*Geophys. J. Int.* (0000) **000**, 000–000

# Elastic symmetry with beachball pictures

Walter Tape[1] and Carl Tape[2]
[1] *Department of Mathematics, University of Alaska, Fairbanks, AK 99775, USA*
[2] *Geophysical Institute and Department of Geosciences, University of Alaska, Fairbanks, AK 99775, USA*
*Corresponding author email: ctape@alaska.edu*

2 May 2021

**SUMMARY**
The elastic map, or generalized Hooke's Law, associates stress with strain in an elastic material. A symmetry of the elastic map is a reorientation of the material that does not change the map. We treat the topic of elastic symmetry conceptually and pictorially. The elastic map is assumed to be linear, and we study it using standard notions from linear algebra—not tensor algebra. We depict strain and stress using the "beachballs" familiar to seismologists. The elastic map, whose inputs and outputs are strains and stresses, is in turn depicted using beachballs. We are able to infer the symmetries for most elastic maps, sometimes just by inspection of their beachball depictions. Many of our results will be familiar, but our versions are simpler and more transparent than their counterparts in the literature.

Key words: Elasticity, seismic anisotropy. theoretical seismology

## 1 Introduction

Elasticity is about the relation between strain and stress. We refer to the function $\mathbf{T}$ from strain to stress as the elastic map. It expresses the "constitutive relations" of the material under consideration, or the "generalized Hooke's Law" (Aki & Richards 2002).

The map $\mathbf{T}$ describes the strain-stress relation at a particular point $\mathbf{p}$ in the material. A symmetry of $\mathbf{T}$ is a rotation of the material, about $\mathbf{p}$, that does not change $\mathbf{T}$. We present a treatment of elastic symmetry that we think is more conceptual than the usual approach through tensor analysis. Our approach has its beginnings in the work of William Thomson (Lord Kelvin) (1856) in the mid-nineteenth century. According to Helbig (1994, 2013) and Cowin et al. (1991), Kelvin's insights were largely forgotten by the elasticity community until much later, when they were reintroduced by Rychlewski (1984). Most of the ideas—especially the notion of eigensystem of a linear transformation—were already routine for mathematicians and theoretical physicists of the early twentieth century, so it is a bit surprising that they were still regarded as novel in elasticity in the 1980s. Rychlewski himself apparently felt much the same:

> Thus we deal with the linear symmetric operator $\boldsymbol{\alpha} \to \mathbf{C} \cdot \boldsymbol{\alpha}$ acting in a finite-dimensional space with a scalar product.... The situation has been investigated as fully as possible, and it only remains to translate the information available into the language of mechanics. (Rychlewski 1984, p 305)

Thus, although our exposition of elasticity is non-traditional vis-a-vis older expositions, it will be unremarkable to mathematicians and physicists. Our exposition does not use the Voigt matrix (Eq. S13 of the Supporting Information), and it requires no knowledge of tensors. What it does rely on is introductory linear algebra, which we review. Specifically, we rely heavily on orthogonality, on matrix representations of the elastic map $\mathbf{T}$, and on eigensystems of $\mathbf{T}$.

Mathematically, strains and stresses are $3 \times 3$ symmetric matrices and can therefore be depicted as "beachballs," as seismologists do for moment tensors. Because the strain-stress relation is assumed to be linear, any elastic map $\mathbf{T}$ can then be depicted using beachballs. The depiction in principle determines $\mathbf{T}$ completely, but of course one cannot just glance at the depiction and expect to infer $\mathbf{T}$ quantitatively.

In Sections 4–9 we characterize elastic maps $\mathbf{T}$ that have as a symmetry the rotation $Z_\xi$ through angle $\xi$ about the $z$-axis. There are five cases to consider: $\xi = \pm 2\pi/n$ for $n = 1, 2, 3, 4$, as well as $\xi$ regular, meaning none of the preceding; see Fig. 1. For each case there is an intrinsic characterization of $\mathbf{T}$ and a more conventional characterization using matrices. Figs. 6, 9, 10, 11 illustrate the intrinsic characterizations, and Table 1 lists the matrix characterizations.

FIG. 1

In Section 14 we give a relatively elementary proof that any material can be oriented so that its group of elastic symmetries is one of eight reference groups. The proof is largely a matter of looking at the intersections of circles on a sphere, as in Figs. 16–18. Matrix characterizations for elastic maps associated with the reference groups are given in Table 4, and intrinsic characterizations are given in Section 12.1. The simplicity of the matrix characterizations relative to their traditional counterparts (e.g., Nye 1957, 1985, pp 140–141) is due to our use of the basis $\mathbb{B}$ defined in Eq. (3).

Nowhere do we assume that elastic symmetry groups arise from crystallographic symmetry groups. We nevertheless find that if an elastic map $\mathbf{T}$ has a symmetry with rotation axis $\mathbf{v}$ and rotation angle $\xi$, where $\xi$ is regular, then all rotations about $\mathbf{v}$, regardless of rotation angle, are symmetries of $\mathbf{T}$. We also find that if $\mathbf{T}$ has a 3-fold or 4-fold symmetry with axis $\mathbf{v}$, then it has three or four (respectively) 2-fold symmetries with axes perpendicular to $\mathbf{v}$. These facts go into deriving the eight reference groups mentioned above.

In Section 15 we show by example how to find the symmetry group of virtually any elastic map $\mathbf{T}$. We say "virtually," because the method can be defeated by a carefully and maliciously constructed $\mathbf{T}$ (Section 15.8). Our method is related to that of Bóna et al. (2007), but we think that our beachball pictures offer a useful complement to the Bona approach.

Many of our results will be familiar, at least to the experts. Fig. 6, for example, which characterizes elastic maps that have symmetry $Z_\xi$ for some regular $\xi$, would have been immediately recognizable to Rychlewski (1984). Likewise, the number eight for the number of elastic symmetry groups is now generally agreed upon (Forte & Vianello 1996; Chadwick et al. 2001).

An idealized seismic plane wave traveling in an arbitrary direction in an anisotropic elastic material is apt to be neither a P-wave nor an S-wave. That is, the wave's vibration direction is neither parallel nor perpendicular to the direction of travel. If, however, the direction of travel is an elastic symmetry axis, then, with some unlikely exceptions, the wave must indeed be either a P-wave or an S-wave (Fedorov 1968). If also the relevant elastic map $\mathbf{T}$ has for its symmetry group one of the reference subgroups of Section 12, then in most cases both the vibration direction and the speed of the wave are simply related to the intrinsic parameters for $\mathbf{T}$. (We do not treat these topics here.)

Treatments of elasticity can be found in Fedorov (1968); Nye (1957, 1985); Auld (1973); Musgrave (1970); Helbig (1994); Chapman (2004); Slawinski (2015) and many others. A reference for linear algebra is Hoffman & Kunze (1971). Our Appendix G is a glossary of notation.

## 2 The elastic map

### 2.1 The elastic map and the $c_{ijkl}$

Expositions of elasticity are generally based on numbers $c_{ijkl}$, $i, j, k, l = 1, 2, 3$, that are assumed to satisfy

$$c_{ijkl} = c_{jikl} \tag{1a}$$

$$c_{ijkl} = c_{ijlk} \tag{1b}$$

$$c_{ijkl} = c_{klij} \tag{1c}$$

The $c_{ijkl}$ determine a linear mapping $\mathbf{T}$ of the six-dimensional space $\mathbb{M}$ of symmetric matrices to itself:

$$\mathbf{T}(E) = F, \qquad f_{ij} = \sum_{k,\,l=1}^{3} c_{ijkl}\, e_{kl}, \tag{2}$$

where $E = (e_{ij})$ and $F = (f_{ij})$ are $3 \times 3$ symmetric matrices (Aki & Richards 2002, Eq. 2.18). If $E$ is the strain matrix at a point in some hypothetical material described by the $c_{ijkl}$, then $F$ is the corresponding stress matrix. We refer to $\mathbf{T}$ as the elastic map.

Eqs. (1a) and (1b) arise from the symmetry of the strain and stress matrices. Eq. (1c) is due to the assumed existence of a strain-energy function (Aki & Richards 2002).

Since the elastic map $\mathbf{T}$ is linear, we consider its matrix representation $[\mathbf{T}]_{\mathbb{BB}}$ with respect to a basis $\mathbb{B}$ for $\mathbb{M}$. The calculations will be simplest, and $[\mathbf{T}]_{\mathbb{BB}}$ will best express $\mathbf{T}$, if the basis vectors are chosen to be orthonormal. The "vectors" must of course be $3 \times 3$ symmetric matrices, since they are in $\mathbb{M}$. We take $\mathbb{B}$ to be the basis whose elements are

$$B_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad B_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$B_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_5 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad B_6 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

The $B_i$ are indeed orthonormal, that is, $B_i \cdot B_j = \delta_{i\,j}$. Here the dot is the inner product of matrices. The inner product of $3 \times 3$ matrices $M = (m_{ij})$ and $N = (n_{ij})$ is defined by

$$M \cdot N = \sum_{i,\,j=1}^{3} m_{ij}\, n_{ij} \tag{4}$$

(Juxtaposition of matrices, with no dot, signifies matrix multiplication.)

We let $t_{ij}$ be the $ij^{\text{th}}$ entry of the $6 \times 6$ matrix $[\mathbf{T}]_{\mathbb{BB}}$. That is,

$$[\mathbf{T}]_{\mathbb{BB}} = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{16} \\ t_{21} & t_{22} & \dots & t_{26} \\ \vdots & \vdots & & \vdots \\ t_{61} & t_{62} & \dots & t_{66} \end{pmatrix} \tag{5}$$

We will find that $[\mathbf{T}]_{\mathbb{BB}}$ is symmetric. and hence the 21 entries $t_{ij}$ with $j \geq i$, in conjunction with the basis $\mathbb{B}$, are enough to determine $\mathbf{T}$ and thus to specify the elasticity of the material under consideration. We think that those 21 numbers are better parameters to focus on than the $c_{ijkl}$. We nevertheless want to be able to translate between the $t_{ij}$ and the $c_{ijkl}$:

As will be explained in Section 2.2, the entries in the $j^{\text{th}}$ column of the matrix $[\mathbf{T}]_{\mathbb{BB}}$ are the coordinates of $\mathbf{T}(B_j)$ with respect to the basis $\mathbb{B}$. That is,

$$\mathbf{T}(B_j) = t_{1j}\, B_1 + \ldots + t_{6\,j}\, B_6 \tag{6}$$

Since the $B_i$ are orthonormal,

$$\mathbf{T}(B_j) = (\mathbf{T}(B_j) \cdot B_1)\, B_1 + \ldots + (\mathbf{T}(B_j) \cdot B_6)\, B_6 \tag{7}$$

Hence

$$t_{ij} = \mathbf{T}(B_j) \cdot B_i \tag{8}$$

As an example, we calculate $t_{14}$. From Eqs. (2) and (3),

$$\mathbf{T}(B_4) = \frac{1}{\sqrt{2}}\begin{pmatrix} c_{1122} - c_{1111} & c_{1222} - c_{1211} & c_{1322} - c_{1311} \\ c_{2122} - c_{2111} & c_{2222} - c_{2211} & c_{2322} - c_{2311} \\ c_{3122} - c_{3111} & c_{3222} - c_{3211} & c_{3322} - c_{3311} \end{pmatrix} \tag{9}$$

Then

$$\begin{aligned} t_{14} &= \mathbf{T}(B_4)\cdot B_1 \\ &= \frac{1}{2}\left(c_{2322} - c_{2311} + c_{3222} - c_{3211}\right) \\ &= c_{2223} - c_{1123} \end{aligned} \tag{10a}$$

Similarly,

$$t_{46} = \frac{1}{\sqrt{6}}\left(-c_{1111} - c_{1133} + c_{2222} + c_{2233}\right) \tag{10b}$$

$$t_{66} = \frac{1}{3}\left(c_{1111} + c_{2222} + c_{3333} + 2c_{1122} + 2c_{1133} + 2c_{2233}\right) \tag{10c}$$

Treating the other 33 entries $t_{ij}$ the same way, we would eventually have $[\mathbf{T}]_{\mathbb{BB}}$. Eq. (S29), however, in the Supporting Information has a less painful calculation of $[\mathbf{T}]_{\mathbb{BB}}$ from the $c_{ijkl}$. The main point at the moment is that $[\mathbf{T}]_{\mathbb{BB}}$ turns out to be symmetric

The peculiar form of Eqs. (10) may at first be regarded as reflecting poorly on the $t_{ij}$. Conceptually, however, the $t_{ij}$ stand on their own. From Eq. (8), the entry $t_{ij}$ tells how much the stress that is associated with strain $B_j$ resembles the strain or stress $B_i$. If anything, then, the form of Eqs. (10) calls for a conceptual justification of the $c_{ijkl}$, not of the $t_{ij}$. In this paper we deal with $\mathbf{T}$ and $[\mathbf{T}]_{\mathbb{BB}}$ rather than with the $c_{ijkl}$. If the $t_{ij}$ are known—through observation or otherwise—then, from a purely logical point of view, the $c_{ijkl}$ can be dispensed with. If desired, the $c_{ijkl}$ can be found from $[\mathbf{T}]_{\mathbb{BB}}$ using Eqs. (1) and (S28).

### 2.2 Matrix representations of linear transformations of $\mathbb{M}$

Let $\mathbb{F}$ be a basis for $\mathbb{M}$ with elements (matrices) $F_1,\ldots,F_6$. For a matrix $E\in\mathbb{M}$ we denote its $\mathbb{F}$-coordinate vector by $[E]_{\mathbb{F}}$. Thus

$$[E]_{\mathbb{F}} = (x_1,\ldots,x_6) \iff E = x_1F_1 + \ldots + x_6F_6 \tag{11}$$

If the basis $\mathbb{F}$ is orthonormal, then

$$x_i = E\cdot F_i \qquad (\mathbb{F}\text{ orthonormal}) \tag{12}$$

Now let $\mathbf{S}$ be a linear transformation of $\mathbb{M}$, and let $\mathbb{F}$ and $\mathbb{G}$ both be bases for $\mathbb{M}$. We define the $6\times 6$ matrix $[\mathbf{S}]_{\mathbb{GF}}$ to be the matrix that takes the $\mathbb{F}$-coordinate vector of $E$ to the $\mathbb{G}$-coordinate vector of $\mathbf{S}(E)$:

$$[\mathbf{S}]_{\mathbb{GF}}[E]_{\mathbb{F}} = [\mathbf{S}(E)]_{\mathbb{G}} \qquad (E\in\mathbb{M}) \tag{13}$$

(Think of coordinate vectors as column vectors when matrix multiplication is concerned.) We refer to $[\mathbf{S}]_{\mathbb{GF}}$ as the matrix of $\mathbf{S}$ with respect to the bases $\mathbb{F}$ and $\mathbb{G}$.

If $\mathbf{S}_1$ and $\mathbf{S}_2$ are linear transformations of $\mathbb{M}$, and if $\mathbb{F}$, $\mathbb{G}$, $\mathbb{H}$ are bases for $\mathbb{M}$, then

$$[\mathbf{S}_2]_{\mathbb{HG}}\,[\mathbf{S}_1]_{\mathbb{GF}} = [\mathbf{S}_2\circ\mathbf{S}_1]_{\mathbb{HF}} \tag{14a}$$

$$[\mathbf{I}]_{\mathbb{FF}} = I_{6\times 6} \tag{14b}$$

where $\mathbf{I}$ is the identity transformation on $\mathbb{M}$, where $I_{6\times 6}$ is the $6\times 6$ identity matrix, and where the symbol $\circ$ denotes composition of functions:

$$(\mathbf{S}_2\circ\mathbf{S}_1)(E) = \mathbf{S}_2\left(\mathbf{S}_1(E)\right) \tag{15}$$

Thus Eq. (14a) says that matrix multiplication is the matrix analog of composition of functions. Eqs. (13) and (14) look innocent enough, but they are the key to many matrix manipulations. Notice how their form suggests the correct move.

To arrive at a more familiar description of $[\mathbf{S}]_{\mathbb{GF}}$: From Eqs. (11) the coordinate vector for $F_j$ with respect to the basis $\mathbb{F}$ is

$$[F_j]_{\mathbb{F}} = \mathbf{e}_j, \tag{16}$$

where $\mathbf{e}_1,\ldots,\mathbf{e}_6$ are the standard basis for $\mathbb{R}^6$. The $j^{\text{th}}$ column of the matrix $[\mathbf{S}]_{\mathbb{GF}}$ is therefore

$$\begin{aligned} [\mathbf{S}]_{\mathbb{GF}}\,\mathbf{e}_j &= [\mathbf{S}]_{\mathbb{GF}}\,[F_j]_{\mathbb{F}} && \text{(from Eq. 16)} \\ &= [\mathbf{S}(F_j)]_{\mathbb{G}} && \text{(from Eq. 13)} \end{aligned} \tag{17}$$

In words, the $j^{\text{th}}$ column of $[\mathbf{S}]_{\mathbb{GF}}$ consists of the coordinates of $\mathbf{S}(F_j)$ with respect to the basis $\mathbb{G}$.

The diagram below summarizes the relation between the linear transformation $\mathbf{S}$ and its matrix representation.

$$\begin{array}{ccc} E\in\mathbb{M} & \xrightarrow{\ \mathbf{S}\ } & \mathbf{S}(E)\in\mathbb{M} \\ \updownarrow & & \updownarrow \\ [E]_{\mathbb{F}}\in\mathbb{R}^6 & \xrightarrow{[\mathbf{S}]_{\mathbb{GF}}} & [\mathbf{S}(E)]_{\mathbb{G}}\in\mathbb{R}^6 \end{array} \tag{18}$$

Finally, from Eq. (13) with $\mathbb{F}=\mathbb{G}$,

$$\mathbf{S}(E) = \lambda\,E \iff [\mathbf{S}]_{\mathbb{FF}}[E]_{\mathbb{F}} = \lambda\,[E]_{\mathbb{F}} \tag{19}$$

That is, the $3\times 3$ symmetric matrix $E$ is an eigenvector of the transformation $\mathbf{S}$ if and only if the coordinate vector $[E]_{\mathbb{F}}$ is an eigenvector of the matrix $[\mathbf{S}]_{\mathbb{FF}}$. The eigenvalues are the same for both.

#### *2.2.1 In terms of the elastic map* $\mathbf{T}$

We now take $\mathbf{S}$ in Section 2.2 to be the elastic map $\mathbf{T}$, and we take both of the bases $\mathbb{F}$ and $\mathbb{G}$ to be the basis $\mathbb{B}$ of Eq. (3). From Eqs. (3), (11), (12), the coordinate vector of the matrix $E=(e_{ij})\in\mathbb{M}$ with respect to the basis $\mathbb{B}$ is

$$[E]_{\mathbb{B}} = \Big(\sqrt{2}\,e_{23},\ \sqrt{2}\,e_{13},\ \sqrt{2}\,e_{12},\ \frac{e_{22}-e_{11}}{\sqrt{2}},\ \frac{e_{11}+e_{22}-2e_{33}}{\sqrt{6}},\ \frac{e_{11}+e_{22}+e_{33}}{\sqrt{3}}\Big) \tag{20a}$$

The matrix whose $\mathbb{B}$-coordinate vector is $(x_1,\ldots,x_6)$ is

$$x_1B_1 + \ldots + x_6B_6 = \frac{1}{\sqrt{6}}\begin{pmatrix} x_5-\sqrt{3}\,x_4+\sqrt{2}\,x_6 & \sqrt{3}\,x_3 & \sqrt{3}\,x_2 \\ \sqrt{3}\,x_3 & x_5+\sqrt{3}\,x_4+\sqrt{2}\,x_6 & \sqrt{3}\,x_1 \\ \sqrt{3}\,x_2 & \sqrt{3}\,x_1 & \sqrt{2}\,x_6-2x_5 \end{pmatrix} \tag{20b}$$

An important property of the $\mathbb{B}$-coordinate mapping $E\to[E]_{\mathbb{B}}$ is that it preserves the inner product, since the basis $\mathbb{B}$ is orthonormal. Thus

$$[E_1]_{\mathbb{B}}\cdot[E_2]_{\mathbb{B}} = E_1\cdot E_2 \qquad (E_1,\,E_2\in\mathbb{M}) \tag{21}$$

where the dots on the left and right sides of the equation refer to the inner products in $\mathbb{R}^6$ and $\mathbb{M}$, respectively. The inner product of $3\times 3$ matrices was defined in Eq. (4).

#### 2.2.2 *Example: calculation of the* $3 \times 3$ *matrix* $\mathbf{T}(E)$

Let $E$ be the strain matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{22}$$

and let $\mathbf{T}$ be the elastic map with

$$[\mathbf{T}]_{\mathbb{BB}} = \frac{1}{5} \begin{pmatrix} 5 & & & -2 & & \\ & 6 & & & -2 & \\ & & 5 & & & \\ -2 & & & 5 & & \\ & -2 & & & 6 & \\ & & & & & 6 \end{pmatrix} \tag{23}$$

(Blank entries are understood to be zeros.) Then the stress matrix $\mathbf{T}(E)$ is calculated in the following steps, using Eqs. (20):

$$[E]_{\mathbb{B}} = (0,\, 0\,, 0,\, 0,\, -\sqrt{6},\, 2\sqrt{3}) \tag{24a}$$

$$[\mathbf{T}(E)]_{\mathbb{B}} = [\mathbf{T}]_{\mathbb{BB}}[E]_{\mathbb{B}} = \frac{1}{5}\,(0,\, 2\sqrt{6},\, 0,\, 0,\, -6\sqrt{6},\, 12\sqrt{3}) \tag{24b}$$

$$\mathbf{T}(E) = \frac{1}{5} \begin{pmatrix} 6 & 0 & 2\sqrt{3} \\ 0 & 6 & 0 \\ 2\sqrt{3} & 0 & 24 \end{pmatrix} \tag{24c}$$

### 2.3 A picture for the elastic map

Since the elastic map $\mathbf{T} : \mathbb{M} \to \mathbb{M}$ is linear, it is determined by its values on any basis for $\mathbb{M}$. To depict $\mathbf{T}$, it is therefore enough to depict some basis elements $F_1, \ldots, F_6$ together with the corresponding $\mathbf{T}(F_1), \ldots, \mathbf{T}(F_6)$. This is done by means of "beachballs," as explained in Section 2.4. In Fig. 2 the basis is $\mathbb{B}$ of Eq. (3) and $\mathbf{T}$ is the elastic map whose
G. 2 matrix with respect to $\mathbb{B}$ is that of Eq. (23).

### 2.4 Beachballs—a picture for strain and stress

Since the members of $\mathbb{M}$ are $3 \times 3$ symmetric matrices, they can be depicted as beachballs as is done in seismology. The radius of the beachball for $E \in \mathbb{M}$ is made proportional to $\|E\|$ and, for any point $\mathbf{v} = (x, y, z)$ on the surface of the ball,

$$\mathbf{v} \text{ is colored } \begin{cases} \text{red} & \text{if} \quad (E\mathbf{v}) \cdot \mathbf{v} > 0 \\ \text{white} & \text{if} \quad (E\mathbf{v}) \cdot \mathbf{v} < 0 \end{cases} \tag{25}$$

The nodal curves on the ball, which separate red from white, are

$$(E\mathbf{v}) \cdot \mathbf{v} = 0 \tag{26}$$

G. 3 See Fig. 3.

The beachball is thus a contour map of the function $\mathbf{v} \to (E\mathbf{v}) \cdot \mathbf{v}$, but with only one contour, namely the zero contour.

If the eigenvalues of the matrix $E$ are of mixed sign, then the beachball for $E$ shows both red and white, and the size and coloring of the ball determine $E$. If they are all of one sign, though, the ball is all red or all white, and it does not reveal $E$. In our figures, when we show a beachball for a matrix $E$ whose eigenvalues all have the same sign (but not all equal), we therefore show not the beachball for $E$ itself but for the perturbed matrix

$$\|E\| \frac{E + \epsilon I}{\|E + \epsilon I\|}, \tag{27}$$

where $I$ is the $3 \times 3$ identity matrix and where the number $\epsilon$, positive or negative and not necessarily small, is such as to nudge the resulting beachball into the bicolored regime. The ball then is not strictly correct, but it gives a suggestion of the matrix $E$. In Fig. 19 the ball for $G_6$ has been perturbed in this way; instead of being solid red, it has two small white caps. The ball for $G_6$ in Fig. 21 has likewise been perturbed, giving it the narrow white band. (The solid red balls for $B_6$ and $\mathbf{T}(B_6)$ in Fig. 2 are correct, since those matrices are multiples of the identity.) We could have employed a more sophisticated coloring scheme that would have made the perturbations unnecessary, but the existing binary scheme seems enough for what we are trying to show. The perturbations are only for display purposes; all calculations are done with the unperturbed matrices.

### 2.5 Matrix of S with respect to an arbitrary orthonormal basis

Continuing from Section 2.2, we now assume that the basis $\mathbb{G}$ for $\mathbb{M}$ is orthonormal. (An example of $\mathbb{G}$ would be the basis $\mathbb{B}$ of Eq. 3.) Denoting the elements (i.e., matrices) of the basis $\mathbb{F}$ by $F_1, F_2, \ldots, F_6$ and those of $\mathbb{G}$ by $G_1, G_2, \ldots, G_6$, we have, for any matrix $E \in \mathbb{M}$,

$$E = (E \cdot G_1)\, G_1 + \ldots + (E \cdot G_6)\, G_6, \tag{28}$$

The $\mathbb{G}$-coordinate vector for $E$ is therefore

$$[E]_G = (E \cdot G_1, \ldots, E \cdot G_6) \tag{29}$$

From Eq. (17) the $j^{\text{th}}$ column of the matrix $[\mathbf{S}]_{\mathbb{GF}}$ consists of the coordinates of $\mathbf{S}(F_j)$ with respect to the basis $\mathbb{G}$. From Eq. (29), the $j^{\text{th}}$ column is therefore

$$[\mathbf{S}]_{\mathbb{GF}}\, \mathbf{e}_j = \left(\mathbf{S}(F_j) \cdot G_1, \ldots, \mathbf{S}(F_j) \cdot G_6\right), \tag{30}$$

with the 6-tuples thought of as column vectors. Hence the $ij^{\text{th}}$ entry of $[\mathbf{S}]_{\mathbb{GF}}$ is

$$\left([\mathbf{S}]_{\mathbb{GF}}\right)_{i\,j} = \big(\mathbf{S}(F_j) \cdot G_i\big) \qquad (\mathbb{G} \text{ orthonormal}) \tag{31a}$$

Explicitly,

$$[\mathbf{S}]_{\mathbb{GF}} = \begin{pmatrix} \mathbf{S}(F_1) \cdot G_1 & \ldots & \mathbf{S}(F_6) \cdot G_1 \\ \vdots & & \vdots \\ \mathbf{S}(F_1) \cdot G_6 & \ldots & \mathbf{S}(F_6) \cdot G_6 \end{pmatrix} \tag{31b}$$

From Eq. (13) the matrix $[\mathbf{I}]_{\mathbb{GF}}$ takes $\mathbb{F}$-coordinates to $\mathbb{G}$-coordinates. From Eq. (31) with $\mathbf{S} = \mathbf{I}$,

$$[\mathbf{I}]_{\mathbb{GF}} = \begin{pmatrix} F_1 \cdot G_1 & \ldots & F_6 \cdot G_1 \\ \vdots & & \vdots \\ F_1 \cdot G_6 & \ldots & F_6 \cdot G_6 \end{pmatrix} \qquad (\mathbb{G} \text{ orthonormal}) \tag{32}$$

The $j^{\text{th}}$ column of $[\mathbf{I}]_{\mathbb{GF}}$ is thus the $\mathbb{G}$-coordinate 6-tuple of $F_j$.

### 2.6 Two special types of transformation

We consider a linear transformation $\mathbf{S} : \mathbb{V} \to \mathbb{V}$. Although $\mathbb{V}$ can be any finite dimensional (real) inner product space, the only relevant instances here are $\mathbb{V} = \mathbb{R}^3$ and $\mathbb{V} = \mathbb{R}^6$ with the standard inner product, and $\mathbb{V} = \mathbb{M}$ with the inner product defined in Eq. (4).

The adjoint of $\mathbf{S}$ is the linear transformation $\mathbf{S}^* : \mathbb{V} \to \mathbb{V}$ such that, for all $E_1, E_2 \in \mathbb{V}$,

$$\mathbf{S}^*(E_1) \cdot E_2 = E_1 \cdot \mathbf{S}(E_2) \qquad \text{(definition of } \mathbf{S}^*\text{)} \tag{33}$$

From Eq. (31) it follows that for any orthonormal basis $\mathbb{G}$ of $\mathbb{V}$,

$$[\mathbf{S}^*]_{\mathbb{G}\mathbb{G}} = [\mathbf{S}]_{\mathbb{G}\mathbb{G}}^\top \qquad (\mathbb{G} \text{ orthonormal}), \tag{34}$$

where $T^\top = (t_{ji})$ is the transpose of the matrix $T = (t_{ij})$.

#### *2.6.1 Unitary transformations*

A linear transformation $\mathbf{U} : \mathbb{V} \to \mathbb{V}$ is said to be unitary if

$$\mathbf{U} \circ \mathbf{U}^* = \mathbf{I} \qquad \text{(definition of unitary)} \tag{35}$$

The unitary transformations are those that preserve inner products, hence distances and angles. From Eqs. (14) and (34),

$$\mathbf{U} \text{ is unitary} \iff [\mathbf{U}]_{\mathbb{G}\mathbb{G}}\,[\mathbf{U}]_{\mathbb{G}\mathbb{G}}^\top = I \qquad (\mathbb{G} \text{ orthonormal}) \tag{36}$$

For a square matrix to be orthogonal means that its transpose is its inverse. Hence Eq. (36) says that $\mathbf{U}$ is a unitary transformation if and only if $[\mathbf{U}]_{\mathbb{G}\mathbb{G}}$ is an orthogonal matrix.

From Eq. (36), $\det [\mathbf{U}]_{\mathbb{G}\mathbb{G}} = \pm 1$ if $\mathbf{U}$ is unitary. We define $\mathbf{U}$ to be a rotation of $\mathbb{V}$ if $\mathbf{U}$ is unitary and $\det [\mathbf{U}]_{\mathbb{F}\mathbb{F}} = +1$. Here $\mathbb{F}$ can be any basis for $\mathbb{V}$, since changing $\mathbb{F}$ does not change the determinant.

#### *2.6.2 Self-adjoint transformations*

A linear transformation $\mathbf{S} : \mathbb{V} \to \mathbb{V}$ is said to be self-adjoint if

$$\mathbf{S}^* = \mathbf{S} \qquad \text{(definition of self-adjoint)} \tag{37}$$

From Eq. (34),

$$\mathbf{S} \text{ is self-adjoint} \iff [\mathbf{S}]_{\mathbb{G}\mathbb{G}} \text{ is symmetric} \qquad (\mathbb{G} \text{ orthonormal}) \tag{38}$$

Since the matrix $[\mathbf{T}]_{\mathbb{B}\mathbb{B}}$ is symmetric and the basis $\mathbb{B}$ is orthonormal, then the elastic map $\mathbf{T}$ is self-adjoint:

$$\mathbf{T}^* = \mathbf{T} \tag{39}$$

#### *2.6.3 Orthogonality terminology*

Some terminology regarding orthogonality:

Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{V}$ are orthogonal if $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

Subspaces $\mathbb{W}_1$ and $\mathbb{W}_2$ of $\mathbb{V}$ are orthogonal, written $\mathbb{W}_1 \perp \mathbb{W}_2$, if every vector in one subspace is orthogonal to every vector in the other:

$$\mathbb{W}_1 \perp \mathbb{W}_2 \iff \big(\mathbf{v}_1 \in \mathbb{W}_1 \text{ and } \mathbf{v}_2 \in \mathbb{W}_2 \implies \mathbf{v}_1 \cdot \mathbf{v}_2 = 0\big) \tag{40}$$

The orthogonal complement of a subspace $\mathbb{W}$ is

$$\mathbb{W}^\perp = \{\mathbf{v} \in \mathbb{V} \,:\, \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in \mathbb{W}\} \tag{41}$$

The subspace $\langle \mathbb{W}_1, \ldots, \mathbb{W}_n \rangle$ spanned by $\mathbb{W}_1, \ldots, \mathbb{W}_n$ consists of all the linear combinations of vectors from $\mathbb{W}_1, \ldots \mathbb{W}_n$.

A subspace $\mathbb{W}$ is the orthogonal direct sum of subspaces $\mathbb{W}_1, \ldots, \mathbb{W}_n$, written $\mathbb{W} = \mathbb{W}_1 \perp \ldots \perp \mathbb{W}_n$, if $\mathbb{W}$ is the span of $\mathbb{W}_1, \ldots, \mathbb{W}_n$ and if $\mathbb{W}_1, \ldots, \mathbb{W}_n$ are pairwise orthogonal:

$$\begin{aligned} &\mathbb{W} = \mathbb{W}_1 \perp \ldots \perp \mathbb{W}_n \\ &\quad \iff \big(\mathbb{W} = \langle \mathbb{W}_1, \ldots \mathbb{W}_n \rangle \text{ and } \mathbb{W}_i \perp \mathbb{W}_j,\, i \neq j\big) \end{aligned} \tag{42}$$

(The notation $\mathbb{W}_1 \perp \mathbb{W}_2$ is therefore ambiguous, with meanings from both Eqs. (40) and (42). We rely on context to distinguish them.)

A subspace $\mathbb{W}$ of $\mathbb{V}$ is said to be invariant under $\mathbf{U} : \mathbb{V} \to \mathbb{V}$ if $\mathbf{U}(\mathbb{W}) \subset \mathbb{W}$. A non-zero subspace $\mathbb{W}$ is prime for $\mathbf{U}$ if $\mathbb{W}$ is invariant (under $\mathbf{U}$) and has no proper subspaces that are invariant. (The improper subspaces of $\mathbb{W}$ are $\{\mathbf{0}\}$ and $\mathbb{W}$ itself.)

From Lemma 1 below, if $\mathbf{U}$ is unitary then the whole space $\mathbb{V}$ is the orthogonal direct sum of subspaces $\mathbb{W}_1, \ldots, \mathbb{W}_n$ that are prime for $\mathbf{U}$. In that case, $\mathbb{W}_1, \ldots, \mathbb{W}_n$ are said to be prime summands (of $\mathbb{V}$, for $\mathbf{U}$):

$$\begin{aligned} &\mathbb{W}_1, \ldots, \mathbb{W}_n \text{ are prime summands for } \mathbf{U} \\ &\iff \begin{cases} \mathbb{V} = \mathbb{W}_1 \perp \ldots \perp \mathbb{W}_n \\ \mathbb{W}_1, \ldots, \mathbb{W}_n \text{ are prime for } \mathbf{U} \end{cases} \end{aligned} \tag{43}$$

If, for example, $\mathbf{U}$ is rotation through $30^\circ$ about the $z$-axis in $\mathbb{R}^3$, then the prime subspaces would be the $z$-axis and the $xy$-plane. Those two subspaces would also be prime summands. If, however, the rotation is through $180^\circ$ then the $z$-axis and every horizontal line through the origin would be prime subspaces. The three coordinate axes would be prime summands. So also would be the $z$-axis together with the lines $x = y$ and $x = -y$ in the $xy$-plane.

Finally, when $\mathbb{V}$ is the orthogonal direct sum of non-zero subspaces $\mathbb{W}_1, \ldots, \mathbb{W}_n$, we write $\mathbf{T} = \underset{\lambda_1}{\mathbb{W}_1} \perp \ldots \perp \underset{\lambda_n}{\mathbb{W}_n}$ to mean that the linear transformation $\mathbf{T} : \mathbb{V} \to \mathbb{V}$ is multiplication by $\lambda_i$ on $\mathbb{W}_i$:

$$\mathbf{T} = \underset{\lambda_1}{\mathbb{W}_1} \perp \ldots \perp \underset{\lambda_n}{\mathbb{W}_n} \iff \begin{cases} \mathbb{V} = \mathbb{W}_1 \perp \ldots \perp \mathbb{W}_n \\ \mathbf{T}(\mathbf{v}) = \lambda_i \mathbf{v} \qquad (\mathbf{v} \in \mathbb{W}_i) \end{cases} \tag{44}$$

The numbers $\lambda_1, \ldots, \lambda_n$ are then the eigenvalues of $\mathbf{T}$, not necessarily distinct. The eigenspace of $\mathbf{T}$ with eigenvalue $\lambda$ (Section 3.4) is the orthogonal direct sum of the $\mathbb{W}_i$ having $\lambda = \lambda_i$

#### *2.6.4 Orthogonality facts*

From Eqs. (41) and (42), for any subspace $\mathbb{W}$ of $\mathbb{V}$,

$$\mathbb{V} = \mathbb{W} \perp \mathbb{W}^\perp \tag{45}$$

If $\mathbb{W}$ is invariant under a unitary transformation $\mathbf{U} : \mathbb{V} \to \mathbb{V}$, then so is $\mathbb{W}^\perp$:

$$\mathbf{U}(\mathbb{W}) \subset \mathbb{W} \implies \mathbf{U}(\mathbb{W}^\perp) \subset \mathbb{W}^\perp \tag{46}$$

**Lemma 1.** Let $\mathbf{U} : \mathbb{V} \to \mathbb{V}$ be unitary and let $\mathbb{W}$ be a non-zero subspace of $\mathbb{V}$ that is invariant under $\mathbf{U}$. Then $\mathbb{W}$ is the orthogonal direct sum of subspaces of $\mathbb{V}$ that are prime for $\mathbf{U}$.

The lemma is proved in Appendix A.

If $\mathbf{T} = \underset{\lambda_1}{\mathbb{W}_1} \perp \ldots \perp \underset{\lambda_n}{\mathbb{W}_n}$ and if $\mathbf{U} : \mathbb{V} \to \mathbb{V}$ is unitary, then from Eq. (44),

$$\mathbf{U} \circ \mathbf{T} \circ \mathbf{U}^* = \underset{\lambda_1}{\mathbf{U}(\mathbb{W}_1)} \perp \ldots \perp \underset{\lambda_n}{\mathbf{U}(\mathbb{W}_n)} \tag{47}$$

### 2.7 The Spectral Theorem applied to T

The Spectral Theorem (Hoffman & Kunze 1971, p 314) states that for each self-adjoint transformation $\mathbf{S} : \mathbb{V} \to \mathbb{V}$ there is an orthonormal eigenbasis—a basis for $\mathbb{V}$ consisting of orthonormal eigenvectors of $\mathbf{S}$.

Since the elastic map $\mathbf{T}$ is self-adjoint, then, according to the Spectral Theorem, there must be an orthonormal basis for $\mathbb{M}$ consisting of six eigenvectors of $\mathbf{T}$. Since $\mathbf{T} : \mathbb{M} \to \mathbb{M}$, an "eigenvector" is now an element of $\mathbb{M}$—a symmetric $3 \times 3$ matrix.

In terms of strain and stress: For any elastic map $\mathbf{T}$, there will be six independent $3 \times 3$ strain matrices $G_i$ such that each of the corresponding stress matrices $\mathbf{T}(G_i)$ is a scalar multiple of its strain matrix.

Fig. 4 depicts $\mathbf{T}$ as did Fig. 2 but with the basis $\mathbb{B}$
G. 4 replaced by an eigenbasis for $\mathbf{T}$.

#### 2.7.1 *Invertibility of* T

An elastic map $\mathbf{T}$ is invertible if and only if its eigenvalues are all non-zero. In that case the eigenvectors of $\mathbf{T}^{-1}$ are the same as those of $\mathbf{T}$, and the eigenvalues of $\mathbf{T}^{-1}$ are the reciprocals of those of $\mathbf{T}$. For $\mathbf{T}$ as in Fig. 4, the beachball depiction of $\mathbf{T}^{-1}$ would appear just as in the figure except that the radius of each ball $\mathbf{T}(G_i)$ on the top row would change from $\lambda_i$ to $1/\lambda_i$.

#### 2.7.2 *Matrix version of the Spectral Theorem*

The Spectral Theorem implies that an $n \times n$ symmetric matrix $S$ can be written

$$S = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^\top \tag{48}$$

for some numbers $\lambda_1, \ldots, \lambda_n$ and for some $n \times n$ rotation matrix $U$. The $j^{\text{th}}$ column of $U$ is then an eigenvector of $S$ with eigenvalue $\lambda_j$.

### 2.8 Constructing T with a prescribed eigensystem

Given numbers $\lambda_1, \ldots, \lambda_6$ and an orthonormal basis $\mathbb{G} = \{G_1, \ldots, G_6\}$ of $\mathbb{M}$, we can construct an elastic map $\mathbf{T}$ that has $\lambda_1, \ldots, \lambda_6$ as its eigenvalues and has $G_1, \ldots, G_6$ as the corresponding eigenvectors. The matrix $[\mathbf{T}]_{\mathbb{GG}}$ of $\mathbf{T}$ with respect to $\mathbb{G}$ must be diagonal with diagonal entries $\lambda_1, \ldots, \lambda_6$. The matrix with respect to $\mathbb{B}$ is then, from Eq. (14a),

$$\begin{aligned} [\mathbf{T}]_{\mathbb{BB}} &= = [\mathbf{I}]_{\mathbb{BG}}\, [\mathbf{T}]_{\mathbb{GG}}\, [\mathbf{I}]_{\mathbb{GB}} \\ &= [\mathbf{I}]_{\mathbb{BG}} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_6 \end{pmatrix} [\mathbf{I}]_{\mathbb{BG}}^T, \end{aligned} \tag{49}$$

where $[\mathbf{I}]_{\mathbb{BG}}$ is calculated from Eq. (32).

## 3 Symmetries of T

### 3.1 Conjugation by a rotation matrix

Recall that a square matrix $U$ is orthogonal if $UU^\top = I$. If also $\det U = 1$ then $U$ is said to be a rotation matrix. We let $\mathbb{U}$ be the group of all $3 \times 3$ rotation matrices. Examples of matrices in $\mathbb{U}$ would be the $3 \times 3$ rotations $X_\xi$, $Y_\xi$, $Z_\xi$ through angle $\xi$ about the $x$, $y$, $z$ axes, respectively:

$$\begin{aligned} X_\xi &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\xi & -\sin\xi \\ 0 & \sin\xi & \cos\xi \end{pmatrix} \\ Y_\xi &= \begin{pmatrix} \cos\xi & 0 & \sin\xi \\ 0 & 1 & 0 \\ -\sin\xi & 0 & \cos\xi \end{pmatrix} \\ Z_\xi &= \begin{pmatrix} \cos\xi & -\sin\xi & 0 \\ \sin\xi & \cos\xi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{50}$$

For $U \in \mathbb{U}$, we define a linear transformation $\overline{U} : \mathbb{M} \to \mathbb{M}$ by

$$\overline{U}(E) = U E\, U^\top \qquad (U \in \mathbb{U},\, E \in \mathbb{M}) \tag{51}$$

In words, $\overline{U}$ is conjugation by $U$.

From Eq. (51),

$$\overline{U_1} \circ \overline{U_2} = \overline{U_1 U_2} \tag{52}$$

Since

$$\begin{aligned} \overline{U}(E_1) \cdot E_2 &= (U E_1\, U^\top) \cdot E_2 \\ &= E_1 \cdot (U^\top E_2\, U) = E_1 \cdot \overline{U^\top}(E_2), \end{aligned}$$

then by comparison with Eq. (33),

$$\overline{U}^* = \overline{U^\top} \tag{53}$$

That is, $\overline{U}^*$ is conjugation by $U^\top$. Then $\overline{U}$ is unitary (Eq. 35), since

$$\overline{U} \circ \overline{U}^* = \overline{U} \circ \overline{U^\top} = \overline{UU^\top} = \overline{I} = \mathbf{I} \tag{54}$$

#### 3.1.1 *The beachball for* $\overline{U}(E)$

For $E \in \mathbb{M}$ the beachball for $\overline{U}(E)$ is the result of applying the rotation $U$ to the beachball for $E$. To see this, let $E' = \overline{U}(E)$ and $\mathbf{v}' = U\mathbf{v}$, with $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$. Then

$$(E'\mathbf{v}') \cdot \mathbf{v}' = (E\mathbf{v}) \cdot \mathbf{v} \tag{55}$$

Thus, from Eq. (25), the rotated point $\mathbf{v}'$ is red on the ball for $E'$ if and only if the original point $\mathbf{v}$ is red on the ball for $E$. See Fig. 5.

The rotation $U$ of $\mathbb{R}^3$ operates on the material and operates on the beachballs. The rotation $\overline{U}$ of $\mathbb{M}$ operates on
$3 \times 3$ symmetric matrices (strains and stresses). FIG. 5

#### 3.1.2 *The matrix of* $\overline{U}$

Eqs. (14) and (54) imply $[\overline{U}]_{\mathbb{FF}}\, [\overline{U}^*]_{\mathbb{FF}} = I_{6\times6}$, whether or not the basis $\mathbb{F}$ is orthonormal. Hence

$$[\overline{U}^*]_{\mathbb{FF}} = [\overline{U}]_{\mathbb{FF}}^{-1} \tag{56}$$

If $\mathbb{G}$ is an orthonormal basis for $\mathbb{M}$, then from Eq. (36)

the matrix $[\overline{U}]_{\mathbb{GG}}$ is orthogonal, that is, $[\overline{U}]_{\mathbb{GG}}\,[\overline{U}]_{\mathbb{GG}}^\top = I$. Equivalently,

$$[\overline{U}]_{\mathbb{GG}}^{-1} = [\overline{U}]_{\mathbb{GG}}^\top \qquad (\mathbb{G}\text{ orthonormal}) \tag{57}$$

From Eqs. (53), (56), (57),

$$[\overline{U}^*]_{\mathbb{GG}} = [\overline{U^\top}]_{\mathbb{GG}} = [\overline{U}]_{\mathbb{GG}}^\top \qquad (\mathbb{G}\text{ orthonormal}) \tag{58}$$

The matrix $[\overline{U}]_{\mathbb{GG}}$ is found from Eq. (31); its $ij^{\text{th}}$ entry is

$$\left([\overline{U}]_{\mathbb{GG}}\right)_{ij} = \left(UG_jU^\top\right)\cdot G_i \qquad (\mathbb{G}\text{ orthonormal}) \tag{59a}$$

More explicitly,

$$[\overline{U}]_{\mathbb{GG}} = \begin{pmatrix} (UG_1U^\top)\cdot G_1 & \dots & (UG_6U^\top)\cdot G_1 \\ \vdots & & \vdots \\ (UG_1U^\top)\cdot G_6 & \dots & (UG_6U^\top)\cdot G_6 \end{pmatrix} \tag{59b}$$

If, for example, we take $U = Z_\xi$ (Eq. 50) and $\mathbb{G} = \mathbb{B}$ (Eq. 3), then

$$[\overline{Z_\xi}]_{\mathbb{BB}} = \begin{pmatrix} \cos\xi & \sin\xi & & & & \\ -\sin\xi & \cos\xi & & & & \\ & & \cos 2\xi & -\sin 2\xi & & \\ & & \sin 2\xi & \cos 2\xi & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \tag{60}$$

with blank entries understood to be zero.

An arbitrary $U \in \mathbb{U}$ has the form $U = VZ_\xi V^\top$ for some $V \in \mathbb{U}$ and $\xi \in \mathbb{R}$. Then $\overline{U} = \overline{V}\circ\overline{Z_\xi}\circ\overline{V^\top}$ and

$$\begin{aligned}\det[\overline{U}]_{\mathbb{BB}} &= \det[\overline{V}]_{\mathbb{BB}}\,\det[\overline{Z_\xi}]_{\mathbb{BB}}\,\det[\overline{V^\top}]_{\mathbb{BB}} \\ &= \det[\overline{Z_\xi}]_{\mathbb{BB}} = 1\end{aligned} \tag{61}$$

Since $\overline{U}$ is unitary, then it is a rotation (of $\mathbb{M}$). Most rotations $\mathbf{U}$ of $\mathbb{M}$, however, do not have the form $\mathbf{U} = \overline{U}$, as can be seen from Section 3.1.3.

#### *3.1.3 Retrieving $U$ from $\overline{U}$*

Given a rotation $\mathbf{U}$ of $\mathbb{M}$, we ask whether $\mathbf{U} = \overline{U}$ for some $U \in \mathbb{U}$.

Let $E$ be the diagonal matrix with diagonal entries $3, 2, 1$. Let $V \in \mathbb{U}$ be a TBP eigenframe for $\mathbf{U}(E)$, and let $\mu_1, \mu_2, \mu_3$ be the eigenvalues of $\mathbf{U}(E)$ in descending order. (See Section 5 for TBP.) If $\mathbf{U} = \overline{U}$ then

$$\mathbf{U}(E) = \overline{U}(E)$$

$$V\begin{pmatrix}\mu_1 & 0 & 0\\ 0 & \mu_2 & 0\\ 0 & 0 & \mu_3\end{pmatrix}V^\top = U\begin{pmatrix}3 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}U^\top \tag{62}$$

Then

$$(\mu_1, \mu_2, \mu_3) = (3, 2, 1) \tag{63a}$$

$$U = VR \text{ for one of } R = I, X_\pi, Y_\pi, Z_\pi \tag{63b}$$

Eq. (63b) is from Proposition 5 of Tape & Tape (2012). It says that the matrices $U$ and $V$ differ at most by sign changes of two columns.

Thus $\mathbf{U}$ cannot have the form $\mathbf{U} = \overline{U}$ unless the eigenvalues of $\mathbf{U}(E)$ are $3, 2, 1$. And when the eigenvalues are in fact $3, 2, 1$ there are only four candidates for $U$; we need only check to see whether $\mathbf{U} = \overline{VR}$ for $R = I, X_\pi, Y_\pi, Z_\pi$.

Helbig (1994) and Mehrabadi & Cowin (1990) have more complicated approaches to this problem.

### 3.2 How T changes when the material is rotated

Let's be clear that our entire enterprise deals only with a specific point in some material; we are not interested in how the elasticity is changing from one point to another. When we speak of rotations, the rotations should be thought of, intuitively, as rotations about the specified point. Strains and stresses are likewise strains and stresses at the specified point. We imagine the point to be at the origin in $\mathbb{R}^3$.

Suppose now that we use a rotation $U \in \mathbb{U}$ to rotate our material. We want to compare the elastic maps $\mathbf{T}$ and $\mathbf{T}'$ before and after the rotation. Suppose the strains before and after the rotations are $E$ and $E'$. Both of the matrices $E$ and $E'$ operate on vectors in $\mathbb{R}^3$. The output vector assigned to the input vector $\mathbf{v}$ by $E'$ is

$$E'\mathbf{v} = UEU^\top\mathbf{v} \qquad (\mathbf{v}\in\mathbb{R}^3) \tag{64}$$

Since Eq. (64) holds for all $\mathbf{v}$, then, with the analogous fact for stresses included,

$$\begin{aligned}E' &= UEU^\top = \overline{U}(E)\\ F' &= UFU^\top = \overline{U}(F)\end{aligned} \tag{65}$$

The maps $\mathbf{T}$ and $\mathbf{T}'$ take strain matrices to stress matrices: $\mathbf{T}(E) = F$ and $\mathbf{T}'(E') = F'$. Thus,

$$\begin{aligned}\mathbf{T}'(E') &= F'\\ \mathbf{T}'(\overline{U}(E)) &= \overline{U}(F)\\ \mathbf{T}'(\overline{U}(E)) &= \overline{U}(\mathbf{T}(E))\end{aligned} \tag{66}$$

Since Eq. (66) holds for all $E$ then

$$\begin{aligned}\mathbf{T}'\circ\overline{U} &= \overline{U}\circ\mathbf{T}\\ \mathbf{T}' &= \overline{U}\circ\mathbf{T}\circ\overline{U}^*\end{aligned} \tag{67}$$

If $\mathbb{G}$ is an orthonormal basis for $\mathbb{M}$, then, from Eqs. (14a) and (58), the matrix equivalent of Eq. (67) is

$$[\mathbf{T}']_{\mathbb{GG}} = [\overline{U}]_{\mathbb{GG}}\,[\mathbf{T}]_{\mathbb{GG}}\,[\overline{U}]_{\mathbb{GG}}^\top \qquad (\mathbb{G}\text{ orthonormal}) \tag{68}$$

### 3.3 The notion of symmetry for T

We define $V \in \mathbb{U}$ to be a symmetry of an elastic map if the map does not change when the relevant material is rotated by $V$. More precisely, $V$ is a symmetry of $\mathbf{T}$ if the two elastic maps $\mathbf{T}$ and $\overline{V}\circ\mathbf{T}\circ\overline{V}^*$ (from Eq. 67) are the same.

$$V \text{ is a symmetry of } \mathbf{T} \iff \overline{V}\circ\mathbf{T}\circ\overline{V}^* = \mathbf{T} \tag{69}$$

We require $V$ to be a rotation matrix—an orthogonal matrix with determinant $+1$. We could have required $V$ to be only an orthogonal matrix, so that perhaps $\det V = -1$. But if $V$ is orthogonal with $\det V = -1$, then $-V$ is a rotation matrix. Since $\overline{-V} = \overline{V}$, then $V$ being a symmetry of $\mathbf{T}$ would be equivalent to $-V$ being a symmetry of $\mathbf{T}$. Allowing $\det V = -1$ would gain nothing.

#### *3.3.1 The $\Delta$-test for a symmetry of $\mathbf{T}$*

From Eq. (69),

$$V \text{ is a symmetry of } \mathbf{T} \iff \Delta(V,\mathbf{T}) = 0_{6\times 6}, \tag{70a}$$

where $0_{6\times6}$ is the $6\times6$ zero matrix and

$$\Delta(V,\mathbf{T}) = [\overline{V}]_{\mathbb{GG}}\,[\mathbf{T}]_{\mathbb{GG}}\,[\overline{V}]_{\mathbb{GG}}^\top - [\mathbf{T}]_{\mathbb{GG}} \qquad (\mathbb{G}\text{ orthonormal}) \tag{70b}$$

Suppose, for example, that we want to find the matrix $[\mathbf{T}]_{\mathbb{B}\mathbb{B}}$ when $Z_\pi$ is a symmetry of $\mathbf{T}$. If $\mathbf{T}$ were an arbitrary elastic map, its matrix $T$ with respect to $\mathbb{B}$ would be, say,

$$T = \begin{pmatrix} a & g & m & q & t & v \\ g & b & h & n & r & u \\ m & h & c & i & o & s \\ q & n & i & d & j & p \\ t & r & o & j & e & k \\ v & u & s & p & k & f \end{pmatrix} \tag{71}$$

With $V = Z_\pi$ (Eq. 50), we therefore want to find the entries $a, b, c, \ldots$ of $T$ so that $V$ is a symmetry of $\mathbf{T}$. From Eq. (60) with $\xi = \pi$ and from Eq. (71),

$$\Delta(V, \mathbf{T}) = -2 \begin{pmatrix} & & m & q & t & v \\ & & h & n & r & u \\ m & h & & & & \\ q & n & & & & \\ t & r & & & & \\ v & u & & & & \end{pmatrix}, \tag{72}$$

where blank entries are understood to be zero. From Eq. (70a), the rotation $V$ is a symmetry of $\mathbf{T}$ if and only if $\Delta(V, \mathbf{T})$ is the zero matrix. So $h = m = n = q = r = t = u = v = 0$, and $T$ in Eq. (71) becomes $T_{\text{MONO}}$, where

$$T_{\text{MONO}} = \begin{pmatrix} a & g & & & & \\ g & b & & & & \\ & & c & i & o & s \\ & & i & d & j & p \\ & & o & j & e & k \\ & & s & p & k & f \end{pmatrix} \tag{73}$$

### 3.4 Eigenspaces of T and their role in symmetry

For $\lambda \in \mathbb{R}$ we let

$$\mathbb{M}_{\mathbf{T}}(\lambda) = \{E \in \mathbb{M} : \mathbf{T}(E) = \lambda\, E\} \tag{74}$$

If $\lambda$ is an eigenvalue of $\mathbf{T}$, then its eigenspace is $\mathbb{M}_{\mathbf{T}}(\lambda)$; it consists of the zero vector together with the eigenvectors of $\mathbf{T}$ having eigenvalue $\lambda$.

**Theorem 1.** Let $\mathbf{T}$ be an elastic map and let $V$ be a $3 \times 3$ rotation matrix. Then $V$ is a symmetry of $\mathbf{T}$ if and only if all eigenspaces of $\mathbf{T}$ are invariant under $\overline{V}$.

*Proof.* Suppose first that $V$ is a symmetry of $\mathbf{T}$. Then $\mathbf{T} \circ \overline{V} = \overline{V} \circ \mathbf{T}$, from Eq. (69). Hence if $E \in \mathbb{M}_{\mathbf{T}}(\lambda)$, then

$$\mathbf{T}\left(\overline{V}(E)\right) = \overline{V}\left(\mathbf{T}(E)\right) = \overline{V}(\lambda E) = \lambda\, \overline{V}(E),$$

so that $\overline{V}(E) \in \mathbb{M}_{\mathbf{T}}(\lambda)$. Hence $\overline{V}\left(\mathbb{M}_{\mathbf{T}}(\lambda)\right) \subset \mathbb{M}_{\mathbf{T}}(\lambda)$.

Conversely, suppose $\overline{V}\left(\mathbb{M}_{\mathbf{T}}(\lambda)\right) \subset \mathbb{M}_{\mathbf{T}}(\lambda)$ for all eigenvalues $\lambda$ of $\mathbf{T}$. Then if $E$ is an eigenvector of $\mathbf{T}$ with eigenvalue $\lambda$, so is $\overline{V}$(E), and so

$$\begin{aligned} (\mathbf{T} \circ \overline{V})(E) &= \mathbf{T}\left(\overline{V}(E)\right) = \lambda\, \overline{V}(E) \\ &= \overline{V}(\lambda E) = \overline{V}\left(\mathbf{T}(E)\right) = (\overline{V} \circ \mathbf{T})(E) \end{aligned}$$

Since the eigenvectors $E$ of $\mathbf{T}$ span $\mathbb{M}$, then $\mathbf{T} \circ \overline{V} = \overline{V} \circ \mathbf{T}$, by linearity. Then $V$ is a symmetry of $\mathbf{T}$, by Eq. (69). □

Theorem 1 was known to Rychlewski (1984).

If $\mathbf{T}$ is an elastic map, then, from the Spectral Theorem, $\mathbb{M}$ is the orthogonal direct sum of the eigenspaces of $\mathbf{T}$. Thus, if $\mu_1, \ldots, \mu_k$ are the distinct eigenvalues of $\mathbf{T}$,

$$\mathbb{M} = \mathbb{M}_{\mathbf{T}}(\mu_1) \perp \ldots \perp \mathbb{M}_{\mathbf{T}}(\mu_k) \tag{75}$$

If $V$ is a symmetry of $\mathbf{T}$, then each $\mathbb{M}_{\mathbf{T}}(\mu_i)$ is invariant under $\overline{V}$. From Lemma 1, each subspace $\mathbb{M}_{\mathbf{T}}(\mu_i)$ is an orthogonal direct sum (Eq. 42) of subspaces prime for $\overline{V}$. In the hypothetical illustration in Eq. (76), the eigenspace $\mathbb{M}_{\mathbf{T}}(\mu_1)$ is the orthogonal direct sum of the prime subspaces $\mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3$, whereas the eigenspace $\mathbb{M}_{\mathbf{T}}(\mu_k)$ is itself prime.

$$\begin{array}{ccccccc} \mathbb{M} & = & & \mathbb{M}_{\mathbf{T}}(\mu_1) \perp & & \ldots & \perp \mathbb{M}_{\mathbf{T}}(\mu_k) \\ & & \swarrow & \downarrow & \searrow & & \downarrow \\ & \mathbb{W}_1 & & \mathbb{W}_2 & & \mathbb{W}_3 & \mathbb{W}_n \end{array} \tag{76}$$

Thus,

$$\mathbf{T} = \underset{\lambda_1}{\mathbb{W}_1} \perp \ldots \perp \underset{\lambda_n}{\mathbb{W}_n} \tag{77}$$

(In this example, $\lambda_1 = \lambda_2 = \lambda_3 = \mu_1$ and $\lambda_n = \mu_k$.) This means, as in Eq. (44), that $\mathbb{M}$ is the orthogonal direct sum of the $\mathbb{W}_i$, and that on $\mathbb{W}_i$ the linear transformation $\mathbf{T}$ is multiplication by $\lambda_i$. Here, however, the $\mathbb{W}_i$ are prime for $\overline{V}$. The converse is also seen to be true. Thus,

**Theorem 2.** A rotation matrix $V$ is a symmetry of an elastic map $\mathbf{T}$ if and only if, for some numbers $\lambda_1, \ldots, \lambda_n$ and for some subspaces $\mathbb{W}_1, \ldots, \mathbb{W}_n$ of $\mathbb{M}$,

(i) Each $\mathbb{W}_i$ is prime for $\overline{V}$.
(ii) $\mathbf{T} = \underset{\lambda_1}{\mathbb{W}_1} \perp \ldots \perp \underset{\lambda_n}{\mathbb{W}_n}$

Using Eqs. (43) and (44), we can paraphrase conditions (i) and (ii) as

$$\begin{gathered} \mathbb{W}_1, \ldots, \mathbb{W}_n \text{ are prime summands of } \mathbb{M} \text{ for } \overline{V} \\ \mathbf{T}(E) = \lambda_i E \qquad (E \in \mathbb{W}_i) \end{gathered} \tag{78}$$

### 3.5 Some matrices, six-tuples, and subspaces

With $B_1, \ldots, B_6$ the basis $\mathbb{B}$ given in Eq. (3), we define matrices in $\mathbb{M}$ by

$$\begin{aligned} B_{12}(r) &= (\cos r)B_1 + (\sin r)B_2 \\ B_{34}(s) &= (\cos s)B_3 + (\sin s)B_4 \\ B_{56}(t) &= (\cos t)B_5 + (\sin t)B_6 \end{aligned} \tag{79}$$

The corresponding elements of $\mathbb{R}^6$ are

$$\begin{aligned} \mathbf{e}_{12}(r) &= [\mathbb{B}_{12}(r)]_{\mathbb{B}} = (\cos r, \sin r, 0, 0, 0, 0) \\ \mathbf{e}_{34}(s) &= [\mathbb{B}_{34}(s)]_{\mathbb{B}} = (0, 0, \cos s, \sin s, 0, 0) \\ \mathbf{e}_{56}(t) &= [\mathbb{B}_{56}(t)]_{\mathbb{B}} = (0, 0, 0, 0, \cos t, \sin t) \end{aligned} \tag{80}$$

Thus $r, s, t$ are the angular polar coordinates in the respective $x_1x_2, x_3x_4, x_5x_6$-planes.

With $\langle S_1, S_2 \rangle$ denoting the subspace spanned by $S_1$ and $S_2$, we define subspaces of $\mathbb{M}$ by

$$\mathbb{B}_{12} = \langle B_1, B_2 \rangle, \quad \mathbb{B}_{34} = \langle B_3, B_4 \rangle, \quad \mathbb{B}_{56} = \langle B_5, B_6 \rangle \tag{81}$$

The corresponding subspaces of $\mathbb{R}^6$ are

$$\begin{aligned} \mathbb{E}_{12} &= \langle \mathbf{e}_1, \mathbf{e}_2 \rangle = \{(x_1, x_2, 0, 0, 0, 0) : x_1, x_2 \in \mathbb{R}\} \\ \mathbb{E}_{34} &= \langle \mathbf{e}_3, \mathbf{e}_4 \rangle = \{(0, 0, x_3, x_4, 0, 0) : x_3, x_4 \in \mathbb{R}\} \\ \mathbb{E}_{56} &= \langle \mathbf{e}_5, \mathbf{e}_6 \rangle = \{(0, 0, 0, 0, x_5, x_6) : x_5, x_6 \in \mathbb{R}\}, \end{aligned} \tag{82}$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_6$ is the standard basis for $\mathbb{R}^6$.

## 4 The symmetry $Z_\xi$ when $\xi$ is regular

### 4.1 Subspaces of $\mathbb{M}$ invariant under $\overline{Z_\xi}$ when $\xi$ is regular

The notion of a prime subspace was introduced in Section 2.6.3. If $A$ is a $6 \times 6$ matrix, a non-zero subspace $\mathbb{E}$ of $\mathbb{R}^6$ is prime for $A$ if it is invariant (under multiplication by $A$) and if it has no proper invariant subspaces.

For the $6 \times 6$ matrix $A = [\overline{Z_\xi}]_{\mathbb{BB}}$ we can try to guess the prime subspaces from inspection of the matrix, and we will usually be right. From Eq. (60),

$$[\overline{Z_\xi}]_{\mathbb{BB}} = \begin{pmatrix} R(-\xi) & & \\ & R(2\xi) & \\ & & I_{2\times 2} \end{pmatrix}, \tag{83}$$

where $R(\theta)$ is the $2 \times 2$ rotation matrix from Eq. (F.1) of Appendix F, and where $I_{2\times 2}$ is the $2 \times 2$ identity matrix. (Blank entries are understood to be zeros.)

From Eqs. (82) and (83), the subspaces $\mathbb{E}_{12}, \mathbb{E}_{34}, \mathbb{E}_{56}$ of $\mathbb{R}^6$ are invariant under $[\overline{Z_\xi}]_{\mathbb{BB}}$. Since on $\mathbb{E}_{56}$ the matrix $[\overline{Z_\xi}]_{\mathbb{BB}}$ is the identity, then $\mathbb{E}_{56}$ itself is not prime (for $[\overline{Z_\xi}]_{\mathbb{BB}}$), but all of its one-dimensional subspaces are prime. Each has the form $\langle \mathbf{e}_{56}(t) \rangle$ for some $t$.

On the subspace $\mathbb{E}_{12}$ the matrix $[\overline{Z_\xi}]_{\mathbb{BB}}$ is rotation through angle $-\xi$, and on $\mathbb{E}_{34}$ it is rotation through angle $2\xi$, so $\mathbb{E}_{12}$ and $\mathbb{E}_{34}$ are prime for most choices of $\xi$. But are they always prime, and might there be other prime subspaces? Theorem 3 gives some answers, but in the context of $\mathbb{M}$ rather than $\mathbb{R}^6$.

Recall from Fig.1 that $\xi$ is regular if rotations through angle $\xi$ are neither 1-fold, 2-fold. 3-fold, nor 4-fold:

$$\xi \text{ is regular} \iff \xi \neq \pm 2\pi/n \pmod{2\pi},\ n = 1, 2, 3, 4 \tag{84}$$

**Theorem 3.** Regardless of $\xi$, the subspaces $\mathbb{B}_{12}$, $\mathbb{B}_{34}$, $\langle B_{56}(t) \rangle$ of $\mathbb{M}$ are invariant under $\overline{Z_\xi}$. If $\xi$ is regular, they are the prime subspaces for $\overline{Z_\xi}$. (See Section 3.5 for notation.)

*Proof.* The proof relies on the $\mathbb{B}$-coordinate mapping to go back and forth between $\mathbb{M}$ and $\mathbb{R}^6$. Thus $\mathbb{B}_{12}$, $\mathbb{B}_{34}$, $\langle B_{56}(t) \rangle$ are invariant under $\overline{Z_\xi}$ since $\mathbb{E}_{12}, \mathbb{E}_{34}, \langle \mathbf{e}_{56}(t) \rangle$ are invariant under $[\overline{Z_\xi}]_{\mathbb{BB}}$. For $\xi$ regular they are the prime subspaces for $\overline{Z_\xi}$, since $\mathbb{E}_{12}, \mathbb{E}_{34}, \langle \mathbf{e}_{56}(t) \rangle$ are then the prime subspaces of $\mathbb{R}^6$ for $[\overline{Z_\xi}]_{\mathbb{BB}}$; see Lemma 6 of Appendix B, □

### 4.2 Prime summands for $\overline{Z_\xi}$ when $\xi$ is regular

From Eq. (43), subspaces $\mathbb{W}_1, \ldots, \mathbb{W}_n$ of $\mathbb{M}$ are prime summands for $\overline{Z_\xi}$ if they are prime for $\overline{Z_\xi}$ and if their orthogonal direct sum is all of $\mathbb{M}$.

If $\xi$ is regular, there is not much choice about the prime summands for $\overline{Z_\xi}$, due to Theorem 3. They can only be, for some t,

$$\mathbb{B}_{12}, \mathbb{B}_{34}, \langle B_{56}(t) \rangle, \langle B_{56}(t') \rangle, \tag{85a}$$

where

$$t' = t + \pi/2 \tag{85b}$$

Note that, although there is a prime subspace $\langle B_{56}(t) \rangle$ for each $t$, there is little choice regarding $t'$ in Eqs. (85)—it must be $t \pm \pi/2$, since $B_{56}(t)$ and $B_{56}(t')$ are to be orthogonal. The prime summands for $\overline{Z_\xi}$ with $\xi$ regular are shown in Fig. 6.

The figure illustrates the invariance of the prime summands under $\overline{Z_\xi}$. Consider, for example, the beachball at $\theta = 0$ in the $x_1x_2$ plane (the 3:00 position), and rotate it through $\xi = 45°$ about its own vertical axis (perpendicular to the page). The resulting ball is present in the diagram, and the upper left $2 \times 2$ submatrix of $[\overline{Z_\xi}]_{\mathbb{BB}}$ (Eq. 83), which describes a rotation through $-\xi$ about the origin in the $x_1x_2$ plane, tells where to find it. (It is at the 4:30 position.) The balls in the $x_3x_4$-plane and $x_5x_6$-plane work analogously, but in the $x_3x_4$-plane the matrix $[\overline{Z_\xi}]_{\mathbb{BB}}$ is rotation through $2\xi$, and in the $x_5x_6$-plane it is the identity.

The $xyz$ spatial coordinates have no logical relation to the $\mathbb{B}$-coordinates $x_1 \ldots x_6$. In a diagram like Fig. 6, where the $xyz$ directions must be known in order to orient the beachballs, some decision must therefore be made that relates the two coordinate systems. We chose to have $z$ point out of the page and $x$ to the right.

In the same vein, a beachball has no particular location in $xyz$ space. Alternatively, all beachballs can be thought of as centered at the origin in $xyz$ space. The location of a beachball in a diagram like Fig. 6 only serves to indicate the coordinate 6-tuple of the ball. FIG. 6

### 4.3 Elastic maps with symmetry $Z_\xi$ for regular $\xi$

According to Eqs. (78), an elastic map $\mathbf{T}$ having symmetry $V$ is determined by specifying prime summands for $\overline{V}$ and by assigning a number to each of them. If $V = Z_\xi$ with $\xi$ regular, then the prime summands are $\mathbb{B}_{12}, \mathbb{B}_{34}, \langle B_{56}(t) \rangle, \langle B_{56}(t') \rangle$; they depend only on $t$. Hence $\mathbf{T}$ is determined by giving $t$ to specify $\langle B_{56}(t) \rangle$ and $\langle B_{56}(t') \rangle$, and then by assigning respective numbers $\lambda_1, \lambda_3, \lambda_5, \lambda_6$ to $\mathbb{B}_{12}, \mathbb{B}_{34}, \langle B_{56}(t) \rangle, \langle B_{56}(t') \rangle$. That is, $\mathbf{T} = \mathbf{T}^{\Lambda}_{\text{XISO}}(t)$, where, in the notation of Eq. (44),

$$\begin{gathered} \mathbf{T}^{\Lambda}_{\text{XISO}}(t) = \underset{\lambda_1\,\lambda_1}{\mathbb{B}_{12}} \perp \underset{\lambda_3\,\lambda_3}{\mathbb{B}_{34}} \perp \underset{\lambda_5}{\langle B_{56}(t) \rangle} \perp \underset{\lambda_6}{\langle B_{56}(t') \rangle} \\ t' = t + \pi/2, \quad \boldsymbol{\Lambda} = (\lambda_1, \lambda_1, \lambda_3, \lambda_3, \lambda_5, \lambda_6) \end{gathered} \tag{86}$$

The repetitions $\lambda_1\,\lambda_1$ and $\lambda_3\,\lambda_3$ in the orthogonal direct sum are reminders that $\dim \mathbb{B}_{12} = 2$ and $\dim \mathbb{B}_{34} = 2$.

For $\mathbf{T} = \mathbf{T}^{\Lambda}_{\text{XISO}}(t)$ as in Eq. (86), its symmetry $Z_\xi$ is seen in Fig. 6, though the figure itself does not involve $\mathbf{T}$. In Fig. 6b, for example, the effect of $\mathbf{T}$ would be to resize the balls by the constant factor $\lambda_3$. One can rotate a ball through angle $\xi$ about its vertical axis, and then resize it, or one can resize it and then rotate it. The result is the same, but only because the rotated ball is in the same subspace as the original ball, so that the resizing factor does not change.

**Theorem 4. (The matrix for $\mathbf{T}^{\Lambda}_{\text{xiso}}(t)$)**
For $\mathbf{T} = \mathbf{T}^{\Lambda}_{\text{XISO}}(t)$,

$$[\mathbf{T}]_{\mathbb{BB}} = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_1 & & & \\ & & \lambda_3 & & \\ & & & \lambda_3 & \\ & & & & R(t) \begin{pmatrix} \lambda_5 & \\ & \lambda_6 \end{pmatrix} R(t)^\top \end{pmatrix}$$

*Proof.* Theorem 4 is the special case of Theorem 7 in which $r = 0$, $U = \begin{pmatrix} I_{2\times 2} & \\ & R(t) \end{pmatrix}$, $\lambda_1 = \lambda_2$, and $\lambda_3 = \lambda_4$. □

The form of the matrix $[\mathbf{T}]_{\mathbb{BB}}$ in Theorem 4 dictates the definition of the matrix $T_{\text{XISO}}$, namely,

$$T_{\text{XISO}} = \begin{pmatrix} a &&&&& \\ & a &&&& \\ && c &&& \\ &&& c && \\ &&&& e & k \\ &&&& k & f \end{pmatrix} \tag{87}$$

The matrices $T_{\text{XISO}}$ and $[\mathbf{T}]_{\mathbb{BB}}$ are the same when

$$\begin{aligned} &a = \lambda_1, \qquad c = \lambda_3 \\ &e = \lambda_5 \cos^2 t + \lambda_6 \sin^2 t \\ &f = \lambda_6 \cos^2 t + \lambda_5 \sin^2 t \\ &k = (\lambda_5 - \lambda_6) \cos t \sin t \end{aligned} \tag{88}$$

Appealing to Eqs. (F.2) of Appendix F, we see that $T_{\text{XISO}}$ and $[\mathbf{T}]_{\mathbb{BB}}$ are also the same when

$$\begin{aligned} &t = \theta_\infty = \frac{1}{2}\,\widehat{\theta}(e - f,\, 2k) \\ &\lambda_1 = a, \qquad \lambda_3 = c \\ &\lambda_5 = \frac{1}{2}\left(e + f + \sqrt{(e-f)^2 + 4k^2}\right) \\ &\lambda_6 = \frac{1}{2}\left(e + f - \sqrt{(e-f)^2 + 4k^2}\right), \end{aligned} \tag{89}$$

where $\widehat{\theta}(x, y)$ is the ordinary angular polar coordinate of the point $(x, y)$. (If $e = f$ and $k = 0$ then $\theta_\infty$ is undefined, but $t$ can be chosen arbitrarily.)

For $\mathbf{T}$ having symmetry $Z_\xi$ for $\xi$ regular, Eqs. (88) give the matrix entries $a, c, e, f, k$ of $T_{\text{XISO}} = [\mathbf{T}]_{\mathbb{BB}}$ in terms of the "intrinsic" parameters $t, \lambda_1, \lambda_3, \lambda_5, \lambda_6$ of $\mathbf{T}$, and Eqs. (89) do the reverse.

The intrinsic parameters are not unique, but it hardly matters. From Appendix F we see that two 5-tuples of intrinsic parameters give the same $\mathbf{T}$:

| $t$ | $\lambda_1$ | $\lambda_3$ | $\lambda_5$ | $\lambda_6$ |
|---|---|---|---|---|
| $t$ | $\mu_1$ | $\mu_3$ | $\mu_5$ | $\mu_6$ |
| $t + \pi/2$ | $\mu_1$ | $\mu_3$ | $\mu_6$ | $\mu_5$ |

(90)

Thus the expressions for $\lambda_5$ and $\lambda_6$ in Eqs. (89) can be swapped if $t = \theta_\infty$ is replaced by $t = \theta_\infty + \pi/2$, but there is generally no reason to do so. One tuple of intrinsic parameters is enough.

We have shown that for $\xi$ regular the following are equivalent:

$$Z_\xi \text{ is a symmetry of } \mathbf{T} \tag{91a}$$

$$\mathbf{T} = \mathbf{T}^{\Lambda}_{\text{XISO}}(t) \text{ for some } t, \lambda_1, \lambda_3, \lambda_5, \lambda_6 \tag{91b}$$

$$[\mathbf{T}]_{\mathbb{BB}} = T_{\text{XISO}}(a, c, e, f, k) \text{ for some } a, c, e, f, k \tag{91c}$$

We refer to the condition $\mathbf{T} = \mathbf{T}^{\Lambda}_{\text{XISO}}(t)$ as an intrinsic characterization of $\mathbf{T}$, in order to distinguish it from the condition $[\mathbf{T}]_{\mathbb{BB}} = T_{\text{XISO}}$, which involves a basis for $\mathbb{M}$. Although the subspaces $\mathbb{B}_{12}, \mathbb{B}_{34}, \langle B_{56}(t), \langle B_{56}(t')$ in the intrinsic characterization appear to involve the basis $\mathbb{B}$, in fact they can be described without $\mathbb{B}$; see Table 2 for $\mathbb{B}_{12}$ and $\mathbb{B}_{34}$, and see Eq. (93) for $B_{56}(t)$.

Intrinsic characterizations of elastic maps go back at least to Rychlewski (1984). Also see Bóna et al. (2007).

### 4.4 Transverse isotropy

**Theorem 5.** If $Z_\xi$ is a symmetry of an elastic map $\mathbf{T}$ for some regular $\xi$, then $Z_\xi$ is a symmetry of $\mathbf{T}$ for all $\xi$.

*Proof.* Let $Z_\xi$ be a symmetry of $\mathbf{T}$. To show that the rotation $Z_\beta$ is a symmetry of $\mathbf{T}$, we need only show that the eigenspaces of $\mathbf{T}$ are invariant under $\overline{Z_\beta}$ (Theorem 1). To that end, let $\mathbb{W}$ be an eigenspace of $\mathbf{T}$. Then $\mathbb{W}$ is invariant under $\overline{Z_\xi}$, by Theorem 1. Hence $\mathbb{W}$ is an orthogonal direct sum of prime subspaces for $\overline{Z_\xi}$. Since $\xi$ is regular, the prime subspaces for $\overline{Z_\xi}$ are $\mathbb{B}_{12}, \mathbb{B}_{34}, \langle B_{56}(t)\rangle$. Those subspaces, by Theorem 3, are invariant under $\overline{Z_\beta}$, hence so is $\mathbb{W}$. □

Theorem 5 also follows from Eqs. (91), since only Eq. (91a) mentions $\xi$.

An elastic map $\mathbf{T}$ is said to be transverse isotropic (with respect to the $z$-axis) if $Z_\xi$ is a symmetry of $\mathbf{T}$ for all $\xi$. Theorem 5 says that If $Z_\xi$ is a symmetry of $\mathbf{T}$ for some regular $\xi$, then $\mathbf{T}$ is transverse isotropic.

Herman (1945) has a weaker version of Theorem 5. Where our version has $\xi$ regular, Herman has $\xi = 2\pi/n$ for some integer $n > 4$. We need the stronger version in deriving the elastic symmetry groups (Section 14, especially Lemma 3).

## 5 Subspaces of $\mathbb{M}$ described intrinsically

In Table 2 we list some subspaces of $\mathbb{M}$ that will be relevant to elastic symmetry. To describe them we borrow terminology from seismology, which we explain next. We do not intend, however, to discuss applications to seismology.

As always, $\mathbb{M}$ is the space of $3 \times 3$ symmetric matrices. A TBP frame for a matrix $E \in \mathbb{M}$ is a rotation matrix whose first, second, and third columns are eigenvectors T, B, P of $E$ corresponding to the respective largest, intermediate, and smallest eigenvalues of $E$.

A deviatoric matrix in $\mathbb{M}$ is one with trace equal to zero. A double couple is a deviatoric matrix with determinant zero. Its eigenvalues therefore have the form $\mu, 0, -\mu$. The beachball for a double couple has the classic beachball look, with the ball surface divided into four congruent lunes having alternating colors (e.g., $B_2$ in Fig. 3). The fault planes of the double couple are the two planes that define the boundaries of the lunes; the normal vectors to the fault planes are T$\pm$P. The null axis is the intersection of the two fault planes; it is in the direction of B.

A crack matrix is a matrix in $\mathbb{M}$ with two equal eigenvalues (not three). Its $c$-axis is in the direction of the eigenvector with the simple (i.e., non-repeated) eigenvalue. The beachball for a crack matrix has rotational symmetry about the $c$-axis through all angles; if bicolored, it looks more like a striped pool ball than a traditional beachball (e.g., $B_5$ in Fig. 3).

An isotropic matrix in $\mathbb{M}$ is a multiple of the identity. Its beachball is all red or all white.

A generic matrix $E \in \mathbb{M}$ is neither a double couple (DC), a crack matrix, nor an isotropic matrix. Thus

$$\begin{aligned} &E \text{ is generic} \\ &\iff E \text{ has distinct eigenvalues but is not a DC} \end{aligned} \tag{92}$$

The matrix for the beachball in Fig. 7(d) is generic. FIG. 7

The subspace $\mathbb{B}_{12}$ in Eq. (81) and Fig. 6a consists of all double couples having a fault plane horizontal. See also Fig. 7(a).

The subspace $\mathbb{B}_{34}$ in Eq. (81) and Fig. 6b consists of all double couples with null axis vertical.

The subspace $\mathbb{B}_{56}$ in Eq. (81) and Fig. 8 consists of all crack matrices with $c$-axis vertical. They are the diagonal matrices with diagonal entries of the form $p, p, q$. The red or white band on the beachball for such a matrix has angular half-width $\nu$ given by $\tan \nu = \sqrt{-p/q}$ (from Eq. 26). The angular half-width of the band for the crack matrix $B_{56}(t)$ is therefore, from Eqs. (3) and (79),

$$\begin{aligned} \nu(t) &= \tan^{-1} \sqrt{\frac{1}{\sqrt{2}} \cot(t_0 - t)} \\ t_0 &= \angle((1,1,1),(0,0,1)) = \tan^{-1}\sqrt{2} \end{aligned} \tag{93}$$

G. 8

The subspace $\langle B_6 \rangle$ consists of the isotropic matrices. Its orthogonal complement $\langle B_6 \rangle^\perp = \langle B_1, \ldots, B_5 \rangle$ consists of the deviatoric matrices. The subspace $\langle B_4, B_5, B_6 \rangle$ consists of the diagonal matrices, and $\langle B_4, B_5 \rangle$ consists of the deviatoric diagonal matrices.

ble 2

## 6 Elastic maps with symmetry $Z_{\pi/2}$

If an elastic map $\mathbf{T}$ has the symmetry $Z_{\pi/2}$ then it also has the symmetry $Z_\pi$ and so, from Section 3.3.1, its matrix with respect to $\mathbb{B}$ has at least the form of $T_{\text{MONO}}$ in Eq. (73). With $V = Z_{\pi/2}$, we therefore look for the entries $a, b, c, \ldots$ of $T_{\text{MONO}}$ such that $V$ is a symmetry of $\mathbf{T}$. From Eqs. (60) and (70b),

$$\Delta(V, \mathbf{T}) = -\begin{pmatrix} a-b & 2g & & & & \\ 2g & b-a & & & & \\ & & & & 2o & 2s \\ & & & & 2j & 2p \\ & & 2o & 2j & & \\ & & 2s & 2p & & \end{pmatrix} \tag{94}$$

The rotation $V$ is a symmetry of $\mathbf{T}$ if and only if $\Delta(V, \mathbf{T})$ is the zero matrix. Setting $b = a$ and $g = j = o = p = s = 0$ in $T_{\text{MONO}}$ gives $[\mathbf{T}]_{\mathbb{BB}} = T_4$ with $T_4$ as in Table 1.

ble 1

We will see momentarily that the intrinsic characterization of elastic maps $\mathbf{T}$ having symmetry $Z_{\pi/2}$ is $\mathbf{T} = \mathbf{T}_4^\Lambda(s,t)$, where

$$\begin{aligned} \mathbf{T}_4^\Lambda(s,t) &= \underset{\lambda_1\,\lambda_1}{\mathbb{B}_{12}} \perp \underset{\lambda_3}{\langle B_{34}(s)\rangle} \perp \underset{\lambda_4}{\langle B_{34}(s')\rangle} \perp \underset{\lambda_5}{\langle B_{56}(t)\rangle} \perp \underset{\lambda_6}{\langle B_{56}(t')\rangle} \\ s' &= s + \pi/2, \qquad t' = t + \pi/2 \\ \mathbf{\Lambda} &= (\lambda_1, \lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \end{aligned} \tag{95}$$

**Theorem 6. (The matrix for $\mathbf{T}_4^\Lambda(s,t)$)**
For $\mathbf{T} = \mathbf{T}_4^\Lambda(s,t)$,

$$[\mathbf{T}]_{\mathbb{BB}} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & R(s)\begin{pmatrix} \lambda_3 & \\ & \lambda_4 \end{pmatrix} R(s)^\top & \\ & & & R(t)\begin{pmatrix} \lambda_5 & \\ & \lambda_6 \end{pmatrix} R(t)^\top \end{pmatrix}$$

*Proof.* The theorem is the special case of Theorem 7 in which $r = 0$, $U = \begin{pmatrix} R(s) & \\ & R(t) \end{pmatrix}$, and $\lambda_1 = \lambda_2$. $\square$

The matrix $[\mathbf{T}]_{\mathbb{BB}}$ in Theorem 6 is the same as the matrix $T_4$ in Table 1 when

$$\begin{aligned} c &= \lambda_3 \cos^2 s + \lambda_4 \sin^2 s \\ d &= \lambda_4 \cos^2 s + \lambda_3 \sin^2 s \\ i &= (\lambda_3 - \lambda_4) \cos s \sin s \\ & a, e, f, k \text{ are as in Eq. (88)} \end{aligned} \tag{96}$$

The two matrices are also the same when

$$\begin{aligned} s &= \theta_4 = \frac{1}{2}\,\widehat{\theta}(c-d,\, 2i) \\ \lambda_3 &= \frac{1}{2}\left(c + d + \sqrt{(c-d)^2 + 4i^2}\right) \\ \lambda_4 &= \frac{1}{2}\left(c + d - \sqrt{(c-d)^2 + 4i^2}\right) \\ & t, \lambda_1, \lambda_5, \lambda_6 \text{ are as in Eqs. (89)} \end{aligned} \tag{97}$$

For $\mathbf{T}$ having symmetry $Z_{\pi/2}$, Eqs. (96) give the matrix entries $a, c, d, e, f, i, k$ of $T_4 = [\mathbf{T}]_{\mathbb{BB}}$ in terms of the intrinsic parameters $s, t, \lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ of $\mathbf{T}$, and Eqs. (97) do the reverse.

The intrinsic parameters are not unique, but it rarely matters. The following three 7-tuples all give the same $\mathbf{T}$.

| $s$ | $t$ | $\lambda_1$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ | $\lambda_6$ |
|---|---|---|---|---|---|---|
| $s$ | $t$ | $\mu_1$ | $\mu_3$ | $\mu_4$ | $\mu_5$ | $\mu_6$ |
| $s + \pi/2$ | $t$ | $\mu_1$ | $\mu_4$ | $\mu_3$ | $\mu_5$ | $\mu_6$ |
| $s$ | $t + \pi/2$ | $\mu_1$ | $\mu_3$ | $\mu_4$ | $\mu_6$ | $\mu_5$ |

(98)

We have now shown that the following are equivalent:

$Z_{\pi/2}$ is a symmetry of $\mathbf{T}$ (99a)

$[\mathbf{T}]_{\mathbb{BB}} = T_4(a, c, d, e, f, i, k)$ for some $a, c, d, e, f, i, k$ (99b)

$\mathbf{T} = \mathbf{T}_4^\Lambda(s,t)$ for some $s, t, \lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ (99c)

To see that the five subspaces in the orthogonal direct sum (Eq, 95) are invariant under $\overline{Z_{\pi/2}}$, note that invariance has nothing to do with the $\lambda_i$. We can therefore assume for a moment that $\lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ are distinct, so that the five subspaces are eigenspaces of $\mathbf{T}$. By Theorem 1 they are therefore invariant under $\overline{Z_{\pi/2}}$. The four one-dimensional subspaces are then automatically prime for $\overline{Z_{\pi/2}}$. The other subspace $\mathbb{B}_{12}$ is also prime, since its only proper subspaces have the form $\langle B_{12}(r) \rangle$, and they are not invariant under $\overline{Z_{\pi/2}}$ (e.g., Fig. 6a). Prime summands for $\overline{Z_{\pi/2}}$—one quintuple of summands for each $s$ and $t$—are therefore

$$\mathbb{B}_{12},\ \langle B_{34}(s)\rangle,\ \langle B_{34}(s')\rangle,\ \langle B_{56}(t)\rangle,\ \langle B_{56}(t')\rangle \tag{100}$$

They are shown in Fig. 9 for $s = 55^\circ$ and $t = 40^\circ$.

An elastic map $\mathbf{T}$ having symmetry $Z_{\pi/2}$ is determined by specifying $s$ and $t$ to give the five prime summands and by attaching a number $\lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ to each. FIG. 9

## 7 Elastic maps with symmetry $Z_\pi$

Let $U = U_{4\times4}$ be the $4 \times 4$ rotation matrix

$$U = \begin{pmatrix} u_{33} & u_{34} & u_{35} & u_{36} \\ u_{43} & u_{44} & u_{45} & u_{46} \\ u_{53} & u_{54} & u_{55} & u_{56} \\ u_{63} & u_{64} & u_{65} & u_{66} \end{pmatrix} \tag{101}$$

Let $\mathbb{B}_2 = \mathbb{B}_2(r, U)$ be the orthonormal basis of $\mathbb{M}$ whose elements are defined by their $\mathbb{B}$-coordinate 6-tuples as follows:

| $\mathbb{B}$-coordinate 6-tuple | Element of basis $\mathbb{B}_2(r, U)$ |
|---|---|
| $(\cos r, \sin r, 0, 0, 0, 0)$ | $B_{12}(r)$ |
| $(-\sin r, \cos r, 0, 0, 0, 0)$ | $B_{12}(r')$ |
| $(0, 0, u_{33}, u_{43}, u_{53}, u_{63})$ | $B_3(U)$ |
| $(0, 0, u_{34}, u_{44}, u_{54}, u_{64})$ | $B_4(U)$ |
| $(0, 0, u_{35}, u_{45}, u_{55}, u_{65})$ | $B_5(U)$ |
| $(0, 0, u_{36}, u_{46}, u_{56}, u_{66})$ | $B_6(U)$ |

(102)

where $r' = r + \pi/2$. Thus

$$B_j(U) = u_{3j} B_3 + u_{4j} B_4 + u_{5j} B_5 + u_{6j} B_6 \quad (j = 3, 4, 5, 6) \tag{103}$$

We will see momentarily that the intrinsic characterization of elastic maps $\mathbf{T}$ having symmetry $Z_\pi$ is $\mathbf{T} = \mathbf{T}_2^{\Lambda}(r, U)$, where

$$\begin{aligned}\mathbf{T}_2^{\Lambda}(r, U) = &\underset{\lambda_1}{\langle B_{12}(r)\rangle} \perp \underset{\lambda_2}{\langle B_{12}(r')\rangle} \perp \\ &\underset{\lambda_3}{\langle B_3(U)\rangle} \perp \underset{\lambda_4}{\langle B_4(U)\rangle} \perp \underset{\lambda_5}{\langle B_5(U)\rangle} \perp \underset{\lambda_6}{\langle B_6(U)\rangle} \\ r' = &\, r + \pi/2, \qquad \mathbf{\Lambda} = (\lambda_1, \ldots, \lambda_6)\end{aligned} \tag{104}$$

**Theorem 7. (The matrix for $\mathbf{T}_2^{\Lambda}(r, U)$)**
For $\mathbf{T} = \mathbf{T}_2^{\Lambda}(r, U)$,

$$[\mathbf{T}]_{\mathbb{B}\mathbb{B}} = \begin{pmatrix} R(r) \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} R(r)^\top & \\ & U \begin{pmatrix} \lambda_3 & & & \\ & \lambda_4 & & \\ & & \lambda_5 & \\ & & & \lambda_6 \end{pmatrix} U^\top \end{pmatrix}$$

*Proof.* From Eq. (104) the matrix $[\mathbf{T}]_{\mathbb{B}_2\mathbb{B}_2}$ is diagonal with diagonal entries $\lambda_1, \ldots, \lambda_6$. It is related to $[\mathbf{T}]_{\mathbb{B}\mathbb{B}}$ by

$$[\mathbf{T}]_{\mathbb{B}\mathbb{B}} = [\mathbf{I}]_{\mathbb{B}\mathbb{B}_2} \, [\mathbf{T}]_{\mathbb{B}_2\mathbb{B}_2} \, [\mathbf{I}]_{\mathbb{B}_2\mathbb{B}}, \tag{105}$$

where $[\mathbf{I}]_{\mathbb{B}\mathbb{B}_2}$ is the matrix that takes $\mathbb{B}_2$-coordinates to $\mathbb{B}$-coordinates. From Eq. (32) its $j^{\text{th}}$ column is the $\mathbb{B}$-coordinate 6-tuple of the $j^{\text{th}}$ element of $\mathbb{B}_2(r, U)$. Hence from Eq. (102),

$$[\mathbf{I}]_{\mathbb{B}_2\mathbb{B}_2} = \begin{pmatrix} R(r) & \\ & U_{4\times 4} \end{pmatrix} \tag{106}$$

Eq. (105) therefore becomes

$$[\mathbf{T}]_{\mathbb{B}\mathbb{B}} = \begin{pmatrix} R(r) & \\ & U_{4\times 4} \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_6 \end{pmatrix} \begin{pmatrix} R(r) & \\ & U_{4\times 4} \end{pmatrix}^\top, \tag{107}$$

which is the same as in the theorem. □

The matrix $[\mathbf{T}]_{\mathbb{B}\mathbb{B}}$ in Theorem 7 has the same form as $T_{\text{MONO}}$ in Table 1, but when are the two matrices equal? From the theorem it is obvious how to find the entries $a, b, c, \ldots$ of $T_{\text{MONO}}$ in terms of the intrinsic parameters $r, U, \lambda_1, \ldots, \lambda_6$ of $\mathbf{T}$. Conversely, one gets $r, \lambda_1, \lambda_2$ from the submatrix $\begin{pmatrix} a & g \\ g & b \end{pmatrix}$ of $T_{\text{MONO}}$ using Eqs. (F.2) or (F.3), and, in principle, one gets $U$ and $\lambda_3, \lambda_4, \lambda_5, \lambda_6$ from an eigensystem for the lower right $4 \times 4$ submatrix of $T_{\text{MONO}}$. Getting the eigensystem symbolically, however, is not appealing, since the characteristic polynomial is quartic. (Finding it numerically is not a problem.)

In Section 3.3.1 we showed that an elastic map $\mathbf{T}$ has symmetry $Z_\pi$ if and only if its matrix with respect to $\mathbb{B}$ has the form $T_{\text{MONO}}$. The following are therefore equivalent:

$$Z_\pi \text{ is a symmetry of } \mathbf{T} \tag{108a}$$

$$[\mathbf{T}]_{\mathbb{B}\mathbb{B}} = T_{\text{MONO}}(a, b, \ldots) \text{ for some } a, b, \ldots \tag{108b}$$

$$\mathbf{T} = \mathbf{T}_2^{\Lambda}(r, U) \text{ for some } r, U, \lambda_1, \ldots, \lambda_6 \tag{108c}$$

Reasoning as we did from Eq. (95), we find from Eq. (104) that the prime summands for $\overline{Z_\pi}$—one sextuple for each choice of $r$ and $U$—are

$$\langle B_{12}(r)\rangle, \ \langle B_{12}(r')\rangle, \ \langle B_3(U)\rangle, \ \langle B_4(U)\rangle, \ \langle B_5(U)\rangle, \ \langle B_6(U)\rangle \tag{109}$$

Fig. 10 shows the prime summands for one choice of $r$ and $U$. The 180° symmetry is obvious in the figure. (The color reversal produced by 180° rotation of the first and second beachballs is acceptable, since the matrix $-E$ is always in the subspace $\langle E\rangle$.)

An elastic map $\mathbf{T}$ having symmetry $Z_\pi$ is determined by a number $r$ and a $4 \times 4$ rotation matrix $U$ to specify the six prime summands, and by numbers $\lambda_1, \ldots, \lambda_6$ to be assigned to them. FIG. 10

## 8 The symmetry $Z_{2\pi/3}$

### 8.1 Prime summands for $\overline{Z_{2\pi/3}}$

Motivated by Eq. (C.1) of Appendix C, we define the matrix $B(\theta, u, v)$ by

$$\begin{aligned} B(\theta, u, v) = &\frac{\cos\theta}{\sqrt{2}} \begin{pmatrix} -\sin u \sin v & \sin u \cos v & 0 \\ \sin u \cos v & \sin u \sin v & \cos u \\ 0 & \cos u & 0 \end{pmatrix} \\ &+ \frac{\sin\theta}{\sqrt{2}} \begin{pmatrix} -\sin u \cos v & -\sin u \sin v & \cos u \\ -\sin u \sin v & \sin u \cos v & 0 \\ \cos u & 0 & 0 \end{pmatrix} \end{aligned} \tag{110}$$

Then

$$B(\theta, u, v) = (\cos\theta)\, B(0, u, v) + (\sin\theta)\, B(\pi/2, u, v) \tag{111}$$

For each $u$ and $v$ there is a subspace of $\mathbb{M}$ spanned by $B(0, u, v)$ and $B(\pi/2, u, v)$, namely,

$$\mathbb{B}(u, v) = \{rB(\theta, u, v) : r, \theta \in \mathbb{R}\} \tag{112}$$

The $\mathbb{B}$-coordinate vector of $B(\theta, u, v)$ is

$$(B(\theta, u, v))_{\mathbb{B}} = \mathbf{e}(\theta, u, v) \tag{113}$$

where $\mathbf{e}(\theta, u, v)$ is from Eq. (C.1). Using Eq. (113) to translate between $\mathbb{M}$ and $\mathbb{R}^6$, we conclude from Lemma 7 of Appendix C:

**Theorem 8.** The subspaces of $\mathbb{M}$ that are prime for $\overline{Z_{2\pi/3}}$ are $\mathbb{B}(u, v)$ and $\langle B_{56}(t)\rangle$ (any $t, u, v$).

The prime summands of $\mathbb{M}$ for $\overline{Z_{2\pi/3}}$ are then

$$\mathbb{B}(u, v), \ \mathbb{B}(u', v), \ \langle B_{56}(t)\rangle, \ \langle B_{56}(t')\rangle \tag{114a}$$

where

$$t' = t + \pi/2, \qquad u' = u + \pi/2 \tag{114b}$$

The prime summands for $\overline{Z_\xi}$ with $\xi = 2\pi/3$ are a generalization of those for regular $\xi$ in the sense that

$$\mathbb{B}(0, v) = \mathbb{B}_{12}, \qquad \mathbb{B}(\pi/2, v) = \mathbb{B}_{34} \tag{115}$$

We let $\mathbb{B}(t, u, v)$ be the orthonormal basis of $\mathbb{M}$ whose elements are

$$\begin{aligned} &B(0, u, v), B(\pi/2, u, v) && \text{(a basis for } \mathbb{B}(u, v)) \\ &B(0, u', v), B(\pi/2, u', v) && \text{(a basis for } \mathbb{B}(u', v)) \\ &B_{56}(t), B_{56}(t') && \text{(a basis for } \mathbb{B}_{56}) \end{aligned} \tag{116}$$

When feasible, we abbreviate $\mathbb{B}(t, u, v)$ to $\mathbb{B}_3$.

For $\xi = 2\pi/3$ the $6 \times 6$ matrix of $\overline{Z_\xi}$ with respect to $\mathbb{B}(t, u, v)$ is found from Eqs. (3) and (59) to be the same as $[\overline{Z_\xi}]_{\mathbb{B}\mathbb{B}}$ in Eq. (83); it is independent of $t\,u\,v$. Since for $\xi = 2\pi/3$ a rotation through $2\xi$ is the same as a rotation through $-\xi$,

$$[\overline{Z_\xi}]_{\mathbb{B}_3\mathbb{B}_3} = \begin{pmatrix} R(-\xi) & & \\ & R(-\xi) & \\ & & I_{2\times 2} \end{pmatrix} \qquad (\xi = 2\pi/3) \tag{117}$$

In Fig. 11 the coordinate planes are for coordinates with respect to the basis $\mathbb{B}(t, u, v)$. The figure illustrates the prime summands and their invariance under $\overline{Z_{2\pi/3}}$. In Fig. 11(b), for example, if the ball at $\theta = 0$ (the 3:00 position) is rotated through an angle of $2\pi/3$ about its own vertical axis (perpendicular to the page), the resulting ball is present in the diagram. According to the middle $2 \times 2$ submatrix in Eq. (117), it should be the ball at $\theta = -2\pi/3$ (the 7:00
G. 11 position).

## 8.2 The remarkable subspaces $\mathbb{B}(u, v)$

Whereas the subspace $\mathbb{B}_{12}$ consists of the double couples having a fault plane horizontal, and $\mathbb{B}_{34}$ consists of the double couples with null axis vertical, the subspaces $\mathbb{B}(u.v)$ are more subtle and intriguing.

Since the matrices $B(0, u, v)$ and $B(\pi/2, u, v)$ are both orthogonal to $B_5$ and $B_6$ in the basis $\mathbb{B}$ (Eqs. 3), the subspace $\mathbb{B}(u, v)$ is a (two-dimensional) subspace of $\mathbb{B}_{12} \perp \mathbb{B}_{34}$. The matrices in $\mathbb{B}(u, v)$ are therefore deviatoric, that is, each has trace zero.

Recall that a double couple matrix is a deviatoric matrix with determinant zero. From Eq. (110),

$$\det B(\theta, u, v) = \frac{1}{\sqrt{8}} \cos^2 u \sin u \sin(v + 3\theta) \tag{118}$$

Hence, for $u \neq n\pi/2$, a matrix $B(\theta, u, v)$ is a double couple if and only if $\theta = -v/3 + n\pi/3$.

From Eq. (110),

$$Z_\beta\, B(\theta, u, v)\, Z_\beta^\top = B(\theta - \beta, u, v + 3\beta) \tag{119}$$

A beachball pattern is determined by the eigenvalue triple $\mathbf{\Lambda}$ of the beachball matrix, with the entries of $\mathbf{\Lambda}$ being in descending order. Conjugating a matrix preserves its eigenvalues, hence, with $\beta = -v/3$ in Eq. (119),

$$\mathbf{\Lambda}(B(\theta, u, v)) = \mathbf{\Lambda}(B(\theta + v/3, u, 0)), \tag{120}$$

and then

$$\mathbf{\Lambda}\big(\mathbb{B}(u, v)\big) = \mathbf{\Lambda}\big(\mathbb{B}(u, 0)\big) \tag{121}$$

Thus the totality of beachball patterns in the subspace $\mathbb{B}(u, v)$ is not affected by $v$.

To explain how $u$ affects the patterns, we use the parameter $\gamma(\mathbf{\Lambda})$, which for a deviatoric eigenvalue triple $\mathbf{\Lambda}$ is the signed angle between $(1, 0, -1)$ and $\mathbf{\Lambda}$ (Tape & Tape 2013, Section 2.3.2). In general, $\gamma$ varies between $-\pi/6$ and $\pi/6$, with $\gamma = 0$ for double couples and $|\gamma| = \pi/6$ for CLVDs—crack matrices that are deviatoric. For matrices in $\mathbb{B}(u, v)$, however, $\gamma$ varies between values $-\gamma_{\text{MAX}}(u)$ and $\gamma_{\text{MAX}}(u)$. As shown in Fig. 12, the number $\gamma_{\text{MAX}}(u)$ ranges from zero at $u = n\pi/2$ to $\pi/6$ at $u = \pm u_0 + n\pi$, where $u_0 = (1/2)\tan^{-1}\sqrt{8} = 35.3°$. Thus, for $u$ near $n\pi/2$ the matrices in the subspace $\mathbb{B}(u, v)$ all resemble double couples, while for $u$ near $\pm u_0 + n\pi$, they vary from double couples nearly to CLVDs.

The subspace $\mathbb{B}(u_0, v)$ also has the special property that its matrices all have a common eigenframe. See Section S3. FIG. 12

## 8.3 Elastic maps with symmetry $Z_{2\pi/3}$

We now give an intrinsic characterization of elastic maps $\mathbf{T}$ that have symmetry $Z_{2\pi/3}$. The reasoning is the same as for regular $\xi$ in Section 4.3, but now the prime summands are as in Eq. (114). From Eqs. (78), the map $\mathbf{T}$ is determined by giving $t, u, v$ to specify the prime summands, and by assigning respective numbers $\lambda_1, \lambda_3, \lambda_5, \lambda_6$ to them. That is, $\mathbf{T} = \mathbf{T}_3^\Lambda(t, u, v)$, where

$$\begin{gathered} \mathbf{T}_3^\Lambda(t, u, v) = \underset{\lambda_1\,\lambda_1}{\mathbb{B}(u, v)} \perp \underset{\lambda_3\,\lambda_3}{\mathbb{B}(u', v)} \perp \underset{\lambda_5}{\langle B_{56}(t)\rangle} \perp \underset{\lambda_6}{\langle B_{56}(t')\rangle} \\ \mathbf{\Lambda} = (\lambda_1, \lambda_1, \lambda_3, \lambda_3, \lambda_5, \lambda_6) \end{gathered} \tag{122}$$

**Theorem 9. (The matrix of $\mathbf{T}_3^\Lambda(t, u, v)$)**
For $\mathbf{T} = \mathbf{T}_3^\Lambda(t, u, v)$,

$$[\mathbf{T}]_{\mathbb{B}\mathbb{B}} = [\mathbf{I}]_{\mathbb{B}\,\mathbb{B}_3} \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_1 & & & & \\ & & \lambda_3 & & & \\ & & & \lambda_3 & & \\ & & & & \lambda_5 & \\ & & & & & \lambda_6 \end{pmatrix} [\mathbf{I}]_{\mathbb{B}_3\mathbb{B}} \tag{123}$$

where $[\mathbf{I}]_{\mathbb{B}\,\mathbb{B}_3}$ is the $6 \times 6$ matrix

$$[\mathbf{I}]_{\mathbb{B}\,\mathbb{B}_3} = \begin{pmatrix} (\cos u) I_{2\times 2} & -(\sin u) I_{2\times 2} & \\ (\sin u) R(v) & (\cos u) R(v) & \\ & & R(t) \end{pmatrix} \tag{124}$$

*Proof.* From Eq. (122) the matrix of $\mathbf{T}$ with respect to $\mathbb{B}(t, u, v)$ is diagonal with diagonal entries $\lambda_1, \lambda_1, \lambda_3, \lambda_3, \lambda_5, \lambda_6$. It is related to $[\mathbf{T}]_{\mathbb{B}\mathbb{B}}$ by $[\mathbf{T}]_{\mathbb{B}\mathbb{B}} = [\mathbf{I}]_{\mathbb{B}\mathbb{B}_3}\,[\mathbf{T}]_{\mathbb{B}_3\mathbb{B}_3}\,[\mathbf{I}]_{\mathbb{B}_3\mathbb{B}}$, where $[\mathbf{I}]_{\mathbb{B}\mathbb{B}_3}$ is the matrix that takes $\mathbb{B}(t, u, v)$-coordinates to $\mathbb{B}$-coordinates. From Eqs. (32), (3), (116), the matrix $[\mathbf{I}]_{\mathbb{B}\mathbb{B}_3}$ is as stated in Eq, (124). $\square$

The matrix $[\mathbf{T}]_{\mathbb{B}\mathbb{B}}$ in Theorem 9 dictates the definition of the matrix $T_3$ in Table 1. The two matrices are the same when the entries of $T_3$ are

$$\begin{aligned} a &= \lambda_1 \cos^2 u + \lambda_3 \sin^2 u \\ c &= \lambda_3 \cos^2 u + \lambda_1 \sin^2 u \\ h &= \frac{1}{2}(\lambda_3 - \lambda_1)\sin 2u \sin v \\ m &= \frac{1}{2}(\lambda_1 - \lambda_3)\sin 2u \cos v \\ &e, f, k \text{ are as in Eqs. (88)} \end{aligned} \tag{125}$$

The two matrices are also the same if $h^2 + m^2 \neq 0$ and

$$\begin{aligned} u &= \theta_u = \frac{1}{2}\widehat{\theta}\big(a - c,\, 2\sqrt{h^2 + m^2}\big) \\ v &= \theta_v = \widehat{\theta}(m, -h) \\ \lambda_1 &= \frac{1}{2}\left(a + c + \sqrt{(a-c)^2 + 4(h^2+m^2)}\right) \\ \lambda_3 &= \frac{1}{2}\left(a + c - \sqrt{(a-c)^2 + 4(h^2+m^2)}\right) \\ &t, \lambda_5, \lambda_6 \text{ are as in Eqs. (89)} \end{aligned} \tag{126}$$

Verification of Eqs. (126) is just a calculation, though best done by computer. (One might nevertheless wonder where the equations come from. See Appendix D). For the case $h = m = 0$ that is ruled out in Eqs. (126), the matrix $T_3$ becomes $T_{\text{XISO}}$ and hence is covered by Eqs. (89).

For $\mathbf{T}$ having symmetry $Z_{2\pi/3}$, Eqs. (125) give the matrix entries $a, c, e, f, h, k, m$ of $T_3 = [\mathbf{T}]_{\mathbb{BB}}$ in terms of the intrinsic parameters $t, u, v, \lambda_1, \lambda_3, \lambda_5, \lambda_6$ of $\mathbf{T}$, and Eqs. (126) do the reverse. As usual, the intrinsic parameters are not unique. The following 7-tuples of intrinsic parameters all give the same $\mathbf{T}$. One 7-tuple is usually enough, however.

| $t$ | $u$ | $v$ | $\lambda_1$ | $\lambda_3$ | $\lambda_5$ | $\lambda_6$ |
|---|---|---|---|---|---|---|
| $t$ | $u$ | $v$ | $\mu_1$ | $\mu_3$ | $\mu_5$ | $\mu_6$ |
| $t$ | $-u$ | $v+\pi$ | $\mu_1$ | $\mu_3$ | $\mu_5$ | $\mu_6$ |
| $t$ | $u+\pi/2$ | $v$ | $\mu_3$ | $\mu_1$ | $\mu_5$ | $\mu_6$ |
| $t+\pi/2$ | $u$ | $v$ | $\mu_1$ | $\mu_3$ | $\mu_6$ | $\mu_5$ |

(127)

We now have the three equivalent conditions:

$$Z_{2\pi/3} \text{ is a symmetry of } \mathbf{T} \tag{128a}$$

$$\mathbf{T} = \mathbf{T}_3^{\Lambda}(t,u,v) \text{ for some } t, u, v, \lambda_1, \lambda_3, \lambda_5, \lambda_6 \tag{128b}$$

$$[\mathbf{T}]_{\mathbb{BB}} = T_3(a,c,e,f,h,k,m) \text{ for some } a,c,e,f,h,k,m \tag{128c}$$

## 9 Elastic maps with symmetry $Z_\xi$ when $\xi = 0$

It remains to treat $\xi = 0$. The matrix $Z_\xi$ is then the identity matrix $I$. Since $\overline{I}$ is the identity transformation, all subspaces of $\mathbb{M}$ are invariant under $\overline{I}$, hence all one-dimensional subspaces are prime for $\overline{I}$. Any six one-dimensional and mutually orthogonal subspaces are therefore prime summands for $\overline{I}$. Basis elements for the subspaces can be specified by a $6 \times 6$ rotation matrix $U$; the columns of $U$ are the $\mathbb{B}$-coordinate vectors for the basis elements, call them $B_1(U), \ldots, B_6(U)$.

An elastic map $\mathbf{T}$ with symmetry $I$—that is, any elastic map whatsoever—is therefore determined by specifying $U$ to give the prime summands $\langle B_1(U)\rangle, \ldots, \langle B_6(U)\rangle$ and by specifying numbers $\lambda_1, \ldots, \lambda_6$ to be assigned to them:

$$\mathbf{T} = \underset{\lambda_1}{\langle B_1(U)\rangle} \perp \ldots \perp \underset{\lambda_6}{\langle B_6(U)\rangle} \qquad (U = U_{6\times 6}) \tag{129}$$

This is not new. The numbers $\lambda_1, \ldots, \lambda_6$ are the eigenvalues of $\mathbf{T}$, and $B_1(U), \ldots, B_6(U)$ are the eigenvectors.

The group of $6 \times 6$ rotation matrices has dimension 15, and so 15 real parameters would be required to specify $U$.

## 10 How the symmetries change when the material is rotated

Elastic maps $\mathbf{T}$ and $\mathbf{T}'$ are defined to be equivalent if there is a matrix $U \in \mathbb{U}$ such that

$$\mathbf{T}' = \overline{U} \circ \mathbf{T} \circ \overline{U}^* \tag{130}$$

Section 3.2 gave some motivation for the definition; the maps $\mathbf{T}$ and $\mathbf{T}'$ can be regarded as describing the elasticity in a material before and after rotating the material using $U$. Section S2 has a test for equivalence of elastic maps whose eigenvalues are simple.

We denote the group of symmetries of $\mathbf{T}$ by $\mathcal{S}_{\mathbf{T}}$:

$$\mathcal{S}_{\mathbf{T}} = \{V \in \mathbb{U} \,:\, V \text{ is a symmetry of } \mathbf{T}\} \tag{131}$$

Then a group $\mathcal{U}$ of rotations is said to be an elastic symmetry group if $\mathcal{U} = \mathcal{S}_{\mathbf{T}}$ for some elastic map $\mathbf{T}$.

For $U$ and $V$ both in $\mathbb{U}$, and with $\mathbf{T}' = \overline{U} \circ \mathbf{T} \circ \overline{U}^*$,

$$\overline{V} \circ \mathbf{T} \circ \overline{V}^* = \mathbf{T} \iff \overline{UVU^\top} \circ \mathbf{T}' \circ (\overline{UVU^\top})^* = \mathbf{T}' \tag{132}$$

Thus $V$ is a symmetry of $\mathbf{T}$ if and only if $UVU^\top$ is a symmetry of $\mathbf{T}'$.

If $\mathbf{T}$ and $\mathbf{T}'$ are equivalent, then their symmetry groups $\mathcal{S}_{\mathbf{T}}$ and $\mathcal{S}_{\mathbf{T}'}$ are conjugate. More precisely, if $\mathbf{T}' = \overline{U} \circ \mathbf{T} \circ \overline{U}^*$ then

$$\mathcal{S}_{\mathbf{T}'} = U \mathcal{S}_{\mathbf{T}} U^\top, \tag{133}$$

where $U\mathcal{S}_{\mathbf{T}}U$ consists of all matrices $UVU^\top$, $V \in \mathcal{S}_{\mathbf{T}}$.

### 10.1 Orientation information in $\mathbf{T}_4^{\Lambda}$, $\mathbf{T}_3^{\Lambda}$, $\mathbf{T}_2^{\Lambda}$

From here up until Section 16, virtually all of the matrix representations are with respect to the basis $\mathbb{B}$. When feasible we therefore drop the subscript and write $[\mathbf{T}]$ for $[\mathbf{T}]_{\mathbb{BB}}$.

Recall that conjugation of $\mathbf{T}$ by $\overline{U}$ formally expresses the effect on $\mathbf{T}$ of rotating the material using the matrix $U \in \mathbb{U}$. Recall also that $\mathbf{T}_4^{\Lambda}(s,t)$, $\mathbf{T}_3^{\Lambda}(t,u,v)$, $\mathbf{T}_2^{\Lambda}(r,U)$, and $\mathbf{T}_{\text{XISO}}^{\Lambda}(t)$ are the most general elastic maps having the respective symmetries $Z_{\pi/2}$, $Z_{2\pi/3}$, $Z_\pi$, and $Z_\xi$ for $\xi$ regular. Since $Z_\beta$ is a symmetry of $\mathbf{T}_{\text{XISO}}^{\Lambda}$, rotating by $Z_\beta$ has no effect on $\mathbf{T}_{\text{XISO}}^{\Lambda}$, but for most $\beta$ it does impact $\mathbf{T}_4^{\Lambda}$, $\mathbf{T}_3^{\Lambda}$, and $\mathbf{T}_2^{\Lambda}$. Thus

$$\overline{Z_\beta} \circ \mathbf{T}_4^{\Lambda}(s,t) \circ \overline{Z_\beta}^* = \mathbf{T}_4^{\Lambda}(s + 2\beta,\, t) \tag{134}$$

$$\overline{Z_\beta} \circ \mathbf{T}_3^{\Lambda}(t,u,v) \circ \overline{Z_\beta}^* = \mathbf{T}_3^{\Lambda}(t,\, u,\, v + 3\beta) \tag{135}$$

$$\overline{Z_\beta} \circ \mathbf{T}_2^{\Lambda}(r,U) \circ \overline{Z_\beta}^* = \mathbf{T}_2^{\Lambda}\left(r - \beta,\, \begin{pmatrix} R(2\beta) & \\ & I_{2\times 2} \end{pmatrix} U\right) \tag{136}$$

Eqs. (134) and (136) follow by inspection of the matrix $[\overline{Z_\beta}]_{\mathbb{BB}}$ (Eq. 83) and the matrices of $\mathbf{T}_4^{\Lambda}$ and $\mathbf{T}_2^{\Lambda}$ in Theorems (6) and (7). Eq. (135) follows from Theorem 9 and from the fact that (from Eqs. 83 and 124)

$$\begin{aligned} &[\overline{Z_\beta}]_{\mathbb{BB}}\, [\mathbf{I}]_{\mathbb{B}\,\mathbb{B}(t,u,v)} \\ &\quad = [\mathbf{I}]_{\mathbb{B}\,\mathbb{B}(t,u,v+3\beta)} \begin{pmatrix} R(-\beta) & & \\ & R(-\beta) & \\ & & I_{2\times 2} \end{pmatrix} \end{aligned} \tag{137}$$

From Eq. (134), the elastic mappings $\mathbf{T}_4^{\Lambda}(s,t)$ and $\mathbf{T}_4^{\Lambda}(s+2\beta,t)$ are equivalent. They describe a material having symmetry $Z_{\pi/2}$, before and after being rotated by $Z_\beta$. The elastic maps $\mathbf{T}_3^{\Lambda}(t,u,v)$ and $\mathbf{T}_3^{\Lambda}(t,\,u,\,v+3\beta)$ likewise describe a material having symmetry $Z_{2\pi/3}$, before and after being rotated by $Z_\beta$. Section S1 has the very simple matrix equivalents of Eqs. (134) and (135).

## 11 Unanticipated but unavoidable symmetries

We have considered elastic maps $\mathbf{T}$ that have symmetry $Z_\xi$. Except when $\xi = n\pi$, the map $\mathbf{T}$ turns out to have other (non-trivial) symmetries as well, perhaps unexpected. Fig. 6, for example, showed prime summands for $\overline{Z_\xi}$ when $\xi$ is regular. The beachballs for the prime summands obviously have every horizontal (i.e., in the plane of the paper) axis as a 2-fold axis of symmetry. Thus,

**Theorem 10. (A regular axis requires an orthogonal 2-fold axis.)**
If an elastic map $\mathbf{T}$ has the symmetry $Z_\xi$ for some regular $\xi$, then it also has as symmetries all 2-fold rotations about horizontal axes. (It also has all rotations about the $z$-axis as symmetries, by Theorem 5.)

### 11.1 Symmetries accompanying a 4-fold rotation

G. 13

What about a map $\mathbf{T}$ that has symmetry $Z_{\pi/2}$? Prime summands for $\overline{Z_{\pi/2}}$ are shown in Fig. 9, and here again a horizontal 2-fold axis is obvious, now that we think to look for it. There are four of them, and Fig. 13 shows how to find them; they are at $\theta = s/2 + n\pi/4$. Thus,

**Theorem 11. (A 4-fold axis requires an orthogonal 2-fold axis.)**
If an elastic map $\mathbf{T}$ has the symmetry $Z_{\pi/2}$, so that $\mathbf{T} = \mathbf{T}_4^{\Lambda}(s,t)$ for some $s,t,\mathbf{\Lambda}$, then it has four 2-fold symmetries with horizontal axes at $\theta = s/2 + n\pi/4$.

In terms of the entries $a, c, \ldots$ of the matrix $T_4$ the horizontal 2-fold axes of $\mathbf{T}$ are at $\theta = \theta_4/2 + n\pi/4$, where, from Eq. (97), $\theta_4 = \frac{1}{2}\,\widehat{\theta}(c-d,\,2i)$.

As one might expect, there is nothing special about the 4-fold axis for $\mathbf{T}$ being vertical. If $\mathbf{T}$ has a 4-fold symmetry with axis in the direction of $\mathbf{v} \in \mathbb{R}^3$ then it also has four 2-fold symmetries with axes perpendicular to $\mathbf{v}$.

### 11.2 Symmetries accompanying a 3-fold rotation

We consider $\mathbf{T} = \mathbf{T}_3^{\Lambda}(t,u,v)$—the most general elastic map having a 3-fold symmetry with vertical axis. A 2-fold symmetry with horizontal axis would have the form $V = Z_\beta Y_\pi Z_\beta^\top$ for some $\beta$. Using Eqs. (59) and (116), we calculate the matrix $[\overline{V}]_{\mathbb{B}_3\mathbb{B}_3} = (t_{ij})$ of $\overline{V}$ with respect to the basis $\mathbb{B}_3 = \mathbb{B}(t,u,v)$ and find that it has the unwanted entry

$$t_{31} = \sin 2u \sin(v-3\beta)\sin(v-\beta) \tag{138}$$

We therefore try setting $\beta = v/3$. The matrix $[\overline{V}]_{\mathbb{B}_3\mathbb{B}_3}$ simplifies to

$$\begin{pmatrix} -\cos\frac{2v}{3} & \sin\frac{2v}{3} & & & & \\ \sin\frac{2v}{3} & \cos\frac{2v}{3} & & & & \\ & & -\cos\frac{2v}{3} & \sin\frac{2v}{3} & & \\ & & \sin\frac{2v}{3} & \cos\frac{2v}{3} & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \tag{139}$$

The prime summands $\mathbb{B}(u,v)$, $\mathbb{B}(u',v)$, $\langle B_{56}(t)\rangle$, $\langle B_{56}(t')\rangle$—and hence the eigenspaces of $\mathbf{T}$—are therefore invariant under $\overline{V}$, where $V = Z_{v/3} Y_\pi Z_{v/3}^\top = Z_\theta X_\pi Z_\theta^\top$ is now the $180^\circ$ rotation about the horizontal axis in the $\theta = \pi/2 + v/3$ direction. The rotation $V$ must be a symmetry of $\mathbf{T}$, by Theorem 1. Thus,

**Theorem 12. (A 3-fold axis requires an orthogonal 2-fold axis.)**
If an elastic map $\mathbf{T}$ has the symmetry $Z_{2\pi/3}$, so that $\mathbf{T} = \mathbf{T}_3^{\Lambda}(t,u,v)$ for some $t,u,v,\mathbf{\Lambda}$, then it has three 2-fold symmetries with horizontal axes at $\theta = \pi/2 + v/3 + n\pi/3$.

In terms of the entries $a, c, \ldots$ of the matrix $T_3$, the 2-fold axes of $\mathbf{T}$ are at $\theta = \pi/2 + \theta_v/3 + n\pi/3$, where $\theta_v = \widehat{\theta}(m,-h)$ (Eq. 126).

There must be a 2-fold axis (three in fact) in Fig. 11, but it does not give itself away. Fig. 14 offers some help.

Consistent with Theorems 11 and 12, Fedorov (1968) recognized that the distinctions between what are effectively our matrices $T_4$ and $T_{\text{TET}}$, and between our $T_3$ and $T_{\text{TRIG}}$, are only distinctions in orientation; it is a matter of where the 2-fold axes fall. (See Table 4 for $T_{\text{TET}}$ and $T_{\text{TRIG}}$.)

> A baseless division into two groups of the classes in the tetragonal and trigonal systems is used in many works. . . (Fedorov 1968, p 31)

FIG. 14

### 11.3 A 2-fold axis requires no other 2-fold axes

An elastic map $\mathbf{T}$ with the 2-fold symmetry $Z_\pi$ may fail to have a horizontal 2-fold axis, but it has an orientation marker nonetheless. For $\mathbf{T} = \mathbf{T}_2^{\Lambda}(r,U)$, the eigenvector $B_{12}(r)$ is a double couple with a horizontal fault plane. Its null axis, necessarily horizontal, is in the direction $\theta = -r$, as for example in Fig. 10, where $r = 30^\circ$.

### 11.4 The meaning of the parameters $r, s, t, u, v$

Eqs. (91b), (99c), (108c), (128b) were supposed to give conceptual characterizations of elastic maps having various rotational symmetries about the $z$-axis. We can now fulfill that promise by explaining the parameters $r, s, t, u, v$ that appear in the equations. The numbers $s$ and $v$ are orientation parameters; they locate the respective 2-fold axes of $\mathbf{T}_4^{\Lambda}(s,t)$ and $\mathbf{T}_3^{\Lambda}(t,u,v)$, as described in Sections 11.1 and 11.2. The number $r$ is also an orientation parameter, as explained in Section 11.3, but rotating the hypothetical material about the $z$-axis changes both $r$ and $U$, as seen in Eq. (136). The number $u$ affects beachball patterns in the subspace $\mathbb{B}(u,v)$, as explained in Section 8.2. The number $t$ determines the pattern on the beachball for the crack matrix $B_{56}(t)$; see Eq. (93).

## 12 The reference subgroups of $\mathbb{U}$

Table 3 lists eight "reference" subgroups of $\mathbb{U}$; they are $\mathcal{U}_1, \mathcal{U}_{\text{MONO}}, \ldots, \mathcal{U}_{\text{ISO}}$. In Section 14 we will see that, for any elastic map $\mathbf{T}$, the group $\mathcal{S}_{\mathbf{T}}$ of its symmetries is a conjugate of one of the reference groups. In that sense there are only
ble 3 eight elastic symmetry groups.

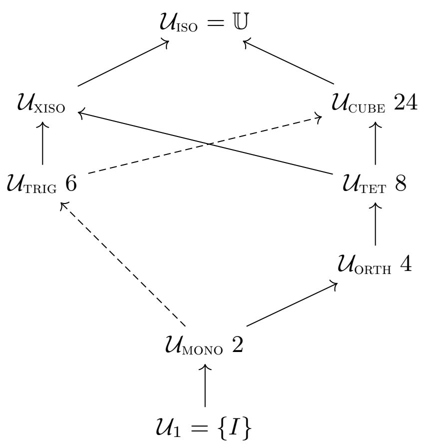
To elaborate on the reference groups:

The matrices in $\mathcal{U}_{\text{XISO}}$ are the rotational symmetries of a vertical cylinder.

The 24 rotational symmetries of any cube (the "gyroid" group) are the 4-fold rotations about the face centers of the cube, the 2-fold rotations about the midpoints of the edges, and the 3-fold rotations about the vertices. For $\mathcal{U}_{\text{CUBE}}$ the cube is oriented with its face centers on the $xyz$ coordinate axes. The matrices in $\mathcal{U}_{\text{CUBE}}$ are the $3 \times 3$ rotation matrices having exactly one non-zero entry in each row and column, and with that entry being $\pm 1$.

The 8 members of $\mathcal{U}_{\text{TET}}$ are the (rotational) symmetries of a square prism. The 6 members of $\mathcal{U}_{\text{TRIG}}$ are the symmetries of an equilateral triangular prism. The four members of $\mathcal{U}_{\text{ORTH}}$ are the symmetries of a brick. The two members of $\mathcal{U}_{\text{MONO}}$ are the symmetries of a wedge—an isosceles triangular prism.

From the third column of Table 3, the containments among the reference groups are



(140)

Solid arrows mean "is a subgroup of" and dashed arrows mean "is a subgroup of a group conjugate to." The integers give the number of elements in the group, if finite.

The subscripts MONO, ORTH, TET, TRIG are for the terms monoclinic, orthorhombic, tetragonal, and trigonal, which are relics from an era—not yet completely past—when crystallographic symmetries were thought to determine elastic symmetries. More informative terms would be wedge-like, brick-like, square-prismatic, and (equilateral) triangular-prismatic. A material whose elastic symmetry is square-prismatic, for example, can be sculpted into a square prism whose geometric symmetries are the same as its elastic symmetries. The term transverse isotropic would become
ble 4 cylindrical, and isotropic would become spherical.

### 12.1 Elastic maps for each reference group

For each reference group $\mathcal{U} = \mathcal{U}_1, \mathcal{U}_{\text{MONO}}, \ldots, \mathcal{U}_{\text{ISO}}$ we now give both an intrinsic and a matrix characterization of elastic maps whose symmetries are at least those in $\mathcal{U}$. The matrix characterizations are the "reference" matrices $T_1, T_{\text{MONO}}, \ldots, T_{\text{ISO}}$ in Table 4.

Each matrix characterization can be verified using the $\Delta$-test of Eqs. (70). In most cases the intrinsic characterization can then be found just by inspection of the matrix characterization. A fancier approach is to appeal to Theorem 7, which pertains to the vertical 2-fold symmetry $Z_\pi$. Since all of the reference groups except $\mathcal{U}_1$ and $\mathcal{U}_{\text{TRIG}}$ contain $Z_\pi$, then the intrinsic characterizations associated with the other six reference groups are special cases of that for $\mathcal{U}_{\text{MONO}}$ (Eq. 147b). (Their six reference matrices are likewise special cases of $T_{\text{MONO}}$, as seen in Table 4.)

The intrinsic characterizations are indeed intrinsic, in the sense that the subspaces in their orthogonal direct sums can be described without mentioning $\mathbb{B}$ or any other basis of $\mathbb{M}$. See Table 2.

We have talked about the notion of prime summands for an individual rotation matrix. The notion also makes sense for a group of rotation matrices; "invariant" then means invariant under $\overline{V}$ for all $V$ in the group. In each of the intrinsic characterizations below, the subspaces in the orthogonal direct sum are prime summands for the relevant reference group.

Recall that $\mathcal{S}_{\mathbf{T}}$ is the group of symmetries of the elastic map $\mathbf{T}$.

#### *12.1.1 Reference group $\mathcal{U}_{\text{ISO}}$*

The condition $\mathcal{S}_{\mathbf{T}} = \mathcal{U}_{\text{ISO}}$ is equivalent to each of

$$[\mathbf{T}] = T_{\text{ISO}}(a, f) \text{ for some } a \text{ and } f \tag{141a}$$

$$\mathbf{T} = \underset{\lambda_1\,\lambda_1\,\lambda_1\,\lambda_1\,\lambda_1}{\langle B_6\rangle^\perp} \perp \underset{\lambda_6}{\langle B_6\rangle} \text{ for some } \lambda_1 \text{ and } \lambda_6 \tag{141b}$$

The subspace $\langle B_6\rangle^\perp$ is $\langle B_1, B_2, B_3, B_4, B_5\rangle$ in Table 2; it consist of the deviatoric matrices.

#### *12.1.2 Reference group $\mathcal{U}_{\text{XISO}}$*

The condition $\mathcal{S}_{\mathbf{T}} \supset \mathcal{U}_{\text{XISO}}$ is equivalent to each of

$$[\mathbf{T}] = T_{\text{XISO}}(a, c, e, f, k) \text{ for some } a, c, e, f, k \tag{142a}$$

$$\mathbf{T} = \underset{\lambda_1\,\lambda_1}{\mathbb{B}_{12}} \perp \underset{\lambda_3\,\lambda_3}{\mathbb{B}_{34}} \perp \underset{\lambda_5}{\langle B_{56}(t)\rangle} \perp \underset{\lambda_6}{\langle B_{56}(t')\rangle}$$
$$\text{for some } t, \lambda_1, \lambda_3, \lambda_5, \lambda_6 \tag{142b}$$

#### *12.1.3 Reference group $\mathcal{U}_{\text{CUBE}}$*

The condition $\mathcal{S}_{\mathbf{T}} \supset \mathcal{U}_{\text{CUBE}}$ is equivalent to each of

$$[\mathbf{T}] = T_{\text{CUBE}}(a, d, f) \text{ for some } a, d, f \tag{143a}$$

$$\mathbf{T} = \underset{\lambda_1\,\lambda_1\,\lambda_1}{\langle B_1, B_2, B_3\rangle} \perp \underset{\lambda_4\,\lambda_4}{\langle B_4, B_5\rangle} \perp \underset{\lambda_6}{\langle B_6\rangle} \text{ for some } \lambda_1, \lambda_4, \lambda_6 \tag{143b}$$

#### *12.1.4 Reference group $\mathcal{U}_{\text{TET}}$*

The condition $\mathcal{S}_{\mathbf{T}} \supset \mathcal{U}_{\text{TET}}$ is equivalent to each of

$$[\mathbf{T}] = T_{\text{TET}}(a,c,d,e,f,k) \text{ for some } a,c,d,e,f,k \quad (144a)$$

$$\mathbf{T} = \underset{\lambda_1\,\lambda_1}{\mathbb{B}_{12}} \perp \underset{\lambda_3}{\langle B_3\rangle} \perp \underset{\lambda_4}{\langle B_4\rangle} \perp \underset{\lambda_5}{\langle B_{56}(t)\rangle} \perp \underset{\lambda_6}{\langle B_{56}(t')\rangle}$$
$$\text{for some } t, \lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \quad (144b)$$

The matrix $T_{\text{TET}}$ is the special case of $T_4$ where the horizontal two-fold axes are at $\theta = n\pi/4$.

#### *12.1.5 Reference group $\mathcal{U}_{\text{ORTH}}$*

Here $U$ is a $3\times3$ rotation matrix $U = (u_{ij})_{i,j=4,5,6}$. Matrices $B_j(U)$ are defined by

$$B_j(U) = u_{4j}B_4 + u_{5j}B_5 + u_{6j}B_6 \qquad (j = 4,5,6) \quad (145)$$

Then the condition $\mathcal{S}_{\mathbf{T}} \supset \mathcal{U}_{\text{ORTH}}$ is equivalent to each of

$$[\mathbf{T}] = T_{\text{ORTH}}(a,b,\ldots) \text{ for some } a,b,\ldots \quad (146a)$$

$$\mathbf{T} = \underset{\lambda_1}{\langle B_1\rangle} \perp \underset{\lambda_2}{\langle B_2\rangle} \perp \underset{\lambda_3}{\langle B_3\rangle} \perp \underset{\lambda_4}{\langle B_4(U)\rangle} \perp \underset{\lambda_5}{\langle B_5(U)\rangle} \perp \underset{\lambda_6}{\langle B_6(U)\rangle}$$
$$\text{for some } U, \lambda_1, \ldots \lambda_6 \quad (146b)$$

Eq. (146b) is simpler than it appears, since the matrices $B_4(U), B_5(U), B_6(U)$ are a basis for the subspace $\langle B_4, B_5, B_6\rangle$ consisting of the diagonal matrices (Table 2).

#### *12.1.6 Reference group $\mathcal{U}_{\text{MONO}}$*

The condition $\mathcal{S}_{\mathbf{T}} \supset \mathcal{U}_{\text{MONO}}$ is equivalent to each of

$$[\mathbf{T}] = T_{\text{MONO}}(a,b,\ldots) \text{ for some } a,b,\ldots, \quad (147a)$$

$$\mathbf{T} = \underset{\lambda_1}{\langle B_{12}(r)\rangle} \perp \underset{\lambda_2}{\langle B_{12}(r')\rangle} \perp \underset{\lambda_3}{\langle B_3(U)\rangle} \perp \underset{\lambda_4}{\langle B_4(U)\rangle} \perp$$
$$\underset{\lambda_5}{\langle B_5(U)\rangle} \perp \underset{\lambda_6}{\langle B_6(U)\rangle} \text{ for some } r, U, \lambda_1, \ldots \lambda_6 \quad (147b)$$

where in Eq. (147b) the matrix $U$ is now a $4\times4$ rotation matrix and where the $B_j(U)$ are as in Eq. (103). This is a repetition of Eqs. (108).

#### *12.1.7 Reference group $\mathcal{U}_{\text{TRIG}}$*

The condition $\mathcal{S}_{\mathbf{T}} \supset \mathcal{U}_{\text{TRIG}}$ is equivalent to each of

$$[\mathbf{T}] = T_{\text{TRIG}}(a,c,e,f,k,m) \text{ for some } a,c,e,f,k,m \quad (148a)$$

$$\mathbf{T} = \underset{\lambda_1\,\lambda_1}{\mathbb{B}(u,0)} \perp \underset{\lambda_3\,\lambda_3}{\mathbb{B}(u',0)} \perp \underset{\lambda_5}{\langle B_{56}(t)\rangle} \perp \underset{\lambda_6}{\langle B_{56}(t')\rangle}$$
$$\text{for some } t, u, \lambda_1, \lambda_3, \lambda_5, \lambda_6 \quad (148b)$$

The matrix $T_{\text{TRIG}}$ is the special case of $T_3$ where the horizontal 2-fold axes are at $\theta = \pi/2 + n\pi/3$. Thus the $y$-axis, not the $x$-axis, is one of the 2-fold axes.

#### *12.1.8 Reference group $\mathcal{U}_1 = \{I\}$*

The condition $\mathcal{S}_{\mathbf{T}} \supset \mathcal{U}_1$ is satisfied for all $\mathbf{T}$.

#### *12.1.9 Only a one-way test for $\mathcal{S}_{\mathbf{T}} \supset \mathcal{U}_{\text{MONO}}$*

Let

$$T^0_{\text{MONO}} = \begin{pmatrix} a &&&&& \\ & b &&&& \\ && c & i & o & s \\ && i & d & j & p \\ && o & j & e & k \\ && s & p & k & f \end{pmatrix} \quad (149)$$

The matrix $T^0_{\text{MONO}}$ is the matrix of $\mathbf{T}$ when $\mathcal{S}_{\mathbf{T}} \supset \mathcal{U}_{\text{MONO}}$ and when the double couple eigenvectors $B_{12}(r)$ and $B_{12}(r')$ of $\mathbf{T}$ are $B_1$ and $B_2$ (in either order), so that their null axes are in the $x$ and $y$ coordinate directions. The condition $[\mathbf{T}] = T^0_{\text{MONO}}$ implies $\mathcal{S}_{\mathbf{T}} \supset \mathcal{U}_{\text{MONO}}$, but the converse is false.

To describe all elastic maps having 2-fold symmetry with axis vertical, one wants the matrix $T_{\text{MONO}}$. On the other hand, every $\mathbf{T}$ having 2-fold symmetry is equivalent to an elastic map whose matrix with respect to $\mathbb{B}$ is $T^0_{\text{MONO}}$ (for some $a, b, \ldots$).

The matrices $T_{\text{MONO}}$ and $T^0_{\text{MONO}}$ are comparable to the matrices in Eqs. (3.29) of Helbig (1994).

## 13 Symmetry for a subspace of $\mathbb{M}$

When a subspace $\mathbb{W}$ of $\mathbb{M}$ is invariant under $\overline{V}$ we will also say that $V$ is a symmetry of $\mathbb{W}$:

$$V \text{ is a symmetry of } \mathbb{W} \iff \overline{V}(\mathbb{W}) \subset \mathbb{W} \quad (150)$$

We thus have two notions of symmetry: one for an elastic map $\mathbf{T}$ (Eq. 69), and one for a subspace $\mathbb{W}$ of $\mathbb{M}$. Theorem 13, next, relates the two notions. Due to the close relation, subspace symmetry will be our key to identifying the symmetry of elastic maps, in Section 15. For example, a consequence of Theorems 13 and 16 is that if an elastic map $\mathbf{T}$ has a simple eigenvalue whose eigenvector ($3\times3$ matrix) is generic (Fig. 7d), then the symmetry of $\mathbf{T}$ can only be orthorhombic, monoclinic, or trivial. Thus, trigonal, tetragonal, cubic, transverse isotropic, and isotropic symmetry can often be ruled out by casual inspection of the eigensystem for $\mathbf{T}$.

**Theorem 13.** A rotation $V \in \mathbb{U}$ is a symmetry of an elastic map $\mathbf{T}$ if and only if $V$ is a symmetry of each eigenspace of $\mathbf{T}$.

*Proof.* The theorem is just a paraphrase of Theorem 1. □

When $V$ is a symmetry of a one-dimensional subspace $\mathbb{W} = \langle E\rangle$, we will also say that $V$ is a symmetry of $E$ itself. Theorems 14 and 16 show that the symmetries of $E$ are easy to recognize from the beachball for $E$.

**Theorem 14.** A rotation $V \in \mathbb{U}$ is a symmetry of $E \in \mathbb{M}$ if and only if $\overline{V}(E) = \pm E$.

*Proof.* First suppose $V$ is a symmetry of $E$. Since $E \in \langle E\rangle$ then so is $\overline{V}(E)$. Since $\langle E\rangle$ is one-dimensional, then, for some number $t$,

$$\overline{V}(E) = t\,E \quad (151)$$

Since $\overline{V}$ is unitary, then $\|\overline{V}(E)\| = \|E\|$. Hence

$$\begin{aligned}\|\overline{V}(E)\| &= \|tE\|\\ \|E\| &= |t|\,\|E\|\\ t &= \pm 1\end{aligned} \tag{152}$$

Hence $\overline{V}(E) = \pm E$.

Conversely, suppose $\overline{V}(E) = \pm E$. If $F \in \langle E\rangle$, then $F = tE$ for some $t$, and $\overline{V}(F) = \overline{V}(tE) = t\overline{V}(E) = \pm tE \in \langle E\rangle$. Hence $V$ is a symmetry of $\langle E\rangle$. $\square$

The condition $\overline{V}(E) = -E$ severely constrains $E$. If $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $E$ in descending order, then $-\mu_3, -\mu_2, -\mu_1$ are the eigenvalues of $-E$ in descending order. Since for any $V \in \mathbb{U}$ the matrices $E$ and $\overline{V}(E)$ have the same eigenvalues, then $\overline{V}(E) = -E$ implies $\mu_3 = -\mu_1$ and $\mu_2 = 0$. Thus,

$$\overline{V}(E) = -E \implies E \text{ is a double couple} \tag{153}$$

We mentioned in Section 3.1.1 that the beachball for $\overline{V}(E)$ is the result of applying the rotation $V$ to the beachball for $E$. Informally, Theorem 14 says that $V$ is a symmetry of $E$ if and only if the rotated ball differs from the original ball by at most a swapping of red with white. This of course assumes that the ball for $E$ is bicolored, not just one solid color.

Fig. 9 illustrates Theorem 14 and Eq. (153). The rotation $Z_{\pi/2}$ is a symmetry of each of the five subspaces in the figure. The one-dimensional subspaces are $\langle B_{34}(s)\rangle$, $\langle B_{34}(s')\rangle$, $\langle B_{56}(t)\rangle$, and $\langle B_{56}(t')\rangle$. Using $Z_{\pi/2}$ to rotate the beachballs for the matrices $B_{56}(t)$ and $B_{56}(t')$ has no effect on the appearance of the balls. Doing the same for $B_{34}(s)$ and $B_{34}(s')$, which are double couples, has the effect of reversing red and white on each ball.

We denote the group of symmetries of $E \in \mathbb{M}$ by $\mathcal{S}(E)$:

$$\mathcal{S}(E) = \{V \in \mathbb{U} : V \text{ is a symmetry of } E\} \tag{154}$$

For a subspace $\mathbb{W}$ of $\mathbb{M}$, we likewise use the notation $\mathcal{S}(\mathbb{W})$ to refer to the group of symmetries of $\mathbb{W}$. Given $\mathbb{W}$, we can consider the elastic map $\mathbf{T}$ such that

$$\mathbf{T} = \underset{1}{\mathbb{W}} \perp \underset{2}{\mathbb{W}^\perp} \tag{155}$$

Since the symmetries of $\mathbb{W}$ are the same as those of $\mathbb{W}^\perp$ (Eq. 46), they are also the symmetries of $\mathbf{T}$, by Theorem 13. The group $\mathcal{S}(\mathbb{W})$ is therefore an elastic symmetry group.

There are not many possibilities for a symmetry $V$ of $E$.

**Theorem 15.** **($\mathcal{S}(E)$ for diagonal $E$)**

$$E = \begin{pmatrix}\mu & & \\ & \mu & \\ & & \mu\end{pmatrix} \implies \mathcal{S}(E) = \mathcal{U}_{\text{ISO}} \tag{156a}$$

$$E = \begin{pmatrix}\mu & & \\ & \mu & \\ & & \mu_3\end{pmatrix} \implies \mathcal{S}(E) = \mathcal{U}_{\text{XISO}} \quad (\mu \neq \mu_3) \tag{156b}$$

$$E = \begin{pmatrix}\mu & & \\ & -\mu & \\ & & 0\end{pmatrix} \implies \mathcal{S}(E) = \mathcal{U}_{\text{TET}} \tag{156c}$$

If $E$ is diagonal and generic (Eq. 92) then $\mathcal{S}(E) = \mathcal{U}_{\text{ORTH}}$ (156d)

*Proof.* The theorem should seem plausible just from beachball pictures. For algebraic proofs of Eqs. (156b) and (156d) see Appendix A of Tape & Tape (2012). A variation of the argument for Eq. (156d) shows that if $E$ is a double couple then the rotations $V$ that give $\overline{V}(E) = -E$ are the two 180° rotations about the fault plane normals, together with the ±90° rotations about the null axis. Together with Eq. (156d), this gives Eq. (156c). $\square$

From Theorem 15 we get, more generally,

**Theorem 16.** **($\mathcal{S}(E)$ for arbitrary $E$)**

(i) If $E$ is generic then its symmetry group $\mathcal{S}(E)$ is conjugate to $\mathcal{U}_{\text{ORTH}}$. The non-trivial members of $\mathcal{S}(E)$ are the three 2-fold rotations about the principal axes of $E$.

(ii) If $E$ is a double couple, then $\mathcal{S}(E)$ is conjugate to $\mathcal{U}_{\text{TET}}$. The null axis of $E$ is the 4-fold axis of $\mathcal{S}(E)$, and the T and P axes of $E$ are two of the 2-fold axes of $\mathcal{S}(E)$.

(iii) If $E$ is a crack matrix, then $\mathcal{S}(E)$ is conjugate to $\mathcal{U}_{\text{XISO}}$. The $c$-axis of $E$ is the regular axis of $\mathcal{S}(E)$.

Thus the symmetry of $E$ is obvious from the (perhaps perturbed) beachball for $E$.

Symmetry groups $\mathcal{S}(\mathbb{W})$ for selected subspaces $\mathbb{W}$ of $\mathbb{M}$ were given in Table 2. As an example, we derive $\mathcal{S}(\mathbb{W})$ for $\mathbb{W} = \langle B_4, B_5\rangle$. The subspace $\mathbb{W}$ is $\langle B_6\rangle^\perp \cap \langle B_4, B_5, B_6\rangle$—the diagonal matrices that are deviatoric. Since conjugation preserves matrix trace, then $\mathcal{S}\langle B_4, B_5, B_6\rangle \subset \mathcal{S}(\mathbb{W})$. Conversely, $\mathcal{S}(\mathbb{W}) \subset \mathcal{S}\langle B_4, B_5, B_6\rangle$, since if $V \in \mathcal{S}(\mathbb{W})$ and $F \in \langle B_4, B_5, B_6\rangle$, then $F = E + tI$ for some $E \in \mathbb{W}$ and $t \in \mathbb{R}$, and

$$\overline{V}(F) = \overline{V}(E+tI) = \overline{V}(E)+t\overline{V}(I) = \underbrace{\overline{V}(E)}_{\in \mathbb{W}}+tI \in \langle B_4, B_5, B_6\rangle \tag{157}$$

Hence $\mathcal{S}(\mathbb{W}) = \mathcal{S}\langle B_4, B_5, B_6\rangle$. From Proposition 1 of Tape & Tape (2016) we know that $\mathcal{S}\langle B_4, B_5, B_6\rangle = \mathcal{U}_{\text{CUBE}}$. Thus

$$\mathcal{S}\langle B_4, B_5\rangle = \mathcal{S}\langle B_4, B_5, B_6\rangle = \mathcal{U}_{\text{CUBE}} \tag{158}$$

**Theorem 17.** The eight reference groups are elastic symmetry groups. That is, for each reference group $\mathcal{U}$ (Table 3) there is an elastic map $\mathbf{T}$ for which $\mathcal{S}_{\mathbf{T}} = \mathcal{U}$.

*Proof.*

(i)–(v) Each of $\mathcal{U}_{\text{ISO}}, \mathcal{U}_{\text{XISO}}, \mathcal{U}_{\text{TET}}, \mathcal{U}_{\text{ORTH}}, \mathcal{U}_{\text{CUBE}}$ is the symmetry group of a subspace of $\mathbb{M}$ (Theorem 15 and Eq. 158) and hence is an elastic symmetry group; see Eq. (155).

(vi) For $\mathcal{U} = \mathcal{U}_{\text{MONO}}$. We can take

$$\mathbf{T} = \underset{\lambda_1}{\langle B_1\rangle} \perp \underset{\lambda_2}{\langle B_2\rangle} \perp \underset{\lambda_3}{\langle B_{34}(s)\rangle} \perp \underset{\lambda_4}{\langle B_{34}(s')\rangle} \perp \underset{\lambda_5}{\langle B_{56}(t)\rangle} \perp \underset{\lambda_6}{\langle B_{56}(t')\rangle}, \tag{159}$$

where, say, $s = 55°$, $t = 40°$, and where $\lambda_1, \ldots, \lambda_6$ are distinct, so that each of $\langle B_1\rangle, \ldots, \langle B_{56}(t')\rangle$ is an eigenspace of $\mathbf{T}$. The beachballs for $B_1, \ldots, B_{56}(t')$ appear in Fig. 9, the balls for $B_1$ and $B_2$ being at $(x_1, x_2) = (1, 0)$ and $(x_1, x_2) = (0, 1)$. The symmetries common to all six beachballs are $Z_\pi$ and $I$, hence $\mathcal{S}_{\mathbf{T}} = \mathcal{U}_{\text{MONO}}$.

(vii) For $\mathcal{U} = \mathcal{U}_{\text{TRIG}}$. We can take $\mathbf{T}$ as in Eq. (148b) with $\lambda_1 = \lambda_6 = 1$, $\lambda_3 = 2$, $\lambda_5 = 3$, $t = v = 0$, and $u = \pi/4$. Then $\langle B_5\rangle$ is an eigenspace of $\mathbf{T}$ and has symmetry group $\mathcal{U}_{\text{XISO}}$, hence $\mathcal{S}_{\mathbf{T}} \subset \mathcal{U}_{\text{XISO}}$. The members of $\mathcal{U}_{\text{XISO}}$ are the rotations $Z_\xi$ (any $\xi$) and the horizontal 2-fold rotations $Z_\theta X_\pi Z_\theta^\top$ (any $\theta$). Which of them are in $\mathcal{S}_{\mathbf{T}}$? The matrix of $\mathbf{T}$ with respect to

$\mathbb{B}$ is

$$[\mathbf{T}] = \frac{1}{2}\begin{pmatrix} 3 & & -1 & & & \\ & 3 & & -1 & & \\ -1 & & 3 & & & \\ & -1 & & 3 & & \\ & & & & 6 & \\ & & & & & 2 \end{pmatrix} \tag{160}$$

Using the Δ-test, we find that $Z_\xi$ is a symmetry of $\mathbf{T}$ if and only if $\xi = n2\pi/3$, and $Z_\theta X_\pi Z_\theta^\top$ is a symmetry of $\mathbf{T}$ if and only if $\theta = \pi/2 + n\pi/3$. Hence $\mathcal{S}_\mathbf{T} = \mathcal{U}_{\text{TRIG}}$.

(viii) For $\mathcal{U} = \mathcal{U}_1$. Let $\mathbf{T}$ be as in Eq. (23). The eigenvalues of $\mathbf{T}$ are $3/5, 4/5, \ldots, 8/5$. Eigenvectors for eigenvalues $3/5$ and $4/5$ are

$$G_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 1 & 0 \\ \sqrt{3} & 0 & -2 \end{pmatrix} \tag{161}$$

The matrices $G_1$ and $G_2$ are generic and have no principal axis in common. The non-trivial symmetries of any generic matrix are the three 2-fold rotations about its principal axes, so the only symmetry common to the eigenspaces $\langle G_1 \rangle$ and $\langle G_2 \rangle$ of $\mathbf{T}$ is the identity. Hence $\mathcal{S}_\mathbf{T} = \mathcal{U}_1$. Fig. 4 is the beachball picture for $\mathbf{T}$.

## 14 The elastic symmetry groups

In Theorem 18 below we show that the symmetry group of every elastic map is the conjugate of some reference group (Table 3). Together with Theorem 17 this means that the elastic symmetry groups are exactly the conjugates of the eight reference groups.

In a tour de force in their Section 6, Forte & Vianello (1996) detail the long history of the problem of determining the number of elastic symmetry groups. We cannot possibly do justice to their recounting of it. We only mention that in the older literature the seemingly natural groups $\{I, Z_{\pi/2}, Z_\pi, Z_{3\pi/2}\}$ and $\{I, Z_{2\pi/3}, Z_{4\pi/3}\}$ were incorrectly considered to be elastic symmetry groups (e.g., Nye 1957, 1985; Cowin et al. 1991). (The discussions were not explicitly in terms of elastic symmetry groups, so our paraphrase is loose.) This would bring the number of elastic symmetry groups to ten, not eight. This of course counts conjugate groups as the same.

Forte & Vianello (1996) gave a proof concluding that the correct number was eight, and other proofs appeared later, also concluding eight (e.g., Chadwick et al. 2001; Bóna et al. 2007). Our proof, also concluding eight, may nevertheless be of interest, due to its pedestrian approach. It mainly involves circles on a sphere, as in Figs. 16, 17, 18. The proof is tedious, however, in that Lemma 4 requires consideration of various cases.

Fortunately, the ideas in the proof of Lemma 4 are not needed elsewhere in the paper, so the proof can be skipped if desired. Neither Theorem 17 nor Theorem 18, however, should be regarded as mere formalities. Their conclusions are not obvious, as illustrated by the historical confusion over the two groups $\{I, Z_{\pi/2}, Z_\pi, Z_{3\pi/2}\}$ and $\{I, Z_{2\pi/3}, Z_{4\pi/3}\}$ alluded to above.

In connection with Lemma 2, a point $\mathbf{v}$ on the unit sphere is a "regular axis" for a group $\mathcal{U}$ of rotations if all rotations about $\mathbf{v}$ are in $\mathcal{U}$. Thus $\mathbf{v} = \pm 001$ are the regular axes for $\mathcal{U}_{\text{XISO}}$.

**Lemma 2.** If a group $\mathcal{U}'_{\text{XISO}}$ is conjugate to $\mathcal{U}_{\text{XISO}}$, then there is no group $\mathcal{U}$ strictly between $\mathcal{U}'_{\text{XISO}}$ and $\mathcal{U}_{\text{ISO}}$. That is,

$$\mathcal{U}'_{\text{XISO}} \subset \mathcal{U} \subset \mathcal{U}_{\text{ISO}} \implies (\mathcal{U} = \mathcal{U}'_{\text{XISO}} \text{ or } \mathcal{U} = \mathcal{U}_{\text{ISO}})$$

*Proof.* Suppose $\mathcal{U}'_{\text{XISO}} \subset \mathcal{U} \subset \mathcal{U}_{\text{ISO}}$. If $\mathcal{U} = \mathcal{U}'_{\text{XISO}}$, then we are done. If not, there is a rotation $U$ in $\mathcal{U} - \mathcal{U}'_{\text{XISO}}$. If $\mathbf{v}_1$ is one of the two regular axes for $\mathcal{U}'_{\text{XISO}}$, then both $\mathbf{v}_1$ and $\mathbf{v}_2 = U\mathbf{v}_1$ are regular axes for $\mathcal{U}$, with $\mathbf{v}_1 \neq \pm\mathbf{v}_2$. Then $\mathcal{U}$ is all of $\mathcal{U}_{\text{ISO}}$, as illustrated in Fig. 15. FIG. 15

**Lemma 3.** Let $\mathcal{U}$ be an elastic symmetry group containing distinct 2-fold rotations $V_1$ and $V_2$. Let $\alpha$ be the angle between their rotation axes, here considered as lines rather than vectors, so that $\alpha \leq 90°$. If $\alpha \neq 45, 60, 90°$, then $\mathcal{U}$ is either $\mathcal{U}_{\text{ISO}}$ or a conjugate of $\mathcal{U}_{\text{XISO}}$.

*Proof.* Since $V_1$ and $V_2$ are 2-fold, the product rotation $V_1 V_2$ has rotation angle $2\alpha$. (So does $V_2 V_1$; as vectors, the rotation axes of $V_1 V_2$ and $V_2 V_1$ are oppositely directed.) Since $\alpha \neq 0, 45, 60, 90°$, then $2\alpha$ is regular. Since $V_1 V_2 \in \mathcal{U}$, then $\mathcal{U}$ has a subgroup $\mathcal{U}'_{\text{XISO}}$ conjugate to $\mathcal{U}_{\text{XISO}}$, by Theorem 10. Thus $\mathcal{U}'_{\text{XISO}} \subset \mathcal{U}$, and so $\mathcal{U}$ must be $\mathcal{U}'_{\text{XISO}}$ or $\mathcal{U}_{\text{ISO}}$, from Lemma 2. □

We define a point $\mathbf{v}$ of the unit sphere to be an available 2-fold point for a group $\mathcal{U}$ of rotations if the angular distances between $\mathbf{v}$ and the axes of all 2-fold rotations in $\mathcal{U}$ are 45°, 60°, or 90°. Note that if $\mathbf{v}$ is an available 2-fold point for $\mathcal{U}$ then so is $-\mathbf{v}$.

**Lemma 4.** For a subgroup $\mathcal{U}$ of $\mathbb{U}$ and for a 2-fold rotation $V$, let $\mathcal{U}(V)$ be the smallest elastic symmetry group that contains $\mathcal{U}$ and $V$. Then if $\mathcal{U}$ is a conjugate of a reference group (Table 3), so is $\mathcal{U}(V)$.

*Proof.* Let $\mathbf{v}$ be one of the two points where the rotation axis of $V$ intersects the unit sphere.

(i) The case where $\mathbf{v}$ is not an available 2-fold point for $\mathcal{U}$. There is a 2-fold rotation $V' \in \mathcal{U}$ with rotation axis $\mathbf{v}'$ such that $\mathbf{v}' \cdot \mathbf{v} \geq 0$ and $\angle(\mathbf{v}, \mathbf{v}') \neq 45, 60, 90°$.

If $V' = V$, then $V \in \mathcal{U}$ and $\mathcal{U}(V) = \mathcal{U}$. (Note that $\mathcal{U}$ is itself an elastic symmetry group, by Theorem 17 and Eq. 133.)

If $V' \neq V$, then applying Lemma 3 to $\mathcal{U}(V)$ shows that $\mathcal{U}(V)$ is either $\mathcal{U}_{\text{ISO}}$ or a conjugate of $\mathcal{U}_{\text{XISO}}$.

(ii) The case where $\mathbf{v}$ is an available 2-fold point for $\mathcal{U}$.

If $\mathcal{U} = \mathcal{U}_1$, the group $\mathcal{U}(V)$ is a conjugate of $\mathcal{U}_{\text{MONO}}$.

If $\mathcal{U} = \mathcal{U}_{\text{MONO}}$, then the point $\mathbf{v}$ is 45°, 60°, or 90° from the north or south pole, and the group $\mathcal{U}(V)$ is a conjugate of $\mathcal{U}_{\text{TET}}$, $\mathcal{U}_{\text{TRIG}}$, or $\mathcal{U}_{\text{ORTH}}$, respectively. (If $\mathcal{U}$ is only a conjugate of $\mathcal{U}_{\text{MONO}}$ rather than being $\mathcal{U}_{\text{MONO}}$ itself, the conclusion does not change.)

If $\mathcal{U} = \mathcal{U}_{\text{ORTH}}$ the available 2-fold points for $\mathcal{U}$ are shown in Fig. 16; the point $\mathbf{v}$ must be one of them. From the figure, the points $\mathbf{v}_1 = 101$ and $\mathbf{v}_2 = 11\sqrt{2}$ are the only two essentially different possibilities for $\mathbf{v}$.

The case $\mathbf{v} = \mathbf{v}_1$: Since $\angle(\mathbf{v}, 100) = 45°$ and $\mathbf{v} \times 100 \propto 010$, then $\mathcal{U}(V)$ has a 4-fold axis at 010. The group $\mathcal{U}(V)$ is then the conjugate of $\mathcal{U}_{\text{TET}}$ that has 2-fold axes at $10\bar{1}$, 100, 101, and 001. (It is a subgroup of the group of symmetries of the dashed cube in Fig. 16b.)

The case $\mathbf{v} = \mathbf{v}_2$: Since $\mathbf{v}$ and the three 2-fold axes for

$\mathcal{U}$ are edge midpoints or face centers of the dashed cube in Fig. 16c, then $\mathcal{U}(V)$ must be a subgroup of the rotational symmetry group of the cube. Since $\angle(\mathbf{v}, 001) = 45^\circ$ and $\mathbf{v} \times 001 \propto 1\bar{1}0$, then $\mathcal{U}(V)$ has a 4-fold rotation with axis at (the face center) $1\bar{1}0$. Since $\angle(\mathbf{v}, 100) = 60^\circ$ and $\mathbf{v} \times 100 \propto 0\sqrt{2}\,\bar{1}$, then $\mathcal{U}(V)$ has a 3-fold rotation with axis at (the lower right cube vertex) $0\sqrt{2}\,\bar{1}$. The group $\mathcal{U}(V)$ is therefore
G. 16 a conjugate of $\mathcal{U}_{\text{CUBE}}$.

If $\mathcal{U} = \mathcal{U}_{\text{TET}}$ the group $\mathcal{U}(V)$ is a conjugate of $\mathcal{U}_{\text{CUBE}}$. The argument is similar to that for $\mathcal{U} = \mathcal{U}_{\text{ORTH}}$.

If $\mathcal{U} = \mathcal{U}_{\text{TRIG}}$ then $\mathcal{U}(V)$ is a conjugate of $\mathcal{U}_{\text{CUBE}}$ or $\mathcal{U}_{\text{XISO}}$.
G. 17 See Fig. 17.
G. 18 If $\mathcal{U} = \mathcal{U}_{\text{CUBE}}$ there are no available 2-fold points for $\mathcal{U}$; see Fig. 18.

If $\mathcal{U} = \mathcal{U}_{\text{XISO}}$ there are also no available 2-fold points for $\mathcal{U}$. □

**Theorem 18. (The elastic symmetry groups are conjugates of the reference groups.)**
For any elastic map $\mathbf{T}$ the group $\mathcal{S}_{\mathbf{T}}$ of its symmetries is a conjugate of one of the eight reference groups $\mathcal{U}_1, \mathcal{U}_{\text{MONO}}, \ldots, \mathcal{U}_{\text{ISO}}$ in Table 3. That is, for each $\mathbf{T}$ there is a reference group $\mathcal{U}$ and a rotation matrix $U$ such that $\mathcal{S}_{\mathbf{T}} = U\mathcal{U}U^\top$.

*Proof.* The idea of the proof is to start with the trivial group $\{I\}$ and add 2-fold rotations from $\mathcal{S}_{\mathbf{T}}$ one-by-one, and then to see what groups are generated.

More precisely, we construct subgroups $\mathcal{U}^k$ of $\mathbb{U}$ by

$$\begin{aligned} \mathcal{U}^1 &= \{I\} \\ \mathcal{U}^{k+1} &= \mathcal{U}^k(V_{k+1}), \quad V_{k+1} \in \mathcal{S}_{\mathbf{T}} - \mathcal{U}^k, \quad V_{k+1} \text{ is 2-fold} \end{aligned} \tag{162}$$

The construction terminates when $\mathcal{S}_{\mathbf{T}} - \mathcal{U}^k$ contains no 2-fold rotation $V_{k+1}$ to add. Until then, we have

$$\mathcal{U}^1 \subset \mathcal{U}^2 \subset \mathcal{U}^3 \subset \ldots \tag{163}$$

Each $\mathcal{U}^k$ is a subgroup of $\mathcal{S}_{\mathbf{T}}$, and each $\mathcal{U}^k$ is a conjugate of some reference group, by Lemma 4. Then, since the subgroup containments in Eq. (163) are strict, there can be at most eight of the $\mathcal{U}^k$. (If $\mathcal{U}^k$ is a conjugate of $\mathcal{U}_{\text{XISO}}$, then $\mathcal{U}^{k+1} = \mathcal{U}_{\text{ISO}}$, by Lemma 2). Thus, for some $k \le 8$,

$$\mathcal{U}^1 \subset \mathcal{U}^2 \subset \ldots \subset \mathcal{U}^k \tag{164a}$$

$$\mathcal{S}_{\mathbf{T}} - \mathcal{U}^k \text{ contains no 2-fold rotations} \tag{164b}$$

Theorems 10, 11, 12 then tell us that the set $\mathcal{S}_{\mathbf{T}} - \mathcal{U}^k$ is not just devoid of 2-fold rotations, it is in fact empty. Then $\mathcal{S}_{\mathbf{T}} = \mathcal{U}^k$ and so $\mathcal{S}_{\mathbf{T}}$ is a conjugate of a reference group. □

With $\mathcal{S}_{\mathbf{T}} = U\mathcal{U}U^\top$ as in the theorem, we refer to $\mathcal{U}$ as the reference group for $\mathbf{T}$. We then call the symmetry of $\mathbf{T}$ trivial, monoclinic, …, isometric according to whether the reference group is $\mathcal{U}_1, \mathcal{U}_{\text{MONO}}, \ldots, \mathcal{U}_{\text{ISO}}$. Although $\mathbf{T}$ uniquely determines its reference group, the rotation matrix $U$ is not unique, since $U$ can always be replaced by $UV$, where $V\mathcal{U}V^\top = \mathcal{U}$.

For each elastic map $\mathbf{T}$ there is a "characteristic solid" whose group of (rotational) geometric symmetries is $\mathcal{S}_{\mathbf{T}}$. If the solid is sculpted out of the material whose elasticity is described by $\mathbf{T}$, without reorienting it, then its elastic symmetries are the same as its geometric symmetries. In Fig. 20 (next section), for example, the characteristic solid for $\mathbf{T}'$ is the brick at the upper right. The elastic symmetries of $\mathbf{T}'$ are obvious from the brick.

If the material being considered is reoriented, its elastic map $\mathbf{T}$ is apt to change, and its elastic symmetry group $\mathcal{S}_{\mathbf{T}} = U\mathcal{U}U^\top$ is apt to change, but its reference group $\mathcal{U}$ will not.

## 15 Finding the symmetries of elastic maps

In Sections 15.1–15.7 we find the symmetries of seven elastic maps $\mathbf{T}'$. To get an impression of the method, it is enough to read just one or two of the seven sections. Readers wanting to use the method themselves, however, will want the full repertoire of seven examples.

Given an elastic map $\mathbf{T}'$, we know from Theorem 18 that its symmetry group has the form $\mathcal{S}_{\mathbf{T}'} = U\mathcal{U}U^\top$, where $U \in \mathbb{U}$ and where $\mathcal{U}$ is one of the eight reference groups. For most $\mathbf{T}'$ the reference group $\mathcal{U}$ can be found just by inspection of the beachball picture for $\mathbf{T}'$. Initially, we therefore recommend ignoring the main text and just looking at the beachball figures and their captions (e.g., Figs. 19 and 20). First, however, review Theorem 16, so as to be able to recognize beachball symmetries.

In the figures the rotation $U$ gives the orientation of the beachballs. Although $U$ can usually be guessed approximately and informally from the figure, the analytic approach described in the text is needed to find the matrix $U$ explicitly and thus to give a complete description of the symmetry group $\mathcal{S}_{\mathbf{T}'}$.

We treat the entries in our matrices $[\mathbf{T}']$ as exact. Thus we are ignoring the important practical problem of how to incorporate observational uncertainties into our analyses. See, for example, Danek et al. (2015).

Many authors have treated the problem of identifying the symmetries of given elastic maps. See Backus (1970); Helbig (1994); Baerheim (1998); Bóna et al. (2007); Abramian et al. (2019).

### 15.1 Example: monoclinic

We will find the symmetry group $\mathcal{S}_{\mathbf{T}'}$ of the elastic map $\mathbf{T}'$ whose matrix with respect to $\mathbb{B}$ is

$$[\mathbf{T}'] = \frac{1}{80}\begin{pmatrix} 222 & -12\sqrt{6} & -82 & 21\sqrt{6} & -39\sqrt{2} & 0 \\ .. & 196 & 12\sqrt{6} & 30 & 10\sqrt{3} & -16 \\ .. & .. & 222 & 9\sqrt{6} & -51\sqrt{2} & 0 \\ .. & .. & .. & 242 & -6\sqrt{3} & -24 \\ .. & .. & .. & .. & 254 & -8\sqrt{3} \\ .. & .. & .. & .. & .. & 304 \end{pmatrix} \tag{165}$$

An eigensystem of $\mathbf{T}'$ is shown in Fig. 19. From the figure,

$$\mathbf{T}' = \underset{1}{\langle G_1\rangle} \perp \underset{2}{\langle G_3\rangle} \perp \underset{3}{\langle G_4\rangle} \perp \underset{4,4,4}{\langle G_2, G_5, G_6\rangle} \tag{166}$$

The one-dimensional eigenspaces are $\langle G_1\rangle, \langle G_3\rangle, \langle G_4\rangle$. The FIG. 19
matrix $G_1$ is a double couple and therefore has tetragonal symmetry, whereas $G_3$ and $G_4$ are generic and therefore have orthorhombic symmetry; see Theorem 16. Orthorhombic symmetry is more informative than tetragonal symmetry, in the sense that it puts more constraints on the symmetry of $\mathbf{T}'$. We will consider $G_3$—the eigenvector of $\mathbf{T}'$ with eigenvalue equal to 2.

From Eq. (165),

$$G_3 = \begin{pmatrix} -\sqrt{3} & \sqrt{2} & -3\sqrt{3} \\ \sqrt{2} & 2\sqrt{3} & -\sqrt{2} \\ -3\sqrt{3} & -\sqrt{2} & -\sqrt{3} \end{pmatrix} \tag{167}$$

(We have omitted the normalizing factor $1/(4\sqrt{5})$, which is inessential.) Diagonalizing gives $G_3 = UH_3U^\top$, where

$$H_3 = 2\begin{pmatrix} \sqrt{3}+1 & & \\ & \sqrt{3}-1 & \\ & & -2\sqrt{3} \end{pmatrix} \tag{168a}$$

$$U = \frac{1}{2}\begin{pmatrix} -1 & 1 & \sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 0 \\ 1 & -1 & \sqrt{2} \end{pmatrix} \tag{168b}$$

The matrix of the elastic map $\mathbf{T} = \overline{U}^* \circ \mathbf{T}' \circ \overline{U}$ with respect to $\mathbb{B}$ is

$$[\mathbf{T}] = [\overline{U}]^\top [\mathbf{T}'] [\overline{U}]$$
$$= \frac{1}{20}\begin{pmatrix} 50+15\sqrt{3} & -15 & & & & \\ -15 & 50-15\sqrt{3} & & & & \\ & & 64 & 0 & 0 & -8 \\ & & 0 & 76 & 12 & 0 \\ & & 0 & 12 & 44 & 0 \\ & & -8 & 0 & 0 & 76 \end{pmatrix}, \tag{169}$$

And from Eqs. (166) and (47),

$$\mathbf{T} = \underset{1}{\langle H_1\rangle} \perp \underset{2}{\langle H_3\rangle} \perp \underset{3}{\langle H_4\rangle} \perp \underset{4,4,4}{\langle H_2, H_5, H_6\rangle}$$
$$(H_i = U^\top G_i\, U) \tag{170}$$

From Eq. (168a) the matrix $H_3$ is diagonal and generic. Hence from Eq. (156d) the group $\mathcal{S}(H_3)$ of symmetries of $H_3$ is $\mathcal{U}_{\text{ORTH}} = \{I, X_\pi, Y_\pi, Z_\pi\}$. Since $\langle H_3\rangle$ is an eigenspace of $\mathbf{T}$, then from Theorem 13,

$$\mathcal{S}_\mathbf{T} \subset \underbrace{\{I, X_\pi, Y_\pi, Z_\pi\}}_{\mathcal{S}(H_3)}, \tag{171}$$

Using Eq. (169) and applying the $\Delta$-test to $X_\pi$, $Y_\pi$, $Z_\pi$, we find that only $I$ and $Z_\pi$ are symmetries of $\mathbf{T}$. Thus $\mathcal{S}_\mathbf{T} = \{I, Z_\pi\} = \mathcal{U}_{\text{MONO}}$, and then $\mathcal{S}_{\mathbf{T}'} = U\mathcal{U}_{\text{MONO}}\, U^\top$; the symmetry of $\mathbf{T}'$ is monoclinic. (Here $U$ can be replaced by the more transparent matrix $Y_{\pi/4}$, since $UZ_{3\pi/4} = Y_{\pi/4}$.) The two matrices in the group $\mathcal{S}_{\mathbf{T}'}$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The wedge at the upper right in Fig. 19 is the characteristic solid for $\mathbf{T}'$. If it had been sculpted out of the hypothetical material under consideration, without reorienting the material, then its geometric symmetries would be the same as its elastic symmetries. (All symmetries are understood to be rotational, as usual.)

## 15.2 Example: orthorhombic

We next find the symmetry group $\mathcal{S}_{\mathbf{T}'}$ of the elastic map $\mathbf{T}'$ whose matrix with respect to $\mathbb{B}$ is

$$[\mathbf{T}'] = \frac{1}{10}\begin{pmatrix} 38 & 18 & 3\sqrt{2} & -5\sqrt{2} & -\sqrt{6} & -\sqrt{8} \\ .. & 38 & 3\sqrt{2} & 5\sqrt{2} & -\sqrt{6} & -\sqrt{8} \\ .. & .. & 41 & 0 & -7\sqrt{3} & 6 \\ .. & .. & .. & 20 & 0 & 0 \\ .. & .. & .. & .. & 27 & -2\sqrt{3} \\ .. & .. & .. & .. & .. & 56 \end{pmatrix} \tag{172}$$

An eigensystem of $\mathbf{T}'$ is shown in Fig. 20. From the figure,

$$\mathbf{T}' = \underset{1}{\langle G_1\rangle} \perp \underset{2}{\langle G_2\rangle} \perp \underset{3}{\langle G_3\rangle} \perp \underset{4}{\langle G_4\rangle} \perp \underset{6,6}{\langle G_5, G_6\rangle} \tag{173}$$

The one-dimensional eigenspaces are $\langle G_1\rangle, \langle G_2\rangle, \langle G_3\rangle, \langle G_4\rangle$. FIG. 20 The matrices $G_1, G_2, G_3$ are double couples and therefore have tetragonal symmetry, whereas $G_4$ is generic and therefore has the more informative orthorhombic symmetry. We therefore consider $G_4$— the eigenvector of $\mathbf{T}'$ with eigenvalue equal to 4.

From Eq. (172),

$$G_4 = \begin{pmatrix} \sqrt{8}+\sqrt{3} & -\sqrt{27} & \sqrt{6} \\ -\sqrt{27} & \sqrt{8}+\sqrt{3} & \sqrt{6} \\ \sqrt{6} & \sqrt{6} & \sqrt{8}-\sqrt{12} \end{pmatrix} \tag{174}$$

Diagonalizing gives $G_4 = UH_4U^\top$, where

$$H_4 = \sqrt{8}\begin{pmatrix} 1+\sqrt{6} & & \\ & 1 & \\ & & 1-\sqrt{6} \end{pmatrix} \tag{175a}$$

$$U = \frac{1}{2}\begin{pmatrix} \sqrt{2} & 1 & -1 \\ -\sqrt{2} & 1 & -1 \\ 0 & \sqrt{2} & \sqrt{2} \end{pmatrix} \tag{175b}$$

The matrix of the elastic map $\mathbf{T} = \overline{U}^* \circ \mathbf{T}' \circ \overline{U}$ with respect to $\mathbb{B}$ is

$$[\mathbf{T}] = [\overline{U}]^\top [\mathbf{T}'] [\overline{U}]$$
$$= \frac{1}{5}\begin{pmatrix} 10 & & & & & \\ & 15 & & & & \\ & & 5 & & & \\ & & & 28 & 2\sqrt{3} & 2 \\ & & & 2\sqrt{3} & 24 & -2\sqrt{3} \\ & & & 2 & -2\sqrt{3} & 28 \end{pmatrix}, \tag{176}$$

And from Eqs. (173) and (47),

$$\mathbf{T} = \underset{1}{\langle H_1\rangle} \perp \underset{2}{\langle H_2\rangle} \perp \underset{3}{\langle H_3\rangle} \perp \underset{4}{\langle H_4\rangle} \perp \underset{6,6}{\langle H_5, H_6\rangle}$$
$$(H_i = U^\top G_i\, U) \tag{177}$$

The matrix $H_4$ is diagonal and generic (Eq. 175a), and so the group $\mathcal{S}(H_4)$ of symmetries of $H_4$ is $\mathcal{U}_{\text{ORTH}} = \{I, X_\pi, Y_\pi, Z_\pi\}$, from Eq. (156d). Then

$$\mathcal{U}_{\text{ORTH}} \subset \mathcal{S}_\mathbf{T} \subset \underbrace{\mathcal{U}_{\text{ORTH}}}_{\mathcal{S}(H_4)} \tag{178}$$

The first subset containment is due to the matrix $[\mathbf{T}]$ in Eq. (176) having the form of the reference matrix $T_{\text{ORTH}}$ in Table 4, and the second is due to $\langle H_4\rangle$ being an eigenspace of $\mathbf{T}$; see Theorem 13. From Eq. (178) we have $\mathcal{S}_\mathbf{T} = \mathcal{U}_{\text{ORTH}}$ and then $\mathcal{S}_{\mathbf{T}'} = U\mathcal{U}_{\text{ORTH}}U^\top$; the symmetry of $\mathbf{T}'$ is orthogonal.

The brick at the upper right in Fig. 20 is the characteristic solid for $\mathbf{T}'$. If it had been sculpted out of the hypothetical material under consideration, without reorienting the material, then its geometric symmetries would be the same as its elastic symmetries.

### 15.3 Example: tetragonal

Next we find the symmetry group $\mathcal{S}_{\mathbf{T}'}$ for the map $\mathbf{T}'$ whose matrix with respect to $\mathbb{B}$ is

$$[\mathbf{T}'] = \frac{1}{64}\begin{pmatrix} 168 & 4\sqrt{6} & -40 & 6\sqrt{6} & 6\sqrt{2} & 0 \\ .. & 324 & -4\sqrt{6} & -42 & -14\sqrt{3} & 16\sqrt{3} \\ .. & .. & 168 & -6\sqrt{6} & -6\sqrt{2} & 0 \\ .. & .. & .. & 233 & 35\sqrt{3} & -8\sqrt{3} \\ .. & .. & .. & .. & 163 & -8 \\ .. & .. & .. & .. & .. & 352 \end{pmatrix} \tag{179}$$

G. 21 An eigensystem of $\mathbf{T}'$ is shown in Fig. 21. From the figure,

$$\mathbf{T}' = \underset{2,2}{\langle G_1, G_2\rangle} \perp \underset{3}{\langle G_3\rangle} \perp \underset{4}{\langle G_4\rangle} \perp \underset{5}{\langle G_5\rangle} \perp \underset{6}{\langle G_6\rangle} \tag{180}$$

The one-dimensional eigenspaces are $\langle G_3\rangle, \langle G_4\rangle, \langle G_5\rangle, \langle G_6\rangle$. The matrices $G_3$ and $G_4$ are double couples, and $G_5$ and $G_6$ are crack matrices. The double couples are more informative than the crack matrices. We consider $G_3$—the eigenvector of $\mathbf{T}'$ with eigenvalue equal to 3.

From Eq. (179),

$$G_3 = \begin{pmatrix} -1 & \sqrt{6} & 1 \\ \sqrt{6} & 2 & -\sqrt{6} \\ 1 & -\sqrt{6} & -1 \end{pmatrix}, \tag{181}$$

Diagonalizing gives $G_3 = UH_3U^\top$, where

$$H_3 = \begin{pmatrix} 4 & & \\ & -4 & \\ & & 0 \end{pmatrix}, \quad U = \frac{1}{\sqrt{8}}\begin{pmatrix} 1 & -\sqrt{3} & 2 \\ \sqrt{6} & \sqrt{2} & 0 \\ -1 & \sqrt{3} & 2 \end{pmatrix} \tag{182}$$

The matrix of the elastic map $\mathbf{T} = \overline{U}^* \circ \mathbf{T}' \circ \overline{U}$ with respect to $\mathbb{B}$ is

$$\begin{aligned} [\mathbf{T}] &= [\overline{U}]^\top [\mathbf{T}'] [\overline{U}] \\ &= \frac{1}{2}\begin{pmatrix} 4 & & & & & \\ & 4 & & & & \\ & & 8 & & & \\ & & & 6 & & \\ & & & & 11 & -1 \\ & & & & -1 & 11 \end{pmatrix}, \end{aligned} \tag{183}$$

And from Eqs. (180) and (47),

$$\mathbf{T} = \underset{2,2}{\langle H_1, H_2\rangle} \perp \underset{3}{\langle H_3\rangle} \perp \underset{4}{\langle H_4\rangle} \perp \underset{5}{\langle H_5\rangle} \perp \underset{6}{\langle H_6\rangle}$$
$$(H_i = U^\top G_i U) \tag{184}$$

From Eq. (182) the matrix $H_3$ is a double couple of the form in Eq. (156c), and so $\mathcal{S}(H_3) = \mathcal{U}_{\text{TET}}$. Then

$$\mathcal{U}_{\text{TET}} \subset \mathcal{S}_{\mathbf{T}} \subset \underbrace{\mathcal{U}_{\text{TET}}}_{\mathcal{S}(H_3)} \tag{185}$$

The first subset containment is due to the matrix $[\mathbf{T}]$ in Eq. (183) having the form of the reference matrix $T_{\text{TET}}$, and the second is due to $\langle H_3\rangle$ being an eigenspace of $\mathbf{T}$; see Theorem 13. From Eq. (185) we have $\mathcal{S}_{\mathbf{T}} = \mathcal{U}_{\text{TET}}$, and then $\mathcal{S}_{\mathbf{T}'} = U\mathcal{U}_{\text{TET}}U^\top$; the symmetry of $\mathbf{T}'$ is tetragonal.

For example, a 4-fold rotation in $\mathcal{S}_{\mathbf{T}'}$ is

$$UZ_{\pi/2}U^\top = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix} \tag{186}$$

The square prism in Fig. 21 is the characteristic solid for $\mathbf{T}'$.

### 15.4 Example: transverse isotropic

Next we find the symmetry group $\mathcal{S}_{\mathbf{T}'}$ of the map $\mathbf{T}'$ whose matrix with respect to $\mathbb{B}$ is

$$[\mathbf{T}'] = \frac{1}{128}\begin{pmatrix} 532 & 92\sqrt{3} & 122 & -2\sqrt{3} & -60 & 48 \\ .. & 348 & -2\sqrt{3} & 126 & -20\sqrt{3} & 16\sqrt{3} \\ .. & .. & 349 & 31\sqrt{3} & -126 & 24 \\ .. & .. & .. & 287 & -42\sqrt{3} & 8\sqrt{3} \\ .. & .. & .. & .. & 212 & -16 \\ .. & .. & .. & .. & .. & 704 \end{pmatrix} \tag{187}$$

An eigensystem of $\mathbf{T}'$ is shown in Fig. 22. From the figure, FIG. 22

$$\mathbf{T}' = \underset{1,1}{\langle G_1, G_2\rangle} \perp \underset{3,3}{\langle G_3, G_4\rangle} \perp \underset{5}{\langle G_5\rangle} \perp \underset{6}{\langle G_6\rangle} \tag{188}$$

The one-dimensional eigenspaces are $\langle G_5\rangle$ and $\langle G_6\rangle$. Both $G_5$ and $G_6$ are crack matrices. We consider $G_5$—the eigenvector of $\mathbf{T}'$ with eigenvalue equal to 5.

From Eq. (187),

$$G_5 = \begin{pmatrix} 8\sqrt{2}+5 & -\sqrt{27} & -6 \\ -\sqrt{27} & 8\sqrt{2}-1 & -6\sqrt{3} \\ -6 & -6\sqrt{3} & 8\sqrt{2}-4 \end{pmatrix}, \tag{189}$$

Diagonalizing gives $G_5 = UH_5U^\top$, where

$$H_5 = 8\begin{pmatrix} \sqrt{2}+1 & & \\ & \sqrt{2}+1 & \\ & & \sqrt{2}-2 \end{pmatrix} \tag{190a}$$

$$U = \frac{1}{\sqrt{8}}\begin{pmatrix} 1 & -\sqrt{6} & 1 \\ \sqrt{3} & \sqrt{2} & \sqrt{3} \\ -2 & 0 & 2 \end{pmatrix} \tag{190b}$$

We always require $U$ to be a rotation matrix, but here, with $G_5$ being a crack matrix, independent eigenvectors of $G_5$ are not necessarily orthogonal, so some care was required in getting $U$.

The matrix of the elastic map $\mathbf{T} = \overline{U}^* \circ \mathbf{T}' \circ \overline{U}$ with respect to $\mathbb{B}$ is

$$\begin{aligned} [\mathbf{T}] &= [\overline{U}]^\top [\mathbf{T}'] [\overline{U}] \\ &= \frac{1}{2}\begin{pmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 6 & & & \\ & & & 6 & & \\ & & & & 11 & -1 \\ & & & & -1 & 11 \end{pmatrix}, \end{aligned} \tag{191}$$

And from Eqs. (188) and (47),

$$\mathbf{T} = \underset{1,1}{\langle H_1, H_2\rangle} \perp \underset{3,3}{\langle H_3, H_4\rangle} \perp \underset{5}{\langle H_5\rangle} \perp \underset{6}{\langle H_6\rangle}$$
$$(H_i = U^\top G_i U) \tag{192}$$

From Eq. (190a) the matrix $H_5$ has the form in Eq. (156b), and so $\mathcal{S}(H_5) = \mathcal{U}_{\text{XISO}}$. Then

$$\mathcal{U}_{\text{XISO}} \subset \mathcal{S}_{\mathbf{T}} \subset \underbrace{\mathcal{U}_{\text{XISO}}}_{\mathcal{S}(H_5)} \tag{193}$$

The first subset containment is due to the matrix $[\mathbf{T}]$ in Eq. (191) having the form of the reference matrix $T_{\text{XISO}}$, and the second is due to $\langle H_5 \rangle$ being an eigenspace of $\mathbf{T}$. From Eq. (193) we have $\mathcal{S}_{\mathbf{T}} = \mathcal{U}_{\text{XISO}}$, and then $\mathcal{S}_{\mathbf{T}'} = U\mathcal{U}_{\text{XISO}}U^{\top}$; the map $\mathbf{T}'$ is transverse isotropic.

The cylinder in Fig. 22 is the characteristic solid for $\mathbf{T}'$. If it had been sculpted out of the hypothetical material under consideration, without reorienting the material, then its geometric symmetries would be the same as its elastic symmetries.

If the matrix $[\mathbf{T}]$ in Eq. (191) had not had the form of $T_{\text{XISO}}$, we would not have had the benefit of the first containment in Eq. (193). We would then test the rotations in $\mathcal{U}_{\text{XISO}}$ to see which are symmetries of $\mathbf{T}$. The proof of Theorem 17(vii) describes a comparable calculation.

### 15.5 Example: cubic

Next we find the symmetry group $\mathcal{S}_{\mathbf{T}'}$ of the map $\mathbf{T}'$ whose matrix with respect to $\mathbb{B}$ is

$$[\mathbf{T}'] = \frac{1}{36}\begin{pmatrix} 52 & 4 & 16 & -6 & -2\sqrt{3} & 0 \\ .. & 64 & 4 & 12 & 4\sqrt{3} & 0 \\ .. & .. & 52 & -6 & -2\sqrt{3} & 0 \\ .. & .. & .. & 45 & 3\sqrt{3} & 0 \\ .. & .. & .. & .. & 39 & 0 \\ .. & .. & .. & .. & .. & 108 \end{pmatrix} \tag{194}$$

G. 23 One eigensystem of $\mathbf{T}'$ is shown in Fig. 23 and given in Appendix E. From the figure,

$$\mathbf{T}' = \underset{1,1,1}{\mathbb{W}'} \perp \underset{2,2}{\langle G_4, G_5 \rangle} \perp \underset{3}{\langle G_6 \rangle}, \tag{195}$$

where $\mathbb{W}' = \langle G_1, G_2, G_3 \rangle$. The lone one-dimensional eigenspace is $\langle G_6 \rangle = \langle I \rangle$, whose symmetry puts no constraints on the symmetry of $\mathbf{T}'$. All is not lost, however. From Fig. 23, the matrices $G_4$ and $G_5$ appear to have a common eigenframe $U$. Analytically, we find from Eq. (194) that

$$G_4 = U H_4 U^{\top}, \qquad G_5 = U H_5 U^{\top} \tag{196a}$$

where $H_4$ and $H_5$ are given in Eqs. (E.1) and where

$$U = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & -1 & \sqrt{2} \\ 0 & 2 & \sqrt{2} \\ -\sqrt{3} & -1 & \sqrt{2} \end{pmatrix} \tag{196b}$$

Since $H_4$ and $H_5$ are diagonal, then $U$ is indeed a common frame for $G_4$ and $G_5$.

Letting $\mathbf{T} = \overline{U}^* \circ \mathbf{T}' \circ \overline{U}$, we have, from Eqs. (195) and (47),

$$\mathbf{T} = \underset{1,1,1}{\mathbb{W}} \perp \underset{2,2}{\langle H_4, H_5 \rangle} \perp \underset{3}{\langle I \rangle} \qquad (H_i = U^{\top} G_i U) \tag{197}$$

The two-dimensional subspace $\langle H_4, H_5 \rangle$, being orthogonal to $\langle I \rangle$, consists of deviatoric matrices, and since they are diagonal, then $\langle H_4, H_5 \rangle$ must be $\langle B_4, B_5 \rangle$, whose symmetry group is $\mathcal{U}_{\text{CUBE}}$ (Eq. 158). Then $\mathbb{W}$ must be $\langle B_4, B_5, B_6 \rangle^{\perp}$, whose symmetry group is also $\mathcal{U}_{\text{CUBE}}$. Writing the appropriate symmetry group above each summand, we have, from Eq. (197),

$$\mathbf{T} = \overset{\mathcal{U}_{\text{CUBE}}}{\underset{1,1,1}{\mathbb{W}}} \perp \overset{\mathcal{U}_{\text{CUBE}}}{\underset{2,2}{\langle B_4, B_5 \rangle}} \perp \overset{\mathcal{U}_{\text{ISO}}}{\underset{3}{\langle I \rangle}} \tag{198}$$

Thus $\mathcal{S}_{\mathbf{T}} = \mathcal{U}_{\text{CUBE}}$ (Theorem 13) and then $\mathcal{S}_{\mathbf{T}'} = U\mathcal{U}_{\text{CUBE}}U^{\top}$; the map $\mathbf{T}'$ is cubic. The cube in Fig. 23 is the characteristic solid for $\mathbf{T}'$.

The matrix $[\mathbf{T}]$, not used here, would be diagonal with diagonal entries $1, 1, 1, 2, 2, 3$.

#### *15.5.1 Cubic symmetry in general*

We have now found the symmetry group of $\mathbf{T}'$ in Eq. (194). We can see from that example how cubic symmetry can arise more generally. For an arbitrary $\mathbf{T}'$ and for $U \in \mathbb{U}$, the map $\mathbf{T}'$ has symmetry group $\mathcal{S}_{\mathbf{T}'} = U\mathcal{U}_{\text{CUBE}}U^{\top}$ if one of the following holds (for some subspace $\mathbb{W}'$ and some numbers $\lambda_1, \lambda_2, \lambda_3$):

$$\mathbf{T}' = \underset{\lambda_1\,\lambda_1\,\lambda_1}{\mathbb{W}'} \perp \underset{\lambda_2\,\lambda_2}{U\langle B_4, B_5 \rangle U^{\top}} \perp \underset{\lambda_3}{\langle I \rangle} \qquad (\lambda_1, \lambda_2, \lambda_3 \text{ distinct}) \tag{199a}$$

$$\mathbf{T}' = \underset{\lambda_1\,\lambda_1\,\lambda_1}{\mathbb{W}'} \perp \underset{\lambda_2\,\lambda_2\,\lambda_2}{U\langle B_4, B_5, B_6 \rangle U^{\top}} \qquad (\lambda_1 \neq \lambda_2) \tag{199b}$$

$$\mathbf{T}' = \underset{\lambda_1\,\lambda_1\,\lambda_1\,\lambda_1}{\mathbb{W}'} \perp \underset{\lambda_2\,\lambda_2}{U\langle B_4, B_5 \rangle U^{\top}} \qquad (\lambda_1 \neq \lambda_2) \tag{199c}$$

The equations are not as daunting as they appear, since matrices in $U\langle B_4, B_5, B_6 \rangle U^{\top}$ and $U\langle B_4, B_5 \rangle U^{\top}$ all have the common eigenframe $U$. The subspace $U\langle B_4, B_5, B_6 \rangle U^{\top}$ consists of all such matrices, and $U\langle B_4, B_5 \rangle U^{\top}$ consists of those that are deviatoric. Both subspaces are therefore relatively easy to recognize.

Eqs. (199) are the only possibilities for cubic symmetry, as follows from Eqs. (47), (143b), and Theorem 13. Hence, from Eq. (199a), if an elastic map $\mathbf{T}'$ has exactly three eigenspaces, a necessary and sufficient condition for $\mathcal{S}'_{\mathbf{T}}$ to be a conjugate of $\mathcal{U}_{\text{CUBE}}$ is that one of the eigenspaces be $\langle I \rangle$ and that another be two-dimensional and consist of matrices all with a common eigenframe. (Fig. 23 is typical.) This is Theorem 4.2 of Bóna et al. (2007).

Similarly, if $\mathbf{T}'$ has exactly two eigenspaces, a necessary and sufficient condition for $\mathcal{S}'_{\mathbf{T}}$ to be a conjugate of $\mathcal{U}_{\text{CUBE}}$ is that one of the eigenspaces be three-dimensional and consist of matrices having a common eigenframe (Eq. 199b), or that one of the eigenspaces be two-dimensional and consist of deviatoric matrices having a common eigenframe (Eq. 199c).

### 15.6 Example: trigonal

Next we find the symmetry group $\mathcal{S}_{\mathbf{T}'}$ of the map $\mathbf{T}'$ whose matrix with respect to $\mathbb{B}$ is $[\mathbf{T}'] =$

$$\frac{1}{16}\begin{pmatrix} 4(8-\sqrt{3}) & -8 & 3\sqrt{2} & -6\sqrt{2} & -3\sqrt{6} & 0 \\ .. & 4(8+\sqrt{3}) & 3\sqrt{2} & 6\sqrt{2} & -3\sqrt{6} & 0 \\ .. & .. & 76 & -2\sqrt{3} & 12\sqrt{3} & 4\sqrt{3} \\ .. & .. & .. & 24 & 6 & 0 \\ .. & .. & .. & .. & 52 & 4 \\ .. & .. & .. & .. & .. & 88 \end{pmatrix} \tag{200}$$

G. 24 An eigensystem of $\mathbf{T}'$ is shown in Fig. 24. From the figure,

$$\mathbf{T}' = \underset{1,1}{\langle G_1, G_2\rangle} \perp \underset{3,3}{\langle G_3, G_4\rangle} \perp \underset{5}{\langle G_5\rangle} \perp \underset{6}{\langle G_6\rangle} \tag{201}$$

The one-dimensional eigenspaces are $\langle G_5\rangle$ and $\langle G_6\rangle$. Both $G_5$ and $G_6$ are crack matrices. We consider $G_5$—the eigenvector of $\mathbf{T}'$ with eigenvalue equal to 5.

From Eq. (200),

$$G_5 = \begin{pmatrix} \sqrt{8}-1 & -3 & 0 \\ -3 & \sqrt{8}-1 & 0 \\ 0 & 0 & \sqrt{8}+2 \end{pmatrix} \tag{202}$$

Diagonalizing gives

$$G_5 = U_0 H_5 U_0^\top, \tag{203a}$$

where

$$H_5 = 2\begin{pmatrix} \sqrt{2}+1 & & \\ & \sqrt{2}+1 & \\ & & \sqrt{2}-2 \end{pmatrix} \tag{203b}$$

$$U_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ -\sqrt{2} & 0 & 0 \end{pmatrix} \tag{203c}$$

The matrix of the elastic map $\mathbf{T}_0 = \overline{U_0}^* \circ \mathbf{T}' \circ \overline{U_0}$ with respect to $\mathbb{B}$ is

$$\begin{aligned} [\mathbf{T}_0] &= [\overline{U_0}]^\top [\mathbf{T}'] [\overline{U_0}] \\ &= \frac{1}{4}\begin{pmatrix} 6 & 0 & 3 & \sqrt{3} & & \\ 0 & 6 & -\sqrt{3} & 3 & & \\ 3 & -\sqrt{3} & 10 & 0 & & \\ \sqrt{3} & 3 & 0 & 10 & & \\ & & & & 22 & -2 \\ & & & & -2 & 22 \end{pmatrix}, \end{aligned} \tag{204}$$

The matrix $[\mathbf{T}_0]$ has the form of $T_3$ in Table 1, so $\mathbf{T}_0$ has a vertical 3-fold axis. Its horizontal 2-fold axes, from Theorem 12, are at $\theta = \pi/2 + \pi/18 + n\pi/3$. We therefore let

$$U = U_0 Z_t = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sin t & -\cos t & 1 \\ \sin t & \cos t & 1 \\ -\sqrt{2}\cos t & \sqrt{2}\sin t & 0 \end{pmatrix} \quad (t = \pi/18) \tag{205}$$

and

$$\begin{aligned} \mathbf{T} &= \overline{Z_t}^* \circ \mathbf{T}_0 \circ \overline{Z_t} \qquad (t = \pi/18) \\ &= \overline{U}^* \circ \mathbf{T}' \circ \overline{U} \end{aligned} \tag{206}$$

The map $\mathbf{T}$ has its horizontal 2-fold axes at $\theta = \pi/2 + n\pi/3$. Its matrix is

$$\begin{aligned} [\mathbf{T}] &= [\overline{U}]^\top [\mathbf{T}'] [\overline{U}] \\ &= \frac{1}{2}\begin{pmatrix} 3 & & \sqrt{3} & & & \\ & 3 & & \sqrt{3} & & \\ \sqrt{3} & & 5 & & & \\ & \sqrt{3} & & 5 & & \\ & & & & 11 & -1 \\ & & & & -1 & 11 \end{pmatrix} \end{aligned} \tag{207}$$

And from Eq. (201),

$$\begin{aligned} \mathbf{T} &= \underset{1,1}{\langle H_1, H_2\rangle} \perp \underset{3,3}{\langle H_3, H_4\rangle} \perp \underset{5}{\langle H_5\rangle} \perp \underset{6}{\langle H_6\rangle} \\ &\qquad (H_i = U^\top G_i U) \end{aligned} \tag{208}$$

Eq. (203a) remains correct when $U$ is substituted for $U_0$, since $Z_\xi H_5 Z_\xi^\top = H_5$. (In changing $U_0$ to $U$, we are only rotating the eigenframe for the crack matrix $G_5$ about its $c$-axis.) From Eq. (203b) the matrix $H_5$ has the form in Eq. (156b), and so $\mathcal{S}(H_5) = \mathcal{U}_{\text{XISO}}$. Then

$$\mathcal{S}_\mathbf{T} \subset \underbrace{\mathcal{U}_{\text{XISO}}}_{\mathcal{S}(H_5)} \tag{209}$$

Using the $\Delta$-test and Eq. (207) we then examine the members of $\mathcal{U}_{\text{XISO}}$ to see which are symmetries of $\mathbf{T}$. (The proof of Theorem 17(vii) describes a comparable calculation.) The result is $\mathcal{S}_\mathbf{T} = \mathcal{U}_{\text{TRIG}}$. The map $\mathbf{T}'$ is therefore trigonal, with $\mathcal{S}_{\mathbf{T}'} = U\mathcal{U}_{\text{TRIG}}U^\top$. The triangular prism in Fig. 24 is the characteristic solid for $\mathbf{T}'$.

### 15.7 Example: trivial symmetry

Let $\mathbf{T}$ be the elastic map with $[\mathbf{T}]_{\mathbb{BB}}$ as in Eq. (23). In item (viii) in the proof of Theorem 17 we noted that the eigenvalues $\lambda_1$ and $\lambda_2$ of $\mathbf{T}$ were simple and that their eigenvectors $G_1$ and $G_2$ were generic, with no principal axis in common. Hence $\mathbf{T}$ had only the trivial symmetry.

Fig. 4 is the beachball picture for $\mathbf{T}$. The characteristic solid for $\mathbf{T}$, not shown, would be an irregularly shaped solid.

A sufficient condition for trivial symmetry of an arbitrary elastic map is that it have simple eigenvalues $\lambda_i$ and $\lambda_j$ with eigenvectors $G_i$ and $G_j$ that have only the trivial symmetry in common.

### 15.8 Example: a defeat

Here is an example of an elastic map $\mathbf{T}'$ where our method fails to identify its symmetry. The matrix of $\mathbf{T}'$ with respect to $\mathbb{B}$ is

$$\frac{1}{160}\begin{pmatrix} 252 & -12\sqrt{6} & -52 & 16\sqrt{6} & -24\sqrt{2} & 0 \\ .. & 180 & 12\sqrt{6} & 6 & 2\sqrt{3} & -32 \\ .. & .. & 252 & 4\sqrt{6} & -36\sqrt{2} & 0 \\ .. & .. & .. & 211 & -23\sqrt{3} & -48 \\ .. & .. & .. & .. & 257 & -16\sqrt{3} \\ .. & .. & .. & .. & .. & 288 \end{pmatrix} \tag{210}$$

Two eigensystems of $\mathbf{T}'$ are shown in Fig. 25, one with orthonormal eigenvectors $G_1, \ldots, G_6$. the other with orthonormal eigenvectors $J_1, \ldots, J_6$. From the figure, FIG. 25

$$\mathbf{T}' = \underset{1,1,1}{\mathbb{W}_1} \perp \underset{2,2,2}{\mathbb{W}_2}, \tag{211a}$$

where

$$\begin{aligned} \mathbb{W}_1 &= \langle G_1, G_2, G_3\rangle = \langle J_1, J_2, J_3\rangle \\ \mathbb{W}_2 &= \langle G_4, G_5, G_6\rangle = \langle J_4, J_5, J_6\rangle \end{aligned} \tag{211b}$$

As always, the symmetry group $\mathcal{S}_{T'}$ of $\mathbf{T}'$ is the intersection of the symmetry groups of the eigenspaces of $\mathbf{T}'$. Neither of the eigenspaces $\mathbb{W}_1$ and $\mathbb{W}_2$, however, is one-dimensional, which makes their symmetries harder to recognize. In fact in Fig. 25(a) we do not recognize either $\langle G_1, G_2, G_3\rangle$ or $\langle G_4, G_5, G_6\rangle$ as conjugates of any of the subspaces in Table 2, whose symmetry groups are known and would have helped. With only Fig. 25(a) to work with, we are at a dead end.

In Fig. 25(b), however, 2-fold symmetry for $\mathbf{T}'$ is clear. Given analytic expressions for $J_1, \ldots, J_6$, we can confirm that the symmetry of $\mathbf{T}'$ is monoclinic, with $\mathcal{S}_{T'} = U\mathcal{U}_{\text{MONO}}U^\top$ and $U = Y_{\pi/4}$—the same as for $\mathbf{T}'$ in Section 15.1. Mathematical software, however, when asked for eigenvectors of $\mathbf{T}'$ here, is not apt to be so kind as to return $J_1, \ldots, J_6$. (We only know $J_1, \ldots, J_6$ because we ourselves constructed $\mathbf{T}'$ from them.) Our method would therefore fail to find the symmetry of $\mathbf{T}'$.

## 16 Stability

An elastic map $\mathbf{T}$ is said to be stable if

$$\mathbf{T}(E) \cdot E > 0 \qquad (E \in \mathbb{M},\ E \neq 0_{3\times 3}) \tag{212a}$$

Equivalently, the matrix of $\mathbf{T}$ with respect to an orthonormal basis $\mathbb{G}$ should satisfy

$$([\mathbf{T}]_{\mathbb{G}\mathbb{G}}\,\mathbf{w}) \cdot \mathbf{w} > 0 \qquad (\mathbf{w} \in \mathbb{R}^6,\ \mathbf{w} \neq \mathbf{0}), \tag{212b}$$

Either of Eqs. (212) is equivalent to the eigenvalues of $\mathbf{T}$ being positive. Thus the elastic map $\mathbf{T}$ in Fig. 4 is stable, as seen from its eigenvalues. Had it been unstable, there would have been a color reversal between the beachballs for $G_i$ and $\mathbf{T}(G_i)$ for at least one of the eigenvectors $G_i$.

Using intrinsic characterizations of elastic maps—e.g., Eqs. (143b) or (147b)—we can easily make up examples of stable elastic maps $\mathbf{T}$ that have prescribed symmetries; see Eq. (49).

Attempting the same using matrix characterizations will usually fail. If, for example, we choose each matrix entry $a, b, \ldots, p$ of $T_{\text{ORTH}}$ randomly between $-1$ and 1, the probability of getting a stable matrix is only $\approx 0.001$. We can get some insight into why this should be so by considering the probability of getting a stable matrix $T = \begin{pmatrix} a & g \\ g & b \end{pmatrix}$ when choosing each of the entries $a, b, g$ randomly between $-1$ and 1. The fractional volume of the unit $abg$-cube occupied by stable matrices $T$ can be visualized and then found to be only about 0.1. If the same experiment is performed with the arbitrary $6\times 6$ symmetric $T = T_1$ of Table 4, thus choosing each entry $a, b, \ldots, v$ randomly between $-1$ and 1, the probability of getting a stable $T$ is for all practical purposes zero; you cannot construct a stable matrix that way.

Either of Eqs. (212) is equivalent to the more traditional characterization of stability in terms of the $6 \times 6$ Voigt matrix $C$. That is, $\mathbf{T}$ is stable if and only if

$$C\mathbf{w} \cdot \mathbf{w} > 0 \qquad (\mathbf{w} \in \mathbb{R}^6,\ \mathbf{w} \neq \mathbf{0}), \tag{213}$$

where $C$ is from Eqs. (S13). Slawinski (2015) explains the physical meaning of Eq. (213).

## 17 Summary and afterthoughts

Two reminders: All of our elastic symmetries are rotational. The vector space $\mathbb{M}$ consists of all $3 \times 3$ symmetric matrices; its members can be thought of as strains or stresses.

The elastic map $\mathbf{T} : \mathbb{M} \to \mathbb{M}$, assumed to be linear, relates strain and stress at a point $\mathbf{p}$ in some material. A symmetry of $\mathbf{T}$ is a rotation of the material about $\mathbf{p}$ that leaves $\mathbf{T}$ unchanged. Given an arbitrary $\mathbf{T}$, we wish to find the group $\mathcal{S}_{\mathbf{T}}$ of all its symmetries.

In Sections 4–9 we describe elastic maps having the symmetry $Z_\xi$—rotation through angle $\xi$ about the $z$-axis. In Section 11, however, we find that the seemingly natural group $\{(Z_\xi)^n : n \in \mathbb{Z}\}$ of integral powers of $Z_\xi$ is not an elastic symmetry group unless the angle $\xi$ is 0 or $\pi$. That is, unless $\xi = 0$ or $\pi$, there is no elastic map $\mathbf{T}$ such that $\{(Z_\xi)^n : n \in \mathbb{Z}\} = \mathcal{S}_{\mathbf{T}}$.

This raises the question of what in fact are the elastic symmetry groups. That is, when is a group of rotations also the group of symmetries of some elastic map? The answer is given in Theorems 17 and 18: The elastic symmetry groups are the conjugates of the eight reference groups in Table 3. The proof of this fact does not assume that elastic symmetries arise from crystallographic symmetries; it is purely mathematical.

We have two notions of symmetry for a rotation. One is as a symmetry of an elastic map, as above, and the other is as a symmetry of a subspace $\mathbb{W}$ of $\mathbb{M}$. In beachball terms, a rotation $V$ is a symmetry of $\mathbb{W}$ if using $V$ to rotate beachballs whose matrices are in $\mathbb{W}$ gives only beachballs whose matrices are also in $\mathbb{W}$. The symmetries of an elastic map $\mathbf{T}$ turn out to be the symmetries that are common to its eigenspaces (Theorem 13). Since the symmetries of a subspace are often relatively easy to recognize, we are usually able to realize our original goal of finding the group $\mathcal{S}_{\mathbf{T}}$ of symmetries of $\mathbf{T}$ (Section 15).

For more of a summary than that, we recommend the introduction. In this concluding section we only add a few comments that would not have made sense in the introduction.

The orthonormal basis $\mathbb{B}$ of $\mathbb{M}$ (Eq. 3) makes the reference matrices simple (Table 4). In the literature, one encounters the basis $\Phi$ defined in our Eq. (S23); see, for example, Eq. (2.5) of Mehrabadi & Cowin (1990) or Eqs. (4) and (5) of Bóna et al. (2007). The basis $\Phi$ plays the same role as $\mathbb{B}$, but it is less suited than $\mathbb{B}$ for the study of symmetry. Historically, $\Phi$ arose because it was orthonormal and because the matrix $[\mathbf{T}]_{\Phi\Phi}$ was closely related to the Voigt matrix for $\mathbf{T}$. The traditional Voigt matrix, defined in Eq. (S13), is still used by some authors, but it is undesirable for reasons explained in Section S4.5.3. It has been an obstacle to understanding.

The fact that the groups $\{I, Z_{\pi/2}, Z_\pi, Z_{3\pi/2}\}$ and $\{I, Z_{2\pi/3}, Z_{4\pi/3}\}$ are not elastic symmetry groups was not always recognized and can cause some confusion. Nye (1957, 1985), for example, has ten matrices, not eight, that would be the analogs of our reference matrices.

The significance of Theorem 5 is apt to be missed. For an elastic map $\mathbf{T}$, the theorem says that if $Z_\xi$ is a symmetry of $\mathbf{T}$ for some regular $\xi$ (Fig. 1), then $Z_\xi$ is a symmetry of $\mathbf{T}$ for all $\xi$. We used the theorem in deriving the elastic symmetry groups. The proof of the theorem looks easy, but the work had already been done in Lemmas 5 and 6 of Appendix B.

We are intrigued by the prime subspaces $\mathbb{B}(u.v)$ for $\overline{Z_\xi}$, $\xi = 2\pi/3$. The contrast with their tame counterparts for $\xi = \pi/2$ is striking; compare Fig. 11 with Fig. 9. We suspect that we are still missing some insights.

The results in this paper depend only on the elastic map being linear and self-adjoint. No other assumptions are involved.

## DATA AVAILABILITY STATEMENT

There are no new data associated with this article. Mathematica notebook files for generating the beachball pictures for elastic maps are available at https://github.com/carltape/mtbeach.

### Acknowledgments

We thank editor Duncan Agnew, reviewer Naofumi Aso, and an anonymous reviewer for their constructive suggestions. Chris Chapman and Michael Slawinski read early drafts of the paper and made helpful comments. C. Tape was supported by National Science Foundation grant EAR 1829447.

### Supporting Information

Section S1. Matrix versions of Eqs. (134) and (135)
Section S2. A test for equivalence of elastic maps
Section S3. Cubic symmetry from trigonal.
Section S4. Comparison with the traditional Voigt approach.

## REFERENCES

Abramian, S., Desmorat, B., Desmorat, R., Kolev, B., & Olive, M., 2019. Recovering the normal form of an elasticity tensor, *hal-02410330*, pp. 1–22.

Aki, K. & Richards, P. G., 2002. *Quantitative Seismology*, University Science Books, San Francisco, Calif., USA, 2nd edn., 2009 corrected printing.

Auld, B., 1973. *Acoustic fields and waves in solids*, vol. 1, John Wiley & Sons, New York.

Backus, G., 1970. A geometrical picture of anisotropic elastic tensors, *Reviews of Geophysics and Space Physics*, **8**(3), 633–671.

Baerheim, R., 1998. Classification of symmetry by means of Maxwell multipoles, *The Quarterly Journal of Mechanics and Applied Mathematics*, **51**, 73–103.

Bóna, A., Bucataru, I., & Slawinski, M. A., 2007. Coordinate-free characterization of the symmetry classes of elastic tensors, *J Elasticity*, **87**, 109–132.

Chadwick, P., Vianello, M., & Cowin, S. C., 2001. A new proof that the number of linear elastic symmetries is eight, *Journal of the Mechanics and Physics of Solids*, **49**, 2471–2492.

Chapman, C. H., 2004. *Fundamentals of Seismic Wave Propagation*, Cambridge University Press, Cambridge, United Kingdom.

Cowin, S., Mehrabadi, M., & Sadegh, A., 1991. Kelvin's Formulation of the Anisotropic Hooke's Law, in *Modern theory of anisotropic elasticity and applications / edited by Julian J. Wu, T.C.T Ting, David M. Barnett*, pp. 340–356, SIAM.

Danek, T., Kochetov, M., & Slawinski, M. A., 2015. Effective elasticity tensors in context of random errors, *Journal of Elasticity*, **121**, 55–67.

Fedorov, F. I., 1968. *Theory of Elastic Waves in Crystals*, Plenum Press, New York.

Forte, S. & Vianello, M., 1996. Symmetry classes for elasticity tensors, *Journal of Elasticity*, **43**, 81–108.

Helbig, K., 1994. *Foundations of Anisotropy for Exploration Scientists*, Pergamon.

Helbig, K., 2013. Review paper: What Kelvin might have written about Elasticity, *Geophys. Prosp.*, **61**, 1–20, doi: 10.1111/j.1365-2478.2011.01049.x.

Herman, B., 1945. Some theorems of the theory of anisotropic media, *Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS*, **48**(2), 89–92.

Hoffman, K. & Kunze, R., 1971. *Linear Algebra*, Prentice-Hall, Englewood Cliffs, New Jersey., USA, 2nd edn.

Mehrabadi, M. M. & Cowin, S. C., 1990. Eigentensors of linear anisotropic elastic materials, *Q. J. Mech. Appl. Math.*, **43**(1), 15–41.

Musgrave, M., 1970. *Crystal Acoustics*, Holden-Day, San Fransisco, Calif., USA.

Nye, J., 1957, 1985. *Physical Properties of Crystals*, Oxford University Press, Oxford.

Rychlewski, J., 1984. On Hooke's Law, *PMM U.S.S.R*, **48**(3), 303–314.

Slawinski, M. A., 2015. *Waves and Rays in Elastic Continua*, World Scientific, Singapore, 3rd edn.

Tape, W. & Tape, C., 2012. A geometric setting for moment tensors, *Geophys. J. Int.*, **190**, 476–498, doi:10.1111/j.1365-246X.2012.05491.x.

Tape, W. & Tape, C., 2013. The classical model for moment tensors, *Geophys. J. Int.*, **195**, 1701–1720, doi:10.1093/gji/ggt302.

Tape, W. & Tape, C., 2016. A confidence parameter for seismic moment tensors, *Geophys. J. Int.*, **205**, 938–953, doi: 10.1093/gji/ggw057.

William Thomson (Lord Kelvin), 1856. Elements of a Mathematical Theory of Elasticity, *Philosophical Transactions of the Royal Society*, **166**, 481–498.

## APPENDIX A: Proof of Lemma 1 of Section 4.3

**Lemma 1** Let $\mathbf{U} : \mathbb{V} \to \mathbb{V}$ be unitary and let $\mathbb{W}$ be a non-zero subspace of $\mathbb{V}$ that is invariant under $\mathbf{U}$. Then $\mathbb{W}$ is the orthogonal direct sum of subspaces of $\mathbb{V}$ that are prime for $\mathbf{U}$.

*Proof.* The proof is by induction. Let $\mathrm{P}(n)$ be the statement that if $\mathbb{W}$ is a non-zero subspace of $\mathbb{V}$ with dim $\leq n$ and if $\mathbb{W}$ is invariant under $\mathbf{U}$, then $\mathbb{W}$ is the orthogonal direct sum of subspaces that are prime for $\mathbf{U}$.

The statement $\mathrm{P}(1)$ is true, vacuously.

Assume $\mathrm{P}(n)$ and prove $\mathrm{P}(n+1)$: Let $\mathbb{W}$ be a non-zero subspace with dim $\leq n+1$ that is invariant (under $\mathbf{U}$). If $\mathbb{W}$ is prime, then the sought-after orthogonal direct sum is $\mathbb{W} = \mathbb{W}$. If $\mathbb{W}$ is not prime, then, among the non-zero invariant subspaces of $\mathbb{W}$, let $\mathbb{W}_1$ be one of smallest dimension. We note: (i) The orthogonal complement $\mathbb{W}_1^\perp$ of $\mathbb{W}_1$ in $\mathbb{W}$ is invariant by Eq. (46). (ii) $\dim \mathbb{W}_1^\perp \leq n$. (iii) $\mathbb{W}_1^\perp \neq \{\mathbf{0}\}$. Thus $\mathrm{P}(n)$ can be applied to $\mathbb{W}_1^\perp$, so that $\mathbb{W}_1^\perp$ is the orthogonal direct sum of prime subspaces $\mathbb{W}_2, \ldots, \mathbb{W}_j$. Since $\mathbb{W}_1$ is also prime, then

$$\mathbb{W} = \mathbb{W}_1 \perp \mathbb{W}_1^\perp = \mathbb{W}_1 \perp \mathbb{W}_2 \perp \ldots \perp \mathbb{W}_j$$
$$\mathbb{W}_1, \ldots \mathbb{W}_j \text{ are prime} \tag{A.1}$$

□

## APPENDIX B: Prime subspaces for $[\overline{Z_\xi}]_{\mathbb{BB}}$ when $\xi$ is regular

Let $A$ be a $6 \times 6$ matrix and let $\mathbf{w} \in \mathbb{R}^6$ be non-zero. In Appendices B and C we will be considering the smallest subspace $\widehat{\mathbf{w}}$ of $\mathbb{R}^6$ that contains $\mathbf{w}$ and that is invariant under $A$. If $\mathbf{w}, A\mathbf{w}, \ldots, A^j\mathbf{w}$ are linearly dependent, then

$$\widehat{\mathbf{w}} = \langle \mathbf{w}, A\mathbf{w}, \ldots, A^{j-1}\mathbf{w} \rangle \tag{B.1}$$

We will also need the matrix $K(k)$ whose rows are $\mathbf{w}, A\mathbf{w}, \ldots, A^k\mathbf{w}$. For $A = [\overline{Z_\xi}]_{\mathbb{BB}}$ and $\mathbf{w} = (a, b, c, d, e, f) \in \mathbb{R}^6$, it is (from Eq. 83)

$$K(k) = \begin{pmatrix} a & b & c & d & e & f \\ a\cos\xi + b\sin\xi & b\cos\xi - a\sin\xi & c\cos 2\xi - d\sin 2\xi & d\cos 2\xi + c\sin 2\xi & e & f \\ a\cos 2\xi + b\sin 2\xi & b\cos 2\xi - a\sin 2\xi & c\cos 4\xi - d\sin 4\xi & d\cos 4\xi + c\sin 4\xi & e & f \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a\cos k\xi + b\sin k\xi & b\cos k\xi - a\sin k\xi & c\cos 2k\xi - d\sin 2k\xi & d\cos 2k\xi + c\sin 2k\xi & e & f \end{pmatrix} \tag{B.2}$$

**Lemma 5.** Let $\xi$ be regular, that is, $\xi \neq \pm 2\pi/n$, $n = 1, 2, 3, 4$. Let the subspace $\mathbb{E}$ of $\mathbb{R}^6$ be invariant under (multiplication by) $[\overline{Z_\xi}]_{\mathbb{BB}}$, and let $\dim \mathbb{E} \leq 3$. If $\mathbf{w} = (a, b, c, d, e, f) \in \mathbb{E}$ then $a = b = 0$ or $c = d = 0$.

*Proof.* Let $A = [\overline{Z_\xi}]_{\mathbb{BB}}$, and consider the matrix with rows $\mathbf{w}, A\mathbf{w}, A^2\mathbf{w}, A^3\mathbf{w}$—it is $K(3)$ in Eq. (B.2). The determinant of its left-hand $4 \times 4$ submatrix is found to be

$$-16(a^2 + b^2)(c^2 + d^2)(1 + 2\cos\xi)^2 \sin 2\xi \sin\xi \sin^4 \frac{\xi}{2} \tag{B.3}$$

Since $\mathbb{E}$ is invariant, then $\mathbf{w}, A\mathbf{w}, A^2\mathbf{w}, A^3\mathbf{w}$ are all in $\mathbb{E}$, and since $\dim \mathbb{E} \leq 3$ they are linearly dependent. The determinant in Eq. (B.3) must therefore be zero. The factors involving $\xi$, however, are non-zero, since $\xi$ is regular. Hence $a = b = 0$ or $c = d = 0$. $\square$

**Lemma 6.** If $\xi$ is regular, the prime subspaces of $\mathbb{R}^6$ for (multiplication by) $[\overline{Z_\xi}]_{\mathbb{BB}}$ are $\mathbb{E}_{12}$, $\mathbb{E}_{34}$, and $\langle \mathbf{e}_{56}(t) \rangle$ (any $t$).

*Proof.* Let $A = [\overline{Z_\xi}]_{\mathbb{BB}}$. We look first for the subspaces of $\dim \leq 3$ that are prime for $A$. From Eq. (83) we can see that the following subspaces are invariant under $A$.

$$\begin{array}{llll} \dim 3 & \mathbb{E}_{12} \perp \langle \mathbf{e}_{56}(t) \rangle & \mathbb{E}_{34} \perp \langle \mathbf{e}_{56}(t) \rangle & \\ \dim 2 & \mathbb{E}_{12} & \mathbb{E}_{34} & \mathbb{E}_{56} \\ \dim 1 & & & \langle \mathbf{e}_{56}(t) \rangle \\ \dim 0 & & \{\mathbf{0}\} & \end{array}, \tag{B.4}$$

But are they prime, and have we found all of them?

To that end, suppose that a subspace $\mathbb{E}$ has $\dim \leq 3$ and is prime (for $A$). Since it is prime, it contains a non-zero element $\mathbf{w} = (a, b, c, d, e, f)$. Again since $\mathbb{E}$ is prime, the smallest invariant subspace $\widehat{\mathbf{w}}$ containing $\mathbf{w}$ must be all of $\mathbb{E}$. Since $\dim \mathbb{E} \leq 3$ then $\widehat{\mathbf{w}} = \langle \mathbf{w}, A\mathbf{w}, A^2\mathbf{w} \rangle$, from Eq. (B.1). The rows $\mathbf{w}, A\mathbf{w}, A^2\mathbf{w}$ of the matrix $K(2)$ in Eq. (B.2) therefore span $\mathbb{E}$, and the same will be true for the rows of any matrix that is row equivalent to $K(2)$.

(i) The case $c^2 + d^2 \neq 0$ (and necessarily $a = b = 0$, from Lemma 5): The matrix $K(2)$ is row equivalent to

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & f \end{pmatrix} \tag{B.5}$$

The subspace $\mathbb{E}$ is therefore spanned by $\mathbf{e}_3$, $\mathbf{e}_4$, and $e\,\mathbf{e}_5 + f\,\mathbf{e}_6$. If $e^2 + f^2 \neq 0$, then $\mathbb{E}$ is the three-dimensional subspace $\mathbb{E}_{34} \perp \langle \mathbf{e}_{56}(t)$ for some $t$, but it is not prime, since it has proper invariant subspaces. If $e = f = 0$, then $\mathbb{E}$ is the two-dimensional space $\mathbb{E}_{34}$.

(ii) The case $a^2 + b^2 \neq 0$: Similar to (i). The only candidate for a prime subspace is $\mathbb{E}_{12}$.

(iii) The case $a = b = c = d = 0$: The subspace $\mathbb{E}$ must be $\langle \mathbf{e}_{56}(t) \rangle$ for some $t$.

Thus the only possible prime subspaces of $\dim \leq 3$ are $\mathbb{E}_{12}$, $\mathbb{E}_{34}$, and $\langle \mathbf{e}_{56}(t) \rangle$. Since no one of them contains another, they are indeed prime.

Could there be a prime subspace with $\dim > 3$? If so, it would have dimension 4 or 5 ($\mathbb{R}^6$ is not prime) and it would then have to be the orthogonal complement of an invariant subspace of dimension 1 or 2. But the invariant subspaces of dimension 1 and 2 are now known (bottom two rows of Eq. B.6), and their orthogonal complements, shown above them, are not prime:

$$\begin{array}{llll} \dim 5 & & & \mathbb{E}_{12} \perp \mathbb{E}_{34} \perp \langle \mathbf{e}_{56}(t') \rangle \\ \dim 4 & \mathbb{E}_{34} \perp \mathbb{E}_{56} & \mathbb{E}_{12} \perp \mathbb{E}_{56} & \mathbb{E}_{12} \perp \mathbb{E}_{34} \\ \dim 2 & \mathbb{E}_{12} & \mathbb{E}_{34} & \mathbb{E}_{56} \\ \dim 1 & & & \langle \mathbf{e}_{56}(t) \rangle \end{array}, \tag{B.6}$$

## APPENDIX C: Prime subspaces for $[\overline{Z_\xi}]_{\mathbb{BB}}$ when $\xi = 2\pi/3$

We define the unit vector $\mathbf{e}(\theta, u, v)$ in $\mathbb{R}^6$ by

$$\begin{aligned} \mathbf{e}(\theta, u, v) = {} & (\cos\theta)\,(\cos u,\ 0,\ \sin u \cos v,\ \sin u \sin v,\ 0,\ 0) + \\ & (\sin\theta)\,(0,\ \cos u,\ -\sin u \sin v,\ \sin u \cos v,\ 0,\ 0) \end{aligned} \tag{C.1}$$

Then

$$\mathbf{e}(\theta, u, v) = (\cos\theta)\,\mathbf{e}(0, u, v) + (\sin\theta)\,\mathbf{e}(\pi/2, u, v) \tag{C.2}$$

For each $u$ and $v$ we define $\mathbb{E}(u, v)$ to be the subspace of $\mathbb{R}^6$ spanned by the orthonormal vectors $\mathbf{e}(0, u, v)$ and $\mathbf{e}(\pi/2, u, v)$:

$$\mathbb{E}(u, v) = \{r\,\mathbf{e}(\theta, u, v) : r, \theta \in \mathbb{R}\} \tag{C.3}$$

Note that in the two-dimensional space $\mathbb{E}(u, v)$ the angle $\theta$ is indeed the usual angular polar coordinate with respect to the basis vectors $\mathbf{e}(0, u, v)$ and $\mathbf{e}(\pi/2, u, v)$. And on $\mathbb{E}(u, v)$ multiplication by $[\overline{Z_\xi}]_{\mathbb{BB}}$ is rotation through angle $-\xi$:

$$[\overline{Z_\xi}]_{\mathbb{BB}}(\mathbf{e}(\theta, u, v)) = \mathbf{e}(\theta - \xi, u, v) \qquad (\xi = 2\pi/3) \tag{C.4}$$

Hence $\mathbb{E}(u, v)$ is not only invariant under $[\overline{Z_\xi}]_{\mathbb{BB}}$, it is prime.

**Lemma 7.** Let $\xi = 2\pi/3$. The subspaces of $\mathbb{R}^6$ that are prime for (multiplication by) $[\overline{Z_\xi}]_{\mathbb{BB}}$ are $\mathbb{E}(u, v)$ and $\mathbf{e}_{56}(t)$ (any $t, u, v$).

*Proof.* Let $\mathbb{E}$ be prime for $A = [\overline{Z_\xi}]_{\mathbb{BB}}$ with $\xi = 2\pi/3$, and choose a non-zero point $\mathbf{w} = (a, b, c, d, e, f)$ in $\mathbb{E}$. Since $\mathbb{E}$

is prime then $\widehat{\mathbf{w}} = \mathbb{E}$, and since $A^3\mathbf{w} = \mathbf{w}$ (Eq. 83) then $\widehat{\mathbf{w}} = \langle \mathbf{w}, A\mathbf{w}, A^2\mathbf{w}\rangle$. As in the proof of Lemma 6, we consider the matrix $K = K(2)$ whose rows are $\mathbf{w}, A\mathbf{w}, A^2\mathbf{w}$. Its rows span $\mathbb{E}$ and hence so do the rows of any matrix that is row equivalent to it. From Eq. (B.2) with $\xi = 2\pi/3$, the matrix $K$ is

$$\begin{pmatrix} a & b & c & d & e & f \\ \frac{-a+\sqrt{3}\,b}{2} & \frac{-\sqrt{3}\,a-b}{2} & \frac{-c+\sqrt{3}\,d}{2} & \frac{-\sqrt{3}\,c-d}{2} & e & f \\ \frac{-a-\sqrt{3}\,b}{2} & \frac{\sqrt{3}\,a-b}{2} & \frac{-c-\sqrt{3}\,d}{2} & \frac{\sqrt{3}\,c-d}{2} & e & f \end{pmatrix} \tag{C.5}$$

The matrix $K$ is row equivalent to

$$K' = \begin{pmatrix} a^2+b^2 & 0 & ac+bd & ad-bc & 0 & 0 \\ 0 & a^2+b^2 & bc-ad & ac+bd & 0 & 0 \\ 0 & 0 & 0 & 0 & e & f \end{pmatrix} \tag{C.6}$$

The rows of $K'$ are

$$\begin{aligned} (a^2+b^2,\, 0,\, ac+bd,\, ad-bc,\, 0,\, 0) &= h\,\mathbf{e}(0,u,v) \\ (0,\, a^2+b^2,\, bc-ad,\, ac+bd,\, 0,\, 0) &= h\,\mathbf{e}(\pi/2,u,v) \\ (0,0,0,0,e,f) &= \sqrt{e^2+f^2}\,\mathbf{e}_{56}(t) \end{aligned} \tag{C.7}$$

where

$$\begin{aligned} h &= \sqrt{(a^2+b^2)(a^2+b^2+c^2+d^2)} \\ u &= \widehat{\theta}(\sqrt{a^2+b^2}, \sqrt{c^2+d^2}) \\ v &= \widehat{\theta}(ac+bd, ad-bc) \\ t &= \widehat{\theta}(e,f) \end{aligned} \tag{C.8}$$

and where $\widehat{\theta}(x,y)$ is the usual angular polar coordinate of a point $(x,y)$ in the plane.

The subspace $\mathbb{E}$ is therefore spanned by $h\,\mathbf{e}(0,u,v)$, $h\,\mathbf{e}(\pi/2,u,v)$, and $\sqrt{e^2+f^2}\,\mathbf{e}_{56}(t)$. If $h = 0$ then $\mathbb{E} = \langle \mathbf{e}_{56}(t)\rangle$ ($\mathbf{w}$ was non-zero), which is prime. If $e^2+f^2 = 0$ then $\mathbb{E} = \langle \mathbf{e}(0,u,v), \mathbf{e}(\pi/2,u,v)\rangle = \mathbb{E}(u,v)$, which is prime by Eq. (C.4). If $h$ and $e^2+f^2$ are both non-zero, then $\mathbb{E}$ is three dimensional and has the proper invariant subspaces $\mathbb{E}(u,v)$ and $\langle \mathbf{e}_{56}(t)\rangle$, so $\mathbb{E}$ is not prime. □

## APPENDIX D: Motivation for Eqs. (89) and (126)

To arrive at Eqs. (89) and (126), we assume $[\mathbf{T}]_{\mathbb{BB}} = T_3$ and then look for $t, u, v$ such that $[\mathbf{T}]_{\mathbb{B}_3\mathbb{B}_3}$ is diagonal. (The basis $\mathbb{B}_3$ is $\mathbb{B}(t,u,v)$ as usual.) We find

$$\begin{aligned} [\mathbf{T}]_{\mathbb{B}_3\mathbb{B}_3} &= [\mathbf{I}]_{\mathbb{B}_3\mathbb{B}}\,[\mathbf{T}]_{\mathbb{BB}}\,[\mathbf{I}]_{\mathbb{BB}_3} \\ &= [\mathbf{I}]_{\mathbb{B}_3\mathbb{B}}\,T_3\,[\mathbf{I}]_{\mathbb{BB}_3} \\ &= \begin{pmatrix} \begin{matrix} c_{11} & 0 & c_{13} & -c_{23} \\ 0 & c_{11} & c_{23} & c_{13} \\ c_{13} & c_{23} & c_{33} & 0 \\ -c_{23} & c_{13} & 0 & c_{33} \end{matrix} & \\ & R(-t)\begin{pmatrix} e & k \\ k & f \end{pmatrix}R(t) \end{pmatrix}, \end{aligned} \tag{D.1a}$$

where, after some manipulations,

$$\begin{aligned} c_{11} &= \frac{1}{2}\big(a+c+\rho_u\cos 2(u-\theta_u)\big) - \rho_v(1-\cos(v-\theta_v))\sin 2u \\ c_{33} &= \frac{1}{2}\big(a+c-\rho_u\cos 2(u-\theta_u)\big) + \rho_v(1-\cos(v-\theta_v))\sin 2u \\ c_{13} &= -\frac{1}{2}\,\rho_u\sin 2(u-\theta_u) - \rho_v\,(1-\cos(v-\theta_v))\,\cos 2u \\ c_{23} &= \rho_v\sin(v-\theta_v), \end{aligned} \tag{D.1b}$$

where $\theta_u$ and $\theta_v$ are from Eqs. (126) and

$$\begin{aligned} \rho_u(T) &= \sqrt{(a-c)^2+4(h^2+m^2)} \\ \rho_v(T) &= \sqrt{h^2+m^2} \end{aligned} \tag{D.2}$$

From Eqs. (F.5) and (D.1a) the condition $t = \theta_\infty$ from Eq. (89) is enough to diagonalize the lower right $2\times 2$ submatrix of $[\mathbf{T}]_{\mathbb{B}_3\mathbb{B}_3}$. From Eqs. (D.1b) the conditions $u = \theta_u$ and $v = \theta_v$ are enough to make $c_{13} = c_{23} = 0$ and thus to diagonalize the upper left $4\times 4$ submatrix. With $[\mathbf{T}]_{\mathbb{B}_3\mathbb{B}_3}$ diagonalized, $\mathbb{B}(t,u,v)$ is an eigenbasis for $\mathbf{T}$, and the diagonal entries are the corresponding eigenvalues.

## APPENDIX E: The eigenbasis for $\mathbf{T}'$ in Fig. 23

With $U$ as in Eq. (196b) the eigenvectors of $\mathbf{T}'$ in Fig. 23 are $G_i = UH_iU^\top$, where

$$\begin{aligned} H_1 &= \begin{pmatrix} 0 & \sqrt{2} & 2-\sqrt{3} \\ \sqrt{2} & 0 & 2+\sqrt{3} \\ 2-\sqrt{3} & 2+\sqrt{3} & 0 \end{pmatrix} \\ H_2 &= \begin{pmatrix} 0 & -\sqrt{2} & 2+\sqrt{3} \\ -\sqrt{2} & 0 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2-\sqrt{3} & 0 \end{pmatrix} \\ H_3 &= \begin{pmatrix} 0 & \sqrt{6} & 1 \\ \sqrt{6} & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \\ H_4 &= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1+\sqrt{3} & 0 \\ 0 & 0 & 1-\sqrt{3} \end{pmatrix} \\ H_5 &= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1-\sqrt{3} & 0 \\ 0 & 0 & 1+\sqrt{3} \end{pmatrix} \\ H_6 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{E.1}$$

(Normalizing factors have been omitted.) Because the eigenvalues $\lambda_1$ and $\lambda_4$ of $\mathbf{T}'$ are not simple, there are infinitely many essentially different possibilities for the $G_i$ and hence for the $H_i$. Note also that initially $U$ is not known; the eigenvectors $G_i$ of $\mathbf{T}'$ come first, and then $U$ is found from them.

## APPENDIX F: Some diagonalization

We review the diagonalization of a $2\times 2$ symmetric matrix $S = (s_{ij})$. We let $R(\theta)$ be the $2\times 2$ rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{F.1}$$

Then

$$S = R(\bar\theta)\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix}R(\bar\theta)^\top, \tag{F.2a}$$

where

$$\mu_1 = \frac{1}{2}(s_{11}+s_{22}+\bar\rho), \quad \mu_2 = \frac{1}{2}(s_{11}+s_{22}-\bar\rho), \tag{F.2b}$$

and where, with $\widehat\theta(x,y)$ being the ordinary angular polar coordinate of a point $(x,y)$,

$$\begin{aligned}\bar\rho = \bar\rho(S) &= \sqrt{(s_{11}-s_{22})^2+4s_{12}^2}\\ \bar\theta = \bar\theta(S) &= \frac{1}{2}\,\widehat\theta(s_{11}-s_{22},\,2s_{12})\end{aligned} \tag{F.2c}$$

Thus $\bar\rho$ and $2\bar\theta$ are the polar coordinates of the point $(s_{11}-s_{22},\,2s_{12})$. From Eq. (F.2a) the numbers $\mu_1$ and $\mu_2$ are the eigenvalues of $S$, and the columns of $R(\bar\theta)$ are the corresponding eigenvectors. The parameter $\bar\theta$ is thus the angular polar coordinate of the first eigenvector. The parameter $\bar\rho$ is zero if and only if $S$ is a multiple of the identity matrix.

An alternative to Eq. (F.2a) is

$$S = R(\bar\theta+\pi/2)\begin{pmatrix}\mu_2 & 0\\ 0 & \mu_1\end{pmatrix}R(\bar\theta+\pi/2)^\top \tag{F.3}$$

The first and second eigenvalues are now $\mu_2$ and $\mu_1$. The two triples $\bar\theta,\mu_1,\mu_2$ and $\bar\theta+\pi/2,\mu_2,\mu_1$ are equally valid descriptions of $S$. The former is the one with the first eigenvalue greater than or equal to the second.

From Eq. (F.2b),

$$\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix} = \frac{s_{11}+s_{22}}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{\bar\rho}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{F.4}$$

Since the operation of conjugation is linear, and since it leaves the identity matrix unchanged, conjugation of Eq. (F.4) by $R(-\theta+\bar\theta)$ gives

$$R(-\theta)SR(\theta) = \frac{s_{11}+s_{22}}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{\bar\rho}{2}\begin{pmatrix}\cos 2(\theta-\bar\theta) & -\sin 2(\theta-\bar\theta)\\ -\sin 2(\theta-\bar\theta) & -\cos 2(\theta-\bar\theta)\end{pmatrix} \tag{F.5}$$

## APPENDIX G: Glossary of selected notation

$\mathbb{M}$ = all $3\times 3$ symmetric matrices.

$\langle \mathbf{w}_1,\dots,\mathbf{w}_n\rangle$ = subspace spanned by $\mathbf{w}_1,\dots,\mathbf{w}_n$

6-tuples:

| | |
|---|---|
| $[E]_{\mathbb{F}}$ | Eq. 11 |
| $\mathbf{e}_1,\dots,\mathbf{e}_6$ = standard basis | |
| $\mathbf{e}_{12}(r),\mathbf{e}_{34}(s),\mathbf{e}_{56}(t)$ | Eq. 80 |
| $\mathbf{e}(\theta,u,v)$ | Eq. C.1 |

$3\times 3$ matrices:

| | |
|---|---|
| $X_\xi, Y_\xi, Z_\xi$ | Eq. 50 |
| $B_i$ | Eq. 3 |
| $B_{12}(r), B_{34}(s), B_{56}(t)$ | Eq. 79 |
| $B(\theta,u,v)$ | Eq. 110 |
| $B_j(U_{4\times 4})$ | Eq. 103 |
| $B_j(U_{3\times 3})$ | Eq. 145 |

$6\times 6$ matrices:

| | |
|---|---|
| $T_{\text{MONO}}, T_3, T_4, T_{\text{XISO}}$ | Table 1 |
| $T_1, T_{\text{MONO}}, T_{\text{ORTH}}, T_{\text{TET}},\dots$ | Table 4 |
| $[\mathbf{S}]_{\mathbb{GF}}$ (e.g., $[\mathbf{T}]_{\mathbb{BB}}$) | Eqs. 13, 17 |
| $[\mathbf{I}]_{\mathbb{GF}}$ | Eq. 32 |

Subspaces of $\mathbb{R}^6$:

| | |
|---|---|
| $\mathbb{E}_{12},\mathbb{E}_{34},\mathbb{E}_{56}$ | Eq. 82 |
| $\mathbb{E}(u,v)$ | Eq. C.3 |

Subspaces of $\mathbb{M}$:

| | |
|---|---|
| $\mathbb{B}_{12},\mathbb{B}_{34},\mathbb{B}_{56}$ | Eq. 81 |
| $\mathbb{B}(u,v)$ | Eq. 112 |

Bases for $\mathbb{M}$:

| | |
|---|---|
| $\mathbb{B}$ | Eq. 3 |
| $\mathbb{B}(t,u,v) = \mathbb{B}_3$ | Eq. 116 |

Linear transformations of $\mathbb{M}$:

| | |
|---|---|
| $\mathbf{S}_2\circ\mathbf{S}_1$ | Eq. 15 |
| $\mathbf{S}^*$ | Eq. 33 |
| $\overline{U}$ | Eq. 51 |
| $\mathbf{T}_2^{\Lambda}(r,U)$ | Eq. 104 |
| $\mathbf{T}_3^{\Lambda}(t,u,v)$ | Eq. 122 |
| $\mathbf{T}_4^{\Lambda}(s,t)$ | Eq. 95 |
| $\mathbf{T} = \underset{\lambda_1}{\mathbb{W}_1}\perp\dots\perp\underset{\lambda_n}{\mathbb{W}_n}$ | Eq. 44 |

Orthogonality:

| | |
|---|---|
| $\mathbb{W}_1\perp\mathbb{W}_2$ | Eq. 40 |
| $\mathbb{W}^\perp$ | Eq. 41 |
| $\mathbb{W} = \mathbb{W}_1\perp\dots\perp\mathbb{W}_n$ | Eq. 42 |

Groups of rotation matrices:

| | |
|---|---|
| $\mathbb{U}$ | Section 3.1 |
| $\mathcal{U}_1,\mathcal{U}_{\text{MONO}},\mathcal{U}_{\text{ORTH}},\mathcal{U}_{\text{TET}},\dots$ | Table 4 |
| $\mathcal{S}_{\mathbf{T}}$ | Eq. 131 |
| $\mathcal{S}(E)$ | Eq. 154 |
| $\mathcal{U}(V)$ | Lemma 4 |

$$T_{\text{MONO}} \ \ (\xi = \pi) \qquad \begin{pmatrix} a & g & & & & \\ g & b & & & & \\ & & c & i & o & s \\ & & i & d & j & p \\ & & o & j & e & k \\ & & s & p & k & f \end{pmatrix} \qquad T_{\text{XISO}} \ \ (\xi \text{ regular}) \qquad \begin{pmatrix} a & & & & & \\ & a & & & & \\ & & c & & & \\ & & & c & & \\ & & & & e & k \\ & & & & k & f \end{pmatrix}$$

$$T_4 \ \ (\xi = \pi/2) \qquad \begin{pmatrix} a & & & & & \\ & a & & & & \\ & & c & i & & \\ & & i & d & & \\ & & & & e & k \\ & & & & k & f \end{pmatrix} \qquad T_3 \ \ (\xi = 2\pi/3) \qquad \begin{pmatrix} a & 0 & m & -h & & \\ 0 & a & h & m & & \\ m & h & c & 0 & & \\ -h & m & 0 & c & & \\ & & & & e & k \\ & & & & k & f \end{pmatrix}$$

**Table 1.** Matrices $[\mathbf{T}]_{\mathbb{BB}}$ for elastic maps $\mathbf{T}$ having rotational symmetry $Z_\xi$ for $\xi$ as indicated. The rotation $Z_{\pi/2}$, for example, is a symmetry of $\mathbf{T}$ if and only if $[\mathbf{T}]_{\mathbb{BB}} = T_4$ for some $a, c, d, e, f, i, k$ (Section 6). If the basis for the matrix representations in the table is changed from $\mathbb{B}$ to the basis $\Phi$ defined in Eq. (S23), the matrices analogous to $T_3$ and $T_4$ are consistent with Eqs. (5.9) and (5.26) of Mehrabadi & Cowin (1990). Blank entries are understood to be zeros.

| Subspace $\mathbb{W}$ | Defining feature for members of $\mathbb{W}$ | Member $E = (e_{i\,j})$ of $\mathbb{W}$ | $\mathcal{S}(\mathbb{W})$ |
|---|---|---|---|
| $\langle B_1, B_2, B_3, B_4, B_5\rangle$ | Deviatoric | $\sum e_{ii} = 0$ | $\mathcal{U}_{\text{ISO}}$ |
| $\langle B_3, B_4, B_5, B_6\rangle$ | $z$-axis is a principal axis | $e_{23} = e_{13} = 0$ | $\mathcal{U}_{\text{XISO}}$ |
| $\langle B_3, B_4, B_5\rangle$ | Deviatoric, and $z$-axis is a principal axis | $e_{23} = e_{13} = \sum e_{ii} = 0$ | $\mathcal{U}_{\text{XISO}}$ |
| $\langle B_4, B_5, B_6\rangle$ | $xyz$-axes are principal axes | $E$ is diagonal | $\mathcal{U}_{\text{CUBE}}$ |
| $\langle B_1, B_2, B_3\rangle$ | | $e_{11} = e_{22} = e_{33} = 0$ | $\mathcal{U}_{\text{CUBE}}$ |
| $\langle B_4, B_5\rangle$ | Deviatoric, and $xyz$-axes are principal axes | $E$ is diagonal, $\sum e_{ii} = 0$ | $\mathcal{U}_{\text{CUBE}}$ |
| $\langle B_1, B_2\rangle = \mathbb{B}_{12}$ | Double couple, and $z$-axis is a fault normal | $e_{11} = e_{22} = e_{33} = e_{12} = 0$ | $\mathcal{U}_{\text{XISO}}$ |
| $\langle B_3, B_4\rangle = \mathbb{B}_{34}$ | Double couple, and $z$-axis is the null axis | $e_{13} = e_{23} = e_{33} = 0$, $e_{11} = -e_{22}$ | $\mathcal{U}_{\text{XISO}}$ |
| $\langle B_5, B_6\rangle = \mathbb{B}_{56}$ | Crack matrix, and $z$-axis is the $c$-axis | $E$ is diagonal, $e_{11} = e_{22}$ | $\mathcal{U}_{\text{XISO}}$ |
| $\langle B_6\rangle$ | Isotropic | $E = t\, I_{3\times 3}$ | $\mathcal{U}_{\text{ISO}}$ |

**Table 2.** TWO-COLUMN WIDTH. Selected subspaces $\mathbb{W}$ of $\mathbb{M}$ relevant to elastic symmetry. The subspace $\langle B_3, B_4, B_5\rangle$, for example, is the set of all deviatoric matrices with a principal axis (i.e., eigendirection) vertical. The matrices $B_1, \ldots, B_6$ are as in Eq. (3). The descriptions in the second and third columns are intrinsic; they do not involve the basis $\mathbb{B}$ or any other basis of $\mathbb{M}$. The last column gives the symmetry group $\mathcal{S}(\mathbb{W})$ of $\mathbb{W}$, to be explained in Section 13. Subspaces that are orthogonal complements of each other, such as $\langle B_1, B_2, B_3\rangle$ and $\langle B_4, B_5, B_6\rangle$, have the same symmetry group.