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The Löwner Function of a Log-Concave Function

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Abstract
We introduce the notion of Löwner (ellipsoid) function for a log-concave function and show that it is an extension of the Löwner ellipsoid for convex bodies. We investigate its duality relation to the recently defined John (ellipsoid) function (Alonso-Gutiérrez et al. in J Geom Anal 28:1182–1201, 2018). For convex bodies, John and Löwner ellipsoids are dual to each other. Interestingly, this need not be the case for the John function and the Löwner function.

Keywords John ellipsoid · Löwner ellipsoid · Log-concave functions

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1 Introduction

Asymptotic convex geometry studies the properties of convex bodies with emphasis on the dependence of geometric and analytic invariants on the dimension. The convexity assumption enforces concentration of volume in a canonical way and it is a main

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question if under natural normalizations the answers to fundamental questions are independent of the dimension.

The most classical normalizations of convex bodies arise as solutions of extremal problems. These normalizations include the isotropic position, which arose from classical mechanics of the 19th century and which is related to a famous open problem in convex geometry, the hyperplane conjecture (see, e.g., the survey [29]). The best results currently available there are due Bourgain [11] and Klartag [28].

Other positions are the John position, also called maximal volume ellipsoid position and the Löwner position, also called minimal volume ellipsoid position. The right choice of a position is important for the study of affinely invariant quantities and their related isoperimetric inequalities. For instance, John and Löwner position are related to the Brascamp–Lieb inequality and its reverse [8,10], to Ball's sharp reverse isoperimetric inequality [9], to the notion of volume ratio [45,47], which is defined as the n-th root of the volume of a convex body divided by the volume of its John ellipsoid and which finds applications in functional analysis and Banach space theory [12,21,42,47]. John and Löwner position are even relevant in quantum information theory [5,6,46]. Since a position may be seen as a choice of a special ellipsoid, and since an ellipsoid entails a Euclidean structure of the underlying space, John and Löwner ellipsoids provide a way to measure how far a normed space is from Euclidean space [22,26]. For a detailed discussion of the John and the Löwner ellipsoid and its connections to functional analysis we refer the reader to [2,13,41] and the survey [25].

John proved in [26] that among all ellipsoids contained in a convex body $K \in \mathbb{R}^n$, there is a unique ellipsoid of maximal volume, now called the John ellipsoid of K. The Löwner ellipsoid of K is the unique ellipsoid of minimal volume containing K. These two notions are closely related by polarity (see, e.g., [13,33]): A 0-symmetric ellipsoid $\mathcal{E}$ is the ellipsoid of maximal volume inside K if and only if $\mathcal{E}^\circ$ is the ellipsoid of minimal volume outside K°, where $K^\circ = \{y \in \mathbb{R}^n : \langle y, x\rangle \leq 1 \text{ for all } x \in K\}$ is the polar of K.

Probabilisitic methods have become extremely useful in convex geometry. In this context, log-concave functions arise naturally from the uniform measure on convex bodies. A function $f(x)$ is said to be log-concave, if it is of the form $f(x) = \exp(-\psi(x))$ where $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex. Extensive research has been devoted within the last ten years to extend the concepts and inequalities from convex bodies to the setting of functions. In fact, it was observed early that the Prékopa–Leindler inequality (see, e.g., [20,37]) is the functional analog of the Brunn–Minkowski inequality (see, e.g., [19]) for convex bodies. Much progress has been made since and functional analogs of many other geometric inequalities were established. Among them are the functional Blaschke-Santaló inequality [3,7,17,32] and its reverse [18], a functional affine isoperimetric inequality for log-concave functions which can be viewed as an inverse log-Sobolev inequality for entropy [4,14] and a theory of valuations, an important concept for convex bodies (e.g., [24,30,31,43,44]), is currently being developed in the functional setting, e.g., [15,16,36].

It was only recently that the notion of a John (ellipsoid) function of a log-concave function was established by Alonso-Gutiérrez et al. [1]. However, the notion of a Löwner ellipsoid function for log-concave functions has been missing till now. In this paper we put forward such a notion and we investigate, among other things, its relation to the John ellipsoid function of [1].

Our main result reads as follows. We denote by $\mathcal{A}$ the set of all invertible affine transformations and by $\|\cdot\|_2$ denote the Euclidean norm on $\mathbb{R}^n$. We say that a function is nondegenerate if $\mathrm{int}(\mathrm{supp} f) \neq \emptyset$.

Theorem *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nondegenerate integrable log-concave function. There exists a unique pair $(A_0, t_0) \in \mathcal{A} \times \mathbb{R}$ such that*

$$\int_{\mathbb{R}^n} e^{-\|A_0 x\|_2 + t_0} dx = \min \left\{ \int_{\mathbb{R}^n} e^{-\|Ax\|_2 + t} dx : t \in \mathbb{R}, A \in \mathcal{A}, e^{-\|Ax\|_2 + t} \geq f(x) \right\}.$$

The uniqueness of A_0 is up to left orthogonal transformations.

We then call $e^{-\|A_0 x\|_2 + t_0}$ the Löwner function of f and denote it by

$$L(f)(x) = e^{-\|A_0 x\|_2 + t_0}.$$

The function $L(f)$ is a functional analog of the Löwner ellipsoid for log-concave functions. Indeed, we show that if $\mathbb{1}_K(x)$ is the characteristic function of a convex body $K \in \mathbb{R}^n$, then the super-level set $\{L(\mathbb{1}_K) \geq 1\}$ is exactly the Löwner ellipsoid of K. If, in addition, 0 is the center of the Löwner ellipsoid of K, then it holds by polarity via the Legendre transform that the polar of the Löwner function is the John function of $(\mathbb{1}_K)^\circ$. This is the exact analog of the above quoted polarity relation of John and Löwner ellipsoids for a convex body and its polar. While in the case of convex bodies the two notions of John and Löwner ellipsoid are always dual to each other, interestingly, in the functional setting this need no longer be the case. It holds when the functions are even or characteristic functions of convex bodies.

The paper is structured as follows. In Sect. 2 we introduce the basic facts and preliminaries. In Sect. 3.1 we define the notion of Löwner function $L(f)$ for a log-concave function f and we prove its existence and uniqueness. In Sect. 4, we recover the John function of [1] and discuss the duality between these two notions.

2 Notation and Preliminaries

Throughout the paper we will use the following notations. The set of all non-singular affine transformations on $\mathbb{R}^n$ is written as $\mathcal{A}$,

$$\mathcal{A} = \{A = T + a : T \in GL(n), a \in \mathbb{R}^n\}.$$

Let S_+ be the set of symmetric positive definite matrices. Then

$$\mathcal{SA} = \{A = T + a : T \in S_+, a \in \mathbb{R}^n\}.$$

For $b \in \mathbb{R}^n$ fixed, put

$$\mathcal{A}(b) = \{A = T + a : T \in GL(n), a \in \mathbb{R}^n, T^{-1}a = b\}.$$

Let $\mathcal{SA}(b) = \mathcal{A}(b) \cap \mathcal{SA}$. Clearly, $\mathcal{A} = \cup_{b\in\mathbb{R}^n}\mathcal{A}(b)$ and $\mathcal{SA} = \cup_{b\in\mathbb{R}^n}\mathcal{SA}(b)$.

The action of an affine transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ on a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as $Af(x) = f(Ax)$.

For $z \in \mathbb{R}^n$, let S_z be a translation of a function by z, that is, for a function f,

$$(S_z f)(x) = f(x+z) \tag{1}$$

For $s \in \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}$, we denote by

$$G_f(s) = \{x \in \mathbb{R}^n : f(x) \geq s\}$$

the super-level sets of f.

2.1 Log-Concave Functions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be log-concave if it is of the form $f(x) = \mathrm{e}^{-\psi(x)}$ where $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a convex function. We always consider in this paper log-concave functions f that are integrable and such that f is nondegenerate, i.e., the interior of the support of f is non-empty, $\mathrm{int}(\mathrm{supp} f) \neq \emptyset$. This then implies that $0 < \int_{\mathbb{R}^n} f \mathrm{d}x < \infty$.

We will also need the Legendre transform which we recall now. Let $z \in \mathbb{R}^n$ and let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a convex function. Then

$$\mathcal{L}_z\psi(y) = \sup_{x\in\mathbb{R}^n} [\langle x-z, y-z\rangle - \psi(x)]$$

is the Legendre transform of ψ with respect to z [3,18] . If $f(x) = \mathrm{e}^{-\psi(x)}$ is log-concave, then

$$f^z(y) = \inf_{x\in\mathrm{supp}(f)} \frac{\mathrm{e}^{-\langle x-z, y-z\rangle}}{f(x)} = \mathrm{e}^{-\mathcal{L}_z\psi(y)} \tag{2}$$

is called the dual or polar function of f with respect to z. In particular, when $z = 0$,

$$f^\circ(y) = \inf_{x\in\mathrm{supp}(f)} \frac{\mathrm{e}^{-\langle x, y\rangle}}{f(x)} = \mathrm{e}^{-\mathcal{L}_0\psi(y)},$$

where $\mathcal{L}_0$, also denoted by $\mathcal{L}$ for simplicity, is the standard Legendre transform.
In the next proposition we collect several well known, easy to verify, properties of the generalized Legendre transform that we will use throughout the paper. They can be found in e.g., [3,17].

Proposition 1 *Let ψ be a convex function. Let S_z be as in* (1). *Then*

(i) *$\mathcal{L}$ and $\mathcal{L}_z$ are involutions, that is, $\mathcal{L}(\mathcal{L}\psi) = \psi$ and $\mathcal{L}_z(\mathcal{L}_z\psi) = \psi$.*
(ii) $\mathcal{L}_z = S_{-z} \circ \mathcal{L} \circ S_z$.
(iii) $\mathcal{L}(S_z\psi)(y) = \mathcal{L}\psi - \langle z, y\rangle$.

(iv) *Legendre transform reverses the oder relation, i.e., if* $\psi_1 \leq \psi_2$*, then* $\mathcal{L}\psi_1 \geq \mathcal{L}\psi_2$.

We now list some basic well-known facts on log-concave functions. A log-concave function is continuous on the interior of its support, e.g., [39].

We include a proof of the first fact for the reader's convenience. More on log-concave functions can be found in e.g., [39].

Fact 1 *If* f *is a nondegenerate integrable log-concave function, then* $G_f(t)$ *is convex and compact for* $0 < t \leq \|f\|_\infty$.

Proof Let $f = \mathrm{e}^{-\psi}$. As ψ is convex and as f is nondegenerate, the super-level set

$$G_f(t) = \{x : f(x) \geq t\} = \{x : -\psi \geq \log t\} = G_{-\psi}(\log t)$$

is convex and closed for all $0 < t \leq \|f\|_\infty$. As $G_f(\|f\|_\infty) \subseteq G_f(t), \forall\, 0 < t \leq \|f\|_\infty$, it remains to show that $G_f(t)$ is bounded for $0 < t < \|f\|_\infty$. It follows from Theorem 7.6 of [39] that every super-level set $G_f(t)$, $0 < t < \|f\|_\infty$, has the same affine dimension as the support of f, which has affine dimension n. Chebyshev inequality then yields

$$\mathrm{vol}_n(G_f(t)) = \mathrm{vol}_n\left(\left\{x \in \mathbb{R}^n : f(x) \geq t\right\}\right) \leq \frac{\|f\|_1}{t} < \infty.$$

Since $G_f(t)$ is a full dimensional convex set with finite volume, it is bounded. Therefore, $G_f(t)$ is compact for $0 < t \leq \|f\|_\infty$. □

The following fact is a direct corollary of the functional Blaschke-Santaló inequality [3,7] and the functional reverse Santaló inequality [17,27].

Fact 2 *Let* $f = \mathrm{e}^{-\psi}$ *be a nondegenerate, integrable, log-concave function such that* 0 *is in the interior of the support of* f*. Then* f° *is again a nondegenerate, integrable log-concave function and thus* $0 < \int_{\mathbb{R}^n} f^\circ(x)\mathrm{d}x < \infty$*. Furthermore,* f^z *is again a nondegenerate, integrable log-concave function, i.e.,* $0 < \int_{\mathbb{R}^n} f^z(x)\mathrm{d}x < \infty$*, provided that* z *is in the interior of* $\mathrm{supp}(f)$.

3 The Löwner Function of a Log-Concave Function

We now define the Löwner function for an integrable, nondegenerate, log-concave function $f = \mathrm{e}^{-\psi}$.

3.1 A Minimization Problem. Definition of the Löwner Function

We consider the following *minimization problem*

$$\min_{(A,t)} \int_{\mathbb{R}^n} \mathrm{e}^{-\|Ax\|_2 + t}\mathrm{d}x \tag{3}$$

subject to

$$\|Ax\|_2 - t \le \psi(x), \quad \text{for all } x \in \mathbb{R}^n, \tag{4}$$

where the minimum is taken over all nonsingular affine maps $A \in \mathcal{A}$ and all $t \in \mathbb{R}$. A change of variables leads to

$$\min_{(A,t)} \int_{\mathbb{R}^n} \mathrm{e}^{-\|Ax\|_2+t} \mathrm{d}x = \min_{(A,t)} e^t \int_{\mathbb{R}^n} \mathrm{e}^{-\|Ax\|_2} \mathrm{d}x = \min_{(A,t)} \frac{e^t}{|\det A|} \int_{\mathbb{R}^n} \mathrm{e}^{-\|y\|_2} \mathrm{d}y$$
$$= n!\, \mathrm{vol}(B_2^n)\ \min_{(A,t)} \frac{e^t}{|\det A|}.$$

Geometrically this means that we minimize the integral of an *ellipsoidal* function $\mathrm{e}^{-\|Ax\|_2+t}$ "outside" f which is exactly what is done when one considers the Löwner ellipsoid of a convex body K: it minimizes the volume of the ellipsoids containing K. The next theorem is the main result of this section.

Theorem 1 *Let $f : \mathbb{R}^n \to \mathbb{R}^+$, $f(x) = \mathrm{e}^{-\psi(x)}$ be a nondegenerate, integrable log-concave function. Then there exists a unique solution modulo $O(n)$ to the minimization problem (3) and (4). That is, there exists a pair (A_0, t_0) satisfying (4) such that*

$$\min_{(A,t)} \int_{\mathbb{R}^n} \mathrm{e}^{-\|Ax\|_2+t} \mathrm{d}x = n!\, \mathrm{vol}(B_2^n)\ \frac{\mathrm{e}^{t_0}}{|\det A_0|}.$$

The number t_0 is unique and the affine map A_0 is unique up to left orthogonal transformations.
We then call $\mathrm{e}^{-\|A_0 x\|_2+t_0}$ the Löwner function of f and denote it by

$$L(f)(x) = \mathrm{e}^{-\|A_0 x\|_2+t_0}.$$

Examples
1. The Löwner function is an extension of the concept of Löwner ellipsoid for convex bodies. Indeed, let

$$f(x) = \mathbb{1}_K(x) = \mathrm{e}^{-I_K(x)}, \quad \text{where} \quad I_K(x) = \begin{cases} \infty, & x \notin K, \\ 0, & x \in K, \end{cases}$$

be the characteristic function of a convex set $K \subset \mathbb{R}^n$. Without loss of generality we may assume that 0 is the center of the Löwner ellipsoid $L(K)$ of K. Then

$$L(\mathbb{1}_K)(x) = \mathrm{e}^{-n\left(\|T_{L(K)}^{-1} x\|_2 - 1\right)}, \tag{5}$$

where $T_{L(K)}$ is the linear map such that $T_{L(K)} B_2^n = L(K)$. To see this, observe that for $A \in \mathcal{A}$, $t \in \mathbb{R}$, the level sets of the map $\varphi(x) = \|Ax\|_2 - t$ are ellipsoids. As 0 is the center of the Löwner ellipsoid of K, $A = T + a$ is such that $a = 0$. Thus we get in particular, that the level set

$$\{x : \varphi(x) = 0\} = \{x : \|Tx\|_2 = t\} = t\, T^{-1} B_2^n.$$

As we require that $\|Ax\|_2 - t \le 0$, for all $x \in K$, the smallest ellipsoid that satisfies this is the Löwner ellipsoid $L(K)$ of K, i.e., $t\, T^{-1} B_2^n = L(K)$. Thus

$$|\det T| = t^n \, \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_n(L(K))}$$

and $\min_{(T,t)} \frac{e^t}{|\det T|}$ is achieved for $t_0 = n$. This means that $T_0 = n T_{L(K)}^{-1}$ and hence $L(\mathbb{1}_K)(x) = \mathrm{e}^{-n\left(\|T_{L(K)}^{-1} x\|_2 - 1\right)}$.

2. It is easy to see that the Löwner function of the Gaussian $g(x) = \mathrm{e}^{-\|x\|_2^2/2}$ is given by

$$L(g)(x) = \mathrm{e}^{-\sqrt{n}\|x\|_2 + \frac{n}{2}}.$$

3. More generally, let $f(x) = \mathrm{e}^{-\psi(x)}$ be a log-concave function where the convex function ψ depends only on the Euclidean norm of x, $\psi(x) = \varphi(\|x\|)$. Then by symmetry $A \in \mathcal{A}$ is of the form $A_0 = a\,\mathrm{Id}$. We compute that a and t_0 are determined by

$$a = \varphi'\left(\frac{n}{a}\right), \quad t_0 = n - \varphi\left(\frac{n}{a}\right)$$

and thus

$$L(f)(x) = \mathrm{e}^{-a\|x\|_2 + n - \varphi\left(\frac{n}{a}\right)}.$$

We will prove Theorem 1 in several steps. The first one is to give an equivalent simplified version of the minimization problem via a reduction argument.

3.2 A Reduction Argument

Let $f = \mathrm{e}^{-\psi}$ be a log-concave function. Let $A = T + a \in \mathcal{A}$. By the polar decomposition theorem, $T \in GL(n)$ can be written as $T = O\, R$, where R is a symmetric positive definite matrix and $O \in O(n)$, the set of orthogonal matrices. Then

$$\begin{aligned}
\min_{\|Ax\|_2 \le \psi(x)+t} \frac{e^t}{|\det A|} &= \min_{\|Tx+a\|_2 \le \psi(x)+t} \frac{e^t}{|\det T|} \\
&= \min_{\|ORx+a\|_2 \le \psi(x)+t} \frac{e^t}{\det R} \\
&= \min \|Rx + O^t a\|_2 \le \psi(x) + t \frac{e^t}{\det R} \\
&= \min_{\|Ax\|_2 \le \psi(x)+t} \frac{e^t}{\det A},
\end{aligned}$$

where $A \in \mathcal{SA}$. Thus we may assume that $A = T + a$, where T is symmetric and positive definite, i.e., $T \in S_+$. We put $b = T^{-1}a$ and re-write the last expression further.

$$
\begin{aligned}
\min_{\|Ax\|_2 \le \psi(x)+t} \frac{e^t}{\det A} &= \left(\max_{\|Ax\|_2 \le \psi(x)+t} \mathrm{e}^{-t} \det A \right)^{-1} \\
&= \left(\max_t \max_{\|Ax\|_2 \le \psi(x)+t} \mathrm{e}^{-t} \det A \right)^{-1} \\
&= \left(\max_t \max_{\|Tx+a\|_2 \le \psi(x)+t} \mathrm{e}^{-t} \det T \right)^{-1} \\
&= \left(\max_t \max_{\|T(x+b)\|_2 \le \psi(x)+t} \mathrm{e}^{-t} \det T \right)^{-1} \\
&= \left(\max_{t\in\mathbb{R}} \max_{b\in\mathbb{R}^n} \max_{\|Tx\|_2 \le \psi(x-b)+t} \mathrm{e}^{-t} \det T \right)^{-1} \\
&= \left(\max_{b\in\mathbb{R}^n} \max_{t\in\mathbb{R}} \max_{\|Tx\|_2 \le \psi(x-b)+t} \mathrm{e}^{-t} \det T \right)^{-1} \\
&= \min_{b\in\mathbb{R}^n} \left(\max_{t\in\mathbb{R}} \max_{\|Tx\|_2 \le \psi(x-b)+t} \mathrm{e}^{-t} \det T \right)^{-1}, \qquad (6)
\end{aligned}
$$

where $T \in S_+$.

This leads us to first consider an optimization problem for fixed $b \in \mathbb{R}^n$.

Proposition 2 *Fix $b \in \mathbb{R}^n$. Let $f = \mathrm{e}^{-\psi}$ be a nondegenerate, integrable log-concave function on $\mathbb{R}^n$. There exists a unique solution, up to left orthogonal transformations, to the maximization problem*

$$
\max_{T\in S_+, t\in\mathbb{R}^n} \mathrm{e}^{-t} \det T \quad \textit{subject to} \quad \|Tx\|_2 - t \le \psi(x-b) \ \forall x \in \mathbb{R}^n. \qquad (7)
$$

Before we prove Proposition 2, we re-write the constraint condition of (7).

For any function $h : \mathbb{R}^n \to \mathbb{R}$ we define its diametral with respect to the point w as

$$
h_{\mathrm{dia},w}(-x + 2w) = h(x).
$$

For a convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ we define its symmetral $\psi_{\mathrm{sym},w}$ with respect to the point w as the greatest, convex function that is smaller than $\max\{\psi, \psi_{\mathrm{dia},w}\}$. In the same way we define the symmetral $f_{\mathrm{sym},w} = \mathrm{e}^{-\psi_{\mathrm{sym},w}}$ of a log-concave function $f = \mathrm{e}^{-\psi}$.

Since for all $x \in \mathbb{R}^n$

$$
\|T(x+b)\|_2 - t = \|T((-x-2b)+b)\|_2 - t
$$

the condition

$$
\forall x \in \mathbb{R}^n : \quad \|T(x+b)\|_2 - t \le \psi(x)
$$

is equivalent to the condition

$$\forall x \in \mathbb{R}^n : \quad \|T(x+b)\|_2 - t \le \psi_{\mathrm{sym},-b}(x).$$

Therefore, we can assume that the convex function ψ is symmetric with respect to $-b$. By Proposition 1 and Fact 2, taking the Legendre transform on both sides yields the equivalent condition

$$\mathcal{L}(\|Tx\|_2 - t)(y) \ge \mathcal{L}\left(\psi(x-b)\right)(y) = \mathcal{L} \circ S_{-b}\psi(y). \tag{8}$$

Observe that

$$\begin{aligned}
\mathcal{L}(\|Tx\|_2 - t)(y) &= \sup_x \langle x, y\rangle - \|Tx\|_2 + t = t + \sup_x \langle x, y\rangle - \|Tx\|_2 \\
&= t + \sup_z \langle T^{-1}z, y\rangle - \|z\|_2 = t + \sup_z \langle z, (T^{-1})^t y\rangle - \|z\|_2 \\
&= t + \begin{cases} \infty & \|(T^{-1})^t y\|_2 > 1, \\ 0 & \|(T^{-1})^t y\|_2 \le 1, \end{cases} \\
&= t + \begin{cases} \infty & y \notin TB_2^n \\ 0 & y \in TB_2^n, \end{cases}
\end{aligned}$$

where from the second to the third equality we have put $z = Tx$. It follows that

$$\mathrm{e}^{-\mathcal{L}(\|Tx\|_2 - t)(y)} = \mathrm{e}^{-t} \mathbb{1}_{TB_2^n}.$$

If we set $f_b = S_{-b} f$, then (8) is equivalent to

$$\mathrm{e}^{-t} \mathbb{1}_{T^t B_2^n} \le (f_b)^\circ.$$

Note that by Fact 2, $(f_b)^\circ$ is an integrable log-concave function, provided $b \in \mathrm{int}\,(\mathrm{supp} f)$. When $b \notin \mathrm{int}\,(\mathrm{supp} f)$, we replace f by $f_{\text{sym,-b}}$ and by the above considerations the minimization problem remains the same.

Moreover, shifting by a vector b does not affect the existence and uniqueness of the solution to the optimization problem in Proposition 2 and hence proving Proposition 2 is equivalent to proving the case $b = 0 \in \mathrm{int}\,(\mathrm{supp} f)$, possibly replacing f by f_{sym}, i.e., we need to show that there is a unique solution modulo $O(n)$ to the maximization problem

$$\max_{T \in S_+, t \in \mathbb{R}} \mathrm{e}^{-t} \det T \quad \text{subject to} \quad \mathrm{e}^{-t} \mathbb{1}_{TB_2^n} \le f^\circ. \tag{9}$$

By Proposition 1 and the Fact 2, to prove (9), and hence Proposition 2, it is enough to prove the following proposition.

Proposition 3 *Let $f = \mathrm{e}^{-\psi}$ be a nondegenerate, integrable log-concave function. Then there exists a unique solution $(t_0, T_0) \in \mathbb{R} \times S_+$, up to right orthogonal transformations, to the maximization problem*

$$\max_{T \in S_+, t \in \mathbb{R}} \mathrm{e}^{-t} \det T \quad \textit{subject to} \quad \mathrm{e}^{-t} \mathbb{1}_{TB_2^n} \leq f. \tag{10}$$

3.3 Proof of Proposition 3

To prove Proposition 3, we introduce, for $0 < s \leq \|f\|_\infty$,

$$\xi_f(s) := s \max_{\{T \in S_+ : TB_2^n \subset G_f(s)\}} \det T.$$

Then we can re-write (10) in terms of ξ_f, namely,

$$\max\{\mathrm{e}^{-t} \ \det T \ : \ T \in S_+, \ t \in \mathbb{R}, \ \mathrm{e}^{-t} \mathbb{1}_{TB_2^n} \ \leq \ f\} = \max_{0 < s \leq \|f\|_\infty} \xi_f(s). \tag{11}$$

Indeed, putting $s = \mathrm{e}^{-t}$,

$$\begin{aligned} &\max\{\mathrm{e}^{-t} \ \det T \ : \ T \in S_+, \ t \in \mathbb{R}, \ \mathrm{e}^{-t} \mathbb{1}_{TB_2^n} \ \leq \ f\} \\ &\quad = \max\{s \ \det T \ : \ T \in S_+, \ s > 0, \ s \mathbb{1}_{TB_2^n} \ \leq \ f\}. \end{aligned}$$

Note that $s \mathbb{1}_{TB_2^n} \leq f \iff TB_2^n \subset G_f(s)$. Thus we may restrict our attention to the set

$$\cup_{s>0} \{T \in S_+ : \ TB_2^n \subset G_f(s)\}.$$

When $s > \|f\|_\infty$, $\{T : \ TB_2^n \subset G_f(s)\} = \emptyset$. Thus we consider

$$\bigcup_{0<s} \{T \in S_+ : \ TB_2^n \subset G_f(s)\} = \bigcup_{0<s \leq \|f\|_\infty} \{T \in S_+ : \ TB_2^n \subset G_f(s)\}.$$

Therefore,

$$\begin{aligned} \max\{s \det T : T \in S_+, s > 0, s \mathbb{1}_{TB_2^n} \leq f\} &= \max_{0<s \leq \|f\|_\infty} s \max_{\{T \in S_+ : \ TB_2^n \subset G_f(s)\}} \det T \\ &= \max_{0<s \leq \|f\|_\infty} \xi_f(s) \end{aligned} \tag{12}$$

We shall show in the next lemma that $\lim_{s \to 0} \xi_f(s) = 0$ and in Corollary 1 below that the map $s \to \xi_f(s)$ is continuous. We then can conclude that the maximizer in Proposition 3 exists.

The next lemma and its proof is similar to Lemma 2.1 in [1]. We include a proof for completeness.

Lemma 1 *Let* $f = \mathrm{e}^{-\psi}$ *be an integrable, nondegenerate, log-concave function on* $\mathbb{R}^n$. *For any* $s_1, s_2 \in (0, \|f\|_\infty]$ *and* $0 \le \lambda \le 1$,

$$\xi_f\left(s_1^{1-\lambda} s_2^{\lambda}\right) \ge \xi_f(s_1)^{1-\lambda} \xi_f(s_2)^{\lambda}. \tag{13}$$

Moreover, $\lim_{s \to 0} \xi_f(s) = 0$.

Proof As the set $\{T \in S_+ : TB_2^n \subset G_f(s)\}$ is compact (e.g., in the operator topology), and as the determinant is continuous, there are T_0, T_1 and T_2 such that $\xi_f(s_1^{1-\lambda} s_2^{\lambda}) = s_1^{1-\lambda} s_2^{\lambda} \cdot \det T_0$, $\xi_f(s_1) = s_1 \cdot \det T_1$ and $\xi_f(s_2) = s_2 \cdot \det T_2$. Then, as f is log-concave,

$$\begin{aligned} G_f\left(s_1^{1-\lambda} s_2^{\lambda}\right) &= \left\{x : f(x) \ge s_1^{1-\lambda} s_2^{\lambda}\right\} \supset (1-\lambda)\{x : f(x) \ge s_1\} + \lambda\{x : f(x) \ge s_2\} \\ &= (1-\lambda) G_f(s_1) + \lambda G_f(s_2) \supset (1-\lambda) T_1 B_2^n + \lambda T_2 B_2^n \\ &\quad \supset ((1-\lambda) T_1 + \lambda T_2) B_2^n. \end{aligned}$$

Hence $\det T_0 \ge \det[(1-\lambda)T_1 + \lambda T_2)]$. Moreover, we have $\det T_0 \ge (\det T_1)^{1-\lambda} (\det T_2)^{\lambda}$. Indeed, by Minkowski's determinant inequality for positive definite matrices (see, e.g., [38]),

$$\begin{aligned} \det T_0 &\ge \det[(1-\lambda)T_1 + \lambda T_2)] \\ &\ge \left((1-\lambda)(\det T_1)^{1/n} + \lambda(\det T_2)^{1/n}\right)^n \qquad (14) \\ &\ge (\det T_1)^{1-\lambda} (\det T_2)^{\lambda}. \qquad (15) \end{aligned}$$

The last inequality follows from the arithmetic-geometric mean inequality. Therefore,

$$s_1^{1-\lambda} s_2^{\lambda} \det T_0 \ge (s_1 \det T_1)^{1-\lambda} (s_2 \det T_2))^{\lambda}.$$

In [1], the authors introduce, for $t > 0$, a function $\phi_f(t)$,

$$\phi_f(t) = \max_{\{A \in \mathcal{A}:\ AB_2^n \subset G_f(t)\}} t \cdot |\det A|.$$

They showed that $\lim_{t \to 0} \phi_f(t) = 0$. It is clear that $\xi_f(s) \le \phi_f(s)$ for all s. Hence $\lim_{s \to 0} \xi_f(s) = 0$. □

Next we state a John-type result which is well known. We include a proof for completeness. We recall the Hausdorff metric, which for two convex bodies K and L is defined as

$$d_H(K, L) = \min\{\lambda \ge 0 : K \subseteq L + \lambda B_2^n;\ L \subseteq K + \lambda B_2^n\}.$$

Lemma 2 *Let* $\mathcal{K}^n$ *be the set of convex bodies in* $\mathbb{R}^n$, *equipped with the Hausdorff metric. The map*

$$K \to \max_{\{T \in S_+:\ TB_2^n \subset K\}} \det T$$

is continuous in K. *Moreover, let* T_K *be a maximizer, i.e.,*

$$\det T_K = \max_{\{T \in S_+ :\ T B_2^n \subset K\}} \det T.$$

Then T *is unique up to an orthogonal transformation.*

Proof First note that if $0 \notin int(K)$, then $\{T \in S_+ : \ T B_2^n \subset K\} = \emptyset$. For K with $0 \in int(K)$, let T_K be such that $\det T_K = \max_{\{T \in S_+ : T B_2^n \subset K\}} \det T$ and let $\hat{K} = K \cap (-K)$. Then

$$T_K B_2^n \subset \hat{K} = K \cap (-K) \subset K.$$

As $K \cap (-K)$ is centrally symmetric, the center of the ellipsoid of maximal volume contained in $K \cap (-K)$ is also centered at 0. Therefore the ellipsoid $T_K B_2^n$ is the ellipsoid of largest volume or John ellipsoid $J(\hat{K})$ contained in $\hat{K} = K \cap (-K)$. It follows that T_K is unique, modulo $O(n)$, as $J(\hat{K})$ is unique, e.g., [20].
Now notice that if K and L are such that $d_H(K, L) < \delta$, then $d_H(\hat{K}, \hat{L}) < 2\delta$. In fact, on the one hand,

$$\hat{L} \subset L \subset K + \delta B_2^n$$
$$\hat{L} \subset -L \subset -K + \delta B_2^n,$$

hence

$$\hat{L} \subset K \cap (-K) + 2\delta B_2^n = \hat{K} + 2\delta B_2^n.$$

The other direction follows similarly. Let $K \in \mathcal{K}^n$. The map $\hat{K} \to J(\hat{K})$ is continuous, see e.g., [23]. Hence, for all $\varepsilon > 0$ there exists δ such that for all $L \in \mathcal{K}^n$ with $d_H(\hat{K}, \hat{L}) < \delta$ we have $d_H(J(\hat{K}), J(\hat{L})) < \varepsilon$. It follows that for all L with $d_H(K, L) < \delta/2$, we get

$$d_H\big(T_K B_2^n, T_L B_2^n\big) < \varepsilon.$$

□

Corollary 1 *The map* $s \to \xi_f(s)$ *is continuous in* s.

Proof Note that the map

$$s \to \max_{\{T \in S_+ : T B_2^n \subset G_f(s)\}} \det T$$

is continuous in s as it is the composition of the continuous maps $s \to G_f(s)$ and $K \to \max_{\{T \in S_+ : T B_2^n \subset K\}} \det T$. Hence,

$$s \to s \cdot \max_{\{T \in S_+ : T B_2^n \subset G_f(s)\}} \det T = \xi_f(s)$$

is continuous in s. □

Now we are ready for the proof of Proposition 3 .

Proof As $\lim_{s\to 0}\xi_f(s) = 0$ by Lemma 1, and as $\xi_f(s)$ is continuous on $(0, \|f\|_\infty]$, $\xi_f(s)$ attains its maximum for some $s_0 \in (0, \|f\|_\infty]$ and $T_0 \in S_+$. In other words, $t_0 = -\log s_0$ and T_0 solve the maximization problem in Proposition 3. To see the uniqueness modulo $O(n)$, it suffices to show uniqueness in s. Uniqueness in T modulo $O(n)$ then follows from Lemma 2.
Suppose there are s_1, s_2 such that $s_1 > s_2$ and $\xi_f(s_1) = \xi_f(s_2)$. Then it follows from (13) and the definition of ξ_f that for $0 \le \lambda \le 1$,

$$\xi_f\left(s_1^{1-\lambda}s_2^{\lambda}\right) = \xi_f(s_1)^{1-\lambda}\xi_f(s_2)^{\lambda}.$$

As in the proof of Lemma 1, let T_0, T_1 and T_2 be such that

$$\xi_f\left(s_1^{1-\lambda}s_2^{\lambda}\right) = s_1^{1-\lambda}s_2^{\lambda}\cdot\det T_0, \quad \xi_f(s_1) = s_1\cdot\det T_1, \quad \xi_f(s_2) = s_2\cdot\det T_2.$$

Then

$$\det T_0 = (\det T_1)^{1-\lambda}(\det T_2)^{\lambda}.$$

In other words, we have equality in the Minkowski determinant inequality and in the arithmetic-geometric mean inequality, (14) and (15), which implies that $\det T_1 = \det T_2$. Thus

$$\xi_f(s_1) = s_1\det T_1 = s_1\det T_2 > s_2\det T_2 = \xi_f(s_2),$$

which is contradiction. □

3.4 Proof of Theorem 1

We need several more lemmas. Some of them are well known. We include a proof for the reader's convenience.

Lemma 3 *Let $\{f_m\}$, f be nondegenerate integrable log-concave functions such that $f_m \to f$ pointwise. Then the super-level sets converge in Hausdorff metric, that is,*

$$G_{f_m}(k) \to G_f(k) \text{ in Hausdorff, for } 0 < k < \|f\|_\infty.$$

Proof Since f_m, f are non-degenerate, integrable log-concave functions, they are continuous on their support and by Fact 1, $G_f(k)$ is a convex body for $0 < k < \|f\|_\infty$ and $G_{f_m}(k)$ is a convex body for $0 < k < \|f_m\|_\infty$ and all $m \ge 1$.
We fix k. By e.g., Theorem 1.8.8 of [41], convergence of $G_{f_m}(k) \to G_f(k)$ in the Hausdorff metric is equivalent to the following two properties to hold:

(i) the limit of any convergent subsequence $(x_{m_j})_{j\in\mathbb{N}}$ with $x_{m_j} \in G_{f_{m_j}}(k)$ for all j, belongs to $G_f(k)$;
(ii) each point in $G_f(k)$ is the limit of a sequence $(x_m)_{\in\mathbb{N}}$ with $x_m \in G_{f_m}(k)$ for all $m \in \mathbb{N}$.

We show (i). Let $(x_{m_j})_{j\in\mathbb{N}}$ be a sequence with $x_{m_j} \in G_{f_{m_j}}(k)$ for all j and let $x = \lim_{j\to\infty} x_{m_j}$. Let $D = \overline{\text{co}}[\{x_{m_j} : j \in \mathbb{N}\}]$ be the closed convex hull of $\{x_{m_j} : j \in \mathbb{N}\}$. Then D is compact and convex and as $f_{m_j} \to f$ pointwise on $\mathbb{R}^n$, $f_{m_j} \to f$ uniformly on D, by e.g., Theorem 10.8 of [39]. Therefore, for j large enough,

$$|f_{m_j}(x_{m_j}) - f(x)| \leq |f_{m_j}(x_{m_j}) - f(x_{m_j})| + |f(x_{m_j}) - f(x)| < 2\varepsilon. \tag{16}$$

The first estimate holds by the uniform convergence and the second by continuity of f. Inequality (16) says exactly that $f_{m_j}(x_{m_j}) \to f(x)$. As $f_{m_j}(x_{m_j}) \geq k$, we thus get that $f(x) \geq k$ and hence $x \in G_f(k)$.

Now we show (ii). By definition, for $0 < k < \|f\|_\infty$,

$$G_f(k) = \{x : f(x) \geq k\} = \{x : \psi(x) \leq -\log k\} = E_\psi(l),$$

where we have put $l = -\log k$. Similarly, we rewrite $G_{f_m}(k) = E_{\psi_m}(l)$ and then need to show that every $x \in E_\psi(l)$ is the limit of a sequence $(x_m)_{\in\mathbb{N}}$ with $x_m \in E_{f_m}(k)$ for all m. We can assume that $\psi(x) = l$. As f is integrable, there is x_0 in $\mathbb{R}^n$ such that $\psi(x_0) = \min_{x\in\mathbb{R}^n} \psi(x)$. We assume without loss of generality that $x_0 = 0$ and consider the 2-dimensional plane spanned by x and $e_{n+1} = (0, \ldots, 1)$. As $k < \|f\|_\infty$, $l > \psi(x_0) = \psi(0)$. Let $0 < 2\varepsilon^{\frac{1}{2}} < \psi(x) - \psi(0)$. As $f_m \to f$ pointwise, $\psi_m \to \psi$ pointwise and therefore we have for all $m \geq m_0$ that

$$|\psi(x) - \psi_m(x)| < \varepsilon \quad \text{and} \quad |\psi(0) - \psi_m(0)| < \varepsilon.$$

Let L be the line determined by $(0, \psi(0) + \varepsilon)$ and $(x, \psi_m(x))$ and let

$$x_m = \frac{l - (\psi(0) + \varepsilon)}{\psi_m(x) - (\psi(0) + \varepsilon)}\, x,$$

that is x_m is such that the value of L at x_m is l. Then

$$\|x_m - x\|_2 = \|x\|_2 \frac{|l - \psi_m(x)|}{|\psi_m(x) - (\psi(0) + \varepsilon)|} \leq \frac{\varepsilon}{|\psi_m(x) - (\psi(0) + \varepsilon)|} \leq \frac{\varepsilon^{\frac{1}{2}}}{2(1 - \varepsilon^{\frac{1}{2}})}.$$

The last inequality holds as $|\psi_m(x) - (\psi(0) + \varepsilon)| = |\psi_m(x) - \psi(0) - \varepsilon| > 2\varepsilon^{\frac{1}{2}} - 2\varepsilon$. By convexity of ψ_m we have for all y in the line segment $[0, x]$ that $\psi_m(y) \leq L(y)$. If $\psi_m(x) \geq \psi(x)$ for all $m \geq m_0$, then $x_m \in [0, x]$ and thus

$$\psi_m(x_m) \leq L(x_m) \leq l,$$

which means that $x_m \in E_{\psi_m}(l)$ and we are done. If there exists $m_1 \geq m_0$ such that $\psi_{m_1}(x) < \psi(x) = l$, then $x \in E_{\psi_{m_1}}(l)$ and we take $x_{m_1} = x$. Thus, for all $m > m_1$, either $\psi_m(x) \geq \psi(x)$ and then we put x_m as above or $\psi_m(x) < \psi(x)$ and then we put $x_m = x$. □

Lemma 4 *Let $\{f_m\}$, f be nondegenerate integrable log-concave functions such that $f_m \to f$ pointwise. Then $\|f_m\|_\infty \to \|f\|_\infty$.*

Proof As f is integrable and log-concave, there is $x_0 \in \mathbb{R}^n$ such that $f(x_0) = \|f\|_\infty$. Thus for an arbitrary $\varepsilon > 0$, there exists m_1 such that

$$f_m(x_0) \geq f(x_0) - \varepsilon,$$

whenever $m > m_1$. So $\|f_m\|_\infty \geq f_m(x_0) \geq f(x_0) - \varepsilon$ whenever $m > m_1$. Thus

$$\liminf \|f_m\|_\infty \geq \|f\|_\infty. \tag{17}$$

On the other hand, fix an arbitrary $0 < \varepsilon < \frac{1}{4}\|f\|_\infty$. By log-concavity of f, there exists $\delta > 0$ such that

$$G_f\left(\frac{1}{2}\|f\|_\infty - \varepsilon\right) \subset G_f\left(\frac{1}{2}\|f\|_\infty\right) + \delta B_2^n.$$

By Lemma 3, there exists m_2 such that

$$G_{f_m}\left(\frac{1}{2}\|f\|_\infty\right) \subset G_f\left(\frac{1}{2}\|f\|_\infty\right) + \delta B_2^n, \tag{18}$$

whenever $m > m_2$. It follows that $f_m(x) < \frac{1}{2}\|f\|_\infty$ for all $x \notin G_f\left(\frac{1}{2}\|f\|_\infty\right) + \delta B_2^n$ and whenever $m > m_2$. In other words,

$$\sup_{x \notin G_f\left(\frac{1}{2}\|f\|_\infty\right) + \delta B_2^n} f_m(x) \leq \frac{1}{2}\|f\|_\infty, \tag{19}$$

whenever $m > m_2$. Moreover, since $f_m(x) \to f(x)$ pointwise on $G_f\left(\frac{1}{2}\|f\|_\infty\right) + \delta B_2^n$ and $G_f\left(\frac{1}{2}\|f\|_\infty\right) + \delta B_2^n$ is a compact set, we have $f_m \to f$ uniformly on $G_f\left(\frac{1}{2}\|f\|_\infty\right) + \delta B_2^n$, by e.g., Theorem 10.8 of [39]. That is, for the same ε, there exists m_3 such that

$$f_m(x) \leq f(x) + \varepsilon$$

whenever $m > m_3$ and for all $x \in G_f\left(\frac{1}{2}\|f\|_\infty\right) + \delta B_2^n$. Thus,

$$\sup_{x \in G_f\left(\frac{1}{2}\|f\|_\infty\right) + \delta B_2^n} f_m(x) \leq \|f\|_\infty + \varepsilon, \tag{20}$$

whenever $m > m_3$. Taking $m > \max\{m_2, m_3\}$ and combining (19) and (20), one has

$$\sup_{x\in\mathbb{R}^n} f_m(x) = \|f_m\|_\infty \le \|f\|_\infty + \varepsilon.$$

Hence

$$\limsup \|f_m\|_\infty \le \|f\|_\infty. \tag{21}$$

Finally, combining (17) and (21), one concludes that $\lim \|f_m\|_\infty = \|f\|_\infty$. □

Lemma 5 *Let $\{f_m\}$, f be a nondegenerate integrable log-concave functions and suppose that $f_m \to f$ pointwise. Then*

$$\max_{0<s\le\|f_m\|_\infty} \xi_{f_m}(s) \to \max_{0<s\le\|f\|_\infty} \xi_f(s).$$

Proof For $m \ge 0$, and with the convention that $f_0 = f$, let $T_{m,s}$ be such that

$$\det T_{m,s} = \max_{\{T\in S_+:\ TB_2^n\subset G_{f_m}(s)\}} \det T.$$

By (11) and Proposition 3, there exists a unique $s_0 = \mathrm{e}^{-t_0}$ and a unique, modulo $O(n)$, $T_0 \in S_+$ such that

$$\xi_f(s_0) = \max_{0<s\le\|f\|_\infty} \xi_f(s) = s_0 \det T_0 = \max_{0<s\le\|f\|_\infty} s \max_{\{T\in S_+:\ TB_2^n\subset G_f(s)\}} \det T$$
$$= \max_{0<s\le\|f\|_\infty} s \det T_{0,s}.$$

The third identity holds by definition of ξ_f and the last identity holds by definition of $T_{0,s}$. Thus $\max_{0<s\le\|f\|_\infty} s \det T_{0,s} = s_0 \det T_0 = s_0 \det T_{0,s_0}$. Similarly, for all $m \in \mathbb{N}$, there exist unique s_m and a unique, modulo $O(n)$, $T_{m,s_m} \in S_+$ such that

$$\xi_{f_m}(s_m) = \max_{0<s\le\|f_m\|_\infty} \xi_{f_m}(s) = s_m \det T_{m,s_m}.$$

Since f is integrable and as $\mathrm{int}(\mathrm{supp}(f)) \ne \emptyset$,

$$0 < \int_{\mathbb{R}^n} f(x)\mathrm{d}x = \int_0^{\|f\|_\infty} \mathrm{vol}_n(G_f(s))\mathrm{d}s < \infty.$$

Therefore, for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $0 < \delta < \delta_\varepsilon$,

$$0 < \int_0^\delta \mathrm{vol}_n(G_f(s))\mathrm{d}s < \varepsilon.$$

In particular, for $\varepsilon_0 \leq \min\left\{\frac{s_0 \det T_{0,s_0} \mathrm{vol}_n(B_2^n)}{2+\|f\|_\infty + \frac{10\int_{\mathbb{R}^n} f(s)\mathrm{d}s}{9\|f\|_\infty}}, \frac{\|f\|_\infty}{2}\right\}$ there is δ_{ε_0} such that for all $0 < \delta < \min\{s_0, \delta_{\varepsilon_0}, \|f\|_2\}$,

$$0 < \int_0^\delta \mathrm{vol}_n(G_f(s))\mathrm{d}s < \varepsilon_0.$$

By Lemma 2, the map

$$G_f(s) \to \max_{\{T \in S_+ :\, TB_2^n \subset G_f(s)\}} \det T = \det T_{0,s}$$

is continuous and the map

$$G_f(s) \to \mathrm{vol}_n(G_f(s))$$

is also continuous. Thus, for $0 < \varepsilon_1 \leq \min\left\{\varepsilon_0, \frac{\|f\|_\infty}{10}\right\}$ given, there exists $\eta_1 = \eta(\varepsilon_1, s)$ such that for all $\eta \leq \eta_1$,

$$\left|\det T_{0,s} - \det T_{m,s}\right| < \varepsilon_1, \tag{22}$$

and

$$\left|\mathrm{vol}_n(G_{f_m}(s)) - \mathrm{vol}_n(G_f(s))\right| < \varepsilon_1, \tag{23}$$

whenever $d_H(G_f(s), G_{f_m}(s)) < \eta$.

We fix $0 < \delta < \min\{s_0, \delta_{\varepsilon_0}, \|f\|_\infty - \varepsilon_1\}$. As $f_m \to f$ pointwise, we get, similarly to the proof of Lemma 3, that for all $0 < \alpha$ with $\delta < \|f\|_\infty - \alpha$,

$$G_{f_m}(s) \to G_f(s)$$

in Hausdorff distance, uniformly for all s with $\delta \leq s \leq \|f\|_\infty - \alpha$. Thus, in particular for all s with $\delta \leq s \leq \|f\|_\infty - \varepsilon_1$, for $0 < \eta < \eta_1$, there is m_1 such that for all $m \geq m_1$,

$$d_H\left(G_{f_m}(s), G_f(s)\right) < \eta. \tag{24}$$

By (22) and (23) we therefore get that uniformly for all s with $\delta \leq s \leq \|f\|_\infty - \varepsilon_1$ and for all $m \geq m_1$,

$$\left|\det T_{0,s} - \det T_{m,s}\right| < \varepsilon_1, \tag{25}$$

and

$$\left|\mathrm{vol}_n(G_{f_m}(s)) - \mathrm{vol}_n(G_f(s))\right| < \varepsilon_1. \tag{26}$$

By Lemma 4, $f_m \to f$ pointwise implies that $\|f_m\|_\infty \to \|f\|_\infty$, i.e., there is m_2 such that

$$\|f\|_\infty - \varepsilon_1 < \|f_m\|_\infty < \|f\|_\infty + \varepsilon_1 \tag{27}$$

for all $m \geq m_2$. In addition, by Lemma 3.2 of [3], $\int_{\mathbb{R}^n} f_m \, \mathrm{d}x \to \int_{\mathbb{R}^n} f \, \mathrm{d}x$, i.e. there is m_3 such that for all $m \geq m_3$,

$$\left| \int_{\mathbb{R}^n} f_m(x) \, \mathrm{d}x - \int_{\mathbb{R}^n} f(x) \, \mathrm{d}x \right| < \varepsilon_1. \tag{28}$$

Let $m_0 = \max\{m_1, m_2, m_3\}$. Then, on the one hand, it follows with (25) that for all $m \geq m_0$, all $\delta < \min\{s_0, \delta_{\varepsilon_0}, \|f\|_2\}$, all s_m such that $\delta \leq s_m \leq \|f\|_\infty - \varepsilon_1$,

$$\begin{aligned} \xi_f(s_0) = s_0 \det T_{0,s_0} \geq s_m \det T_{0,s_m} &\geq s_m(\det T_{m,s_m} - \varepsilon_1) \\ &\geq s_m \det T_{m,s_m} - \varepsilon_1(\|f\|_\infty - \alpha) \\ &\geq \xi_{f_m}(s_m) - \varepsilon_1 \|f\|_\infty. \end{aligned} \tag{29}$$

Furthermore, for $m \geq m_0$ and $s_m < \delta$,

$$\begin{aligned} \limsup_{m\to\infty, s_m<\delta} \xi_{f_m}(s_m) = \limsup_{m\to\infty, s_m<\delta} s_m \det T_{m,s_m} &\leq \frac{\varepsilon_0 \left(2 + \|f\|_\infty + \frac{10 \int_{\mathbb{R}^n} f(s) \mathrm{d}s}{9\|f\|_\infty}\right)}{\mathrm{vol}_n(B_2^n)} \\ &\leq s_0 \det T_{0,s_0} = \xi_f(s_0). \end{aligned} \tag{30}$$

The last inequality holds by assumption on ε_0. We now verify the second last inequality. We have for all $s \leq s_m$ that $G_{f_m}(s_m) \subseteq G_{f_m}(s)$ and therefore by definition of T_{m,s_m},

$$\mathrm{vol}_n(G_{f_m}(s)) \geq \mathrm{vol}_n(G_{f_m}(s_m)) \geq \det T_{m,s_m} \mathrm{vol}_n(B_2^n).$$

Thus, as $s_m < \delta$ and also using (27),

$$\begin{aligned} s_m \det T_{m,s_m} &\leq \frac{1}{\mathrm{vol}_n(B_2^n)} \int_0^{s_m} \mathrm{vol}_n(G_{f_m}(s)) \, \mathrm{d}s \leq \frac{1}{\mathrm{vol}_n(B_2^n)} \int_0^{\delta} \mathrm{vol}_n(G_{f_m}(s)) \, \mathrm{d}s \\ &= \frac{1}{\mathrm{vol}_n(B_2^n)} \left(\int_0^{\|f\|_\infty - \varepsilon_1} \mathrm{vol}_n(G_{f_m}(s)) \mathrm{d}s - \int_\delta^{\|f\|_\infty - \varepsilon_1} \mathrm{vol}_n(G_{f_m}(s)) \mathrm{d}s \right) \\ &\leq \frac{1}{\mathrm{vol}_n(B_2^n)} \left(\int_0^{\|f_m\|_\infty} \mathrm{vol}_n(G_{f_m}(s)) \, \mathrm{d}s - \int_\delta^{\|f\|_\infty - \varepsilon_1} \mathrm{vol}_n(G_{f_m}(s)) \, \mathrm{d}s \right) \\ &= \frac{1}{\mathrm{vol}_n(B_2^n)} \left(\int f_m \, \mathrm{d}x - \int_\delta^{\|f\|_\infty - \varepsilon_1} \mathrm{vol}_n(G_{f_m}(s)) \, \mathrm{d}s \right) \\ &\leq \frac{1}{\mathrm{vol}_n(B_2^n)} \left(\int f \, \mathrm{d}x + \varepsilon_1 - \int_\delta^{\|f\|_\infty - \varepsilon_1} \mathrm{vol}_n(G_{f_m}(s)) \, \mathrm{d}s \right) \\ &= \frac{1}{\mathrm{vol}_n(B_2^n)} \left(\int_0^{\|f\|_\infty} \mathrm{vol}_n(G_f(s)) \, \mathrm{d}s + \varepsilon_1 - \int_\delta^{\|f\|_\infty - \varepsilon_1} \mathrm{vol}_n(G_{f_m}(s)) \, \mathrm{d}s \right). \end{aligned}$$

The last inequality follows by (28). Now we use (26) and get that

$$\begin{aligned} s_m \det T_{m,s_m} &\leq \frac{1}{\mathrm{vol}_n(B_2^n)}\Bigg(\int_0^{\|f\|_\infty} \mathrm{vol}_n(G_f(s))\,\mathrm{d}s + \varepsilon_1 - \int_\delta^{\|f\|_\infty-\varepsilon_1} \mathrm{vol}_n(G_f(s))\,\mathrm{d}s \\ &\quad +\varepsilon_1(\|f\|_\infty - \varepsilon_1 - \delta)\Bigg) \\ &\leq \frac{1}{\mathrm{vol}_n(B_2^n)}\Bigg(\int_0^{\delta} \mathrm{vol}_n(G_f(s))\,\mathrm{d}s + \int_{\|f\|_\infty-\varepsilon_1}^{\|f\|_\infty} \mathrm{vol}_n(G_f(s))\,\mathrm{d}s \\ &\quad +\varepsilon_1(1+\|f\|_\infty - \delta - \varepsilon_1)\Bigg) \\ &\leq \frac{1}{\mathrm{vol}_n(B_2^n)}\Big(\varepsilon_0 + \varepsilon_1(1+\|f\|_\infty - \delta - \varepsilon_1) + \varepsilon_1\,\mathrm{vol}_n(G_f(\|f\|_\infty - \varepsilon_1))\Big) \\ &\leq \frac{1}{\mathrm{vol}_n(B_2^n)}\left(\varepsilon_0 + \varepsilon_1\,(1+\|f\|_\infty) + \varepsilon_1 \frac{10\int_{\mathbb{R}^n} f(s)\mathrm{d}s}{9\,\|f\|_\infty}\right) \\ &= \frac{1}{\mathrm{vol}_n(B_2^n)}\left(\varepsilon_0 + \varepsilon_1\left(1+\|f\|_\infty + \frac{10\int_{\mathbb{R}^n} f(s)\mathrm{d}s}{9\,\|f\|_\infty}\right)\right) \\ &\leq \frac{\varepsilon_0}{\mathrm{vol}_n(B_2^n)}\left(2+\|f\|_\infty + \frac{10\int_{\mathbb{R}^n} f(s)\mathrm{d}s}{9\,\|f\|_\infty}\right). \end{aligned}$$

The last inequality follows by choice of ε_1. The second last inequality above follows as for all $\|f\|_\infty - \varepsilon_1 \leq s \leq \|f\|_\infty$, we have that $\mathrm{vol}_n(G_f(s)) \leq \mathrm{vol}_n(G_f(\|f\|_\infty - \varepsilon_1))$ and as

$$\mathrm{vol}_n(G_f(\|f\|_\infty - \varepsilon_1) \leq \frac{\int_{\mathbb{R}^n} f(s)\mathrm{d}s}{\|f\|_\infty - \varepsilon_1} \leq \frac{10\int_{\mathbb{R}^n} f(s)\mathrm{d}s}{9\,\|f\|_\infty},$$

by choice of ε_1. Now we use how ε_0 was chosen and get that for all $s_m < \delta$,

$$s_m \det T_{m,s_m} \leq s_0 \det T_{0,s_0} = \xi_f(s_0).$$

It remains to check when $\|f\|_\infty - \varepsilon_1 \leq s_m \leq \|f\|_\infty + \varepsilon_1$.

$$\begin{aligned} \xi_{f_m}(s_m) &= s_m \det T_{m,s_m} \\ &\leq (\|f\|_\infty + \varepsilon_1)(\det T_{m,\|f\|_\infty - \varepsilon_1}) \\ &\leq (\|f\|_\infty + \varepsilon_1)(\det T_{0,\|f\|_\infty - \varepsilon_1} + \varepsilon_1) \\ &= (\|f\|_\infty - \varepsilon_1 + 2\varepsilon_1)(\det T_{0,\|f\|_\infty - \varepsilon_1} + \varepsilon_1) \\ &= (\|f\|_\infty - \varepsilon_1)\det T_{0,\|f\|_\infty - \varepsilon_1} + 2\varepsilon_1 \det T_{0,\|f\|_\infty - \varepsilon_1} + \varepsilon_1(\|f\|_\infty - \varepsilon_1) + 2\varepsilon_1^2 \\ &\leq \xi_f(s_0) + 2\varepsilon_1 \det T_{0,\|f\|_\infty - \varepsilon_1} + \varepsilon_1\|f\|_\infty + \varepsilon_1^2 \\ &\leq \xi_f(s_0) + 2\varepsilon_0\left(\det T_{0,\frac{\|f\|_\infty}{2}} + \frac{3}{4}\|f\|_\infty\right). \end{aligned}$$

In the first inequality, $s_m < \|f\|_\infty + \varepsilon_1$ by assumption, and $\det T_{m,s_m} \le \det T_{m,\|f\|_\infty - \varepsilon_1}$ since $G_{f_m}(s_m) \subset G_{f_m}(\|f\|_\infty - \varepsilon_1)$. In the second inequality, we apply (25). In the last inequality, we use $\varepsilon_1 < \varepsilon_0$ as assumed, and we also use the assumption on ε_0. Therefore, we have for all $m \ge m_0$ and $\|f\|_\infty - \varepsilon_1 < s_m < \|f\|_\infty + \varepsilon_1$,

$$\xi_f(s_0) \ge \xi_{f_m}(s_m) - 2\varepsilon_0 \left(\det T_{0, \frac{\|f\|_\infty}{2}} + \frac{3}{4}\|f\|_\infty \right) \tag{31}$$

It now follows from (29), (30) and (31) that

$$\xi_f(s_0) \ge \limsup_m \xi_{f_m}(s_m). \tag{32}$$

On the other hand, as $\delta \le s_0$, for ε_1 given, it follows from (25) that for all $m \ge m_0$,

$$\det T_{0,s_0} \le \det T_{m,s_0} + \varepsilon_1.$$

Therefore, for all $m \ge m_0$,

$$s_0 \det T_{0,s_0} \le s_0 \det T_{m,s_0} + s_0\varepsilon_1 \le s_m \det T_{m,s_m} + s_0\varepsilon_1.$$

The last inequality holds as $s_m \det T_{m,s_m} = \max_{\{T \in S_+ : T B_2^n \subset G_{f_m}(s)\}} s \det T$. Consequently, for all $m \ge m_0$,

$$\xi_f(s_0) = s_0 \det T_{0,s_0} \le s_m \det T_{m,s_m} + s_0\varepsilon_1 = \xi_{f_m}(s_m) + s_0\varepsilon_1,$$

and hence

$$\xi_f(s_0) \le \liminf_m \xi_{f_m}(s_m). \tag{33}$$

Altogether, by (32) and (33),

$$\limsup_m \xi_{f_m}(s_m) \le \xi_f(s_0) \le \liminf_m \xi_{f_m}(s_m),$$

and thus

$$\lim_m \xi_{f_m}(s_m) = \xi_f(s_0). \tag{34}$$

By (12), this is equivalent to

$$\lim_m \max_{0 < s \le \|f_m\|_\infty} \xi_{f_m}(s) = \max_{0 < s \le \|f\|_\infty} \xi_f(s).$$

□

In fact, (34) together with (11) says that if $\{f_m\}$, f are integrable, log-concave functions and if $f_m \to f$ pointwise, then

$$\max\{e^{-t} \det T : T \in S_+, \ t \in \mathbb{R}, \ e^{-t}\mathbb{1}_{T B_2^n} \le f_m\} \to \max\{e^{-t} \det T : T \in S_+, \ t \in \mathbb{R}, \ e^{-t}\mathbb{1}_{T B_2^n} \le f\}.$$

Thus we immediately get the following corollary.

Corollary 2 *Let $\{b_m\}$, b in $\mathbb{R}^n$ be such that $b_m \to b$. Then*

$$\max\{e^{-t}\ \det T : T \in S_+,\ t \in \mathbb{R},\ \|Tx\|_2 - t \le \psi(x - b_m)\} \to \\ \max\{e^{-t}\ \det T : T \in S_+,\ t \in \mathbb{R},\ \|Tx\|_2 - t \le \psi(x - b)\}.$$

Proof We have that

$$\forall x \in \mathbb{R}^n : \quad \|Tx\|_2 - t \le \psi(x - b_m)$$

is equivalent to

$$\forall y \in \mathbb{R}^n : \quad \|T(y + b_m)\|_2 - t \le \psi(y).$$

We put

$$B_m = \{(T, t) : \ \|T(y + b_m)\|_2 - t \le \psi(y)\, \forall y \in \mathbb{R}^n\}$$

and

$$B = \{(T, t) : \quad \|T(y + b)\|_2 - t \le \psi(y)\, \forall y \in \mathbb{R}^n\}.$$

Then

$$\max\{e^{-t} \det T : (T, t) \in B_m\} = e^{\|T(b-b_m)\|_2} \max\{e^{-t-\|T(b-b_m)\|_2} \det(T) : (T, t) \in B_m\}.$$

Since

$$\|T(y + b_m)\|_2 \ge \|T(y + b)\|_2 - \|T(b - b_m)\|_2$$

we get

$$\max\{e^{-t} \det T : (T, t) \in B_m\} \le e^{\|T(b-b_m)\|_2} \max\{e^{-s} \det(T) : (T, s) \in B\}.$$

It follows that

$$\limsup_{m\to\infty} \max\{e^{-t} \det T : (T, t) \in B_m\} \le \max\{e^{-s} \det(T) : (T, s) \in B\}.$$

Now we interchange the roles of b and b_m and get

$$\max\{e^{-s} \det T : (T, s) \in B\} \le \liminf_{m\to\infty} \max\{e^{-t} \det(T) : (T, t) \in B_m\}.$$

□

The proof of Theorem 1 is next.

Proof Let $f = e^{-\psi}$ be an integrable log-concave function with positive integral. We put

$$I_f := \min\left\{\int_{\mathbb{R}^n} e^{-\|Ax\|_2+t}dx : A \in \mathcal{A}, t \in \mathbb{R}, \|Ax\|_2 - t \le \psi(x)\right\}$$

and

$$I_f(b) := \min\left\{\int_{\mathbb{R}^n} e^{-\|Ax\|_2+t}dx : A \in \mathcal{A}(b), t \in \mathbb{R}, \|Ax\|_2 - t \le \psi(x)\right\}.$$

It follows from (6) that $I_f = \min_{b\in\mathbb{R}^n} I_f(b)$.
By the reduction arguments in Sect. 3.2,

$$\begin{aligned} I_f(b) &= \min\left\{\int_{\mathbb{R}^n} e^{-\|Ax\|_2+t}dx : A \in \mathcal{A}(b), t \in \mathbb{R}, \|Ax\|_2 - t \le \psi(x)\right\} \\ &= n!\mathrm{vol}_n(B_2^n)\ \min\left\{\frac{e^t}{\det T} : T \in S_+, t \in \mathbb{R}, \|Tx\|_2 - t \le \psi(x-b)\right\} \\ &= n!\mathrm{vol}_n(B_2^n)\ \left\{\max\{e^{-t}\det T : T \in S_+, t \in \mathbb{R}, \|Tx\|_2 - t \le \psi(x-b)\}\right\}^{-1}. \end{aligned}$$

Corollary 2 implies that $I_f(b)$ is continuous in b. To see that the minimum I_f exists, it suffices to show that the minimum is achieved on a compact set.
Let $0 < d_0 < \|f\|_\infty$ be such that $G_f(d_0)$ has positive volume. Let $b_0 \in G_f(d_0)$. Clearly, $I_f \le I_f(b_0)$. Let $r = \frac{I_f(b_0)}{d_0} - \mathrm{vol}_n(G_f(d_0))$. Then $r > 0$ since

$$d_0\mathrm{vol}_n(G_f(d_0)) < \int_{\mathbb{R}^n} f(x)dx \le I_f(b_0).$$

The last inequality holds as

$$\begin{aligned} I_f(b_0) &= \min\left\{\int_{\mathbb{R}^n} e^{-\|Ax\|_2+t}dx : A \in \mathcal{A}(b_0), t \in \mathbb{R}, \|Ax\|_2 - t \le \psi(x)\right\} \\ &= \min\left\{\int_{\mathbb{R}^n} e^{-\|Ax\|_2+t}dx : A \in \mathcal{A}(b_0), t \in \mathbb{R}, e^{-\|Ax\|_2+t} \ge f(x)\right\}. \end{aligned}$$

To finish the existence argument, we need the notion of illumination body of a convex body K. This notion was introduced in [48] as follows. Let $\delta > 0$ be given. The illumination body K^δ of K is

$$K^\delta = \{x \in \mathbb{R}^n : \mathrm{vol}_n(\mathrm{conv}[K, x]) \le \delta + \mathrm{vol}_n(K)\}.$$

The illumination body is always convex, [48]. See, e.g., [34,35] for recent developments.

Let now $G^r = [G_f(d_0)]^r$ be the illumination body of $G_f(d_0)$. We will show that for $b \notin G^r$, $I_f(b) > I_f(b_0)$. Suppose $b \notin G^r$ and let $A_0 \in \mathcal{A}(b)$, $t_0 \in \mathbb{R}$ achieve $I_f(b)$.

Let $h(x) = \mathrm{e}^{-\|A_0 x\|_2 + t_0}$. Since $G_f(d_0) \subseteq G_h(d_0)$ there exists $z \in bd(G^r) \cap G_h(d_0)$ such that

$$\operatorname{conv}[z, G_f(d_0)] \subsetneq G_h(d_0).$$

It follows that

$$\begin{aligned} I_f(b) &= \int_{\mathbb{R}^n} h(x)\mathrm{d}x > d_0 \cdot \operatorname{vol}_n(G_h(d_0)) > d_0 \operatorname{vol}_n(\operatorname{conv}[z, G_f(d_0)]) \\ &= d_0 \cdot (r + \operatorname{vol}_n(G_f(d_0))) = I_f(b_0). \end{aligned}$$

So for the minimization problem, we need only consider $b \in G^r$ where $G^r = [G(d_0)]^r$ is a compact set of $\mathbb{R}^n$. The continuity of $I_f(b)$ gives the existence of a minimizer.

Next we address the uniqueness. Recall that $I_f = \min_b I_f(b)$ and Proposition 2 guarantees that for each $b \in \mathbb{R}^n$ there is a unique, modulo $O(n)$, minimizer. Hence it suffices to show that there is a unique b_0 such that $I_f = \min_b I_f(b) = I_f(b_0)$.

We prove by way of contradiction. Suppose that there are b_1, b_2 such that $I_f = I_f(b_1) = I_f(b_2)$ and $b_1 \neq b_2$. Let the two minimizers corresponding to b_1 and b_2 be $(T_1, t_1) \in S_+ \times \mathbb{R}$ and $(T_2, t_2) \in S_+ \times \mathbb{R}$, respectively. T_1 and T_2 are unique up to an orthogonal transformation. Then for all $x \in \mathbb{R}^n$

$$\|T_1(x + b_1)\|_2 - t_1 \leq \psi(x), \qquad \|T_2(x + b_2)\|_2 - t_2 \leq \psi(x)$$

and

$$\frac{\mathrm{e}^{t_1}}{\det T_1} = \frac{\mathrm{e}^{t_2}}{\det T_2},$$

or, equivalently, taking logarithm on both sides,

$$t_1 - \log\det T_1 = t_2 - \log\det T_2. \tag{35}$$

We distinguish two cases.
Case 1 $T_1 \neq T_2$. Then we consider the function

$$\mathrm{e}^{-\left\|\frac{T_1+T_2}{2}x + \frac{T_1 b_1 + T_2 b_2}{2}\right\|_2 + \frac{t_1+t_2}{2}}.$$

Observe that

$$\begin{aligned} &\left\|\frac{T_1 + T_2}{2}x + \frac{T_1 b_1 + T_2 b_2}{2}\right\|_2 - \frac{t_1 + t_2}{2} \\ &= \left\|\frac{1}{2}T_1(x + b_1) + \frac{1}{2}T_2(x + b_2)\right\|_2 - \frac{t_1 + t_2}{2} \\ &\leq \frac{1}{2}\left(\|T_1(x + b_1)\|_2 - t_1\right) + \frac{1}{2}\left(\|T_2(x + b_2)\|_2 - t_2\right) \leq \psi(x). \end{aligned}$$

But

$$\int_{\mathbb{R}^n} \mathrm{e}^{-\left\| \frac{T_1+T_2}{2}x + \frac{T_1 b_1 + T_2 b_2}{2} \right\|_2 + \frac{t_1+t_2}{2}} \mathrm{d}x = n!\mathrm{vol}_n(B_2^n) \frac{\mathrm{e}^{\frac{t_1+t_2}{2}}}{\det\left(\frac{T_1+T_2}{2}\right)}.$$

And by the Minkowski determinant inequality,

$$\left(\det\left(\frac{T_1+T_2}{2}\right)\right)^{\frac{1}{n}} \geq \frac{1}{2}\left(\det(T_1)^{\frac{1}{n}} + \det(T_2)^{\frac{1}{n}}\right),$$

from which it follows by concavity of the logarithm that

$$\log\det\left(\frac{T_1+T_2}{2}\right) > \frac{1}{2}\left(\log\det T_1 + \log\det T_2\right).$$

The inequality is strict because the function $T \to -\log\det T$ is strictly convex on the set of positive definite matrices. Hence

$$\begin{aligned} \log \frac{\mathrm{e}^{\frac{t_1+t_2}{2}}}{\det\left(\frac{T_1+T_2}{2}\right)} &= \frac{t_1+t_2}{2} - \log\det\left(\frac{T_1+T_2}{2}\right) \\ &< \frac{t_1}{2} - \frac{1}{2}\log\det T_1 + \frac{t_2}{2} - \frac{1}{2}\log\det T_2 \\ &= t_1 - \log\det T_1 = t_2 - \log\det T_2. \end{aligned}$$

It follows that

$$\frac{\mathrm{e}^{\frac{t_1+t_2}{2}}}{\det\left(\frac{T_1+T_2}{2}\right)} < \frac{\mathrm{e}^{t_1}}{\det T_1},$$

which contradicts the fact that the latter is the minimum.

Case 2 $T_1 = T_2$, modulo $O(n)$. It follows from (35) that $t_1 = t_2$. We show $b_1 = b_2$. We put

$$f_1(x) = \mathrm{e}^{-\|T_1(x+b_1)\|_2 + t_1}$$

and

$$f_2(x) = \mathrm{e}^{-\|T_2(x+b_2)\|_2 + t_2} = \mathrm{e}^{-\|T_1(x+b_2)\|_2 + t_1}.$$

We consider super-level sets. For $0 < s < \mathrm{e}^{t_1}$,

$$G_{f_1}(s) = -b_1 + (t_1 - \log s) T_1^{-1} B_2^n$$

and

$$G_{f_2}(s) = -b_2 + (t_1 - \log s) T_1^{-1} B_2^n.$$

For $0 < s < \|f\|_\infty$, one has by definition of f_1 and f_2 that

$$G_f(s) \subset G_{f_1}(s) \cap G_{f_2}(s).$$

Now we claim that

$$G_{f_1}(s) \cap G_{f_2}(s) \subset -\frac{b_1 + b_2}{2} + (t_1 - \log s) T_1^{-1} B_2^n.$$

If $G_{f_1}(s) \cap G_{f_2}(s) = \emptyset$, this inclusion is trivially true. If not, let $x \in G_{f_1}(s) \cap G_{f_2}(s)$. Then there exist $u, v \in B_2^n$ such that

$$x = -b_1 + (t_1 - \log s) T_1^{-1} u = -b_2 + (t_1 - \log s) T_1^{-1} v. \tag{36}$$

Thus

$$x = \frac{x + x}{2} = -\frac{b_1 + b_2}{2} + (t_1 - \log s) T_1^{-1} \left(\frac{u + v}{2} \right). \tag{37}$$

Since $\|(u + v)/2\| \le \|u\|/2 + \|v\|/2 \le 1$,

$$x = -\frac{b_1 + b_2}{2} + (t_1 - \log s) T_1^{-1} \left(\frac{u + v}{2} \right) \in -\frac{b_1 + b_2}{2} + (t_1 - \log s) T_1^{-1} B_2^n.$$

In the following we show that there is $\widetilde{T_1}$ with $\det(\widetilde{T_1}) > \det(T_1)$ satisfying

$$\begin{aligned} G_{f_1}(s) \cap G_{f_2}(s) &\subset -\frac{b_1 + b_2}{2} + (t_1 - \log s) \widetilde{T_1}^{-1} B_2^n \\ &\subset -\frac{b_1 + b_2}{2} + (t_1 - \log s) T_1^{-1} B_2^n. \end{aligned}$$

Both, $G_{f_1}(s) \cap G_{f_2}(s)$ and $-\frac{b_1+b_2}{2} + (t_1 - \log s) T_1^{-1} B_2^n$, are closed sets and centrally symmetric with respect to the same center $-\frac{b_1+b_2}{2}$.

Next we observe that $G_{f_1}(s) \cap G_{f_2}(s)$ does not intersect the boundary of the ellipsoid $-\frac{b_1+b_2}{2} + (t_1 - \log s) T_1^{-1} B_2^n$. Indeed, if $x \in G_{f_1}(s) \cap G_{f_2}(s)$ as represented in (37) is on the boundary of $-\frac{b_1+b_2}{2} + (t_1 - \log s) T_1^{-1} B_2^n$, it follows that $u = v \in S^{n-1}$. Hence by (36), $b_1 = b_2$, a contradiction.

Therefore $G_{f_1}(s) \cap G_{f_2}(s)$ is a convex body such that

$$\big(G_{f_1}(s) \cap G_{f_2}(s)\big) \cap \overline{\left(-\frac{b_1 + b_2}{2} + (t_1 - \log s) T_1^{-1} B_2^n \right)^c} = \emptyset,$$

and thus

$$\operatorname{dist}\left(G_{f_1}(s)\cap G_{f_2}(s), \overline{\left(-\frac{b_1+b_2}{2}+(t_1-\log s)T_1^{-1}B_2^n\right)^c}\right)>0,$$

where $\operatorname{dist}(A,B)=\inf\{\|x-y\|_2, x\in A, y\in B\}$. Hence we may shrink the ellipsoid $-\frac{b_1+b_2}{2}+(t_1-\log s)T_1^{-1}B_2^n$ with respect to the center $-(b_1+b_2)/2$ homothetically to get a new ellipsoid $-\frac{b_1+b_2}{2}+(t_1-\log s)\widetilde{T_1}^{-1}B_2^n$ such that still

$$G_{f_1}(s)\cap G_{f_2}(s)\subset -\frac{b_1+b_2}{2}+(t_1-\log s)\widetilde{T_1}^{-1}B_2^n$$

and such that $-\frac{b_1+b_2}{2}+(t_1-\log s)\widetilde{T_1}^{-1}B_2^n$ intersects the boundary of $G_{f_1}(s)\cap G_{f_2}(s)$. Given such a $\widetilde{T_1}^{-1}$, it follows from

$$G_f(s)\subset G_{f_1}(s)\cap G_{f_2}(s)\subset -\frac{b_1+b_2}{2}+(t_1-\log s)\widetilde{T_1}^{-1}B_2^n$$

that

$$\left\|\widetilde{T_1}\left(x+\frac{b_1+b_2}{2}\right)\right\|_2-t_1\le\psi(x).$$

However

$$\int_{\mathbb{R}^n}\mathrm{e}^{-\left\|\widetilde{T_1}\left(x+\frac{b_1+b_2}{2}\right)\right\|_2+t_1}\mathrm{d}x=n!\operatorname{vol}(B_2^n)\frac{\mathrm{e}^{t_1}}{\det\widetilde{T_1}}<n!\operatorname{vol}(B_2^n)\frac{\mathrm{e}^{t_1}}{\det T_1},$$

which is a contradiction.

Consequently, we have proved that $b_1=b_2$. □

4 John Function and Duality

4.1 The John Function of Alonso–Gutiérrez, Merino, Jiménez, and Villa

A notion of a John ellipsoid function has already been introduced in [1, Theorem 1.1]. We first recall the definition from this work.

Theorem 2 [1] *Let $f:\mathbb{R}^n\to\mathbb{R}$ be an integrable log-concave function. There exists a unique solution $(s_0,A_0)\in\mathbb{R}\times\mathcal{A}$ to the maximization problem*

$$\max\{s|\det A| : s\le\|f\|_\infty, A\in\mathcal{A}\}\quad \textit{subject to}\quad s\,\mathbb{1}_{AB_2^n}\le f. \tag{38}$$

A_0 is unique up to right orthogonal transformations. Then $s_0\,\mathbb{1}_{A_0B_2^n}$ is called the John ellipsoid of f, $J(f)=s_0\,\mathbb{1}_{A_0B_2^n}$.

Note that, as for the Löwner function, $J(f) = s_0 \, \mathbb{1}_{A_0 B_2^n}$ up to an orthogonal transformation.

We show that Theorem 2 can be obtained from Proposition 3 and Lemma 5. However, it seems that Theorem 1 cannot be obtained immediately from Theorem 2 as the optimization in (38) is over all affine maps, i.e., translation is allowed under the constraint that $s\mathbb{1}_{AB_2^n} \le f$. To see how Theorem 2 follows from Proposition 3 and Lemma 5, we first rewrite (38) in Theorem 2. Let $A = T - b$,

$$\begin{aligned} s_0 \det A_0 &= \max\{s \det A : s \le \|f\|_\infty, A \in \mathcal{A}, s\mathbb{1}_{AB_2^n} \le f\} \\ &= \max\{s \det T : s \le \|f\|_\infty, T \in S_+, b \in \mathbb{R}^n, s\mathbb{1}_{TB_2^n - b}(x) \le f(x) \; \forall x \in \mathbb{R}^n\} \\ &= \max\{s \det T : s \le \|f\|_\infty, T \in S_+, b \in \mathbb{R}^n, s\mathbb{1}_{TB_2^n}(x + b) \le f(x) \; \forall x \in \mathbb{R}^n\} \\ &= \max\{s \det T : s \le \|f\|_\infty, T \in S_+, b \in \mathbb{R}^n, s\mathbb{1}_{TB_2^n}(x) \le f(x - b) \; \forall x \in \mathbb{R}^n\} \\ &= \max_{b \in \mathbb{R}^n} \max\{s \det T : s \le \|f\|_\infty, T \in S_+, s\mathbb{1}_{TB_2^n}(x) \le f(x - b) \; \forall x \in \mathbb{R}^n\} \end{aligned}$$

If we put $J = s_0 \det A_0$ and

$$J_f(b) = \max\{s \det T : s \le \|f\|_\infty, T \in S_+, s\mathbb{1}_{TB_2^n}(x) \le f(x - b) \; \forall x \in \mathbb{R}^n\},$$

then $J = \max_{b \in \mathbb{R}^n} J_f(b)$. Note also that $J_f(b)$ is continuous in b by Lemma 5.

We show now that the existence of the John function follows from Proposition 3 and Lemma 5.

Existence of the John function in Theorem 2 Recall that existence and uniqueness of $J_f(b)$ are proved in Proposition 3 . Choose $b' \in \mathbb{R}^n$ such that $J_f(b') > 0$. Now let $\varepsilon = J_f(b')$. Since f is integrable, there exists $\delta(\varepsilon)$ such that

$$\int_0^{\delta(\varepsilon)} \mathrm{vol}_n(G_f(s))\mathrm{d}s < \varepsilon.$$

Then for $b \notin G_f(\delta(\varepsilon))$, $J_f(b) < \varepsilon$. In fact,

$$J_f(b) \le \int_0^{\delta(\varepsilon)} \mathrm{vol}_n(G_f(s))\mathrm{d}s < \varepsilon.$$

Hence

$$\max_{b \in \mathbb{R}^n} J_f(b) = \max_{b \in G_f(\delta(\varepsilon))} J_f(b).$$

Since $G_f(\delta(\varepsilon))$ is compact and $J_f(b)$ is continuous in b by Lemma 5, $\max_{b \in \mathbb{R}^n} J_f(b)$ $= \max_{b \in G_f(\delta(\varepsilon))} J_f(b)$ exists.

We include the uniqueness argument for the reader's convenience.

Uniqueness of the John function in Theorem 2 Suppose that $\max_{b \in \mathbb{R}^n} J_f(b) = J_f(b_1) = J_f(b_2)$ for some $b_1 \neq b_2$. If $b_1 = b_2$, then the solution is unique modulo $O(n)$, by Proposition 3 . Suppose that t_1, t_2, T_1, T_2 are maximizers satisfying

$$J_f(b_1) = t_1 \det T_1 \quad \text{and} \quad J_f(b_2) = t_2 \det T_2.$$
$$f(T_1 v + b_1) \geq t_1 \quad \text{and} \quad f(T_2 v + b_2) \geq t_2, \ \forall\, v \in B_2^n.$$

Thus we have

$$\log t_1 + \log\det T_1 = \log t_2 + \log\det T_2.$$

We may furthermore assume that $t_1 \neq t_2$. Indeed, observe that $T_1 B_2^n + b_1$ is the John ellipsoid of $G_f(t_1)$ and $T_2 B_2^n + b_2$ is the John ellipsoid of $G_f(t_2)$. If $t_1 = t_2$, then $G_f(t_1) = G_f(t_2)$ and by the uniqueness of John ellipsoid of a convex body [20,26,41], $T_1 = T_2$. Hence without loss of generality, we assume $t_1 < t_2$.

Now we consider the function

$$\sqrt{t_1 t_2}\,\mathbb{1}_{\frac{T_1+T_2}{2}B_2^n + \frac{b_1+b_2}{2}}.$$

We first show that

$$\sqrt{t_1 t_2}\,\mathbb{1}_{\frac{T_1+T_2}{2}B_2^n + \frac{b_1+b_2}{2}} \leq f.$$

In fact, by the concavity of $\log f$, we have for any $u \in B_2^n$,

$$\begin{aligned}\log f\left(\frac{T_1+T_2}{2}u + \frac{b_1+b_2}{2}\right) &\geq \frac{1}{2}\log f\,(T_1 u + b_1) + \frac{1}{2}\log f\,(T_2 u + b_2)\\ &\geq \frac{1}{2}\log t_1 + \frac{1}{2}\log t_2 = \log\sqrt{t_1 t_2}.\end{aligned}$$

However, $\sqrt{t_1 t_2}\det(\frac{T_1+T_2}{2}) > J_f(b_1)$. Indeed, using again the strict concavity of the function $T \to \log\det T$ on positive definite operators we have

$$\begin{aligned}\log\left(\sqrt{t_1 t_2}\det\left(\frac{T_1+T_2}{2}\right)\right) &= \frac{1}{2}\log t_1 + \frac{1}{2}\log t_2 + \log\det\left(\frac{T_1+T_2}{2}\right)\\ &> \frac{1}{2}\log t_1 + \frac{1}{2}\log t_2 + \frac{1}{2}\log\det T_1 + \frac{1}{2}\log\det T_2\\ &= \frac{1}{2}(\log t_1 + \log\det T_1) + \frac{1}{2}(\log t_2 + \log\det T_2)\\ &= \log t_1 + \log\det T_1 = \log(J_f(b_1)),\end{aligned}$$

which is a contradiction to the assumption that $\max_{b\in\mathbb{R}^n} J_f(b) = J_f(b_1)$. Consequently, $b_1 = b_2$.

4.2 Duality

Let K be a convex body in $\mathbb{R}^n$ such that 0 is the center of the Löwner ellipsoid $L(K)$. Then it holds that $(L(K))^\circ = J(K^\circ)$, where $J(K^\circ)$ is the John ellipsoid of K°. This duality relation carries over when we consider the convex bodies in the functional setting.

Proposition 4 *Let K be a convex body in $\mathbb{R}^n$. Assume, without loss of generality, that 0 is the center of the Löwner ellipsoid $L(K)$ of K. Then*

$$(L(\mathbb{1}_K))^\circ = J((\mathbb{1}_K)^\circ).$$

Proof It was shown in (5), that $L(\mathbb{1}_K)(x) = \mathrm{e}^{-n\left(\|T_{L(K)}^{-1}x\|_2 - 1\right)}$. Then

$$\begin{aligned}
\mathcal{L}\left(n\left(\|T_{L(K)}^{-1}x\|_2 - 1\right)\right)(y) &= n + \sup_{x\in\mathbb{R}^n} \langle x, y\rangle - n\|T_{L(K)}^{-1}x\|_2 \\
&= n + \sup_{z\in\mathbb{R}^n} \langle z, T_{L(K)}^t y\rangle - n\|z\|_2 \\
&= n + \sup_{z\in\mathbb{R}^n} \|z\|_2 \left(\|T_{L(K)}^t y\|_2 - n\right) \\
&= n + \begin{cases} \infty & y \notin n(T_{L(K)}^t)^{-1} B_2^n \\ 0 & y \in n(T_{L(K)}^t)^{-1} B_2^n. \end{cases}
\end{aligned}$$

Hence,

$$(L(\mathbb{1}_K))^\circ = \mathrm{e}^{-n}\mathbb{1}_{n(T_{L(K)}^t)^{-1}B_2^n} = \mathrm{e}^{-n}\mathbb{1}_{nJ(K^\circ)}.$$

The last identity holds as $L(K) = T_{LK}B_2^n$, and thus $J(K^\circ) = (L(K))^\circ = (T_{LK}B_2^n)^\circ = (T_{L(K)}^t)^{-1}B_2^n$. Now we compute $(\mathbb{1}_K)^\circ = (\mathrm{e}^{-I_K})^\circ$, where

$$I_K(x) = \begin{cases} 0 & x \in K, \\ \infty & x \notin K. \end{cases}$$

The Legendre transform of I_K is

$$\mathcal{L}(I_K)(y) = \sup_{x\in\mathbb{R}^n} \langle x, y\rangle - I_K(x) = \sup_{x\in K}\langle x, y\rangle = h_K(y),$$

where h_K is the support function of K. K° is a convex body since 0 is contained in the interior of K. Thus, $(\mathbb{1}_K)^\circ(y) = \mathrm{e}^{-h_K(y)}$. Next we compute the John function $J((\mathbb{1}_K)^\circ)$ of $(\mathbb{1}_K)^\circ$. For $0 < s \le 1$,

$$\mathrm{e}^{-h_K(y)} \ge s \Leftrightarrow h_K(y) \le -\log s \Leftrightarrow y \in (-\log s)K^\circ.$$

So the super-level set of $(\mathbb{1}_K)^\circ$ at s is $G_{(\mathbb{1}_K)^\circ}(s) = (-\log s)K^\circ$. Moreover,

$$J(-\log s\ K^\circ) = -\log s\ J(K^\circ) = -\log s(L(K))^\circ$$

and $\max_s s(-\log s)^n$ is reached at $s = \mathrm{e}^{-n}$. Thus $J((\mathbb{1}_K)^\circ) = \mathrm{e}^{-n}\mathbb{1}_{nJ(K^\circ)}$. □

In a functional context, we view as *ellipsoidal functions* or, *ellipsoids* in short, functions of the form

$$t\mathbb{1}_{\mathcal{E}} \text{ and } \exp(-\|Tx+a\|_2+t),$$

where $\mathcal{E}$ is an ellipsoid in $\mathbb{R}^n$ and $t \in \mathbb{R}$, $a \in \mathbb{R}^n$, $T \in S_+$.

We want to establish a duality relation between the ellipsoidal functions, similar to the one that holds for convex bodies. As in the case of convex bodies, we can only expect such a duality relation if we take polarity with respect to the proper point. Indeed, let $f = \mathrm{e}^{-\psi}$ be a log-concave function. Let $L(f)(x) = \mathrm{e}^{-\|T_0x+a_0\|+t_0}$ be the Löwner function of f. Let $b \in \mathbb{R}^n$. Then

$$\begin{aligned}
\mathcal{L}_b\left(\|T_0x+a_0\|_2+t_0\right)(y) &= t_0 + \sup_{x\in\mathbb{R}^n} \langle x-b, y-b\rangle - \|T_0x+a_0\|_2 \\
&= t_0 + \sup_{z\in\mathbb{R}^n} \langle T_0^{-1}(z-a_0)-b, y-b\rangle - \|z\|_2 \\
&= t_0 - \langle b, y-b\rangle - \langle T_0^{-1}(a_0), y-b\rangle \\
&\quad + \sup_{z\in\mathbb{R}^n} \langle z, T_0^{-1}(y-b)\rangle - \|z\|_2 \\
&= t_0 - \langle b, y-b\rangle - \langle T_0^{-1}(a_0), y-b\rangle \\
&\quad + \sup_{z\in\mathbb{R}^n} \|z\|_2\left(\|T_0^{-1}(y-b)\|_2 - 1\right) \\
&= t_0 - \langle b, y-b\rangle - \langle T_0^{-1}(a_0), y-b\rangle + \begin{cases}\infty & y \notin T_0B_2^n + b \\ 0 & y \in T_0B_2^n + b\end{cases}
\end{aligned}$$

and $(L(f))^b = \mathrm{e}^{-\mathcal{L}_b(\|T_0x+a_0\|_2+t_0)}$ is again an ellipsoidal function if and only if $b = b_0 = -T_0^{-1}a_0$. In this case

$$(L(f))^{-b_0} = \mathrm{e}^{-\mathcal{L}_{-b_0}(\|T_0x+a_0\|_2+t_0)} = \mathrm{e}^{-t_0}\, \mathbb{1}_{T_0B_2^n-b_0}.$$

For log-concave functions $f = \mathrm{e}^{-\psi}$ that are even, i.e., $\psi(x) = \psi(-x)$, the point $b_0 = 0$ and such a duality relation holds.

Proposition 5 *If $f = \mathrm{e}^{-\psi}$ is an even log-concave function, then $(L(f))^\circ = J(f^\circ)$.*

Proof Let $L(f) = \mathrm{e}^{-\|T_0(x+b_0)\|_2+t_0}$ be the Löwner function of f. By Theorem 1, (T_0, b_0, t_0) are the unique solution, modulo $O(n)$, to the optimization problem

$$n!\mathrm{vol}_n(B_2^n)\ \min_{b\in\mathbb{R}^n}\min\left\{\frac{e^t}{\det T} : T \in S_+, t \in \mathbb{R}, \mathrm{e}^{-t}\mathbb{1}_{TB_2^n}(y) \le (f_b)^\circ(y)\right\},$$

where $f_b(x) = S_{-b}f$. As f is even, $b_0 = 0$. Hence the above minimum is obtained when $b = 0$, that is,

$$\begin{aligned}
&\min_{b\in\mathbb{R}^n}\min\left\{\frac{e^t}{\det T}:T\in S_+,t\in\mathbb{R},\mathrm{e}^{-t}\mathbb{1}_{TB_2^n}(y)\le (f_b)^\circ(y)\right\}\\
&=\min\left\{\frac{e^t}{\det T}:T\in S_+,t\in\mathbb{R},\mathrm{e}^{-t}\mathbb{1}_{TB_2^n}(y)\le f^\circ\right\}\\
&=\left(\max\left\{\mathrm{e}^{-t}\det T:T\in S_+,t\in\mathbb{R},\mathrm{e}^{-t}\mathbb{1}_{TB_2^n}(y)\le f^\circ\right\}\right)^{-1}.
\end{aligned}$$

In other words, (T_0, t_0) also solves

$$\max\left\{\mathrm{e}^{-t}\det T:T\in S_+,t\in\mathbb{R},\mathrm{e}^{-t}\mathbb{1}_{TB_2^n}(y)\le f^\circ\right\}. \tag{39}$$

Now observe that f° is an even function. In fact, since $\psi(x)=\psi(-x)$,

$$\begin{aligned}
\mathcal{L}(\psi)(-y)&=\sup_{x\in\mathbb{R}^n}\langle x,-y\rangle-\psi(x)=\sup_{x\in\mathbb{R}^n}\langle -x,-y\rangle-\psi(-x)=\sup_{x\in\mathbb{R}^n}\langle x,y\rangle-\psi(x)\\
&=\mathcal{L}(\psi)(y).
\end{aligned}$$

Thus, $f^\circ(-y)=\mathrm{e}^{-\mathcal{L}(\psi)(-y)}=\mathrm{e}^{-\mathcal{L}(\psi)(y)}=f^\circ(y)$. By the eveness of f°, the maximum

$$\max_{b\in\mathbb{R}^n}\max\left\{\mathrm{e}^{-t}\det T:T\in S_+,t\in\mathbb{R},\mathrm{e}^{-t}\mathbb{1}_{TB_2^n+b}(y)\le f^\circ\right\} \tag{40}$$

is achieved at the same solution to (39). But the solution to (40) gives the John ellipsoid function of f°. Therefore, $J(f^\circ)=\mathrm{e}^{-t_0}\mathbb{1}_{T_0B_2^n}$. It follows from a routine computation that

$$(L(f))^\circ=\left(\mathrm{e}^{-\|T_0x\|_2+t_0}\right)^\circ=\mathrm{e}^{-t_0}\mathbb{1}_{T_0B_2^n}=J(f^\circ).$$

□

However, it is not true in general that $L(f)^{b_0}=J(f^{b_0})$ or $L(f^{b_0})=J(f)^{b_0}$. We give a 1-dimensional counter example. The higher dimensional counter example is constructed accordingly.

A counter example Let $f(x)=\mathrm{e}^{-\psi(x)}$ be the log-concave function such that

$$\psi(x)=\begin{cases}4x^2 & x\le 0\\ x^2 & x>0.\end{cases}$$

We compute that the Löwner function of f is

$$L(f)=\mathrm{e}^{-\frac{4}{\sqrt5}\left|x-\frac{3}{8\sqrt5}\right|+\frac12}$$

and that the polar of $L(f)$ with respect to $\frac{3}{8\sqrt5}$ is

$$(L(f))^{\frac{3}{8\sqrt5}}=\mathrm{e}^{-\frac12}\mathbb{1}_{\left[-\frac{4}{\sqrt5},\frac{4}{\sqrt5}\right]+\frac{3}{8\sqrt5}}.$$

The polar of f with respect to $\frac{3}{8\sqrt{5}}$ is

$$(f)^{\frac{3}{8\sqrt{5}}} = e^{\frac{3}{8\sqrt{5}}\left(x-\frac{3}{8\sqrt{5}}\right)-\frac{1}{16}\left(x-\frac{3}{8\sqrt{5}}\right)^2} \mathbb{1}_{\left(-\infty,\frac{3}{8\sqrt{5}}\right]} + e^{\frac{3}{8\sqrt{5}}\left(x-\frac{3}{8\sqrt{5}}\right)-\frac{1}{4}\left(x-\frac{3}{8\sqrt{5}}\right)^2} \mathbb{1}_{\left(\frac{3}{8\sqrt{5}},\infty\right)}.$$

To find the John ellipsoid $J\left((f)^{\frac{3}{8\sqrt{5}}}\right)$ of $(f)^{\frac{3}{8\sqrt{5}}}$ we determine the super-level sets of $(f)^{\frac{3}{8\sqrt{5}}}$,

$$\begin{aligned} G_{(f)^{\frac{3}{8\sqrt{5}}}}(s) &= \left\{x : (f)^{\frac{3}{8\sqrt{5}}} \geq s\right\} \\ &= \begin{cases} \left[\frac{3}{8\sqrt{5}} + \frac{3-(9-80\log s)^{\frac{1}{2}}}{\sqrt{5}}, \frac{3}{8\sqrt{5}} + \frac{3+(9-320\log s)^{\frac{1}{2}}}{4\sqrt{5}}\right], & s \leq 1 \\ \left[\frac{3}{8\sqrt{5}} + \frac{3-(9-320\log s)^{\frac{1}{2}}}{4\sqrt{5}}, \frac{3}{8\sqrt{5}} + \frac{3+(9-320\log s)^{\frac{1}{2}}}{4\sqrt{5}}\right], & s \geq 1 \end{cases} \end{aligned}$$

and then maximize the function

$$h(s) = \begin{cases} \frac{s}{4\sqrt{5}}\left(4\,(9-80\log s)^{\frac{1}{2}} + (9-320\log s)^{\frac{1}{2}} - 9\right), & s \leq 1 \\ \frac{s}{2\sqrt{5}}\,(9-320\log s)^{\frac{1}{2}}, & s \geq 1. \end{cases}$$

If it were so that

$$(L(f))^{\frac{3}{8\sqrt{5}}} = e^{-\frac{1}{2}} \mathbb{1}_{\left[-\frac{4}{\sqrt{5}},\frac{4}{\sqrt{5}}\right]+\frac{3}{8\sqrt{5}}} = J\left((f)^{\frac{3}{8\sqrt{5}}}\right),$$

then the function h would have its maximum at $s = e^{-\frac{1}{2}}$ and thus the derivative of h at $s = e^{-\frac{1}{2}}$ should be 0. But $h'\left(e^{-\frac{1}{2}}\right) \simeq -0.3538 < 0$.

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