finite set of elements $\gamma_1, \ldots, \gamma_k \in \pi_1(S)$, the images $\rho_n(\gamma_i)$ converge to $\rho(\gamma_i)$ for every i.

Setting $\Gamma_\rho = \rho(\pi_1(S, x_0))$, the manifold $N_\rho = \mathbb{H}^3/\Gamma_\rho$ is then homeomorphic to $S \times \mathbb{R}$ by work of Bonahon [6]. The limit set Λ_ρ is defined to be the smallest nonempty closed Γ_ρ–invariant set in $\partial\mathbb{H}^3$. The space $\mathrm{DD}(S) \subset \mathrm{AH}(S)$ of **doubly degenerate representations** of $\pi_1(S, x_0)$ is the subspace of $\mathrm{AH}(S)$ consisting of conjugacy classes $[\rho]$ such that $\Lambda_\rho = \partial\mathbb{H}^3$ and such that $\rho(\gamma)$ is parabolic if and only if $\gamma \in \pi_1(S, x_0)$ is peripheral. For such a ρ, the manifold N_ρ has **geometrically infinite** ends, which can be characterized by saying that for each end, there are closed geodesics in N_ρ that exit that end.

By work of Bonahon and Thurston [6, 45], there are unique, distinct **end invariants** $\mathcal{F}^+_\rho, \mathcal{F}^-_\rho \in \mathcal{EL}(S)$ associated to the positive and negative ends of N_ρ, such that if $\{\alpha_i\}$ is a bi-infinite sequence of closed geodesics exiting both ends, then $\alpha_i \to \mathcal{F}^-$ as $i \to \infty$ and $\alpha_i \to \mathcal{F}^+$ as $i \to +\infty$. (Here, as in Section 2.3, we pass freely between foliations and laminations.) Hence we get a well-defined function $\mathcal{E} \colon \mathrm{DD}(S) \to \mathcal{EL}(S) \times \mathcal{EL}(S) - \Delta$, where Δ is the diagonal.

Thurston conjectured that $\mathcal{E}$ is a bijection. This conjecture was proved by Minsky [39] and Brock–Canary–Minsky [8]. Subsequently, Leininger and Schleimer observed that $\mathcal{E}$ is actually a homeomorphism [31, Theorem 6.5].

Theorem 6.1 (Ending Lamination Theorem, parametrized)**.** *The end invariant function* $\mathcal{E} \colon \mathrm{DD}(S) \to \mathcal{EL}(S) \times \mathcal{EL}(S) - \Delta$ *sending* ρ *to the pair* $(\mathcal{F}^-_\rho, \mathcal{F}^+_\rho)$ *is a homeomorphism.*

We now specialize to the case of the punctured surface $\mathring{S}$. Recall from Section 5 that $\mathcal{EQD}_p(\mathring{S})$ is the subspace of $\mathcal{QD}(\mathring{S})$ consisting of quadratic differentials whose foliations have singularities only at punctures and which have no horizontal or vertical saddle connections. As described in Section 2.5, every quadratic differential $q \in \mathcal{EQD}_p(\mathring{S})$ has an associated veering triangulation τ_q of $\mathring{S} \times \mathbb{R}$.

Every $q \in \mathcal{EQD}_p(\mathring{S})$ defines a doubly degenerate hyperbolic structure on $\mathring{S} \times \mathbb{R}$, constructed as follows. By Lemma 5.1, there is a map $\mathcal{F} \colon \mathcal{EQD}_p(\mathring{S}) \to \mathcal{EL}(\mathring{S}) \times \mathcal{EL}(\mathring{S}) - \Delta$ sending q to the pair $(\mathcal{F}^+_q, \mathcal{F}^-_q)$ of filling foliations/laminations. By Theorem 6.1, $\mathcal{E}^{-1}(\mathcal{F}^+_q, \mathcal{F}^-_q)$ is a doubly degenerate representation $\rho_q \colon \pi_1(\mathring{S}) \to \mathrm{PSL}(2, \mathbb{C})$, unique up to conjugation. Then $N_q = \mathbb{H}^3/\rho(\pi_1 \mathring{S})$ is a marked hyperbolic 3–manifold triangulated via τ_q. In summary, the composition

$$\mathcal{E}^{-1} \circ \mathcal{F} : \mathcal{EQD}_p(\mathring{S}) \to \mathrm{DD}(\mathring{S})$$

is a continuous function mapping q to a conjugacy class of doubly degenerate representations, which we denote by $[\rho_q]$. This map is $\mathrm{Mod}(S)$–equivariant by construction and continuous by a combination of [28] and Theorem 6.1.

We remark that the end invariants $(\mathcal{F}^+_q, \mathcal{F}^-_q)$ of N_q can be recovered directly from the edge set of τ_q. By a theorem of Minsky and Taylor [40, Theorem 1.4], the edge set $\mathcal{A}(\tau_q)$ is totally geodesic in $\mathcal{AC}(\mathring{S})$. Furthermore, this edge set is quasi-isometric to a line, which has two endpoints at infinity. Any sequence of edges of $\mathcal{A}(\tau_q)$ whose slope in q approaches ∞ will exit the positive end of N_q and limit to $\mathcal{F}^+_q$, while any sequence in $\mathcal{A}(\tau_q)$ whose slope in q approaches 0 will exit the negative end of N_q and limit to $\mathcal{F}^-_q$.

With this background, we can state and prove the main result of this section.

Proposition 6.2. *Fix* $q \in \mathcal{EQD}_p(\mathring{S})$ *and a finite, connected sub-complex* $K \subset \tau_q$*. Then the following holds for any convergent sequence* $q_n \to q$*, where* $q_n \in \mathcal{EQD}_p(\mathring{S})$*:*

- *For all* $n \gg 0$*,* K *is a sub-complex of the veering triangulation* τ_{q_n}*.*
- *For every tetrahedron* $\mathfrak{t} \subset K$*, the shape of* $\mathfrak{t}$ *in* N_{q_n} *converges to the shape of* $\mathfrak{t}$ *in* N_q *as* $n \to \infty$*.*

In particular, if τ_q *is not geometric, then neither is* τ_{q_n}*, for* n *sufficiently large.*

Proof. Fix $q \in \mathcal{EQD}_p(\mathring{S})$ and a finite, connected sub-complex $K \subset \tau_q$. We may assume without loss of generality that the dual 1–skeleton of K is connected (otherwise, add some number of tetrahedra). Let Y be a maximal tree in the dual 1–skeleton. Fix a base vertex $v_0 \in Y$, which corresponds to the barycenter of an oriented tetrahedron $\mathfrak{t}_0 \subset K$.

By construction, every vertex $v \in Y$ is a barycenter of some tetrahedron $\mathfrak{t} \subset K$, which has 4 ideal vertices at punctures of $\mathring{S}$. We use this structure to construct finitely many group elements in $\pi_1(\mathring{S} \times \mathbb{R}, v_0)$. For every vertex $v \in Y$, follow the unique path in Y from v_0 to v. Walk from v to the neighborhood of an ideal vertex of the ambient tetrahedron $\mathfrak{t}$, walk around the corresponding puncture of $\mathring{S}$, and then return to v and back to v_0.

This construction gives a collection of loops $\gamma_1, \ldots, \gamma_k$, where $k = 4V(Y)$. Not all of these loops are homotopically distinct, but all of them are peripheral in $\mathring{S}$.

Now, let $[\rho_q]$ be the conjugacy class of doubly degenerate representations corresponding to q. For every representation in this conjugacy class, the image of each peripheral element γ_i must be parabolic, with a single fixed point on $\partial\mathbb{H}^3$.

Next, consider a convergent sequence $q_n \to q$, where $q_n \in \mathcal{EQD}_p(\mathring{S})$. By Theorem 6.1, there is a convergent sequence

$$\mathcal{E}^{-1}(\mathcal{F}(q_n)) = [\rho_{q_n}] \to [\rho_q].$$

After choosing a representative $\rho_q \in [\rho_q]$, this means there are choices of representatives $\rho_{q_n} \in [\rho_{q_n}]$ such that $\rho_{q_n}(\gamma_i) \to \rho_q(\gamma_i)$ for $1 \leqslant i \leqslant k$.

Let $\mathfrak{t} \subset K$ be a tetrahedron, with ideal vertices $x_1, \ldots, x_4$. By the above construction, every x_j corresponds to a peripheral group element γ_{i_j} in the chosen collection. Let $p_j^n \in \partial\mathbb{H}^3$ be the parabolic fixed point of $\rho_{q_n}(\gamma_{i_j})$, and let p_j be the parabolic fixed point of $\rho_q(\gamma_{i_j})$. Since $\rho_{q_n}(\gamma_i) \to \rho(\gamma_i)$, we also have convergence of the parabolic fixed points: $p_j^n \to p_j$ as $n \to \infty$.

For every q_n, let τ_{q_n} be the veering triangulation of $N_{q_n} \cong \mathring{S} \times \mathbb{R}$. Since $q_n \to q$, Corollary 5.6 implies that K embeds into τ_{q_n} for all $n \gg 0$. (The marking of N_{q_n} makes this embedding unique.) The shape parameter of $\mathfrak{t}$ in the hyperbolic metric on N_{q_n} is the cross-ratio $[p_1^n, p_2^n, p_3^n, p_4^n]$. As $n \to \infty$, these cross-ratios converge to $[p_1, p_2, p_3, p_4]$, hence the shape of $\mathfrak{t}$ converges as well. □

Remark 6.3. Proposition 6.2 gives a concrete way to see that, with suitably chosen basepoints, the manifolds N_{q_n} converge geometrically to N_q. Let $z \in N_q$ be an arbitrary basepoint, and let $B_R(z) \subset N_q$ be a metric R–ball about z. Since the edges of τ_q eventually exit the ends of N_q, there are only finitely many edges (hence finitely many tetrahedra) in τ_q that intersect $B_R(z)$. By Proposition 6.2, the shapes of these tetrahedra in N_{q_n} converge to the shape in N_q, hence for $n \gg 0$, there is a metric ball in N_{q_n} almost-isometric to $B_R(z)$.

The statement that the algebraic and geometric limits of N_{q_n} agree is due to Canary [9], and is used in the proof of continuity in Theorem 6.1. Hence this remark does not give a new proof of geometric convergence.

We can now complete the proof of Theorem 1.4.

Proof of Theorem 1.4. Let S be a hyperbolic surface, and let $\varphi \in \mathrm{Mod}(S)$ be a principal pseudo-Anosov. Let μ be a probability distribution on $\mathrm{Mod}(S)$ with finite first moment, such that $\langle \mathrm{Supp}(\mu) \rangle_+$ is non-elementary and contains φ. According to Corollary 3.4, for almost every sample path $\omega = (\omega_n)$ of the random walk on $\mathrm{Mod}(S)$ we have for $n \gg 0$,

(1) ω_n is a pseudo-Anosov with Teichmüller geodesic γ_{ω_n} in the principal stratum,
(2) $h_n q_{\omega_n} \to q_\varphi$ in $\mathcal{GQD}(S)$ for quadratic differentials q_{ω_n} along γ_{ω_n} and q_φ along γ_φ, and $h_n \in \mathrm{Mod}(S)$.

Since $h_n q_{\omega_n} = q_{h_n \omega_n h_n^{-1}}$, and ω_n defines the same unmarked mapping torus as $h_n \omega_n h_n^{-1}$, the veering triangulations associated to the ω_n are simplicially isomorphic to those associated to $h_n \omega_n h_n^{-1}$. Let $q_n = h_n q_{\omega_n}$.

By Lemma 4.2, for sufficiently large n, we can pass from the sequence $q_n \to q_\varphi$ to a sequence $\mathring{q}_n \to \mathring{q}_\varphi$ in $\mathcal{QD}_p(\mathring{S})$, where $\mathring{S}$ is the surface obtained by puncturing S at the singularities of q_φ. By construction, $\mathring{q}_\varphi$ is a quadratic differential along the Teichmüller axis of $\mathring{\varphi} \in \mathrm{Mod}(\mathring{S})$, and similarly for the $\mathring{q}_n$. Thus, in fact, $\mathring{q}_n \to \mathring{q}_\varphi \in \mathcal{EQD}_p(\mathring{S})$.

Let $\tau_n = \tau_{q_n}$ be the veering triangulation of $N_{\mathring{q}_n} \cong \mathring{S} \times \mathbb{R}$ associated to the quadratic differential $\mathring{q}_n$, and let τ_{q_φ} be the veering triangulation of $N_{\mathring{q}_\varphi}$ associated to $\mathring{q}_\varphi$. Now let $K \subset \tau_{q_\varphi}$ be any finite connected subcomplex as in the statement of the theorem. Applying Proposition 6.2, we conclude that for n sufficiently large, K is a subcomplex of the veering triangulation τ_{q_n} and that for every tetrahedron $\mathfrak{t} \subset K$, the shape of $\mathfrak{t}$ in $N_{\mathring{q}_n}$ converges to the shape of $\mathfrak{t}$ in $N_{\mathring{q}_\varphi}$ as $n \to \infty$.

Hence, it only remains to show that K embeds as a subcomplex of τ_φ, the veering triangulation of the mapping torus $M_{\mathring{\omega}_n}$. That is, we must show that the covering map $N_{\mathring{q}_n} \to M_{\mathring{\omega}_n}$ is injective on K, once n is sufficiently large. For this, we use a result of Maher [33], which implies that the Teichmüller translation length of ω_n grows linearly in n. Since dilatation, and hence Teichmüller translation length, is unchanged after puncturing along singularities, we also have that the translation length of $\mathring{\omega}_n$ grows linearly in n. If $N_{\mathring{q}_n} \to M_{\mathring{\omega}_n}$ fails to be injective on K then there are edges k_1 and k_2 of K which, when viewed as arcs of $\mathring{S}$, satisfy $(\mathring{\omega}_n)^i k_1 = k_2$ for some $i > 0$. Since these arcs represent saddle connections of $\mathring{q}_n$, this implies that the stretch factor of $\mathring{\omega}_n$ is no more than the quantity

$$\frac{\max_{k \in K^{(1)}} v_{q_n}(k)}{\min_{k \in K^{(1)}} v_{q_n}(k)}, \qquad \text{which converges to} \qquad \frac{\max_{k \in K^{(1)}} v_{q_\varphi}(k)}{\min_{k \in K^{(1)}} v_{q_\varphi}(k)}$$

as $n \to \infty$. Since this implies that the stretch factors of ω_n are eventually bounded, we obtain a contradiction and the proof is complete. □

Remark 6.4. Theorem 1.4 also holds for $S \cong \Sigma_{1,1}$, without the hypothesis that φ is principal. Recall that the principal stratum of $\mathcal{QD}(\Sigma_{1,1})$ is empty. In this setting, Corollary 3.4 holds for every pseudo-Anosov φ. (See Remark 3.5.) There are no interior singularities, so $S = \mathring{S}$ and Section 4 is not needed. Now, the rest of the proof of Theorem 1.4 using Proposition 6.2 applies verbatim.

7. Counting non-geometric veering triangulations

In this section, we prove Theorem 1.2, showing that geometric veering triangulations are atypical from the point of view of counting closed geodesics in moduli space. The proof of this result uses many of the same ingredients as the proof of Theorem 1.4. The main difference is that the appeal to Gadre and Maher's Theorem 3.1 will be replaced with results from Hamenstädt [24] and Eskin–Mirzakhani [13].

Fix a surface S such that $\xi(S) \geqslant 1$. As in the introduction, let $\mathcal{G}(L)$ denote the set of conjugacy classes of pseudo-Anosov mapping classes in $\mathrm{Mod}(S)$ whose Teichmüller translation length is no more than $L \geqslant 0$. Since the veering triangulation τ_φ depends only on the conjugacy class of the pseudo-Anosov, each $[\varphi] \in \mathcal{G}(L)$ uniquely determines a veering triangulation of $\mathring{M}_\varphi$. As in Baik–Gekhtman–Hamenstädt [3], say that a **typical pseudo-Anosov conjugacy class** in $\mathrm{Mod}(S)$ has a property $\mathcal{P}$ if

$$\lim_{L \to \infty} \frac{|\{[\varphi] \in \mathcal{G}(L) \;:\; \varphi \text{ has } \mathcal{P}\}|}{|\mathcal{G}(L)|} = 1.$$

In this terminology, Theorem 1.2 is implied by the following, slightly stronger statement.

Theorem 7.1. *Let S be a surface with $\xi(S) \geqslant 2$. Then a typical pseudo-Anosov conjugacy class $[\varphi] \subset \mathrm{Mod}(S)$ is principal and defines a non-geometric veering triangulation τ_φ.*

For the proof of Theorem 7.1, let $\pi\colon \mathcal{QD}^1(S) \to \mathcal{T}(S)$ be the projection map sending a unit area quadratic differential to its underlying Riemann surface. As in Section 2.2, denote the Techmüller geodesic flow by $\Phi^t\colon \mathcal{QD}^1(S) \to \mathcal{QD}^1(S)$. We will use the same notation to denote the corresponding flow on $\mathcal{MQD}^1(S) = \mathcal{QD}^1(S)/\mathrm{Mod}(S)$, namely the moduli space of unit area quadratic differentials.

Let $g \in \mathrm{Mod}(S)$ be a pseudo-Anosov with Teichmüller axis γ_g. For each $\rho, T > 0$ we define the following subset of $\mathcal{QD}^1(S)$:

$$\widetilde{V}(\gamma_g, \rho, T) = \big\{q \in \mathcal{QD}^1(S) \;:\; \pi \circ \Phi^t(q) \in N_\rho(\gamma_g),\ \forall t \in [-T, T]\big\},$$

where $N_\rho(\cdot)$ denotes an open ρ–neighborhood with respect to the Teichmüller metric. Observe that $\widetilde{V}(\gamma_g, \rho, T)$ is nonempty and open.

Our proof of Theorem 1.4 has the following corollary.

Corollary 7.2. *Let $g \in \mathrm{Mod}(S)$ be a principal pseudo-Anosov with non-geometric veering triangulation. For every $\rho > 0$ there is a number $T = T(\rho, g) > 0$, such that if $\varphi \in \mathrm{Mod}(S)$ is a pseudo-Anosov with an associated quadratic differential q_φ and $\Phi^t(q_\varphi) \in \widetilde{V}(\gamma_g, \rho, T)$ for some $t \in \mathbb{R}$, then φ is principal and the veering triangulation τ_φ is also non-geometric.*

Proof. Fix $\rho > 0$. Once T is larger than the constant $D = D(\rho, g)$ given by [20, Proposition 4.3], every $q_\varphi \in \widetilde{V}(\gamma_g, \rho, T)$ must be principal.

Now, suppose for a contradiction that no $T > 0$ suffices for the other conclusion of the corollary. Then there is a sequence $T_n \to \infty$ and an associated sequence of principal pseudo-Anosovs φ_n, such that the invariant axis γ_{φ_n} is a ρ–fellow traveler of γ_g for distance $2T_n$, but the veering triangulation τ_{φ_n} is geometric. By Lemma 3.2, there is a choice of quadratic differentials q_n associated to points along γ_{φ_n}, which converge to a quadratic differential q associated to γ_g. By Lemma 4.2, for sufficiently large n, we can pass from the sequence $q_n \to q$ to a sequence $\mathring{q}_n \to \mathring{q}$ in $\mathcal{EQD}_p(\mathring{S})$, where $\mathring{S}$ is the surface obtained by puncturing S at the singularities of g. By Proposition 6.2, the veering triangulation τ_{q_n} covering τ_{φ_n} is non-geometric for n sufficiently large. But this contradicts our assumption about φ_n. Thus some $T > 0$ must suffice. ∎

We finish the proof of Theorem 7.1 (hence also Theorem 1.2) with the following argument, whose idea was suggested by I. Gekhtman.

Proof of Theorem 7.1. We identify a conjugacy class of pseudo-Anosovs on S with the corresponding closed orbit of the Teichmüller flow $\Phi^t\colon \mathcal{MQD}^1(S) \to \mathcal{MQD}^1(S)$. Following this identification, it makes sense to refer to typical closed orbits of the Teichmüller flow.

Let $g \in \mathrm{Mod}(S)$ be a principal pseudo-Anosov whose associated veering triangulation is not geometric. (Such a mapping class exists by Theorem 1.3, which will be proved in Sections 8 and 9.) Fix $\rho = 1$, and let $T = T(1, g) > 0$ be given by Corollary 7.2. Finally, let V be the image of $\widetilde{V}(\gamma_g, \rho, T)$ in $\mathcal{MQD}^1(S)$. This set is also open and nonempty. We will show that a typical closed orbit of $\Phi^t\colon \mathcal{MQD}^1(S) \to \mathcal{MQD}^1(S)$ meets V.

Set $h = 2\xi(S) = \dim \mathcal{T}(S)$. For each closed orbit γ of Φ^t, let $\delta(\gamma)$ be the Φ^t–invariant Lebesgue measure $\mathcal{QD}^1(S)$, supported on γ, of total mass $\ell(\gamma)$. Thus, for a Lebesgue measurable set $E \subset \mathcal{QD}^1(S)$, we have $\delta(\gamma)(E) = \ell(\gamma \cap E)$. Hamenstädt [24] proved that as

$L \to \infty$, the measures

$$he^{-hL} \sum_{\gamma \in \mathcal{G}(L)} \delta(\gamma)$$

converge weakly to Masur–Veech measure λ on $\mathcal{MQD}^1(S)$. (See also [23, Theorem 5.1].) The probability measure λ is in the Lebesgue measure class, has full support, and is ergodic for the Teichmüller flow Φ^t [48, 35].

Next, we recall the geodesic counting theorem of Eskin and Mirzakhani [13], which states that as $L \to \infty$,

$$|\mathcal{G}(L)| \cdot hLe^{-hL} \to 1.$$

Combining the above displayed equations, we have that the measures

$$\nu_L = \frac{1}{L|\mathcal{G}(L)|} \sum_{\gamma \in \mathcal{G}(L)} \delta(\gamma)$$

converge weakly to λ. This convergence is also noted in the proof of [3, Proposition 5.1].

Let $A \subset \mathcal{MQD}^1(S)$ be the union of all closed orbits of the flow Φ^t that are disjoint from V. Then the closure $\overline{A}$ is closed, flow-invariant, and disjoint from V because V is open. By the ergodicity of λ, we must have $\lambda(\overline{A}) = 0$. Since $\overline{A}$ is closed, weak convergence $\nu_L \to \lambda$ and the Portmanteau Theorem imply that $\limsup \nu_L(\overline{A}) \leqslant \lambda(\overline{A}) = 0$.

Now, we wish to show that a typical closed orbit is not contained in A. To that end, fix $\epsilon > 0$ and choose $\sigma > 0$ so that $e^{-h\sigma} < \epsilon$. In the following computation for fixed $L > \sigma$, the symbol γ denotes both a pseudo-Anosov conjugacy class and the corresponding closed orbit in $\mathcal{MQD}^1(S)$. We have

$$\begin{aligned}
|\{\gamma \in \mathcal{G}(L) : \gamma \cap V = \varnothing\}| &= |\{\gamma : \gamma \subset A,\, \ell(\gamma) \leqslant L\}| \\
&= |\{\gamma : \gamma \subset A,\, \ell(\gamma) < L - \sigma\}| + |\{\gamma : \gamma \subset A,\, L - \sigma \leqslant \ell(\gamma) \leqslant L\}| \\
&= |\{\gamma : \gamma \subset A,\, \ell(\gamma) < L - \sigma\}| + \sum_{\substack{\gamma \subset A \\ L - \sigma \leqslant \ell(\gamma) \leqslant L}} 1 \\
&\leqslant |\mathcal{G}(L - \sigma)| \quad + \sum_{\substack{\gamma \subset A \\ L - \sigma \leqslant \ell(\gamma) \leqslant L}} \frac{\ell(\gamma)}{L - \sigma} \\
&\leqslant |\mathcal{G}(L - \sigma)| \quad + \sum_{L - \sigma \leqslant \ell(\gamma) \leqslant L} \frac{\ell(\gamma \cap \overline{A})}{L - \sigma} \\
&\leqslant |\mathcal{G}(L - \sigma)| \quad + \frac{1}{L - \sigma} \sum_{\mathcal{G}(L)} \ell(\gamma \cap \overline{A}) \\
&= |\mathcal{G}(L - \sigma)| \; + \; \frac{L|\mathcal{G}(L)|}{L - \sigma} \, \nu_L(\overline{A}).
\end{aligned}$$

Dividing by $|\mathcal{G}(L)|$ and taking limits as $L \to \infty$, we obtain

$$\begin{aligned}
\limsup_{L \to \infty} \frac{|\{\gamma \in \mathcal{G}(L) : \gamma \cap V = \varnothing\}|}{|\mathcal{G}(L)|} &\leqslant \limsup_{L \to \infty} \left(\frac{|\mathcal{G}(L - \sigma)|}{|\mathcal{G}(L)|} \; + \; \frac{L}{L - \sigma} \, \nu_L(\overline{A}) \right) \\
&= e^{-h\sigma} \; + \; \limsup_{L \to \infty} \, \nu_L(\overline{A}) \\
&< \epsilon + 0.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this shows that a typical closed orbit meets V.

By the definition of $\widetilde{V}$ and V, this means that a typical pseudo-Anosov conjugacy class has a representative φ with associated quadratic differential q_φ such that $\Phi^t(q_\varphi) \in \widetilde{V}(\gamma_g, \rho, T)$ for some $t \in \mathbb{R}$. By Corollary 7.2, this implies that a typical pseudo-Anosov conjugacy class $[\varphi]$ is principal and produces a non-geometric veering triangulation τ_φ. □

8. A few non-geometric triangulations, via computer

Our remaining goal in this paper is to prove Theorem 1.3: for every surface S of complexity $\xi(S) \geqslant 2$, there exists a principal pseudo-Anosov map φ so that the veering triangulation of $M_{\mathring{\varphi}}$ is non-geometric. We will prove this result in two stages.

(1) In this section, we use rigorous computer assistance to find a finite collection of non-geometric triangulations. See Proposition 8.4.
(2) In Section 9, we use Thurston norm methods to show that (finite covers of) the finitely many mapping tori described in Proposition 8.4 account for all fibers of complexity at least 2. This will prove Theorem 1.3.

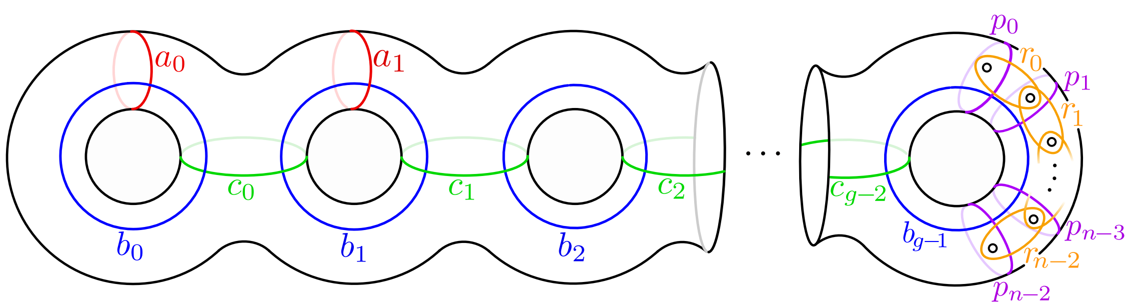

Figure 4. The mapping class group $\mathrm{Mod}(\Sigma_{g,n})$, for $g > 0$, is generated by full Dehn-twists about the a_i, b_i, c_i, and p_i curves, and half-twists about the r_i curves.

We begin by describing how the computer programs `flipper`, `SnapPy`, and `Regina` are used to show that certain triangulations are non-geometric. Given a mapping class φ on a hyperbolic surface Σ, described by a composition of left Dehn twists and/or half twists in the generators shown in Figures 4 and 5, we first use `flipper` to verify the Nielsen–Thurston type of φ. For a pseudo-Anosov mapping class, `flipper` computes the invariant measured foliation associated to φ, and thereby the order of singularities at both interior points and punctures, telling us whether φ is principal. Finally, `flipper` punctures the surface at the singularities of φ and computes the veering triangulation τ_φ of the mapping torus $M_{\mathring{\varphi}}$. All of the above `flipper` computations are rigorous.

The program `SnapPy` can find an approximate solution to the gluing equations for the veering triangulation τ_φ (see discussion below). This approximate solution is a good heuristic indication that τ_φ is not geometric. To rigorously certify that τ_φ is not geometric, we follow the method of Hodgson, Issa, and Segerman [25], relying on Theorems 8.1 and 8.2.

Let τ be an ideal triangulation of a hyperbolic 3-manifold M with k tetrahedra, and let $\vec{z} = (z_1, \ldots, z_k)$ be a vector of complex numbers in bijection with the tetrahedra in τ. Every z_i has an associated **algebraic volume** $\mathrm{Vol}(z_i) \in \mathbb{R}$, computed via the dilogarithm

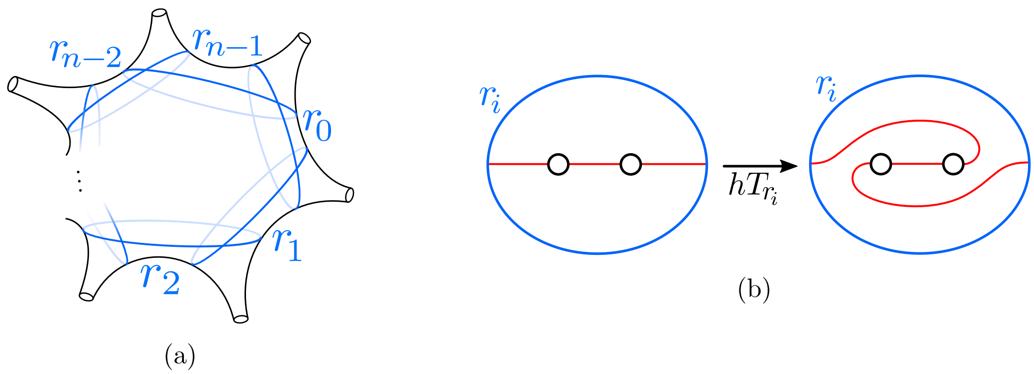

FIGURE 5. Left: The mapping class group of the n–punctured sphere $\Sigma_{0,n}$, for $n \geqslant 4$, is generated by half-twists about the r_i curves shown. Right: A half twist about r_i fixes r_i and transposes the punctures in the twice-punctured disc bounded by r_i.

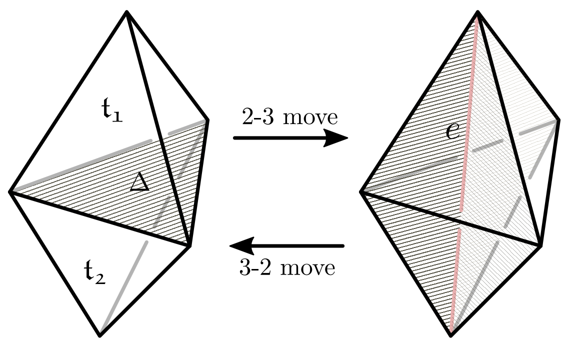

FIGURE 6. In a Pachner 2-3 move, two tetrahedra meeting at a face Δ are replaced by three tetrahedra meeting at an edge dual to Δ. A Pachner 3-2 move is the reverse of a 2-3 move.

function [41]. In particular, $\text{Vol}(z_i)$ has the same sign as $\text{Im}(z_i)$. We define the algebraic volume $\text{Vol}(\vec{z}) = \sum_{i=1}^{k} \text{Vol}(z_i)$.

The **gluing and completeness equations** for τ are a system of polynomial equations in $z_1, \dots, z_k$ [45, Chapter 4]. Any solution $\vec{z} = (z_1, \dots, z_k)$ to this system of equations defines a representation $\rho \colon \pi_1 M \to \text{PSL}(2, \mathbb{C})$, unique up to conjugacy, in which peripheral elements map to parabolics. In the resulting structure on M, the shape parameter of the tetrahedron $\mathfrak{t}_i$ is exactly z_i. If $\vec{z}$ is geometric, meaning $\text{Im}(z_i) > 0$ for each i, then ρ is the discrete, faithful representation that gives the complete hyperbolic metric on M. In general, at most one solution to the gluing and completeness equations corresponds to the complete metric on M. By Francaviglia's theorem [16], this is the solution with the largest algebraic volume. The following statement combines [16, Theorem 5.4.1 and Remark 4.1.20].

Theorem 8.1 (Francaviglia)**.** *Let τ be an ideal triangulation of a hyperbolic 3-manifold M. Then there exists at most one solution $\vec{z}$ to the gluing and completeness equations for τ such that* $\text{Vol}(\vec{z}) = \text{Vol}(M)$, *where* $\text{Vol}(M)$ *is the hyperbolic volume of M.*

Now, suppose the triangulation τ changes as follows. Let Δ be a 2-simplex in τ which is the face of distinct tetrahedra $\mathfrak{t}_1$ and $\mathfrak{t}_2$. We can obtain a new triangulation τ' by replacing Δ by a dual edge e, and adding three faces each meeting e and a distinct vertex of Δ. Then $\tau \to \tau'$ is called a **Pachner 2-3 move**, while the reverse operation $\tau' \to \tau$ is called a **Pachner 3-2 move**. See Figure 6.

An ideal tetrahedron $\mathfrak{t}$ in a hyperbolic 3–manifold M is called **degenerate** if some edge of $\mathfrak{t}$ is homotopic into a cusp of M. If $\mathfrak{t}$ is non-degenerate, its lift to $\widetilde{M} = \mathbb{H}^3$ is homotopic to a straight tetrahedron, hence $\mathfrak{t}$ can be assigned a shape parameter $z_{\mathfrak{t}} \in \mathbb{C} \smallsetminus \{0, 1\}$. A triangulation τ is called non-degenerate if all of its tetrahedra are non-degenerate.

If a triangulation τ is equipped with a solution $\vec{z}$ of the gluing and completeness equations, and τ' is obtained from τ via a Pachner move, then from $\vec{z}$ we get an associated solution $\vec{z}\,'$ to the gluing and completeness equations for τ'. Furthermore, if τ and τ' are both non-degenerate, we get algebraic volume information as well. The following result, due to Neumann and Yang [41, Proposition 10.1], is a consequence of the "five-term relation," an identity of the dilogarithm function.

Theorem 8.2 (Neumann–Yang)**.** *Let M be a hyperbolic 3-manifold with ideal triangulation τ, and let $\vec{z}$ be a solution to the gluing and completeness equations for τ. Let τ' be an ideal triangulation obtained from τ by a Pachner move, with no degenerate tetrahedra. If $\vec{z}\,'$ is the solution to the gluing equations for τ' corresponding to the solution $\vec{z}$ for τ, then* $\mathrm{Vol}(\vec{z}) = \mathrm{Vol}(\vec{z}\,')$.

Combining Theorems 8.1 and 8.2, we get the following corollary:

Corollary 8.3. *Let M be a hyperbolic 3-manifold, and let $\tau_1, \ldots, \tau_n$ be a sequence of non-degenerate ideal triangulations such that τ_i is obtained from τ_{i-1} via a Pachner move. Suppose that τ_1 has a non-geometric solution $\vec{z}_1$ to the gluing and completeness equations, and let $\vec{z}_i$ be the solution for τ_i obtained from $\vec{z}_{i-1}$ via the corresponding Pachner move. If $\vec{z}_n$ is a geometric solution (i.e., τ_n is geometric), then τ_1 is non-geometric.*

To establish that a given veering triangulation τ is non-geometric, we will first find a Pachner path from $\tau = \tau_1$ to a triangulation τ_n that has a geometric solution $\vec{z}_n$. For each tetrahedron in τ_n, `SnapPy` gives a rigorously verified complex interval in $\mathbb{C}$ containing its shape parameter, using an algorithm derived from `HIKMOT` [27]. In other words, we get a box $K_n \subset \mathbb{C}^{|\tau_n|}$ which is guaranteed to contain a geometric solution to the gluing and completeness equations. We then follow the path backwards, from τ_n to τ_1, obtaining for each intermediate triangulation τ_i a box $K_i \subset \mathbb{C}^{|\tau_i|}$ containing the corresponding solution $\vec{z}_i$. This certifies that τ_i is non-degenerate. When τ_1 is reached, we check that the corresponding box K_1 has at least one coordinate (i.e., at least one shape parameter) whose imaginary part is negative. Since K_1 is guaranteed to contain the solution $\vec{z}_1$, it follows that this solution is non-geometric, so by Corollary 8.3 the triangulation $\tau = \tau_1$ is non-geometric.

In practice, a path to a geometric triangulation is found by randomly choosing Pachner moves based at edges and faces of negatively oriented tetrahedra. In our examples, the paths we find range in length from 4 to 18. We use `Regina` to help keep track of the labeling of edges on the reverse path from τ_n to τ_1.

The following proposition gives a list of mapping classes which have been rigorously verified to be principal pseudo-Anosovs with non-geometric veering triangulations. These mapping classes are described as words in the left Dehn twists and half-twists about the curves shown in Figures 4 and 5.

Proposition 8.4. *For each of the following surfaces S_i, the mapping class φ_i described below is a principal pseudo-Anosov. Furthermore, the veering triangulation of $M_{\mathring{\varphi}_i}$ is non-geometric.*

- $S_1 = \Sigma_{2,0}$ *and* $\varphi_1 = T_{a_1} T_{c_0}^{-2} T_{b_0} T_{a_0}^{-1} T_{b_1}^{-1} T_{b_0}^{-1} T_{a_0} T_{b_0}$.
- $S_2 = \Sigma_{2,1}$ *and* $\varphi_2 = T_{c_0} T_{b_0}^{-2} T_{c_0} T_{a_0} T_{b_0} T_{a_1}^{-1} T_{a_0} T_{b_1}$.
- $S_3 = \Sigma_{1,2}$ *and* $\varphi_3 = T_{a_0}^{-2} T_{r_0}^{-1} T_{b_0}^{-1} T_{p_0}^{-1} T_{a_0} T_{b_0} T_{p_0}^{-3}$.
- $S_5 = \Sigma_{0,5}$ *and* $\varphi_5 = T_{r_3}^{-2} T_{r_0}^{-3} T_{r_1}^{2} T_{r_0}^{-1} T_{r_2}^{-1} T_{r_4}$.
- $S_6 = \Sigma_{0,6}$ *and* $\varphi_6 = T_{r_5}^{-1} T_{r_1} T_{r_2}^{-1} T_{r_3} T_{r_4}^{-1} T_{r_0}^{-1} T_{r_1} T_{r_2} T_{r_1}^{2}$.
- $S_7 = \Sigma_{0,7}$ *and* $\varphi_7 = T_{r_5}^{2} T_{r_4} T_{r_3}^{-1} T_{r_1}^{-1} T_{r_4} T_{r_2}^{-1} T_{r_3} T_{r_1} T_{r_2}^{-1}$.

Proof. For each S_i, `flipper` certifies that φ_i is a principal pseudo-Anosov. Then, `SnapPy` combined with Corollary 8.3 certifies that the veering triangulation of $M_{\mathring{\varphi}_i}$ is non-geometric. A detailed certificate of non-geometricity, including a path from the veering triangulation to a geometric triangulation, appears in the ancillary files [19].

We remark that for working with φ_1 in `flipper`, we actually consider this mapping class on $\Sigma_{2,1}$, with the puncture located as in Figure 4. Then we fill the puncture, recovering the closed surface $\Sigma_{2,0}$. For the other φ_i, we work with surface S_i exactly as given. □

9. Non-geometric triangulations via the Thurston norm

In this section, we use Thurston norm theory to show the existence of a mapping class with non-geometric veering triangulation for every hyperbolic surface of complexity at least 2. The eventual result will be that (finite covers of) the mapping tori of the classes $\varphi_1, \ldots, \varphi_7$ from Proposition 8.4 contain fibers homeomorphic to every surface S with $\xi(S) \geqslant 2$. This will imply Theorem 1.3 from the Introduction.

We begin by reviewing some classical results about the Thurston norm, and then proceed to find the desired fiber surfaces in the mapping tori of $\varphi_1, \ldots, \varphi_7$.

9.1. **The Thurston norm.** Let M be a compact orientable 3–manifold with ∂M a possibly empty union of tori, such that the interior of M is hyperbolic. We will pass freely between M and its interior. Thurston [46] showed that there is a norm $\|\cdot\| \colon H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$ on second homology, defined on integral classes by the property

$$\|x\| = \min\left\{-\chi(S) \ : \ S \text{ is an embedded surface without } S^2 \text{ components representing } x\right\}.$$

He proved that this norm, now called the **Thurston norm**, has the following properties:

(1) The unit ball $B = \{x : \|x\| \leqslant 1\}$ is a centrally symmetric polyhedron.
(2) If M is a fibered 3–manifold with fiber F, then the class $[F] \in H_2(M, \partial M)$ lies on a ray from the origin that passes through an open top-dimensional face $\mathbf{F} \subset \partial B$. In this case, $\mathbf{F}$ is called a **fibered face**, and the open cone $\mathbb{R}_+\mathbf{F}$ is called a **fibered cone**.
(3) If $x \in \mathbb{R}_+\mathbf{F}$ is a primitive integral homology class lying in a fibered cone, then x is represented by a fiber surface S. Furthermore, $\|x\| = -\chi(S)$. In particular, if $\dim(H_2(M, \partial M)) \geqslant 2$, then $\mathbb{R}_+\mathbf{F}$ contains infinitely many fibered classes.

When M is fibered with fiber F, the pseudo-Anosov monodromy $\varphi \colon F \to F$ of M induces a **suspension flow** η on M. We also have η–invariant 2–dimensional foliations $\Lambda^\pm$, which are suspensions of the invariant foliations $\mathcal{F}^\pm$ associated to φ. Let $\mathbb{R}_+\mathbf{F}$ be the fibered cone containing F. Then $\mathbf{F}$ determines $\Lambda^\pm$ and the flow η (up to isotopy), independent of the fiber F. Moreover, for every fiber S in $\mathbb{R}_+\mathbf{F}$, the foliations Λ^+ and Λ^- are transverse to $S \subset M$, and the intersections $\Lambda^\pm \cap S$ are isotopic to the stable and unstable foliations $\mathcal{F}^\pm_S$ associated to the monodromy of S. See Fried [17] and McMullen [38] for more details.

A **slope** on a torus T is an isotopy class of simple closed curves, or equivalently an (unsigned) primitive homology class in $H_1(T;\mathbb{Z})$. In a fibered 3–manifold M, with boundary tori $T_1, \ldots, T_m$, any fibration of M determines two slopes on each torus T_i. First, a fiber F must meet every T_i in a union of disjoint, consistently oriented simple closed curves. The isotopy class of these simple closed curves is called the **boundary slope** of F on T_i. Second, the orbit under the flow η of a singular leaf of $\mathcal{F}^\pm$ is a (2–dimensional) singular leaf of $\Lambda^\pm$. Every singular leaf traces out a simple closed curve on some T_i, whose slope is called the **degeneracy slope** on T_i. We emphasize that the degeneracy slope is entirely determined by $\Lambda^\pm$, hence by the fibered cone containing F.

Lemma 9.1. *Let $M = M_\varphi$ be the mapping torus of a principal pseudo-Anosov $\varphi\colon F \to F$. Then, on every component of ∂M, the boundary slope of F intersects the degeneracy slope once. If $S \subset M$ is another fiber surface in the same fibered cone as F, then the monodromy of S is principal if and only if the boundary slope of S intersects the degeneracy slope once.*

Proof. Let $\mathcal{F}^+_F$ be the stable foliation of φ on F. Since φ is principal, $\mathcal{F}^+_F$ has 3–prong singularities at interior points of F. Thus Λ^+ also has 3–prong singularities at interior points of M. In addition, every puncture of F meets exactly one singular leaf of $\mathcal{F}^+_F$. Thus, on every cusp torus $T_i \subset \partial M$, a loop about the puncture of F intersects the degeneracy slope in exactly one point.

Let S be another fiber surface in the same fibered cone as F. As mentioned above, the stable foliation $\mathcal{F}^+_S$ is isotopic to $\Lambda^+ \cap S$, hence has 3–prong singularities at interior points of S. Meanwhile, the singular prongs of $\mathcal{F}^+_S$ at a given puncture of S are in bijective correspondence with points of $\Lambda^+ \cap \gamma_i$, where γ_i denotes a loop about the puncture. Thus the monodromy of S is principal if and only if every γ_i intersects the degeneracy slope once. □

For a pseudo-Anosov $\varphi\colon S \to S$, recall that $\mathring{\varphi}\colon \mathring{S} \to \mathring{S}$ denotes the restricted map obtained by puncturing S at the singularities of φ. The mapping torus $\mathring{M} = M_{\mathring{\varphi}}$ can be constructed by drilling M_φ along the singular flow-lines of $\Lambda^\pm$, i.e. the orbits of the singularities of $\mathcal{F}^\pm$ under the flow η; in particular, $\mathring{M}$ depends only on the face $\mathbf{F}$ containing S. Furthermore, the fibered face of $M_{\mathring{\varphi}}$ whose cone contains $\mathring{S}$ depends only on $\mathbf{F}$.

The following lemma is a special case of [40, Proposition 2.7].

Lemma 9.2 (Agol)**.** *Let M be a fibered hyperbolic 3-manifold with fibered face $\mathbf{F}$. Then any two fibers $F_1, F_2 \in \mathbb{R}_+\mathbf{F}$ produce the same veering triangulation of $\mathring{M}$, up to isotopy.* □

We close this background section with two easy but useful observations that date back to Thurston [46].

Fact 9.3. For $x, y \in H_2(M, \partial M; \mathbb{R})$, the equality $\|x + y\| = \|x\| + \|y\|$ holds if and only if x and y are in the same cone over a face of the unit Thurston norm ball.

This follows by the definition of a norm, combined with the property that the unit ball B is a polyhedron.

Fact 9.4. If S represents $x \in H_2(M, \partial M; \mathbb{Z})$, then $\|x\| \equiv -\chi(S) \mod 2$.

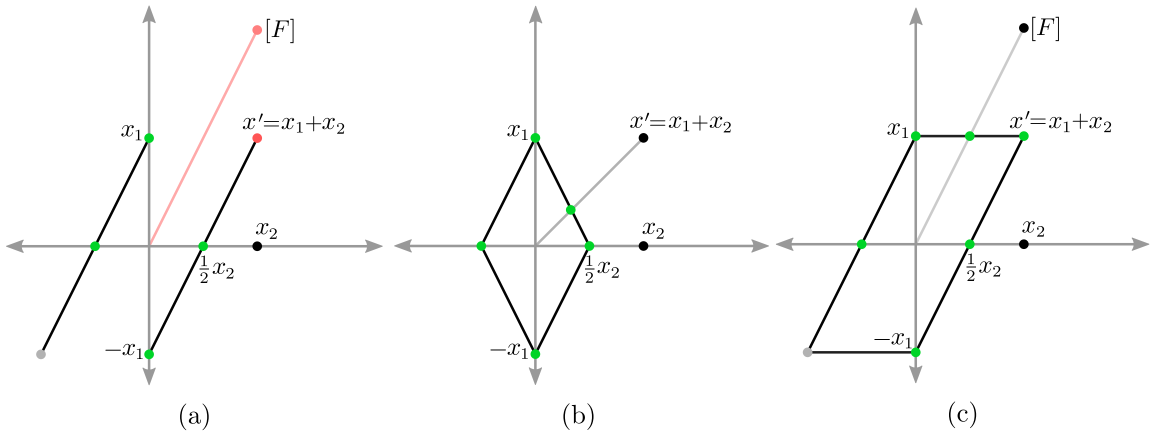

FIGURE 7. (a) For the case $F \cong \Sigma_{2,1}$, the assumption $\|x'\| = 1$ leads to a contradiction. (b) When $F \cong \Sigma_{2,1}$, the Thurston norm ball is the rhombus shown. (c) When $F \cong \Sigma_{1,2}$, the Thurston norm ball is either the polygon shown, or the mirror image of this polygon across the vertical axis.

This follows from the existence of the boundary map $\partial\colon H_2(M, \partial M; \mathbb{Z}) \to H_1(\partial M, \mathbb{Z})$, and the fact that the number of components of an embedded multi-curve representing an element of $H_1(\partial M, \mathbb{Z})$ is invariant mod 2.

9.2. **Finding desired fibers.** The following lemma will be used to find fibers of almost every topological type.

Lemma 9.5. *Let M be a one-cusped fibered hyperbolic manifold with $H_2(M, \partial M; \mathbb{R}) \cong \mathbb{R}^2$, and suppose M contains embedded surfaces $S_1 \cong \Sigma_{1,1}$ and $S_2 \cong \Sigma_{2,0}$ representing non-trivial classes in $H_2(M, \partial M)$.*

(1) *If M has a fiber $F \cong \Sigma_{2,1}$, then the vertices of the unit Thurston norm ball are $\pm[S_1]$ and $\pm\frac{1}{2}[S_2]$. Furthermore, the fibered cone containing F also contains fibers homeomorphic to $\Sigma_{g,n}$ for all $g \geqslant 2$ and $n \geqslant 1$ such that $(g-1, n)$ are relatively prime. All of these fibers have the same boundary slope as F.*

(2) *If M has a fiber $F \cong \Sigma_{1,2}$, then the vertices of the unit Thurston norm ball are $\pm[S_1]$, and either $\pm([S_2] + [S_1])$ or $\pm([S_2] - [S_1])$. Furthermore, the fibered cone containing F also contains fibers homeomorphic to $\Sigma_{1,n}$ for all $n \geqslant 2$. All of these fibers have the same boundary slope as F.*

Proof. Let $x_1 = [S_1]$ and $x_2 = [S_2]$. Since $\chi(\Sigma_{1,1}) = -1$ and $\|x_1\| > 0$, we conclude that $\|x_1\| = 1$. Similarly, since $\chi(\Sigma_{2,0}) = -2$ and $\|x_2\| > 0$, Fact 9.4 implies that $\|x_2\| = 2$. Thus the four classes $\pm x_1$ and $\pm\frac{1}{2}x_2$ all lie in ∂B, where B is the unit ball of the norm.

Recall the boundary homomorphism $\partial\colon H_2(M, \partial M) \to H_1(\partial M)$, and fix the class $l = \partial x_1 \in H_1(\partial M; \mathbb{Z})$. Note that $\pm l$ are the unique primitive classes in $H_1(\partial M; \mathbb{Z})$ that are trivial in $H_1(M)$. Observe as well that $x_1 \neq x_2$ because $\partial x_1 = l \neq 0 = \partial x_2 \in H_1(\partial M; \mathbb{Z})$.

Now consider $x' = x_1 + x_2$ and $x'' = x_1 - x_2$. Since $\|x'\| \leqslant \|x_1\| + \|x_2\| = 3$ and $\|x'\| \equiv \chi(\Sigma_{1,1}) + \chi(\Sigma_{2,0}) \mod 2$, we have $\|x'\| \in \{1, 3\}$. Similarly, $\|x''\| \in \{1, 3\}$.

Case 1: M has a fiber $F \cong \Sigma_{2,1}$, which implies $\partial[F] = \pm l$.

Suppose for a contradiction that $\|x'\| = 1$. Then, by Fact 9.3, the points $-x_1, \frac{1}{2}x_2, x' \in \partial B$ determine a line segment contained in a face of ∂B. Since M has a cusp, and the interior point $\frac{1}{2}x_2$ is represented by (half) the closed surface $S_2 \cong \Sigma_{2,0}$, this cannot be a fibered face. Similarly, the points $x_1, \frac{-1}{2}x_2, -x' \in \partial B$ determine a line segment in a non-fibered face of ∂B. (See Figure 7a.) It follows that the fiber F lies in a cone over some other face, which we may assume is in the first quadrant by changing the orientation of F if necessary. Hence we can write $[F] = ax_1 + bx'$, for some $a, b \in \mathbb{Q}_{>0}$. Since $x' = x_1 + x_2$, we have

$$\partial x' = \partial x_1 + \partial x_2 = \partial x_1 = l \in H_1(\partial M),$$

implying

$$\pm l = \partial[F] = \partial(ax_1 + bx') = (a+b)l.$$

Hence $a + b = \pm 1$, but $a + b \geqslant 0$, so we must have $a + b = 1$. It follows that $[F]$ lies on the line segment joining x_1 and x', which is impossible since $\|[F]\| = 3$. From this contradiction, it follows that $\|x'\| = 3$.

If we consider $x'' = x_1 - x_2$ in place of x', and assume that $\|x''\| = 1$, then the argument above with the obvious modifications again gives a contradiction. Hence $\|x_1 - x_2\| = 3 = \|x_2 - x_1\|$. Therefore the points $\pm x_1$, $\pm\frac{1}{2}x_2$, $\pm\frac{1}{3}x'$, $\pm\frac{1}{3}x''$ all have norm 1, hence Fact 9.3 implies that these points determine the unit norm ball. It follows that the vertices are $\pm v_1, \pm v_2$, where $v_1 = x_1 = [S_1]$ and $v_2 = \frac{1}{2}x_2 = \frac{1}{2}[S_2]$. See Figure 7b.

Now, let $\mathbf{F}$ be the face containing F. Without loss of generality, $\mathbf{F}$ has vertices $\{v_1, v_2\}$. Fix a pair (g, n) where $g \geqslant 2$ and $n \geqslant 1$, and where $\gcd(g-1, n) = 1$. Let $y = ax_1 + bx_2$, where $a = n$ and $b = g - 1$. Since $\gcd(a, b) = 1$, the homology class y is primitive and hence represented by a fiber F'. Since the norm is linear on faces by Fact 9.3, $\|y\| = a\|x_1\| + b\|x_2\| = a + 2b$. Furthermore, F' has exactly $a = n$ boundary components, since S_2 is closed and S_1 has one boundary component. Thus $a + 2b = \|y\| = -\chi(F') = 2g(F') - 2 + a$, giving $g(F') = b + 1 = g$, as desired.

Case 2: $F \cong \Sigma_{1,2}$, which implies $\partial[F] = \pm 2l$.

Suppose, for a contradiction, that $\|x'\| = \|x''\| = 3$. Then the points $\pm x_1$, $\pm\frac{1}{2}x_2$, $\pm\frac{1}{3}x'$, $\pm\frac{1}{3}x''$ all have norm 1, and determine the unit norm ball must be as shown in Figure 7b. Hence, up to changing signs, we may assume that $[F] = ax_1 + b(\frac{1}{2}x_2)$ for some $a, b \in \mathbb{Q}_+$. Then

$$\pm 2l = \partial[F] = \partial\big(ax_1 + b(\tfrac{1}{2}x_2)\big) = al \qquad \Longrightarrow \qquad a = \pm 2.$$

Since the Thurston norm is linear in the cone over a face, we have

$$2 = \|[F]\| = \|ax_1 + b(\tfrac{1}{2}x_2)\| = |a| + |b| = 2 + |b| \qquad \Longrightarrow \qquad b = 0,$$

which is impossible, because the fiber F must be in the interior of a fibered cone. This contradiction implies that either $\|x'\| = 1$ or $\|x''\| = 1$.

If $\|x'\| = 1$, the unit sphere ∂B contains the segments connecting $\pm x'$ and $\mp x_1$, as shown in Figure 7c. As above, we observe that $\frac{1}{2}x_2$ cannot lie in the interior of a fibered face, so (after possibly reversing the orientation on F) we must have $[F] = ax_1 + bx'$ for $a, b \in \mathbb{Q}_+$. Applying the boundary homomorphism gives

$$2l = \partial[F] = \partial(ax_1 + bx') = (a+b)l$$

which implies

$$a + b = 2 = \|[F]\| \leqslant a\|x_1\| + b\|x'\| = a + b.$$

Since the norm is only linear in the cone over a face (Fact 9.3), the segment joining x_1 to x' must lie in a face of ∂B. It follows that $\pm x_1, \pm x'$ are the only vertices.

If $\|x''\| = 1$, an identical argument applies with x' replaced by x''. In this case, the vertices of ∂B are $\pm x_1, \pm x''$. Thus, in both cases, the vertices of the unit norm ball are $\pm v_1$ and $\pm v_2$, where $v_1 = [S_1]$, and v_2 is either $[S_2] + [S_1]$ or $[S_2] - [S_1]$.

Now, let $\mathbf{F}$ be the face containing F. Without loss of generality, say $v_2 = [S_1] + [S_2]$ and $\mathbf{F}$ has vertices $\{v_1, v_2\}$. The norm-realizing surface P representing v_2 has $\chi(P) = -1$ and $\partial[P] = l$. Thus P is either a pair of pants or a one-holed torus. If P is a pair of pants, then two boundary components of P must cancel in $H_1(\partial M)$, which means they can be tubed together to obtain an embedded one-holed torus. Thus, in either case, v_2 is represented by an embedded $\Sigma_{1,1}$. Fix an integer $n \geqslant 2$, and let $y = v_1 + (n-1)v_2 = n[S_1] + (n-1)[S_2]$. As before, y is primitive and therefore represented by a fiber F'. Since the Thurston norm is linear on the fibered cone, $\|y\| = \|v_1\| + (n-1)\|v_2\| = n$. Furthermore, since $\partial S_2 = \varnothing$, F' must have exactly n boundary components. This gives that $n = \|y\| = -\chi(F') = 2g(F') - 2 + n$ which implies $g(F') = 1$. We conclude that $F' \cong \Sigma_{1,n}$, as required. ☐

Before proving Theorem 1.3, we need a straightforward lemma about covers.

Lemma 9.6. *Let $\varphi\colon S \to S$ be a pseudo-Anosov homeomorphism and $f\colon \widehat{S} \to S$ a degree $d < \infty$ covering. Then the following holds.*

(1) There exists a pseudo-Anosov $\widehat{\varphi} : \widehat{S} \to \widehat{S}$ that is a lift of some power φ^k of φ.
(2) The veering triangulation τ_φ is a geometric triangulation of $\mathring{M}_\varphi$ if and only if $\tau_{\widehat{\varphi}}$ is a geometric triangulation of $\mathring{M}_{\widehat{\varphi}}$.
(3) If φ is principal and each peripheral curve of S has d lifts to $\widehat{S}$, then $\widehat{\varphi}$ is also principal.

Proof. Conclusion (1) is standard. Let d be the degree of the cover. Then the finitely many index d subgroups of $\pi_1(S)$ are permuted by the induced isomorphism φ_*. Thus some power of φ_* must stabilize the subgroup $f_*\pi_1(\widehat{S}) \subset \pi_1(S)$, allowing the lifting criterion to be applied.

Conclusion (2) follows from the fact that every simplex in the veering triangulation τ_φ of $\mathring{M}_\varphi$ lifts to a simplex in the veering triangulation $\tau_{\widehat{\varphi}}$ of $\mathring{M}_{\widehat{\varphi}}$, with the same shape.

For conclusion (3), note that since φ is principal, every singularity of φ is either 3–pronged and occurs at an interior point of S or 1–pronged and occurs at a puncture. Since each peripheral curve of S has d lifts to $\widehat{S}$, the same is true for $\widehat{\varphi}$. Thus $\widehat{\varphi}$ is principal. ☐

We can now begin proving Theorem 1.3, case by case.

Proposition 9.7. *Let $S \cong \Sigma_{g,n}$ be a hyperbolic surface of genus $g \geqslant 1$, excluding $\Sigma_{1,1}$. Then there exists a principal pseudo-Anosov $\varphi \in \mathrm{Mod}(S)$ such that the associated veering triangulation of the mapping torus $\mathring{M}_\varphi$ is non-geometric.*

Proof. We consider three different cases.

Case 1: $g = 1$ and $n \geqslant 2$. Let $F = \Sigma_{1,2}$ and let $\varphi = \varphi_3$ be the third mapping class described in Proposition 8.4. By Proposition 8.4, φ is a principal pseudo-Anosov, such that the veering triangulation of $M_{\mathring{\varphi}}$ is non-geometric.

Let M_φ be the mapping torus of $\varphi\colon F \to F$. According to `Regina`, M_φ contains embedded surfaces $S_1 \cong \Sigma_{1,1}$ and $S_2 \cong \Sigma_{2,0}$ which are non-trivial in $H_2(M_\varphi, \partial M_\varphi; \mathbb{R}) \cong \mathbb{R}^2$. To verify this, `Regina` computes the complete list of embedded vertex normal surfaces for M_φ. Among these vertex normal surfaces are $S_1 \cong \Sigma_{1,1}$ and $S_2 \cong \Sigma_{2,0}$. Cutting M_φ along these surfaces ensures that they are homologically non-trivial. The dimension of the homology is also rigorously computed by `Regina`. See the ancillary files [19] for full details.

Thus, by Lemma 9.5, the fibered cone containing F also contains fibers homeomorphic to $\Sigma_{1,n}$ for all $n \geqslant 1$. All of these fibers have the same boundary slope as F, hence the mapping classes of these fibers are all principal by Lemma 9.1. Finally, Lemma 9.2 says that all of these fibers induce the same non-geometric veering triangulation of $M_{\mathring{\varphi}}$.

Case 2: $g \geqslant 2$ and $n = 0$. Let $F = \Sigma_{2,0}$, and let $\varphi = \varphi_1 \in \mathrm{Mod}(F)$ be the first mapping class described in Proposition 8.4. By that proposition, φ is a principal pseudo-Anosov, such that the veering triangulation of $M_{\mathring{\varphi}}$ is non-geometric. Now, recall that every closed hyperbolic surface S is a finite cover of F. Thus Lemma 9.6 gives the desired result for S.

Case 3: $g \geqslant 2$ and $n \geqslant 1$. Let $F = \Sigma_{2,1}$ and let $\varphi = \varphi_2$ be the second mapping class described in Proposition 8.4. By Proposition 8.4, φ is a principal pseudo-Anosov, such that the veering triangulation of $M_{\mathring{\varphi}}$ is non-geometric.

Let M_φ be the mapping torus of $\varphi\colon F \to F$. Using `Regina`, as in Case 1, we check that M_φ contains embedded surfaces $S_1 \cong \Sigma_{1,1}$ and $S_2 \cong \Sigma_{2,0}$ which are non-trivial in $H_2(M_\varphi, \partial M_\varphi; \mathbb{R}) \cong \mathbb{R}^2$. By Lemma 9.5, the fibered cone containing F also contains fibers homeomorphic to $\Sigma_{g,n}$ for all $g \geqslant 2$ and $n \geqslant 1$, where $(g-1, n)$ are relatively prime. All of these fibers have the same boundary slope as F. Thus, by Lemmas 9.1 and 9.2, we obtain the desired conclusion for all $g \geqslant 2$ and $n \geqslant 1$ such that $\gcd(g-1, n) = 1$.

Finally, suppose $S \cong \Sigma_{g,n}$, with $\gcd(g-1, n) = d > 1$. Then $g' - 1 = (g-1)/d$ and $n' = n/d$ are relatively prime, with $g' \geqslant 2$ and $n' \geqslant 1$. Thus, by the above paragraph, the fibered cone of M_φ containing F also contains a fiber $F' \cong \Sigma_{g',n'}$. Observe that S is a d–fold cyclic cover of F' (realize S with d groups of $g'-1$ doughnut holes and n' punctures, arranged symmetrically around a central doughnut hole). By construction, peripheral curves of F' lift to peripheral curves of S. Thus, by Lemma 9.6, a power of the monodromy of F' lifts to a principal pseudo-Anosov on S, and the non-geometric veering triangulation of $M_{\mathring{\varphi}}$ lifts to a non-geometric veering triangulation of the corresponding finite cover of $M_{\mathring{\varphi}}$. □

Proposition 9.8. *Let $S \cong \Sigma_{0,n}$ be a surface of genus $g = 0$, with $n \geqslant 5$ punctures. Then there exists a principal pseudo-Anosov $\varphi \in \mathrm{Mod}(S)$ such that the associated veering triangulation of the mapping torus $\mathring{M}_\varphi$ is non-geometric.*

Proof. If $n = 5$ or $n = 6$, the mapping classes φ_5 and φ_6 described in Proposition 8.4 satisfy the desired conclusion. From now on, we treat planar surfaces with $n \geqslant 7$ punctures.

Let $F = \Sigma_{0,7}$, and let φ be the mapping class

$$\varphi_7 = T_{r_5}^2 T_{r_4} T_{r_3}^{-1} T_{r_1}^{-1} T_{r_4} T_{r_2}^{-1} T_{r_3} T_{r_1} T_{r_2}^{-1}$$

given in Proposition 8.4. By Proposition 8.4, φ is a principal pseudo-Anosov and the veering triangulation of the mapping torus $M_{\mathring{\varphi}}$ is non-geometric. We will show that the fibered cone of $H_2(M_\varphi, \partial M_\varphi)$ containing $[F]$ also contains a fiber homeomorphic to $\Sigma_{0,n}$ for every $n \geqslant 7$. Then, we will show that all of these fibers have principal monodromy.

To begin the proof, we embed $M = M_\varphi$ as a link complement in S^3. Note that the generators T_{r_6} and T_{r_0} do not appear in φ, hence one of the punctures of F is fixed. We can therefore think of φ as a mapping class on the 6−punctured disk. More precisely, let B_k be the braid group on k strands, and consider the natural homomorphism $B_k \to \mathrm{Mod}(\Sigma_{0,k+1})$ defined by $\sigma_i \mapsto T_{r_i}$. Then φ is the mapping class corresponding to the braid

$$\beta = \sigma_5^2 \sigma_4 \sigma_3^{-1} \sigma_1^{-1} \sigma_4 \sigma_2^{-1} \sigma_3 \sigma_1 \sigma_2^{-1}.$$

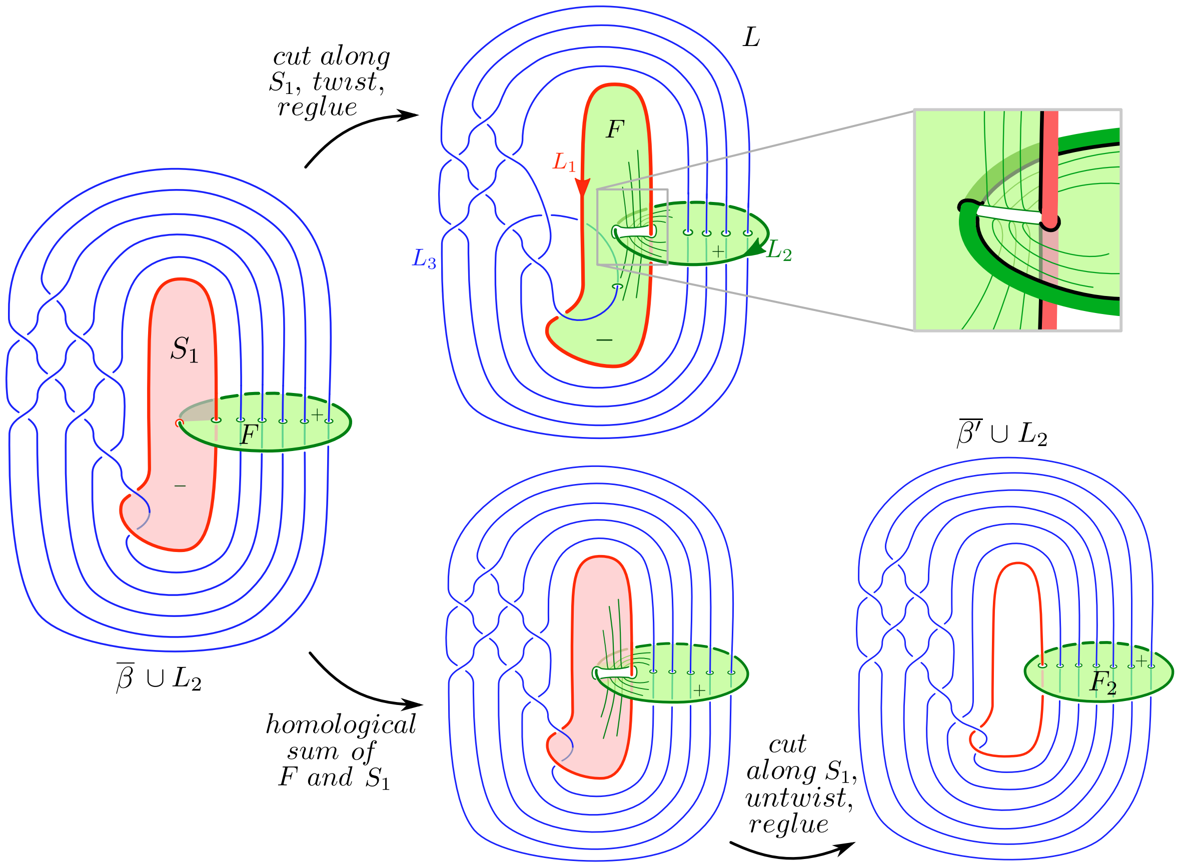

FIGURE 8. The mapping torus M_φ of $\varphi = T_{r_5}^2 T_{r_4} T_{r_3}^{-1} T_{r_1}^{-1} T_{r_4} T_{r_2}^{-1} T_{r_3} T_{r_1} T_{r_2}^{-1}$ has many embeddings as a link complement in S^3. In the left panel, we realize φ as a braid word β, whose braid generators are read from the bottom up. In the top center panel, we cut, twist, and reglue along the twice-punctured disk S_1, giving a re-embedding of M. In the bottom right panel, we twist along S_1 in the opposite direction, making it easier to see the fiber F_2 that is homologous to $F + S_1$ and compute the monodromy of F_2.

Consequently, the mapping torus M_φ is homeomorphic to $S^3 \smallsetminus (\overline{\beta} \cup L_2)$, where $\overline{\beta}$ is the braid closure of β and L_2 is the braid axis. See the left panel of Figure 8. In this embedding of M_φ, the fiber F becomes the 6–punctured disk shown in green.

Next, we re-embed M into S^3 via a Rolfsen twist. That is: cut M along the twice-punctured disk S_1 (colored pink in Figure 8), perform one counter-clockwise full twist from the underside, and re-glue along S_1. After this operation, we have $M_\varphi \cong S^3 \smallsetminus L$, where $L = L_1 \cup L_2 \cup L_3$ is the three-component link in the upper center of Figure 8. The image of the fiber F under this re-embedding is shown again in light green.

The link L allows a clear view of two surfaces that will be important for our homological computations: the 2–punctured disk S_1 bounded by L_1, and the 5–punctured disk S_2 bounded by L_2. The top center of Figure 8 shows their (transverse) orientations: we are looking at the back side of S_1 and the front side of S_2. Then, setting $x_1 = [S_1]$ and

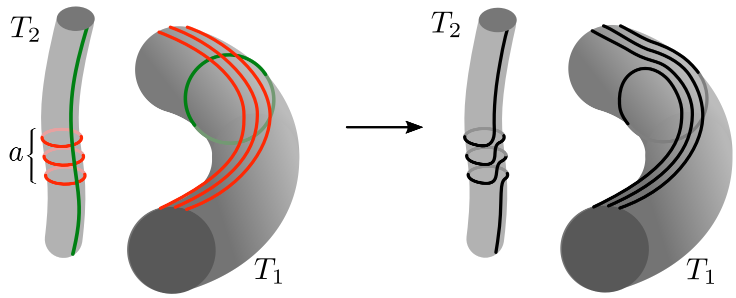

FIGURE 9. The homological sum $a\partial x_1 + \partial x_2 \in H_1(\partial M)$. The left frame shows $a \cdot \partial S_1 \cap (T_1 \cup T_2)$ in red and $\partial S_2 \cap (T_1 \cup T_2)$ in green. After the sum (right), there is a curve of slope $1/a$ on T_1 and a curve of slope a on T_2.

$x_2 = [S_2]$, we have $[F] = x_1 + x_2$. Since

$$5 = \|[F]\| \leqslant \|x_1\| + \|x_2\| \leqslant 1 + 4 = 5,$$

we learn that $\|x_1\| = 1$ and $\|x_2\| = 4$. Furthermore, since the Thurston norm is only linear in the cone over a face (see Fact 9.3), it follows that the segment joining x_1 to x_2 must lie in the fibered cone containing $[F]$.

Now, let $a \in \mathbb{N}$ and consider $y = ax_1 + x_2$. Since y is primitive, it is represented by a fiber F_a. We wish to compute the topological type of F_a, starting with the number of punctures.

Let $\partial\colon H_2(M, \partial M) \to H_1(\partial M)$ be the boundary homomorphism $[S] \mapsto [\partial S]$. To compute $\partial y = \partial(ax_1 + x_2) = a\partial x_1 + \partial x_2$, it suffices to take the homological sum (in $H_1(\partial M)$) of a copies of $[\partial S_1]$ and one copy of $[\partial S_2]$. Let T_i be the torus of ∂M corresponding to link component L_i. Then the only intersections of ∂S_1 with ∂S_2 occur on T_1 and T_2. On the torus T_1, there are a copies of the longitude, coming from $a[\partial S_1]$, and one copy of the meridian, coming from $[\partial S_2]$. The homological sum of these is a single curve of slope $1/a$. The situation on T_2 is similar: there are a copies of the meridian, and one copy of the longitude, giving a single curve of slope a. Figure 9 demonstrates this for $a = 3$. Since L_3 intersects S_1 once and S_2 four times, the boundary of y also contains $a + 4$ copies of the meridian on the torus T_3. Furthermore, the orientations on S_1 and S_2 induce the same orientation on each of these $a+4$ copies of the meridian, so none of them cancel in $H_1(\partial M)$. We conclude that the number of boundary components of F_a is $1 + 1 + (a + 4) = a + 6$.

Since the norm is linear on the cone over a face, we get $\|y\| = a\|x_1\| + \|x_2\| = a + 4$. Furthermore, since $F_a \cong \Sigma_{g,n}$ is a fiber, hence norm-realizing, we have

$$a + 4 = \|y\| = -\chi(F_a) = 2g - 2 + n = 2g - 2 + (a + 6) = 2g + (a + 4).$$

We conclude that the genus of F_a is $g = 0$. Hence $F_a \cong \Sigma_{0,a+6}$. By varying the value of $a \in \mathbb{N}$, we get all surfaces $\Sigma_{0,n}$ for $n \geqslant 7$. By Lemma 9.2, the veering triangulation associated to the monodromy for F_a is the same as the veering triangulation for $F = F_1$, hence non-geometric.

Next, we compute the monodromy of F_2 and show that it is principal. Since $[F] = [F_1] = x_1 + x_2$, we have

$$[F_2] = x_1 + x_1 + x_2 = [S_1] + [F].$$

The fiber F_2 is shown in the bottom center frame of Figure 8. We may visualize the monodromy of F_2 by again re-embedding M into S^3, via a Rolfsen twist in the opposite direction. That is: cut M along the twice-punctured disk S_2, perform a full clockwise twist (from the underside), and reglue. This realizes M as the complement of a new link, shown in Figure 8, bottom right. This link is $\overline{\beta'} \cup L_2$, where $\beta' = \sigma_6^2\sigma_5\sigma_4\sigma_3^{-1}\sigma_1^{-1}\sigma_4\sigma_2^{-1}\sigma_3\sigma_1\sigma_2^{-1} \in B_7$. After the re-embedding, the fiber F_2 becomes the green 7-punctured disk shown, with monodromy ψ corresponding to by the braid word β':

$$\psi = T_{r_6}^2 T_{r_5} T_{r_4} T_{r_3}^{-1} T_{r_1}^{-1} T_{r_4} T_{r_2}^{-1} T_{r_3} T_{r_1} T_{r_2}^{-1}.$$

Using `flipper`, we confirm that ψ is in fact principal.

It remains to show that the monodromy of F_a is principal for every $a \in \mathbb{N}$. We already know this for F_1 and F_2. We finish the proof using Lemma 9.1 and linear algebra. For each cusp torus T_i of M, let δ_i be a simple closed curve realizing the degeneracy slope of F, oriented in the direction of the flow η. Then, for $i \in \{1,2,3\}$, we have a sequence of homomorphisms ($\mathbb{Z}$ coefficients are presumed):

$$H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) \xrightarrow{\pi_i} H_1(T_i) \xrightarrow{\iota(\cdot,\delta_i)} \mathbb{Z},$$

where $\pi_i \colon H_1(\partial M) \to H_1(T_i)$ is the projection map to the i-th component and $\iota(\cdot,\delta_i)$ is the algebraic intersection pairing. The composition of these homomorphisms is a linear functional $\nu_i \colon H_2(M,\partial M) \to \mathbb{Z}$. Consider its values for $[F_1]$ and $[F_2]$.

On the torus T_1, both fibers F_1 and F_2 have a single boundary component (see Figure 9, right). Since the monodromies of F_1 and F_2 are principal, both ∂F_1 and ∂F_2 intersect δ_1 once. With our orientations, $\nu_1([F_1]) = \nu_1([F_2]) = 1$. Thus, by linearity, we have $\nu_1([F_a]) = 1$ for every a. By an identical argument, $\nu_2([F_1]) = \nu_2([F_2]) = 1$, hence $\nu_2([F_a]) = 1$ for every a. Finally, on the torus T_3, we have seen that ∂F_a consists of $a+4$ parallel components whose slope is independent of a. Since F_1 is principal, each of these components intersects δ_3 once. Since every boundary component of F_a intersects the degeneracy slope once, Lemma 9.1 implies that the monodromy of F_a is principal for every $a \in \mathbb{N}$. □

Remark 9.9. `flipper` has the capability to compute degeneracy slopes from a veering triangulation, using [18, Observation 2.9]. A combination of `flipper` and `Snappy` shows that in $M \cong S^3 \smallsetminus (L_1 \cup L_2 \cup L_3)$, the degeneracy slope δ_1 agrees with the longitude of L_1, while δ_2 and δ_3 are meridians of L_2 and L_3, respectively. This fact, combined with Figure 9, shows that every F_a is principal. The above argument using the linear functionals ν_i avoids the need to ever identify δ_i.

Proof of Theorem 1.3. Let S be a hyperbolic surface. If $\xi(S) = 0$, then $S \cong \Sigma_{0,3}$, hence $\mathrm{Mod}(S)$ is finite. If $\xi(S) = 1$, then $S \cong \Sigma_{0,4}$ or $\Sigma_{1,1}$, and the work of Akiyoshi [2], Lackenby [30], and Guéritaud [21] shows that all pseudo-Anosov mapping classes in $\mathrm{Mod}(S)$ have geometric veering triangulations.

Now, assume that $\xi(S) \geqslant 2$. Under this hypothesis, Propositions 9.7 and 9.8 show that there exists a principal pseudo-Anosov $\varphi \in \mathrm{Mod}(S)$ such that the associated veering triangulation of the mapping torus $\mathring{M}_\varphi$ is non-geometric. □

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