Chapter 1
A Heuristic Approach to Convex Integration for the Euler Equations

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Abstract The purpose of these lecture notes is to employ a heuristic approach in designing a convex integration scheme that produces non-conservative weak solutions to the Euler equations.

1.1 Convex Integration as a Mathematical Tool to Resolve Onsager's Conjecture

In these lecture notes, we aim to outline how a convex integration can be used to construct non-conservative weak solutions to the Euler equations:

$$\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p &= 0\,, \\ \operatorname{div} v &= 0\,. \end{aligned} \tag{1.1}$$

We will restrict ourselves to considering the Euler equations on the periodic torus $\mathbb{T}^3$ for times $t \in (-1, 1)$. It is easy to check, after a simple integration by parts, that for smooth solutions to the Euler equation, the kinetic energy, defined by

$$E(t) := \frac{1}{2} \int_{\mathbb{T}^3} |v(t)|^2 \, dx\,,$$

is conserved. This calculation however does not hold for weak solutions. Indeed, the theory of turbulence naturally leads one to study the existence of dissipative

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weak solutions to the Euler equations. If one views the Euler equation as an inviscid limit of the Navier–Stokes equations, then formally, if one takes the inviscid limit of *turbulent* solutions, then one expects to obtain solutions to the Euler equations that dissipate kinetic energy, and are therefore necessarily weak solutions (see for example Section 2 of [5] and the references within for a more detailed discussion). The postulate of dissipation of kinetic energy at the inviscid limit is sometimes referred to in the literature as the *zeroth law of turbulence*.

In [20], Onsager famously conjectured the following dichotomy:

Conjecture 1.1 (Onsager's Conjecture)

(a) Any weak solution v belonging to the Hölder space C^θ for $\theta > \frac{1}{3}$ conserves kinetic energy.
(b) For any $\theta < \frac{1}{3}$ there exist weak solutions $v \in C^\theta$ which dissipate kinetic energy.

Part (a) was resolved by Constantin, E and Titi in [9], following a partial resolution of Eyink in [16] (see also [7, 15] for more refined results). The first result towards proving Part (b) was the construction of non-conservative L^2 weak solution to the Euler equations by Scheffer [21]. While the solutions constructed by Scheffer were non-conservative, they could not be classed a dissipative since they did not satisfy the property of non-increasing energy. The first example of dissipative weak solutions to the Euler equations was due to Shnirelman in [22] (cf. [11, 12]). Motivated in part by the convex integration scheme of Nash, employed in order to construct exotic counter-examples to the C^1 isometric embedding problem [19], De Lellis and Székelyhidi Jr. in [13, 14], made significant progress towards Part (b) by constructing dissipative Hölder $C^{\frac{1}{10}-}$ continuous weak solutions to the Euler equations. Then after a series on advancements [1–3, 10, 17], Isett resolved the conjecture in [18]. However, like the original paper of Scheffer [21], the weak solutions constructed by Isett [18] were not strictly dissipative. This technical issue was resolved in a paper by the authors in collaboration with De Lellis and Székelyhidi Jr. [4], in which the precise statement of Part (b) was proven.

Instead of considering the more difficult problem of proving Part (b), let us consider the simpler problem of constructing non-trivial, non-conservative, Hölder continuous weak solutions:

Theorem 1.1 *For some Hölder exponent β, there a non-trivial weak solution to the Euler equations $v \in C((-1,1); C^\beta(\mathbb{T}^3))$ with compact support in time.*

The purpose of these notes is to provide an outline of how to go about constructing a convex integration scheme in order prove Theorem 1.1. The outline will track closely with the approach taken in Section 5 of the review paper [5], which itself is based on the works [2, 11, 12, 17]. In this presentation, we eschew mathematical rigor in favor rough heuristics. This will allow us to better illustrate the main ideas that go into designing a convex integration of the type pioneered by De Lellis and Székelyhidi Jr. in [13], without getting caught up in the nitty gritty technicalities that a rigorous approach entails.

1.2 The Iteration

The general strategy for proving a theorem such as Theorem 1.1 is to construct a sequence $(v_q, \mathring{R}_q)$ of solutions to the *Euler–Reynolds* system

$$\partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = \operatorname{div} \mathring{R}_q, \qquad \operatorname{div} v_q = 0 \tag{1.2}$$

such that $\mathring{R}_q \to 0$ uniformly and $v_q \to v \in C^\beta$, whereby v is a non-trivial weak solution to the Euler equations (1.1) with compact support in time. The tensor $\mathring{R}_q$ is called the *Reynolds stress*, and is assumed to be symmetric and trace-free. At each inductive step, the perturbation

$$w_{q+1} = v_{q+1} - v_q$$

is designed such that v_{q+1} satisfies (1.2) with a smaller Reynolds stress $\mathring{R}_{q+1}$. It will prove helpful to split the Reynolds stress $\mathring{R}_{q+1}$ into several components.*

$$\begin{aligned}\operatorname{div} \mathring{R}_{q+1} = &\underbrace{\operatorname{div}\left(w_{q+1} \otimes w_{q+1} + \mathring{R}_{q+1}\right) + \nabla(p_{q+1} - p_q)}_{\text{oscillation error}} \\ &+ \underbrace{\partial_t w_{q+1} + v_{q+1} \cdot \nabla w_{q+1}}_{\text{transport error}} + \underbrace{w_{q+1} \cdot \nabla v_q}_{\text{Nash error}} .\end{aligned} \tag{1.3}$$

The Reynolds stress $\mathring{R}_{q+1}$ can be then be solved using a -1 order linear differential operator $\mathcal{R}$, defined as follows:

Definition 1.1 The operator $\mathcal{R}$ is defined on mean zero vector fields by

$$(\mathcal{R}v)^{k\ell} = (\partial_k \Delta^{-1} v^\ell + \partial_\ell \Delta^{-1} v^k - \frac{1}{2}(\delta_{k\ell} + \partial_k \partial_\ell \Delta^{-1}) \operatorname{div} \Delta^{-1} v .$$

The operator $\mathcal{R}$ is formally an inverse of the divergence equation, i.e. $\operatorname{div} \mathcal{R} v = v$ for any smooth, mean zero vector field v. Moreover, the matrix $\mathcal{R}v$ is symmetric and trace free.

Suppose we are given a smooth vector field $b : \mathbb{T}^3 \to \mathbb{R}^3$ and a smooth phase function $\Phi : \mathbb{T}^3 \to \mathbb{T}^3$ satisfying for all $x \in \mathbb{T}^3$ the bound

$$C^{-1} \le |\nabla \Phi(x)| \le C .$$

Since $\mathring{R}_{q+1}$ is a -1 order linear differential operator, then for any $\alpha \in (0, 1)$, and λ sufficiently large, we expect an estimate of the form

$$\left\| \mathcal{R}\left(a e^{i\lambda\xi\cdot\Phi(x)}\right) \right\|_{C^\alpha} \lesssim \frac{\|a\|_{C^0}}{\lambda^{1-\alpha}} + \text{ error} ,$$

where the error can be made arbitrarily small by taking λ sufficiently large. See for example [10, Lemma 2.2]) or [5, Lemma 5.6]), for a mathematically rigorous statement. The implicit constant in the above inequality depends on α. In our scheme we will take α to be sufficiently small, and thus for the matter of heuristics we will ignore the loss of λ^α, as well as the additional 'error', and instead pretend that we have the estimate

$$\left\| \mathcal{R}\left(a e^{i\lambda\xi\cdot\Phi(x)}\right)\right\|_{C^0} \lesssim \frac{\|a\|_{C^0}}{\lambda} . \tag{1.4}$$

Roughly, the perturbation w_{q+1} will be of the form

$$w_{q+1} = \sum_{\xi\in\Lambda} a_\xi W_{\xi,\lambda_{q+1}} \tag{1.5}$$

where Λ is a finite set of directions, the vector fields $W_{\xi,\lambda_{q+1}}$ are oscillatory *building blocks* oscillating in direction ξ, and a_ξ are coefficient functions chosen such that

$$\sum_{\xi\in\Lambda} a_\xi^2 \fint_{\mathbb{T}^3} W_{\xi,\lambda_{q+1}} \mathring{\otimes} W_{\xi,\lambda_{q+1}} = -\mathring{R}_q . \tag{1.6}$$

Here $\mathring{\otimes}$ represents the projection of the outer product onto trace free tensors. The building blocks $W_{\xi,\lambda_{q+1}}$ will oscillate at a frequency parameterized by λ_{q+1}. The cancellation (1.6) will be essential in estimated the oscillation error defined in (1.3). Let us heuristically assume that the frequencies scale geometrically

$$\lambda_q = \lambda^q \tag{1.7}$$

for some large $\lambda \in \mathbb{N}$.[1] Then for v_q to converge in $v \in C^\beta$, we roughly require

$$\left\| w_{q+1}\right\|_{C^0} \leq \lambda_{q+1}^{-\beta} . \tag{1.8}$$

Recalling that $v_q = \sum_{q'=0}^{q} w_{q'}$, where $w_{q'}$ oscillates at frequency $\lambda_{q'} = \lambda^{q'}$; then, (1.8) roughly translates into the estimate

$$\left\| v_q\right\|_{C^1} \leq \sum_{q'=0}^{q} \lambda_{q'}^{1-\beta} \lesssim \lambda_q^{1-\beta} , \tag{1.9}$$

[1] In practice is often simpler to assume that the frequencies grow super-exponentially (cf. [2, 5, 12, 17]). However, for the purpose of heuristics, geometric growth simplifies some of the calculations.

assuming that λ is chosen sufficiently large. In view of (1.8), for such a cancellation of the type (1.6) to occur, we would require the following estimate on $\mathring{R}_q$

$$\left\| \mathring{R}_q \right\|_{C^0} \leq \lambda_{q+1}^{-2\beta} . \tag{1.10}$$

Utilizing that the building blocks $W_{\xi,\lambda_{q+1}}$ oscillate at frequency λ_{q+1}, then by heuristically using an estimate of the type (1.4), it is now possible to attain a heuristic estimate on the Nash error defined in (1.3)

$$\left\| \mathcal{R}(w_{q+1} \cdot \nabla v_q) \right\|_{C^0} \lesssim \frac{\left\| w_{q+1} \right\|_{C^0} \left\| v_q \right\|_{C^1}}{\lambda_{q+1}} \lesssim \lambda_{q+1}^{-1-\beta} \lambda_q^{1-\beta} \lesssim \lambda_{q+2}^{-2\beta} \lambda^{3\beta-1} ,$$

where we used (1.8) and (1.9) in the second inequality, and (1.7) in the last inequality.

Since the Nash error forms part of the Reynolds stress error $\mathring{R}_{q+1}$, in order that (1.10) is satisfied (with q replaced by $q+1$), we require that $\beta < \frac{1}{3}$.

1.2.1 Beltrami Flows

We are yet to define the building blocks $W_{\xi,\lambda_{q+1}}$ used in the definition of the perturbation w_{q+1}. There a number of different options depending on the goals of the convex integration schemes: *Beltrami flows*, were first utilized in the context of a convex integration scheme by De Lellis and Székelyhidi Jr. in [13]; *Mikado flows*, introduced by Daneri and Székelyhidi Jr. in [10], were essential in resolving Onsager's conjecture; *intermittent Beltrami flows*, were used in the first non-uniqueness result for weak solution to the Navier–Stokes equations [6]; and *intermittent jets*, were introduced as an improvement on intermittent Beltrami flows [8]. For the purpose of this note, we will employ Beltrami flows as our building blocks, as in [13].

A stationary divergence free vector field v is called a *Beltrami flow* if it satisfies the *Beltrami condition*:

$$\lambda(x)v(x) = \operatorname{curl} v(x), \qquad \lambda(x) > 0 ,$$

for all x. The function λ is called the *Beltrami coefficient*. For the purpose of these notes, we will assume that the Beltrami coefficient is a constant.

Given a Beltrami flow v, from the divergence free condition we have the following identity

$$\operatorname{div}(v \otimes v) = v \cdot \nabla v = \nabla \frac{|v|^2}{2} - v \times (\operatorname{curl} v) = \nabla \frac{|v|^2}{2} - \lambda v \times v = \nabla \frac{|v|^2}{2} .$$

In particular setting $p := \frac{|v|^2}{2}$, then (v, p) is a stationary solution to the Euler equations. Note that trivially, any linear sum of Beltrami flows with the same Beltrami coefficient λ, is itself a Beltrami flow with the Beltrami coefficient λ. This later property will be used to create a large family Beltrami flows, which will be necessary in order to achieve the cancellation (1.6).

Let us now define $W_{\xi,\lambda_{q+1}}$. We will suppose that $\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$ is such that $\lambda_{q+1}\xi \in \mathbb{Z}^3$. We define $A_\xi \in \mathbb{R}^3$ and $B_\xi \in \mathbb{C}^3$ by

$$A_\xi \cdot \xi = 0, \quad A_{-\xi} = A_\xi \qquad \text{and} \qquad B_\xi = \frac{1}{\sqrt{2}}\left(A_\xi + i\xi \times A_\xi\right) .$$

We then observe that B_ξ satisfies the following properties

$$|B_\xi| = 1, \quad B_\xi \cdot \xi = 0, \quad i\xi \times B_\xi = B_\xi, \quad B_{-\xi} = \overline{B_\xi} .$$

Hence the vector field

$$W_{\xi,\lambda_{q+1}}(x) := B_\xi e^{i\lambda_{q+1}\xi\cdot x} \tag{1.11}$$

is $\mathbb{T}^3$ periodic (using that $\lambda_{q+1}\xi \in \mathbb{Z}^3$), divergence free, and is an eigenfunction of the curl operator with eigenvalue λ. That is, $W_{\xi,\lambda_{q+1}}$ is a complex Beltrami plane wave with Beltrami coefficient λ_{q+1}. A real valued Beltrami plane wave with Beltrami coefficient λ_{q+1} is then attained by summing $W_{\xi,\lambda_{q+1}}$ with its complex conjugate. In view of this, we define

$$W_{-\xi,\lambda_{q+1}} = \overline{W}_{\xi,\lambda_{q+1}} .$$

Then in order to ensure the right-hand-side of (1.5) is real valued, it will suffice that $\overline{a}_\xi = a_{-\xi}$; or more simply, $a_\xi = a_{-\xi}$, if we further assume the coefficients a_ξ to be real valued. Now, for the moment let us assume that the coefficients a_ξ are chosen to be real valued constants—later, we will allow dependence on x. We further suppose that Λ is a finite subset of $\mathbb{S}^2 \cap \mathbb{Q}^3$ such that $-\Lambda = \Lambda$, and moreover $\lambda_{q+1}\Lambda \subset \mathbb{Z}^3$. Then the vector field

$$W(x) = \sum_{\xi\in\Lambda} a_\xi B_\xi e^{i\lambda_{q+1}\xi\cdot x}$$

is a real-valued, divergence-free Beltrami vector field satisfying curl $W = \lambda_{q+1}W$. Moreover, from the identity $B_\xi \otimes B_{-\xi} + B_{-\xi} \otimes B_\xi = 2\mathrm{Re}\,(B_\xi \otimes B_{-\xi}) = \mathrm{Id} - \xi \otimes \xi$, we have

$$\fint_{\mathbb{T}^3} W \otimes W \, dx = \frac{1}{2}\sum_{\xi\in\Lambda} a_\xi^2 \,(\mathrm{Id} - \xi \otimes \xi) . \tag{1.12}$$

We refer the reader to Proposition 3.1 in [13] for additional details regarding the computations above.

1.3 Oscillation Error

In this section, we demonstrate how the Beltrami flows of the previous section can be used in order the minimize the oscillation error defined in (1.3).

In order to maintain notational consistency with previous convex integration schemes in the literature, we introduce the amplitude parameter

$$\delta_{q+1} = \lambda_{q+1}^{-2\beta} .$$

Applying a little bit of linear algebra, it is not difficult to construct a finite set $\Lambda \subset \mathbb{S}^2 \cap \mathbb{Q}^3$ and coefficient functions a_ξ for each $\xi \in \Lambda$ whose amplitude is proportional to the square root of the uniform norm of $\mathring{R}_q$, that is by (1.10) we have $a_\xi = O(\delta_{q+1}^{\frac{1}{2}})$, in such a way as to achieve the cancellation (1.6). More specifically, we define

$$a_\xi(x,t) = \delta_{q+1}^{\frac{1}{2}} \gamma_\xi \left(\mathrm{Id} - \frac{\mathring{R}_q(x,t)}{\delta_{q+1}} \right) \tag{1.13}$$

where γ_ξ are smooth functions whose domain consists of a small ball around the identity matrix within the space of symmetric matrices. We refer the reader to Lemma 3.2 in [13] (alternatively Lemma 1.3 in [2]) for the precise definition of γ_ξ. Technically, in order that the definition (1.13) makes sense, we require a slightly stronger bound than (1.10) in order to ensure that $\mathrm{Id} - \frac{\mathring{R}_q(x,t)}{\delta_{q+1}}$ lies in the image of γ_ξ. For the purpose of this note, we ignore this minor technicality.

Assuming uniform bounds on the functions γ_ξ, we obtain the following bounds on a_ξ

$$\left\| a_\xi \right\|_{C^0} \lesssim \delta_{q+1}^{\frac{1}{2}} \tag{1.14}$$

$$\left\| \nabla a_\xi \right\|_{C^0} \lesssim \delta_{q+1}^{-\frac{1}{2}} \left\| \mathring{R}_q \right\|_{C^1} \tag{1.15}$$

We now define our perturbation w_{q+1} to be

$$w_{q+1} = \sum_{\xi \in \Lambda} a_\xi W_{\xi,q+1} . \tag{1.16}$$

Then by construction, we have (1.6). Let us now compute the term $\operatorname{div}(w_{q+1} \otimes w_{q+1} + \mathring{R}_q)$ that appears in the oscillation error, defined in (1.3)

$$
\begin{aligned}
&\operatorname{div}\left(w_{q+1} \otimes w_{q+1} + \mathring{R}_q\right) \\
&\quad = \sum_{\xi\in\Lambda} \operatorname{div}\left(a_\xi^2(\mathrm{Id} - \xi \otimes \xi) + \mathring{R}_q\right) + \sum_{\xi+\xi'\neq 0,\ \xi,\xi'\in\Lambda} \operatorname{div}\left(a_\xi a_{\xi'} W_\xi \otimes W_{\xi'}\right) \\
&\quad = \nabla r_1 + \underbrace{\sum_{\xi+\xi'\neq 0,\ \xi,\xi'\in\Lambda} \left(W_\xi \otimes W_{\xi'}\right) \nabla\left(a_\xi a_{\xi'}\right)}_{:=I} \\
&\qquad \underbrace{+ \frac{1}{2} \sum_{\xi+\xi'\neq 0,\ \xi,\xi'\in\Lambda} a_\xi a_{\xi'} \operatorname{div}\left(W_\xi \otimes W_{\xi'} - W_{\xi'} \otimes W_\xi\right)}_{:=II}
\end{aligned}
$$

where the pressure r_1 is implicitly defined. Applying the estimate (1.4) with $\Phi(x) = x$ and ξ replaced by $\xi + \xi'$, together with (1.14) and (1.15), we obtain

$$
\begin{aligned}
\|\mathcal{R}(I)\|_{C^0} &\lesssim \frac{1}{\lambda_{q+1}} \sum_{\xi+\xi'\neq 0,\ \xi,\xi'\in\Lambda} \left\|\nabla(a_\xi a_{\xi'})\right\|_{C^0} \\
&\lesssim \frac{1}{\lambda_{q+1}} \sum_{\xi+\xi'\neq 0,\ \xi,\xi'\in\Lambda} \left\|a_\xi\right\|_{C^0} \left\|\nabla a_{\xi'}\right\|_{C^0} \\
&\lesssim \frac{1}{\lambda_{q+1}} \left\|\mathring{R}_q\right\|_{C^1} .
\end{aligned}
$$

Thus, this error can be made small by assuming λ_{q+1} to be sufficiently large. Now consider II. We write

$$
II = \nabla r_2 \underbrace{- \frac{1}{2} \sum_{\xi+\xi'\neq 0,\ \xi,\xi'\in\Lambda} \nabla(a_\xi a_{\xi'}) \left(W_\xi \cdot W_{\xi'}\right)}_{:=III}
$$

with the pressure r_2 again being implicitly defined. Then III can be estimated in the same manner as I. Hence, setting $p_{q+1} = p_q - r_1 - r_2$, we are able to attain suitable bounds on the contribution of the oscillation error to $\mathring{R}_{q+1}$.

An issue with the definition (1.16), is the vector field w_{q+1} is not necessarily divergence free. To fix this, we relabel the right-hand-side of (1.16) to be the *principal perturbation* $w_{q+1}^{(p)}$, i.e.

$$
w_{q+1}^{(p)} := \sum_{\xi\in\Lambda} a_\xi W_{\xi,q+1} . \tag{1.17}
$$

We then define a *corrector* $w_{q+1}^{(c)}$ by

$$w_{q+1}^{(p)} := \frac{1}{\lambda_{q+1}} \sum_{\xi \in \Lambda} \nabla a_\xi \times W_{\xi,q+1} \,. \tag{1.18}$$

Finally, defining

$$w_{q+1} = w_{q+1}^{(p)} + w_{q+1}^{(c)} \,, \tag{1.19}$$

then a simple calculation yields

$$w_{q+1} = \frac{1}{\lambda_{q+1}} \sum_{\xi \in \Lambda} \operatorname{curl} \left(a_\xi W_{\xi,q+1} \right) \,,$$

from which it follows that w_{q+1} is divergence free. Moreover, assuming λ_{q+1} is sufficiently large, the corrector $w_{q+1}^{(c)}$ is small, and hence can be made to have a suitably small contribution to the Reynolds stress $\mathring{R}_{q+1}$ resulting from the perturbation defined in (1.19).

1.4 Transport Error

We now consider the transport error

$$\underbrace{(\partial_t + v_q \cdot \nabla)}_{D_t} w_{q+1} \,,$$

defined in (1.3), where here D_t represents the *material derivative* associated with v_q. Ignoring the contribution of the corrector to the transport error, by definition (1.19), applying (1.4), we heuristically attain

$$\begin{aligned} \left\| \mathcal{R}(D_t w_{q+1}^{(p)}) \right\|_{C^0} \lesssim \frac{1}{\lambda_{q+1}} & \sum_{\xi \in \Lambda} \left\| D_t a_\xi \right\|_{C^0} \left\| W_{\xi,q+1} \right\|_{C^0} \\ & + \frac{1}{\lambda_{q+1}} \sum_{\xi \in \Lambda} \left\| a_\xi \right\|_{C^0} \left\| v_q \cdot \nabla W_{\xi,q+1} \right\|_{C^0} \,. \end{aligned}$$

The second term is unfortunately not small, since $\nabla W_{\xi,q+1} = O(\lambda_{q+1})$. To rectify this issue we will replace $W_{\xi,q+1}$ in the ansatz (1.17) with

$$W_{(\xi)} = W_{\xi,j,q+1} = W_{\xi,q+1} \circ \Phi_j$$

where Φ_j are phase functions solving the transport equation

$$D_t \Phi_j \equiv 0, \qquad \Phi(x, j\ell) = x$$

for some small parameter $\ell > 0$ to be chosen later and $j \in \mathbb{Z}$. With this definition, we have

$$D_t W_{(\xi)} \equiv 0 .$$

The trade-off with using $W_{(\xi)}$ in place of $W_{\xi,q+1}$, is that $W_{(\xi)}$ is no longer an exact eigenfunction of curl . Let us write

$$W_{(\xi)} = B_\xi e^{i\lambda_{q+1}\xi\cdot\Phi_j} = \underbrace{e^{i\lambda_{q+1}\xi\cdot(\Phi_j - x)}}_{:=\Phi_{(\xi)}} W_\xi$$

then

$$\operatorname{curl} W_{(\xi)} = \lambda_{q+1} W_\xi + \nabla\phi_{(\xi)} \times W_{\xi,q+1} . \tag{1.20}$$

Thus in order to quantity how well $W_{(\xi)}$ approximates an eigenfunction of curl , we need to estimate $\nabla\phi_{(\xi)}$. By standard transport estimates we obtain

$$\left\|\nabla\Phi_j - \mathrm{Id}\right\|_{C^0} \leq \exp\left(\int_{j\ell}^t \left\|v_q(s)\right\|_{C^1} ds\right) .$$

In particular, if $|t - j\ell| \leq \left\|v_q\right\|_{C^1}^{-1}$, we have

$$\left\|\nabla\Phi_j - \mathrm{Id}\right\|_{C^0} \lesssim |t - j\ell| \left\|v_q\right\|_{C^1} .$$

From which we deduce

$$\left\|\nabla\phi_{(\xi)}\right\|_{C^0} \lesssim \lambda_{q+1} |t - j\ell| \left\|v_q\right\|_{C^1} . \tag{1.21}$$

Thus $W_{(\xi)}$ is Beltrami like so long that $|t - j\ell|$ is suitably small. To achieve this, we partition time, and replace a_ξ with new coefficient functions $a_{(\xi)}$ with small temporal support. We introduce cut-off functions $\chi_j : (-1, 1) \to \mathbb{R}$ with support contained in the interval $(\ell(j-2), \ell(j+2))$, such that the squares χ_j^2 form a partition of unity, i.e.

$$\sum_j \chi_j^2 \equiv 1 .$$

In place of Λ, we will require two disjoint finite subsets $\Lambda^{(0)}, \Lambda^{(1)} \subset \mathbb{S}^2 \cap \mathbb{Q}^3$. The set $\Lambda^{(0)}$ can be taken to be Λ, and $\Lambda^{(1)}$ can be defined in terms of a rational rotation of Λ. Similarly in place of the family of smooth functions γ_ξ, we will require two families of smooth functions $\{\gamma_\xi^{(0)} \mid \xi \in \lambda^{(0)}\}$ and $\{\gamma_\xi^{(1)} \mid \xi \in \lambda^{(1)}\}$. Again, we refer to Lemma 3.2 in [13] (alternatively Lemma 1.3 in [2]) for the precise definitions of the sets $\Lambda^{(j)}$ and functions $\gamma_\xi^{(j)}$. We then define

$$a_{(\xi)} = \delta_{q+1}^{\frac{1}{2}} \chi_j \gamma_\xi^{(j)} \left(\mathrm{Id} - \frac{\mathring{R}_q(x,t)}{\delta_{q+1}} \right),$$

where by an abuse of notation we write $\gamma_\xi^{(j)} = \gamma_\xi^{(j \bmod 2)}$. With these definitions, we replace the definitions $w_{q+1}^{(p)}$ and $w_{q+1}^{(c)}$ given in (1.17) and (1.18) respectively with the new definitions

$$\begin{aligned} w_{q+1}^{(p)} &:= \sum_j \sum_{\xi \in \Lambda_j} a_{(\xi)} W_{(\xi)} \\ w_{q+1}^{(c)} &:= \frac{1}{\lambda_{q+1}} \sum_j \sum_{\xi \in \Lambda_j} \nabla (a_{(\xi)} \phi_{(\xi)}) W_{(\xi)} \end{aligned} \tag{1.22}$$

where we use the notation $\Lambda_j = \Lambda_{j \bmod 2}$. The functions $\gamma_\xi^{(j)}$ are again defined in such a way that we achieve a cancellation analogous to (1.6). More precisely, in place of (1.6), we have

$$\sum_j \sum_{\xi \in \Lambda_j} a_\xi^2 \fint_{\mathbb{T}^3} W_{\xi,\lambda_{q+1}} \mathring{\otimes} W_{\xi,\lambda_{q+1}} = -\mathring{R}_q .$$

The principal reason for introducing the two families $\Lambda^{(0)}$, $\Lambda^{(1)}$ was to reduce the interactions between the oscillatory Beltrami waves across the neighboring temporal regions where the cut-off functions χ_j overlap.

Due to the small prefactor in the definition of $w_{q+1}^{(c)}$, the term $D_t w_{q+1}^{(p)}$ will be the the main contribution of $D_t w_{q+1}$ to the transport error. Hence, in order to estimate the transport error, we will need bounds on

$$D_t a_{(\xi)} W_{(\xi)} = (D_t a_{(\xi)}) W_{(\xi)} .$$

By definition

$$\left\| D_t \chi_j \right\|_{C^0} \lesssim \ell^{-1} .$$

The material derivative falling on the cut-off is expected to produce the main contribution to the transport error. Conversely, owing to the calculation (1.20), the main contribution to the new oscillation error associated with the new perturbation definition (1.22) occurs when derivatives fall on $\phi_{(\xi)}$. Recalling (1.21), we have

$$\left\|\nabla\phi_{(\xi)}\right\|_{C^0} \lesssim \lambda_{q+1}\,|t-j\ell|\,\left\|v_q\right\|_{C^1}$$

Thus in order to balance the transport and oscillation error, it is necessary to optimize our choice of ℓ. Making the appropriate choice, we can simultaneously obtain effective bounds on the oscillation, transport and Nash errors in (1.3) in order to ensure that (1.10) holds with $q+1$ replacing q.[2]

1.5 Mollification and Loss of Derivative Problem

Recall that $(v_q, \mathring{R}_q)$ are defined inductively. In order to ensure convergence to a solution, one needs to inductively keep track of estimates on $(v_q, \mathring{R}_q)$. As the current scheme is currently defined above, the definition of $\mathring{R}_q$ involves derivatives of $\mathring{R}_{q-1}$ (for example when a derivative falls on $a_{(\xi)}$ in the oscillation error), which in turn involves higher order derivatives on $\mathring{R}_{q-2}$, and so forth. Thus in order for the scheme to close, one would have to keep track of estimates on infinitely many derivatives of $(v_q, \mathring{R}_q)$. To avoid this *loss of derivative problem*, we introduce an addition step where we replace $(v_q, \mathring{R}_q)$ with the mollified $(v_\ell, \mathring{R}_\ell)$ defined by

$$v_\ell = (v_q *_x \psi_\ell)) *_t \varphi_\ell, \qquad \text{and} \qquad \mathring{R}_\ell = (\mathring{R}_q *_x \psi_\ell)) *_t \varphi_\ell$$

where ψ_ℓ and φ_ℓ are standard space and time mollifiers respectively. Then we have

$$\partial v_\ell + \operatorname{div}(v_\ell \otimes v_\ell) + \nabla p_\ell = \operatorname{div}\Big(\mathring{R}_\ell + \underbrace{v_\ell \mathring{\otimes} v_\ell - ((v_q \mathring{\otimes} v_q) *_x \psi_\ell)) *_t}_{R_{\text{commutator}}}\Big)\,.$$

The new error $R_{\text{commutator}}$ can be made small by assuming ℓ to be sufficiently small. With $(v_\ell, \mathring{R}_\ell)$ defined above, in the definition of w_{q+1} described above, we replace all references of v_q and $\mathring{R}_q$ with v_ℓ and $\mathring{R}_\ell$. Then the new velocity field v_{q+1} is defined by

$$v_{q+1} := v_\ell + w_{q+1}\,.$$

With this additional mollification step, we no longer need to keep track of infinitely many derivatives of $(v_q, \mathring{R}_q)$, indeed it will suffice to keep track of C^0 and C^1 estimates on $(v_q, \mathring{R}_q)$.

[2] It should be noted however that in order for the scheme described here to close, one should replace the geometric growth of frequencies λ_q described in (1.7) with superexponential growth. A scheme involving geometric growth of frequencies is slightly more delicate to describe.

1.6 Compact Support in Time

We close out these notes by outlining an argument in order to achieve non-trivial weak solutions with compact support in time.

In order to ensure $v = \lim_q v_q$ has compact support in time, we inductively assume that

$$\operatorname{supp}_t v_q \cup \operatorname{supp}_t \mathring{R}_q \subset \left[-\frac{1}{2} + 2^{-q-2}, \frac{1}{2} - 2^{-q-2}\right]. \tag{1.23}$$

The mollification step, will increase the temporal support, assuming the temporal mollifier φ_ℓ is suitably defined, we have

$$\operatorname{supp}_t v_\ell \cup \operatorname{supp}_t \mathring{R}_\ell \subset \left[-\frac{1}{2} + 2^{-q-2} - \ell, \frac{1}{2} - 2^{-q-2} + \ell\right].$$

Then in order to correct the Reynolds stress R_ℓ, we need only sum j in the definition (1.22), for j satisfying

$$\operatorname{supp}_t \chi_j \subset \left[-\frac{1}{2} + 2^{-q-2} - 4\ell, \frac{1}{2} - 2^{-q-2} + 4\ell\right].$$

Hence choosing ℓ sufficiently small we have

$$\begin{aligned}\operatorname{supp}_t v_{q+1} \cup \operatorname{supp}_t \mathring{R}_{q+1} &\subset \left[-\frac{1}{2} + 2^{-q-2} - 4\ell, \frac{1}{2} - 2^{-q-2} + 4\ell\right] \\ &\subset \left[-\frac{1}{2} + 2^{-q-3}, \frac{1}{2} - 2^{-q-3}\right],\end{aligned}$$

and thus we attain (1.23) with $q + 1$ replacing q. Hence for $v = \lim_q v_q$ we have

$$\operatorname{supp}_t v \subset \left[-\frac{1}{2}, \frac{1}{2}\right].$$

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