Dynamical Methods for Studying Stability and Noise in Frequency Combs Sources

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with

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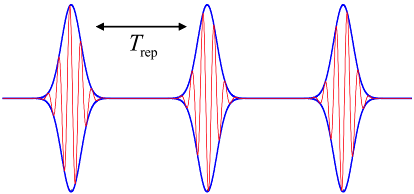
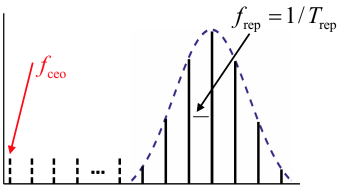
Background

In 1999–2000: Frequency combs were invented

"The excitement surrounding the rapid evolution in these fields since 1999 gives us a hint of what it must have been like after 1927 when the first ideas of quantum mechanics were introduced. . ."

— J. L. Hall and T. W. Hänsch, 2005 Nobel prize winners in Physics





The key advance was electronically locking f_{ceo} and f_{rep}!

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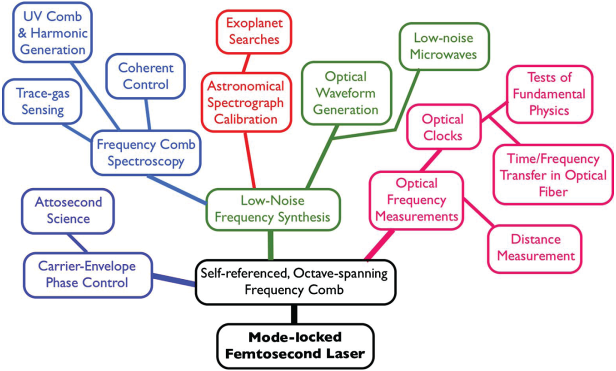
Background

- Frequency and time measurement was revolutionized
- Many new applications opened up[1]



[1] S. Diddams, J. Opt. Soc. Am. B **27**, 51 (2010).

Motivation

- Advances have all been through “cut-and-try” experimentation
- Theoretical tools for analyzing and designing frequency combs are primitive
  - “Brute force” simulations or rough analytical approximations
    - Adequate for post-hoc analysis; inadequate for design
    - yield limited insight into the sources of instability

Key theoretical questions:

- Where in the adjustable parameter space are combs stable?
- What is the noise performance?
- How can we optimize the comb?
  - high output power and/or large bandwidth and/or low noise

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Approach

Combine: 400 years of dynamical systems theory

$+$

modern computers and algorithms

(linear algebra $+$ root-finding)

to answer these questions

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Origins of the Dynamical Approach

These dynamical ideas are very old!

- The pendulum clock:
  (Galileo – 1632; Huygens – 1673; Euler – 1736)
- Stability of the solar system:
  - Two body problem: Newton – 1686
  - Three body problem:
    In general, not solvable, but . . .
    stable fixed points found by Lagrange – 1772 (observed 1906)
- Application to continuous systems . . .
  (described by partial differential equations)

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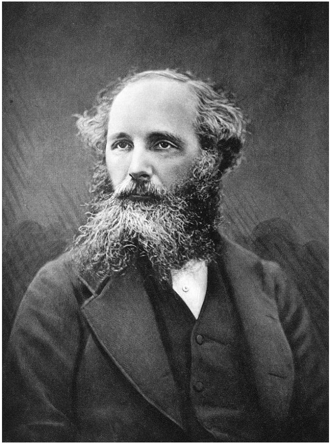
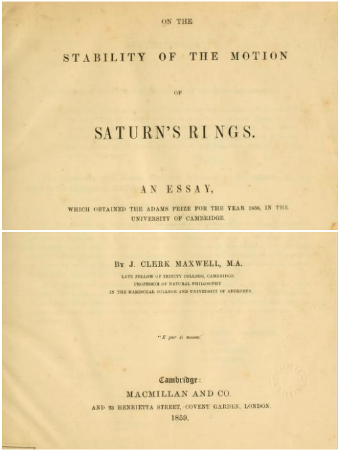
Origins of the Dynamical Approach

Before Maxwell's Equations (1861) . . .

Before the Maxwell-Boltzmann distribution (1865) . . .

Maxwell explained the stability of Saturn's rings (1859)





ON THE

STABILITY OF THE MOTION

OF

SATURN'S RINGS.

AN ESSAY,

WHICH OBTAINED THE ADAMS PRIZE FOR THE YEAR 1856, IN THE UNIVERSITY OF CAMBRIDGE.

By J. CLERK MAXWELL, M.A.

LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE.
PROFESSOR OF NATURAL PHILOSOPHY
IN THE MARISCHAL COLLEGE AND UNIVERSITY OF ABERDEEN.

"E pur si muove."

Cambridge:
MACMILLAN AND CO.
AND 23 HENRIETTA STREET, COVENT GARDEN, LONDON.
1859.

Numerous Applications

- Control of satellite orbits
- Electronic system design
- Plasma systems (tokamaks, ionosphere,. . .)
- Mechanical, chemical, and fluid systems
- Biological systems (heart, animal populations)
- Economic systems
- Lasers/other optical resonators
  - but mostly in highly simplified, almost analytical approximations!

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Frequency Comb Systems

Passively modelocked lasers (slow saturable gain):

$$\frac{\partial u}{\partial T} = \left[-i\phi + t_s \frac{\partial}{\partial t} + \frac{g(|u|)}{2}\left(1 + \frac{1}{2\omega_g^2}\frac{\partial^2}{\partial t^2}\right) - \frac{l}{2} - i\frac{\beta''}{2}\frac{\partial^2}{\partial t^2} + i\gamma|u|^2 - f_{\mathrm{sa}}(|u|)\right] u,$$

$$g(|u|) = g_0\left[1 + w_0/(P_{\mathrm{sat}}T_R)\right]^{-1}$$

cubic-quintic model (fast saturable absorber)[1]

$$f_{\mathrm{sa}}(|u|) = \delta|u|^2 - \sigma|u|^4$$

SESAM model (slow saturable absorber)[2]

$$f_{\mathrm{sa}}(|u|) = -\frac{\rho}{2}n(t,T), \quad \frac{\partial n(t,T)}{\partial T} = \frac{1-n}{T_A} - \frac{|u(t,T)|}{w_A}n$$

[1] S. Wang et al., J. Opt. Soc. Am. B **31**, 2914 (2014).
[2] S. Wang et al., Opt. Lett. **42**, 2362 (2017).

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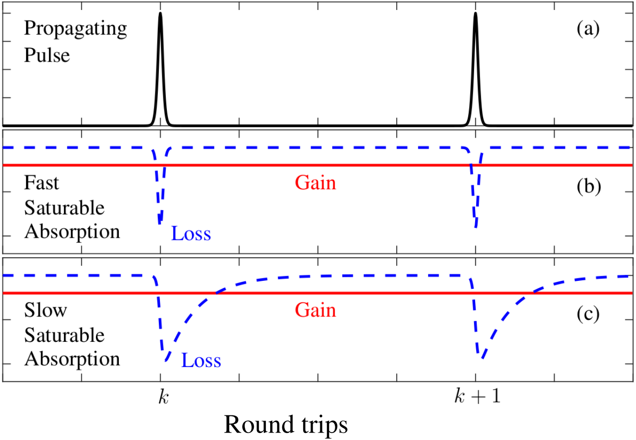
Saturable Absorption



The loss is saturated by the incoming pulse and then recovers

- almost instantaneously with fast absorbers
- slowly compared to the pulse duration with slow absorbers

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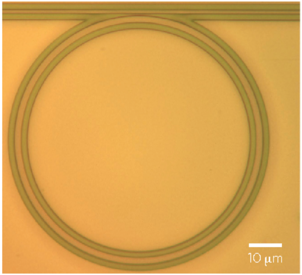
Frequency Comb Systems

Microresonators

The Lugiato-Lefever Equation:[3]

$$\frac{\partial \psi}{\partial t} - i\frac{\partial^2 \psi}{\partial x^2} - i|\psi|^2\psi + (1 + i\alpha)\,\psi - F = 0$$



System Parameters:

α : frequency detuning (-5 to 10)
F : pump amplitude (0 to 4)
L : microresonator length (25 to 200)

[3] Z. Qi et al., Conf. Lasers Elect.-Opt. (2018), paper SF2A.6.

References

- Haus modelocking equation:
  - S. Wang et al., J. Opt. Soc. Am. B **31**, 2914 (2014)
  - Martinez, Fork, and Gordon, J. Opt. Soc. Am. B **2**, 753 (1985)
- Cubic-quintic modelocking equation:
  - Moores, Opt. Comm. **96**, 65 (1993)
  - Soto-Crespo et al., J. Opt. Soc. Am. B **13**, 1439 (1996)
  - Articles in *Dissipative Solitons*, Akhmediev and Ankiewicz ed. (2005)
- SESAM equation:
  - Kärtner et al., IEEE J. Sel. Quantum Electron. **2**, 540 (1996)
- Lugiato-Lefever equation:
  - Lugiato and Lefever, Phys. Rev. Lett. **58**, 2209 (1987)
  - Haelterman et al., Opt. Comm. **91**, 401 (1992)
  - Matsko et al., Opt. Lett. **36**, 2845 (2011)
  - Coen et al., Opt. Lett. **38**, 1790 (2013)
  - Chembo and Menyuk, Phys. Rev. A **87**, 053852 (2013)

Dynamical Systems Methods

Standard, "brute-force" approach

- Solve the evolution equations for many roundtrips
- Use a noisy initial condition
- Convergence $\Longleftrightarrow$ existence + stability
- Change parameters; repeat

Advantages:

- Easy to program
- Intuitive (mimics experiments)

Disadvantages:

- Computationally slow
- Ambiguous near a stability boundary
- Limited insight into sources of instability

This approach is better for analysis than synthesis!

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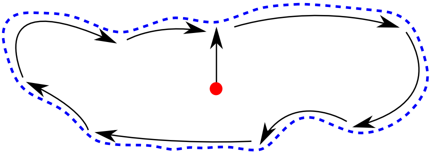
Dynamical Systems Methods

Our Approach

- Solve the evolution equations **once** to find a stationary solution
  - in a highly stable case
- Determine the stationary solution as parameters vary by solving a **root-finding problem**
- In parallel, find the eigenvalues of the linearized evolution equation
  - **The dynamical spectrum**
  - A stable solution has no eigenvalues with positive real parts
- Find parameters where one or more eigenvalues hit the imaginary axis
- Track the stability boundary



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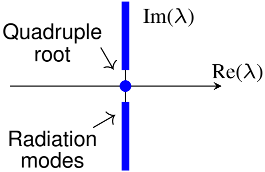
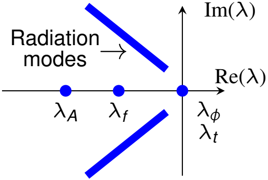
Atlas of Dynamical Spectra

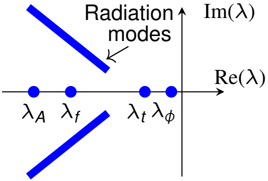
Classical NLS Spectrum



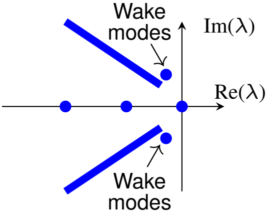
Soliton Laser



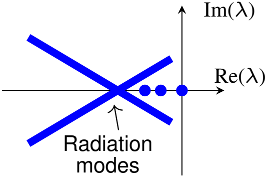
With locking



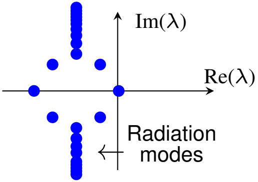
SESAM laser



Normal dispersion lasers



Microresonator



Dynamical Systems Methods

Advantages

- $10^3 - 10^5$ times faster than brute-force solutions
- Unambiguous determination of stable operating parameter regimes
- Allows rapid mapping and optimization of solution properties
  - bandwidth, power, noise. . .
- Yields insight into the sources of instability

Disadvantages

- More difficult to program
- **The concepts are unfamiliar to many optical experimentalists**

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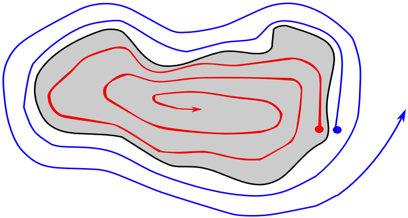
Dynamical Systems Methods

Two important caveats

- Accessibility vs. stability
  - Dynamical methods do not tell you how to access stable solutions
    - Example: single solitons are hard to access in microresonators



Dynamical Systems Methods

Two important caveats

- Accessibility vs. stability
  - Dynamical methods do not tell you how to access stable solutions
    - Example: single solitons are hard to access in microresonators
- Unstable system evolution
  - Dynamical methods do not tell you how an unstable solution evolves
    - Chaos, another stable solution, breathers are all possible

Dynamical and evolutionary methods are complementary!

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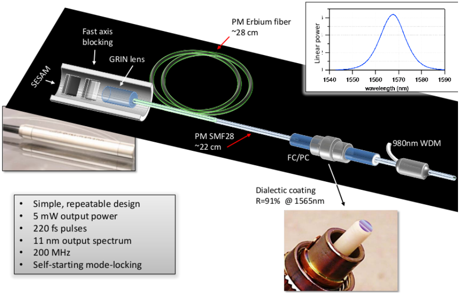
The SESAM Modelocked Fiber Laser

- The system is built using
  - telecom grade polarization-maintaining (PM) components
  - highly-doped erbium-doped fiber
  - highly non-linear PM fibers
  - a semiconductor saturable absorber mirror (SESAM)
- Output: highly stable 200 MHz combs with $P_{av} = 5$ mW



[1] Sinclair et al., Opt. Express **22**, 6996 (2014).

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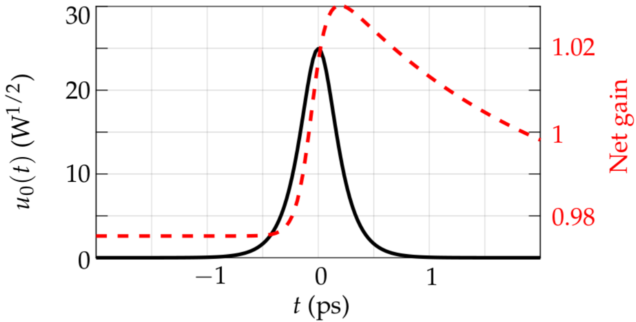
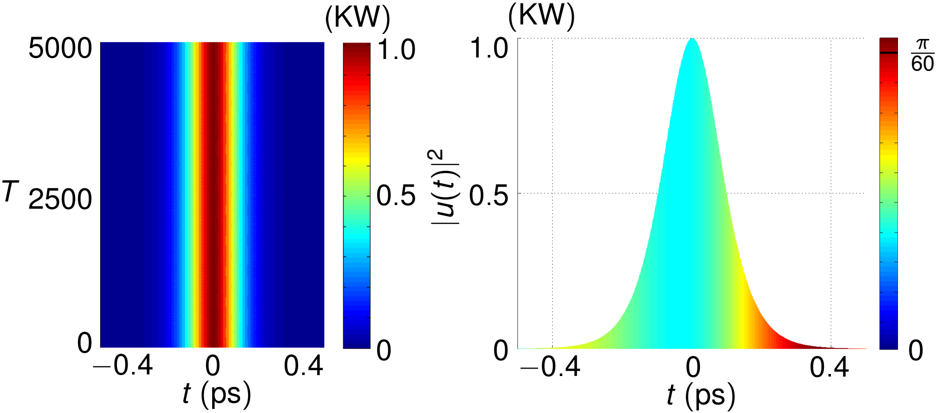
Balance of Energy

- The SESAM and the linear gain open a gain window that allows the pulse to grow
- A soliton wake instability will occur when g_0 becomes sufficiently large or β'' becomes sufficiently small



Stable Operation

The stable pulse is close in shape to a sech pulse
(soliton solution of the nonlinear Schrödinger equation)



Color indicates the pulse power

Color indicates the phase of the pulse

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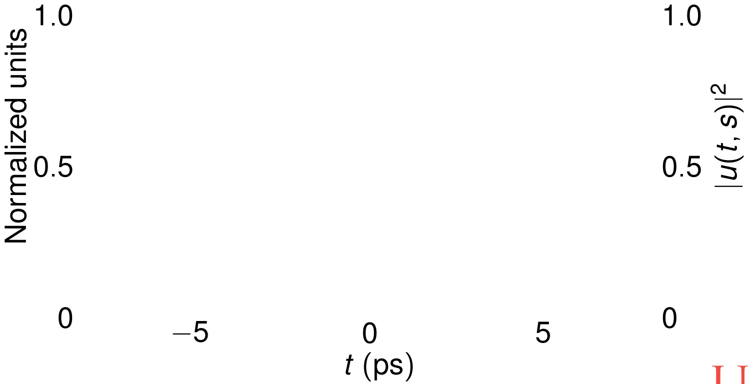
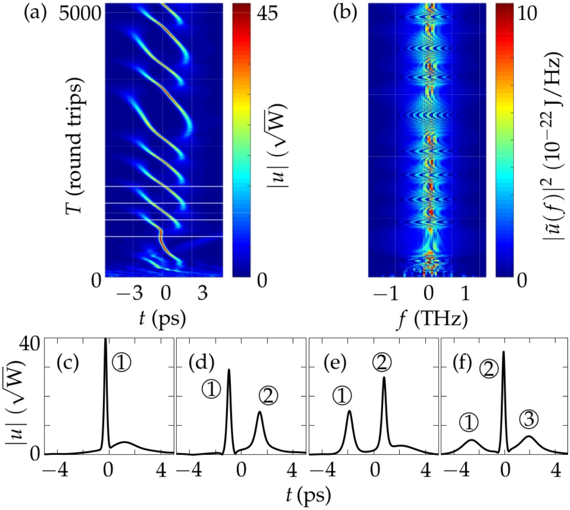
Soliton Wake Instability

Quasi-periodicity is observed in the evolution



Soliton Wake Instability

Quasi-perio Letter is observed in the evolution



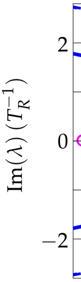


Fig. 4. The
saturated g
$\lambda_{w\pm}$ = −7
9.09 × 10^{-3}
dicate how
The eigenva

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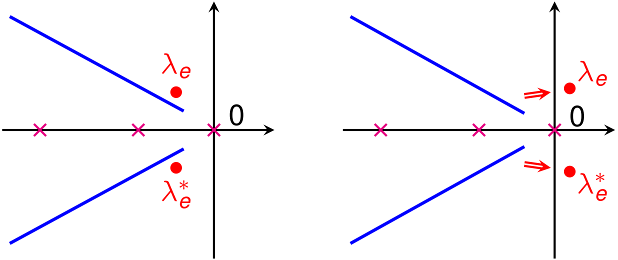
Dynamical Spectrum

- As g_0 increases:
  - a pair of eigenvalues λ_e, λ_e^* emerge from the radiation modes (edge bifurcation)
- Re[λ_e], Re[λ_e^*] become positive, leading to instability (Hopf bifurcation)[1]



[1] S. Wang et al., Opt. Lett. **42**, 2362 (2017).

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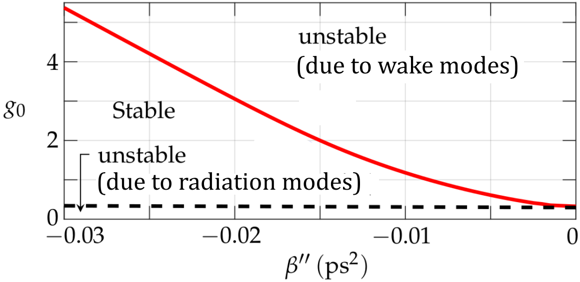
Stable Region

- Continuous modes become unstable when the gain is too low
- The wake mode instability occurs when
  - the unsaturated gain becomes large
  - the group delay dispersion becomes small



Once the stable region is known, optimization is possible!

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Cnoidal Waves in Microresonators

- Solitons are a special case of a broader class of periodic waveforms: **cnoidal waves**
- Cnoidal waves include:
  - Single solitons
  - Soliton crystals
  - Turing rolls
- Cnoidal wave attributes:

  On the one hand, they . . .

  - have clean spectra with evenly spaced comb lines (like single solitons)
  - have analytical solutions that exist in the no-loss limit (like single solitons)[1]
  - can have a broad bandwidth (like solitons)

[1] Z. Qi et al., J. Opt. Soc. Am. B **34**, 785 (2017).

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Cnoidal Waves in Microresonators

- Solitons are a special case of a broader class of periodic waveforms: **cnoidal waves**
- Cnoidal waves include:
  - Single solitons
  - Soliton crystals
  - Turing rolls
- Cnoidal wave attributes:

  On the other hand, they . . .
  - can be easily and deterministically accessed

    (unlike single solitons or soliton crystals)
  - use the pump more efficiently and produce higher power comb lines

BUT

with lines spaced farther apart

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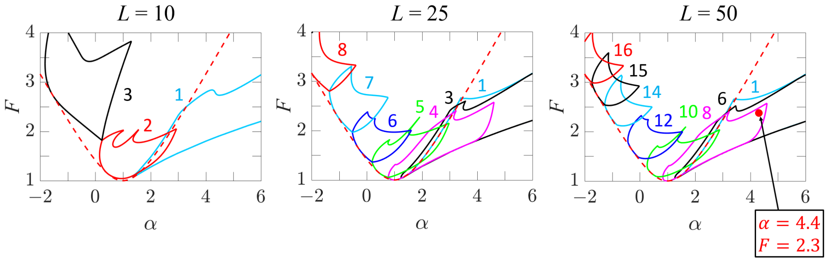
Stable Regions

Stable regions for different periodicities

F = normalized pump power
α = normalized detuning
L = normalized diameter



- Continuous waves are stable below the red-dashed line
- Below $\alpha = 41/30 \simeq 1.37$, cnoidal waves can be easily accessed by raising the pump power
- Cnoidal waves are stable in a U-shaped region in α-F space
- Moving along this region, different values of N_{per} can be deterministically accessed

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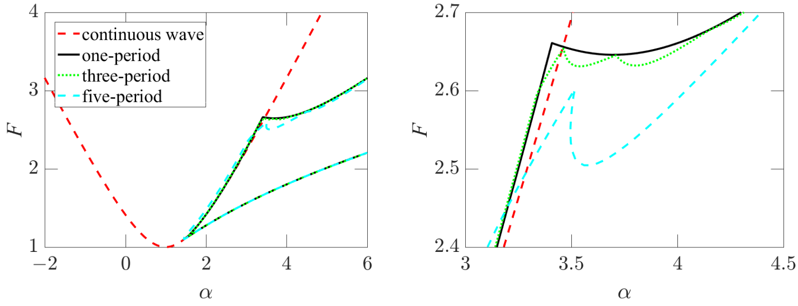
Stable Regions ($L = 50$)

Why are single solitons hard to access



- Substantial overlap with other cnoidal waves
- Almost complete overlap with continuous waves

By contrast the periodicity-8 cnoidal wave can be deterministically accessed!

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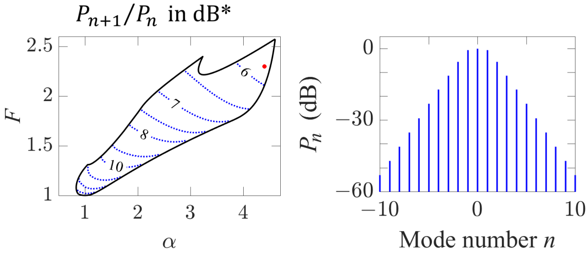
Periodicity-8 Cnoidal Waves ($L = 50$)

Controllability and bandwidth



- With FSR = 125 GHz; 30 dB down bandwidth = 24 THz
- Uses the pump more efficiently than a single soliton

Large bandwidth can be obtained!

---

* large n limit ($n \gtrsim 4$)

Conclusions

- Powerful dynamical methods can be combined with modern computer algorithms to:
  - rapidly determine where in the parameter space stable solutions lie
  - yield important insights into the sources of instability
  - rapidly determine the noise performance
  - optimize the system performance

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Conclusions

- We have applied these methods to:
  - Laser models with slow saturable gain
    - ★ Identified parameters ranges where two stable solutions exist
    - ★ Compare the cubic-quintic model to other models
  - SESAM laser
    - ★ Characterized the wake mode instability
    - ★ Determined where stable solutions exist
    - ★ Explained the appearance of sidebands
    - ★ Characterized the noise performance
    - ★ Optimized the system parameters
  - Microresonators
    - ★ Determined where cnoidal waves are stable
    - ★ Explained why single solitons are hard to access
    - ★ Found broadband, easily accessible cnoidal wave solutions
    - ★ Determined the impact of thermal effect
    - ★ Determined the impact of avoided crossings

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Summary

These methods are an important complement to widely used evolutionary methods and should be commonly used!

Our software is available at:

`http://www.umbc.edu/photonics/software.html`

See: "Dynamical method to evaluate noise. . ."
"Boundary tracking algorithms"
"Stability boundary tracking for cnoidal wave solutions"

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