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# Regularity for Almost Convex Viscosity Solutions of the Sigma-2 Equation

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**Dedicated to Professors Sun-Yung Alice Chang and Paul C. Yang on their 70th birthdays**

**Abstract.** We establish interior regularity for almost convex viscosity solutions of the sigma-2 equation.

**AMS subject classifications**: 35D10, 35J60, 35J99

**Key words**: Regularity, viscosity solution, fully nonlinear, elliptic partial differential equation.

## 1 Introduction

In this paper, we establish regularity for almost convex viscosity solutions of the $\sigma_2$ equation

$$F\left(D^2u\right)=\sigma_2(\lambda)-1=\sum_{1\leq i<j\leq n}\lambda_i\lambda_j-1=0, \tag{1.1}$$

where $\lambda_i's$ are the eigenvalues of the Hessian $D^2u$.

Fully nonlinear equation (1.1) is the quadratic analogue of the Laplace equation $\sigma_1=\Delta u$ and the Monge-Ampère equation $\sigma_n=\det D^2u$. In dimension three, $\sigma_2=1$ if and only if $\sum_{i=1}^3\arctan\lambda_i=\pm\pi/2$, which is the special Lagrangian equation from calibrated geometry. The equation $\sigma_2(\kappa)=1$ prescribes the scalar curvature of a Euclidean hypersurface $(x,u(x))$ with principal curvatures $(\kappa_1,\cdots,\kappa_n)=\kappa$. Complex $\sigma_2$-type equations arise from the Strominger system in string theory, and the $\sigma_2$ function of the Schouten tensor arises in conformal geometry.

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**Theorem 1.1.** *Let $u$ be a semiconvex viscosity solution of $\sigma_2\left(D^2u\right)=1$ on $B_1(0)\subset\mathbb{R}^n$ with $\Delta u>0$ and $D^2u\geq(\delta-K)I$ for some $\delta>0$ and $K=\left[n(n-1)/2\right]^{-1/2}$. Then $u$ is analytic on $B_1(0)$ and has the effective Hessian bound*

$$\left|D^2u(0)\right|\leq C(n)\exp\left[C(n)\operatorname*{osc}_{B_1(0)}u\right]^2.$$

One quick consequence is that every entire almost convex (such as in Theorem 1.1) viscosity solution of (1.1) is a quadratic function; the smooth case was done in [5]. Recall the classic rigidity results for the equations $\triangle u=1$ and $\det D^2u=1$: every entire convex viscosity solution is quadratic. Our result shows that if a singular viscosity solution of (1.1) exists, then it is not convex, or even almost convex.

The interior regularity for (1.1) in general dimensions is a longstanding problem. Sixty years ago, Heinze [8] achieved a priori estimates and regularity for two dimensional Monge-Ampère type equations including (1.1) with $n=2$ by two dimensional techniques. More than ten years ago, a priori estimates and regularity for (1.1) with $n=3$ were obtained via the minimal surface structure of equation (1.1) in the joint work with Warren [17]. Along this "integral" way, Qiu [12] has proved a priori Hessian estimates–then regularity follows–for three dimensional (1.1) with $C^{1,1}$ variable right hand side, and even with left hand side $\lambda$ replaced by the principal curvatures $\kappa$. Hessian estimates for convex smooth solutions of general quadratic Hessian equations in general dimensions have been obtained via a pointwise approach by Guan and Qiu [7]. Hessian estimates for almost convex smooth solutions of (1.1) in general dimensions have been derived by a compactness argument in [10], and recently for semiconvex smooth solutions in [13] using new mean value and Jacobi inequalities.

In contrast, there are Pogorelov-like singular convex viscosity solutions of the symmetric Hessian equations $\sigma_k(\lambda)=1$ with $k\geq3$ in dimension $n\geq3$. Under a strict k-convexity assumption on weak/viscosity solutions of $\sigma_k(\lambda)=1$, a priori Hessian estimates and then regularity were obtained by Pogorelov [11] and Chou-Wang [6], for $k=n$ and $2\leq k<n$ respectively, using Pogorelov's pointwise technique. Lastly, we also mention a priori gradient estimates by Trudinger [14] and a priori Hessian estimates for solutions of $\sigma_k$ as well as $\sigma_k/\sigma_n$ equations in terms of certain integrals of the Hessian by Urbas [15,16], Bao-Chen-Guan-Ji [1].

Extending the above a priori estimates to regularity statements about viscosity solutions of (1.1) is more subtle. In dimensions two and three, one can smoothly solve the Dirichlet problem on interior balls with smoothly approximated boundary data; a limiting procedure combined with the a priori estimates then yields the desired interior regularity for the viscosity solution. However, for dimension $n\geq4$, a priori estimates are not known for general solutions of (1.1). Because the smooth approximations may not satisfy the convexity constraints, we cannot invoke the available a priori estimates while taking the limit and deduce interior regularity.

We circumvent this difficulty using the improved regularity properties of the equation for the Legendre-Lewy transform $\bar{u}(\bar{x})$ found in [5]. By the analytical definition of the transform valid for merely semiconvex functions, we show that $\bar{u}(\bar{x})$ is indeed a viscosity solution of a new concave and uniformly elliptic equation if $u(x)$ is a viscosity solution of (1.1). It follows that $\bar{u}(\bar{x})$ is smooth. Then the boundedness of the original solution $u(x)$ combined with the constant rank of $D^2\bar{u}(\bar{x})$ in [3] implies $u(x)$ is smooth, and in turn, the a priori estimate in [13] provides the explicit estimate in Theorem 1.1. A similar approach has lead to the interior regularity for convex viscosity solutions of the special Lagrangian equation in our recent joint work [4].

It is still unclear to us whether semiconvex viscosity solutions of (1.1) are regular, if only $D^2u \geq -KI$ for some large $K>0$. Unlike in [4], where one can only justify the rotated transform $\bar{u}$ satisfying a phase decreased special Lagrangian equation for convex viscosity solutions $u$, here the Legendre-Lewy transform $\bar{u}$ is still a $C^{1,1}$ viscosity solution of a new uniformly elliptic equation, for any semiconvex viscosity solution. However, as the new equation no longer has convex level set, for large $K$, we are unable to deduce smoothness for $\bar{u}$ at this point. Without the smoothness of $\bar{u}$, we are currently unable to obtain a $C^{1,1}$ version of the constant rank theorem to gain negative definiteness of the negative semidefinite Hessian $D^2\bar{u} \leq 0$, for the $C^{1,1}$ solution $\bar{u}$ of a uniformly elliptic and inversely concave equation. Otherwise, the interior regularity for such semiconvex viscosity solutions of (1.1) would be justified.

## 2 Preliminaries

### 2.1 Smooth functions and solutions

It was shown in [5] that smooth semiconvex solutions $u$ of (1.1) solve a better equation after the Legendre-Lewy transform. First adding a large quadratic to produce uniformly convex $\tilde{u}(x) = u(x) + \frac{K}{2}|x|^2$, we reflect the "gradient" graph $(x, D\tilde{u}(x)) \in \mathbb{R}^n \times \mathbb{R}^n$ to produce another "gradient" graph $(-D\bar{u}(\bar{x}), \bar{x}) = (x, D\tilde{u}(x))$ with potential $\bar{u}(\bar{x})$. This potential can be found using

$$-d\bar{u}(\bar{x}) = -D\bar{u}(\bar{x})\cdot d\bar{x} = x\cdot d\bar{x} = d(x\cdot\bar{x}) - D\tilde{u}(x)\cdot dx = d(x\cdot\bar{x} - \tilde{u}(x)),$$

so up to a constant,

$$\bar{u}(\bar{x}) = -(x\cdot\bar{x} - \tilde{u}(x)), \qquad x\in B_1,$$

which is, in fact, negative the Legendre transform of striclty convex function $f(x) = \tilde{u}(x)$, formulated in extremal form as

$$f^*(y) = \sup_{x\in B_1}[x\cdot y - f(x)].$$

Here, the subdifferential $y \in \partial f(B_1)$. If $f$ is smooth, then $y = Df(x)$ and the analytic definition agrees with the geometric one. We finally add a minus sign, and define the Legendre-Lewy transform of a semiconvex function $u$ with $D^2u \geq (-K+\delta)I$ by

$$\bar{u}(\bar{x}) = -\tilde{u}^*(\bar{x}) = -\left[u(x) + \frac{K}{2}|x|^2\right]^*(\bar{x}) \tag{2.1}$$

for those $\bar{x} \in \partial\tilde{u}(B_1)$.

The Hessians are related by

$$D^2\bar{u}(\bar{x}) = -(D^2\tilde{u}(x))^{-1} = -(D^2u(x)+KI)^{-1},$$

so semiconvexity $D^2u \geq (-K+\delta)I$ and uniform convexity $D^2\tilde{u} \geq \delta I$ imply $\bar{u} \in C^{1,1}$ with

$$-\frac{1}{\delta}I \leq D^2\bar{u}(\bar{x}) < 0. \tag{2.2}$$

Indeed, the tangent planes of $(x, D\tilde{u}(x))$ are formed by $(e, D^2\tilde{u}\cdot e)$ with $e\in\mathbb{R}^n$, so reflection implies that $(-(D^2\tilde{u})^{-1}e, e)$ form tangent planes for $(-D\tilde{u}^*(\bar{x}), \bar{x})$. Consequently, equation (1.1) transforms to

$$\bar{F}(D^2\bar{u}) = \sigma_n\left(-\bar{\lambda}\right)\left[\sigma_2\left(-\frac{1}{\bar{\lambda}}-K\right)-1\right] = \sigma_n\left(-\bar{\lambda}\right)F\left(D^2u\right) = 0, \tag{2.3}$$

where $\bar{\lambda}_i' s$ are the eigenvalues of the Hessian $D^2\bar{u}$. It was shown in [5, p. 661–663] that this equation, with an equivalent conformal factor

$$\frac{1}{\sqrt{(1+\lambda_1^2)\cdots(1+\lambda_n^2)}} \overset{C(n,K,\delta)}{\approx} \sigma_n\left(-\bar{\lambda}\right),$$

is uniformly elliptic for all $K>0$, and has convex level set for $K=[n(n-1)/2]^{-1/2}$. Moreover, for smooth solutions, the constant rank theorem of [3] applies since the "inverse" equation of $\bar{F}$ also has convex level set:

$$\{\sigma_2(M)-1=0\} = \left\{\operatorname{tr}(M)-\sqrt{|M|^2+2}=0\right\}.$$

These favorable properties were used in [10] to find an a priori estimate for smooth solutions.

The first challenge is to show that $\bar{u}(\bar{x})$ is a viscosity solution of (2.3) if $u(x)$ is one for (1.1). The favorable regularity properties of (2.3) will then imply $\bar{u}(\bar{x})$ is a classical solution of (2.3), after which the constant rank theorem will take over.

## 2.2 Convex functions and viscosity solutions

The Legendre-Lewy transform $\bar{u}$ in (2.1) still makes sense if $u \in C^0$ is only semiconvex with $D^2u \geq (\delta - K)I$. Because $\tilde{u} = u + \frac{K}{2}|x|^2$ is uniformly convex $D^2\tilde{u} \geq \delta I$, it follows that the subdifferential map $x \mapsto \partial\tilde{u}(x)$ is "distance increasing", and we can show as in [4, Lemma 2.1] that $\bar{\Omega} = \partial\tilde{u}(B_1)$ is an open connected set.

Moreover, the Legendre transform is order reversing and respects constants: $f \leq g \to f^* \geq g^*$ and $(f+c)^* = f^* - c$. This means the transform respects uniform convergence: if $f - \varepsilon \leq g \leq f + \varepsilon$, then $f^* + \varepsilon \geq g \geq f^* - \varepsilon$. It follows that the Legendre-Lewy transform (2.1) obeys these same properties, except it is now order preserving: if $u-\varepsilon \le v \le u+\varepsilon$, then $\bar{u}-\varepsilon \le \bar{v} \le \bar{u}+\varepsilon$.

By smooth approximation, it follows from (2.2) and the respect for uniform convergence that concave $\bar{u}$ is $C^{1,1}$ from below with

$$-\frac{1}{\delta} I \le D^2 \bar{u} \le 0.$$

Order preservation also implies preservation of the supersolution property.

**Proposition 2.1.** *Let $u$ be a semiconvex viscosity supersolution of* (1.1) *on $B_1(0)$ with $D^2 u \ge (\delta - K)I$. Then the Legendre-Lewy transform $\bar{u}$ in* (2.1) *is a corresponding viscosity supersolution of* (2.3) *on $\bar{\Omega} = \partial \tilde{u}(B_1)$.*

*Proof.* Let $\bar{Q}$ be any quadratic function touching $\bar{u}$ from below locally somewhere on the open set $\bar{\Omega}$, say the origin. Already $D^2\bar{Q} \le D^2\bar{u} \le 0$. By subtracting $\varepsilon|\bar{x}|^2$ from $\bar{Q}$, then taking the limit as $\varepsilon$ goes to 0, we assume $D^2\bar{Q} < 0$. This guarantees the existence of its inverse transform, quadratic function $Q$. From the order preservation of the Legendre-Lewy transform (2.1), which is also valid for the inverse operation, we see that $Q$ touches $u$ from below somewhere. Because $u$ is a supersolution, $F(D^2Q) \le 0$. Recalling $-D^2\bar{Q} > 0$, we conclude $\bar{F}(D^2\bar{Q}) = \sigma_n(-D^2\bar{Q})F(D^2Q) \le 0$. □

The concavity pertaining to (1.1) implies the preservation of subsolutions under the Legendre-Lewy transform.

**Proposition 2.2.** *Let $u$ be a semiconvex viscosity subsolution of* (1.1) *on $B_{1.2}(0)$ with $D^2 u \ge (\delta - K)I$. Then the Legendre-Lewy transform $\bar{u}$ in* (2.1) *is a corresponding viscosity subsolution of* (2.3) *on $\bar{\Omega} = \partial \tilde{u}(B_1)$.*

*Proof.* Step 1. For convenience, we extend the semiconvex $u(x)$ to an entire semiconvex function on $\mathbb{R}^n$. Set the standard convolution $u_\varepsilon(x) = u * \rho_\varepsilon(x)$ with $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$ and nonnegative $\rho(x) = \rho(|x|) \in C_0^\infty(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \rho(x)dx = 1$. Given the $C^0$ uniform continuity of $u$, we have $|u_\varepsilon(x) - u(x)| < o(1)$ for all small enough $\varepsilon$.

We claim that the smooth Legendre-Lewy transform $\bar{u}_\varepsilon$ is defined at least on $\bar{\Omega}$ for all small enough $\varepsilon$. We verify this by showing that for any $\bar{a} \in \partial\tilde{u}(a)$ with $a \in B_1(0)$, there exists $b$ such that $D\tilde{u}_\varepsilon(b) = \bar{a}$ with $\tilde{u}_\varepsilon(x) = \frac{K}{2}|x|^2 + u_\varepsilon(x)$ and $|b-a| \le o(1)$ as $\varepsilon$ goes to 0. Consequently, $\partial\tilde{u}(B_1(0)) \subset D\tilde{u}_\varepsilon(B_{1.1}(0))$ for all small enough $\varepsilon$.

Now for any $\bar{a} \in \partial\tilde{u}(a)$, given the uniform convexity of $\tilde{u}_\varepsilon$, $D^2\tilde{u}_\varepsilon \ge \delta I$, there exists $b \in \mathbb{R}^n$ such that $D\tilde{u}_\varepsilon(b) = \bar{a}$. By subtracting linear function $\bar{a} \cdot x$ from both $\tilde{u}$ and $\tilde{u}_\varepsilon$, we assume $0 \in \partial\tilde{u}(a) \cap \partial\tilde{u}_\varepsilon(b)$. Then coupled with the $\delta$-convexity of $\tilde{u}$ and $\tilde{u}_\varepsilon$, we have

$$\tilde{u}(b) - \tilde{u}(a) \ge \frac{\delta}{2}|b-a|^2 \quad \text{and} \quad \tilde{u}_\varepsilon(a) - \tilde{u}_\varepsilon(b) \ge \frac{\delta}{2}|a-b|^2.$$

For small enough $\varepsilon$, we always have

$$\tilde{u}(a) - \tilde{u}_\varepsilon(a) \ge -|o(1)| \quad \text{and} \quad \tilde{u}_\varepsilon(b) - \tilde{u}(b) \ge -|o(1)|.$$

Adding all the above four inequalities together, we get

$$|b-a|^2 \leq 2|o(1)|/\delta.$$

for small enough $\varepsilon$. Therefore, we have proved that $\bar{u}_\varepsilon$ is defined on $\bar{\Omega} = \partial\tilde{u}(B_1(0)) \subset D\tilde{u}_\varepsilon(B_{1.1}(0))$ for all small enough $\varepsilon$.

Step 2. Note that the equivalent form $\sqrt{\sigma_2(\lambda)} - 1 = 0$ of equation (1.1) is concave. By the well-known result in [2, p. 56], the solid convex average $u * \rho_\varepsilon$ (instead of the hollow spherical one there) is still a subsolution of (1.1) in $B_{1.1}(0)$ for small enough $\varepsilon > 0$. For smooth subsolutions $u_\varepsilon$, the corresponding smooth Legendre-Lewy transform $\bar{u}_\varepsilon$ is a subsolution of (2.3) on $\bar{\Omega}$ from Step 1 and $\sigma_n\left(-D^2\bar{u}_\varepsilon\right) \geq 0$. The viscosity solutions are stable under $C^0$ uniform convergence. Hence uniformly convergent limit $\lim_{\varepsilon\to 0}\bar{u}_\varepsilon = \bar{u}$ is a viscosity subsolution of (2.3) on $\bar{\Omega}$. $\square$

**Remark 2.1.** We shrank $B_{1.2}(0)$ to $B_1(0)$ in the conclusion of Proposition 2.2 for clarity of exposition. If we instead mollify on small balls centered near the boundary, then s-traightforward modifications of the above yield the result on all of $\partial\tilde{u}[B_{1.2}(0)]$, not just on $\partial\tilde{u}[B_1(0)]$.

## 3 Proof of Theorem 1.1

By Propositions 2.1 and 2.2, the Legendre-Lewy transform $\bar{u}(\bar{x})$ is a viscosity solution of transformed equation (2.3) on open and connected set $\bar{\Omega} = \partial\left(u + \frac{1}{2}K|x|^2\right)(B_1(0))$ (we may assume $u$ is defined on $B_{1.2}(0)$ by scaling, $1.2^2 u(x/1.2)$ ). Moreover,

$$-\frac{1}{\delta} I \leq D^2\bar{u} \leq 0.$$

By [5, p. 661–663], equation (2.3) with $K = [n(n-1)/2]^{-1/2}$ is uniformly elliptic and has convex level set, so the Evans-Krylov theorem implies that $\bar{u} \in C^{2,\alpha}$ in $\bar{\Omega}$ (see [2, Theorem 6.6]), hence smooth in $\bar{\Omega}$.

We now show $D^2\bar{u} < 0$ on the open and connected set $\bar{\Omega}$, which then implies that the original $u$ satisfies $D^2 u < +\infty$, and hence is smooth and even analytic on $B_1(0)$. If not, then $D^2\bar{u}$ is not full rank somewhere. By the constant rank theorem of the Hessian $D^2\bar{u}(\bar{x})$ in [3, Theorem 1.1], $D^2\bar{u}$ is nowhere full rank (nowhere negative definite).

But we can arrange a "large" quadratic function $Q = \frac{A}{2}|x|^2 + t$ touching $u$ from above at an interior point of $B_1(0)$. By the order preservation of the Legendre-Lewy transform, it follows that $\bar{Q} = -\frac{1}{2(K+A)}|\bar{x}|^2 + t$ touches $\bar{u}$ from above somewhere. Since $D^2\bar{Q} < 0$, it follows that $D^2\bar{u} < 0$ somewhere on $\bar{\Omega}$, and we obtain a contradiction.

We thus deduce $u$ is smooth on $B_1$, and even analytic [9, p. 203]. The effective Hessian bound then follows from [13].

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