SUBSPACE CONTROLLABILITY OF MULTI-PARTITE SPIN NETWORKS

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Abstract

In a network of spin $\frac{1}{2}$ particles, controlled through an external electro-magnetic field, the gyromagnetic ratio of each spin is a parameter that characterizes the interaction of the spin with the external control field. *Multipartite* networks are such that the spins are divided into subsets according to their gyromagnetic ratio and spins in one set interact in the same way with all spins in another set. Due to the presence of symmetries in this type of systems, the underlying Hilbert state space splits into invariant subspaces for the dynamics. Subspace controllability is verified if every unitary evolution can be generated by the dynamics on these subspaces.

We give an exact characterization, in term of graph theoretic conditions, of subspace controllability for multipartite quantum spin networks. This extends and unifies previous results.

Keywords: Controllability of quantum mechanical systems; Subspace controllability; Networks of spins.

1 Introduction and statement of main result

The dynamics of quantum mechanical systems, subject to a control electromagnetic field, can often be described by the Schrödinger equation in the form

$$\dot{\psi} = A\psi + \sum_{j=1}^{m} B_j u_j \psi, \tag{1}$$

where u_j, $j = 1, ..., m$, are the control variables and $\{A, B_1, ..., B_m\}$ are given operators, with ψ denoting the state of the quantum system, varying in the underlying Hilbert space $\mathcal{H}$. In finite dimensions, the controllability properties of system (1) are usually assessed using the *Lie algebra rank condition* (see, e.g., [8], [9]). One calculates the Lie algebra , $\mathcal{G}$, generated by the matrices $\{A, B_1, ..., B_m\}$, which is called the **dynamical Lie algebra**. Given $e^{\mathcal{G}}$, the connected Lie group associated with it, assumed compact, the condition says that the reachable set $\mathcal{R}_{\psi_0}$, for (1) starting from ψ_0 is given by

$$\mathcal{R}_{\psi_0} = \{X\psi_0 \,|\, X \in e^{\mathcal{G}}\}.$$

In the case of large systems, it is important to find ways to assess controllability which avoid the repeated calculation of commutators of very large matrices in (1). Such controllability criteria should be easily related to the physical structure of the system under consideration. One example of large system is given by *networks* of n interacting spin $\frac{1}{2}$ particles, where the dimension of the Hilbert space $\mathcal{H}$ grows exponentially with n, as 2^n. In some cases, *graph theoretic conditions* have been given to assess the controllability of quantum systems (see, e.g., [2], [14]), and this paper has this objective as well.

In the presence of a *group of symmetries* G, i.e., a (discrete) group of matrices commuting with the matrices $\{A, B_1, ..., B_m\}$ in (1), the underlying Hilbert space $\mathcal{H}$ for the system splits in the direct sum of invariant subspaces for the dynamics (1) and, in an appropriate basis, the matrices $\{A, B_1, ..., B_m\}$ in (1) take a block diagonal form

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[10]. Transitions from one subspace to the other are forbidden and therefore controllability is lost. For a network of n spins, the topology of the network itself often suggests the symmetries to be considered which typically are subgroups of the permutation groups leaving the network unchanged. For example, for the network of Figure 1 the group of permutations on the three spin in the set Cl_3 is a symmetry group for the system. In these cases, it is of interest to investigate whether one has controllability within the invariant subspaces. This property is called **subspace controllability** and it has been investigated in several recent papers (see, e.g., [1], [4], [15], [16]). We shall see that, in general, this property is subspace-dependent, that is, for the same decomposition, there might be some subspaces of dimension D where the restriction of the dynamical Lie algebra is the *full su(D)* (special unitary) Lie algebra of $D \times D$ skew-Hermitian traceless matrices and some others where it is not. In the first case subspace controllability is verified, in the second case it is not.

Identifying the invariant subspaces of a quantum control system is also important when control laws are designed. For these *lower dimensional* spaces typically more powerful design techniques are available (see, e.g., [5]) and states in such subspaces might have special quantum information theoretical properties [4]. Furthermore, control on invariant subspaces can be a way to shield the dynamics from the deteriorating effect of *de-coherence* (see, e.g., [18]).

In this paper, we shall explore subspace controllability for networks of spin $\frac{1}{2}$ particles in the multipartite configuration. This means that spin particles are collected in sets, which we shall call *clusters* according to the value of their *gyromagnetic ratio*, that is the parameter which models the interaction with an external control field. Spins in the same cluster interact in the same way with spins in another cluster. These systems presents a group of symmetries given by permutations of the spins within the same cluster. We shall give in Theorem 1 a necessary and sufficient condition of subspace controllability for such systems in graph theoretic terms. This result will extend the result of [1] which only dealt with the *bipartite* case and with bounds on the number of spin in one cluster. We remove such bounds. The technique we shall use is different from the one in [1] which was based on a direct computation of the dynamical Lie algebra. Here we shall use techniques of representation theory and, in particular, the *Clebsch-Gordan decomposition* (see, e.g.,[7], [12], [17]) of the tensor product representation of $su(2)$. Our result also generalizes the result of [2] which is found as a special case when all the spins have different gyromagnetic ratios.

1.1   Basic notations

We recall the definition of the Pauli matrices (multiplied by $\frac{1}{2}$)

$$\sigma_x := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y := \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma_z := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

which satisfy the basic commutation relations

$$[i\sigma_x, i\sigma_y] = i\sigma_z, \qquad [i\sigma_y, i\sigma_z] = i\sigma_x, \qquad [i\sigma_z, i\sigma_x] = i\sigma_x, \tag{3}$$

and

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \frac{1}{4}\mathbf{1}, \qquad \{\sigma_x, \sigma_y\} = \{\sigma_y, \sigma_z\} = \{\sigma_z, \sigma_x\} = 0, \tag{4}$$

where $\{A, B\}$ is the *anticommutator* of A and B, that is, $\{A, B\} := AB + BA$. Here and in the following $\mathbf{1}$ always denotes the identity matrix or operator, the dimension being understood from the context. The matrices $\{i\sigma_x, i\sigma_y, i\sigma_z\}$ along with the commutation relations (3) form an irreducible 2-dimensional representation of $su(2)$, the *standard representation*. Given a certain positive integer $\tilde{n}$, which is usually determined by the context, the matrices $S_{x,y,z}$ are defined as the sums of $\tilde{n}$ terms where each term is the tensor product of $\tilde{n}$ factors, each being the 2×2 identity except the l-th one which is $\sigma_{x,y,z}$, for $l = 1, 2, ..., \tilde{n}$. So, for example, for $\tilde{n} = 3$,

$$S_x = \sigma_x \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_x \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \sigma_x.$$

We denote by I_{gb} the sum of matrices where each term of the sum is the tensor product of $\tilde{n}$ identities except for one position occupied by σ_g and one occupied by σ_b and viceversa. The sum extends over all possible pairs of locations and therefore contains $\tilde{n}(\tilde{n} - 1)$ terms. For example for $\tilde{n} = 3$, I_{xy} is equal to

$$I_{xy} = \sigma_x \otimes \sigma_y \otimes \mathbf{1} + \sigma_y \otimes \sigma_x \otimes \mathbf{1} + \sigma_x \otimes \mathbf{1} \otimes \sigma_y + \sigma_y \otimes \mathbf{1} \otimes \sigma_x + \mathbf{1} \otimes \sigma_x \otimes \sigma_y + \mathbf{1} \otimes \sigma_y \otimes \sigma_x.$$

We consider a network of spin $\frac{1}{2}$ particles grouped in N **clusters** of indistinguishable spins. Clusters are defined as sets of spin particles which have the same gyromagnetic ratios. Moreover, we assume that each spin in a cluster interacts in the same way with spins in a different cluster and do not interact with each other. Any permutation of the spins belonging to the same cluster will leave the dynamics unchanged.

If a network has N clusters, with the k-cluster having n_k spin particles, we denote by A^j a matrix which is the tensor product of N identity matrices, where in the k-cluster the identity has dimension 2^{n_k}, except in the position j which is occupied by the matrix A, a $2^{n_j} \times 2^{n_j}$ matrix. Examples of these types of matrices we shall often use are $S_{x,y,z}^j$, $j = 1, \ldots, N$. So, for example:

$$S_x^2 = \mathbf{1} \otimes S_x \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1},$$

where S_x has dimension 2^{n_2} and the identity matrix in the first position has dimension 2^{n_1}, the one in the third position has dimension 2^{n_3}, and so on.

Extending this notation, the matrices of the form $A^j B^k$, with $j \neq k$, $j, k \in \{1, 2, ..., N\}$, can be seen as the product of A^j and B^k but also as tensor products of identities, with various dimensions, except in the positions j and k occupied by A and B, respectively, of dimensions 2^{n_j} and 2^{n_k}. This notation is naturally extended to any number of factors in the product besides two.

1.2   The model

The quantum control system model we shall study in this paper is a network of n spin $\frac{1}{2}$ particles interacting with each other. We have grouped the n spins in N clusters of indistinguishable spins, each interacting with the same coupling constant with spins in other clusters. The interaction is assumed to be of the Ising $Z - Z$ form $(S_z^j S_z^k)$ (although the results will be extended to every other type of two body interaction (coupling) in section 3). The network is represented by a **connectivity graph** where each node represents a cluster of equivalent spins and there is an edge connecting two nodes if there is a non zero interaction between spins in the corresponding clusters. We assume the interactions between spin in two different clusters all equal. For example, the network of Figure 1 consists of a total of eight spin $\frac{1}{2}$ particles, two of them in the first cluster (Cl_1), two of them in the second one (Cl_2), three of them in the third one (Cl_3) and one in the fourth one (Cl_4). The lines represent nonzero interactions which are assumed to be the same for spins belonging to the same couple of clusters. The connectivity graph for such a network is given in Figure 2.
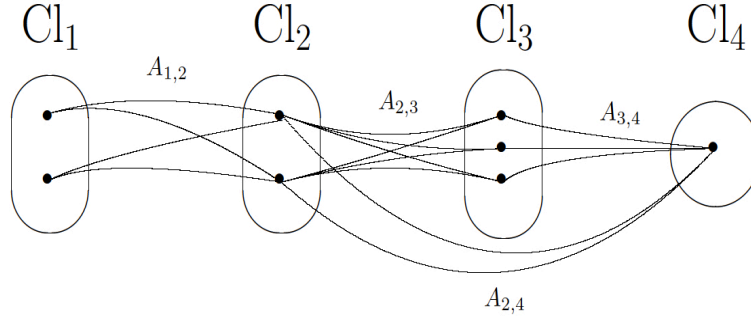

Figure 1: Example of a multipartite spin network

The Schrödinger equation which models the dynamics takes the form (1) with

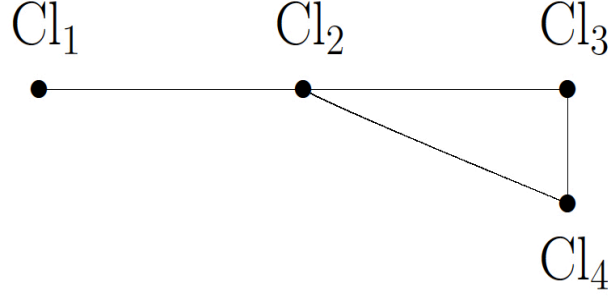$$iA := \sum_{1 \leq j < k \leq N} A_{j,k} S_z^j S_z^k, \tag{5}$$

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Figure 2: Connectivity graph for the network of Figure 1

with $A_{j,k}$ the *coupling constants* and

$$iB_{x,y,z} = \sum_{j=1}^{N} \gamma_j S^j_{x,y,z}, \tag{6}$$

where γ_j are the *gyromagnetic ratios* of the spins in the cluster j, assuming an isotropic type of interaction with the three components of the electro-magnetic field $u_{x,y,z}$. Notice that our model assumes that *all* the components of the electro-magnetic field are available for control. If only one component is available, the dynamical Lie algebra would be a subalgebra of the Lie algebra that we will describe in this paper. The decomposition in invariant subspaces for the dynamics will still hold but within each invariant subspace further decomposition is possible (cf. also subsection 3.4).

We assume that some of the coupling constants $A_{j,k}$ are different from zero so that the *connectivity graph associated with the network is connected.* This is done without loss of generality since if the graph has several connected components we can repeated the analysis we shall perform on each one of them.

The dynamical Lie algebra $\mathcal{G}$, for this type of systems, is generated by $\{A, B_x, B_y, B_z\}$, in (5) and (6).

A crucial observation for our development is that, with n spin particles, $\{iS_x, iS_y,$ and $iS_z\}$ span a 2^n-dimensional representation of $su(2)$ since they satisfy (cf. (3))

$$[iS_x, iS_y] = iS_z, \qquad [iS_y, iS_z] = iS_x, \qquad [iS_z, iS_x] = iS_x. \tag{7}$$

This representation coincides with the tensor product of n copies of the standard representations (see, e.g., [17]) as it will be further elaborated upon in the following.

By using (7), we have that

$$[B_x, B_y] = \sum_{j=1}^{N} \gamma_j^2 S^j_z, \quad [B_y, B_z] = \sum_{j=1}^{N} \gamma_j^2 S^j_x, \quad [B_z, B_x] = \sum_{j=1}^{N} \gamma_j^2 S^j_y,$$

belong to $\mathcal{G}$, and by iterating the Lie brackets, we have that all

$$\sum_{j=1}^{N} \gamma_j^l S^j_z, \qquad \sum_{j=1}^{N} \gamma_j^l S^j_x, \qquad \sum_{j=1}^{N} \gamma_j^l S^j_y,$$

for $l \geq 1$, belong to $\mathcal{G}$. Using a Vandermonde determinant type of argument and assuming, as we will, that the γ_j's are all different from zero (besides being different from each other), it follows that $iS^j_{x,y,z}$ for $j = 1, ..., N$, also belong to $\mathcal{G}$. Therefore, the dynamical Lie algebra $\mathcal{G}$ is generated by $\mathcal{S}$ and A,

$$\mathcal{S} := \{iS^j_x, iS^j_y, iS^j_z | j = 1, ..., N\}, \qquad A = -i \sum_{1 \leq j < k \leq N} A_{j,k} S^j_z S^k_z.$$

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We also have

Lemma 1.1. The dynamical Lie algebra $\mathcal{G}$ is the same as the one generated by $\mathcal{S}$ and by all the $iS_z^j S_z^k$ such that $A_{j,k} \neq 0$.

Proof. Set $j = 1$ and $k = 2$, without loss of generality and assume $A_{1,2} \neq 0$. We want to show that $iS_z^1 S_z^2$ belongs to $\mathcal{G}$. Start with $[A, iS_x^1]$ to obtain $H_1 := -i\sum_{l>1} A_{1l} S_y^1 S_z^l$. Then take $[H_1, iS_x^1]$ to obtain $H_2 = i\sum_{l>1} A_{1l} S_z^1 S_z^l$. Then take $[H_2, iS_x^2]$ to obtain $H_3 = -iA_{12} S_z^1 S_y^2$. Then take $[H_3, iS_x^2]$ to obtain $iA_{12} S_z^1 S_z^2$. Since $A_{12} \neq 0$, we obtain the result. □

1.3 Decomposition in invariant subspaces and subspace controllability

Let n_j denote the number of spins in the j-th cluster. According to the postulates of quantum mechanics the subsystem corresponding to the j-th cluster lives in a Hilbert space $(V^1)^{\otimes n_j}$ where V^1 denotes the two dimensional (spin $\frac{1}{2}$) carrier of the standard representation of $su(2)$. The full Hilbert state space is therefore

$$\mathcal{H} = (V^1)^{\otimes n_1} \otimes (V^1)^{\otimes n_2} \otimes \cdots \otimes (V^1)^{\otimes n_N}. \tag{8}$$

Extending the above notation, let us denote by V^l the spin $\frac{l}{2}$ irreducible representation of $su(2)$. Here V^l has (complex) dimension $l + 1$.

Using (iteratively) the **Clebsch-Gordan decomposition** (see, e.g., [17]) we have that $(V^1)^{\otimes n_j}$ decomposes in the direct sum of a number of (possibly repeated) subspaces V^{n_j}, V^{n_j-2},...., where the last term is V^0 or V^1 according to whether n_j is even or odd, respectively. It is not important for our purposes how many copies of the same V^l are present. This will be determined on a case by case basis according to the iteration for the given cluster. For a fixed cluster j, the matrices $S_{x,y,z}^j$ act on each space V^l as the $\frac{l}{2}$ irreducible representation of $su(2)$. In particular when $l = 0$ they have value equal to zero. This will be used in the following.

Example 1.2. Consider the network of spins of Figure 1 and the first cluster for which the Clebsch-Gordan decomposition gives $V^1 \otimes V^1 = V^{1+1} \oplus V^{1+1-2} = V^2 \oplus V^0$. For the third cluster the Clebsch-Gordan decomposition gives

$$V^1 \otimes V^1 \otimes V^1 = (V^2 \oplus V^0) \otimes V^1 = (V^2 \otimes V^1) \oplus (V^0 \otimes V^1) = V^3 \oplus V^1 \oplus V^1.$$

For the second cluster, we have $V^1 \otimes V^1 = V^2 \oplus V^0$ and for the fourth cluster, we have V^1.

We consider as *invariant* subspaces of the full system of N clusters of spins the spaces

$$S = F_1 \otimes F_2 \otimes \cdots \otimes F_N, \tag{9}$$

where F_j, $j = 1, .., N$, is one of the spaces V^{n_j}, V^{n_j-2},..... The spaces (9) are indeed invariant under the dynamical Lie algebra $\mathcal{G}$ since they are invariant under the generators. We shall see later (see Remark 1.7) that they are *minimal* invariant, that is, they contain no proper nontrivial invariant subspaces. In the language of representation theory, they carry *irreducible* representations of the dynamical Lie algebra $\mathcal{G}$.

As a result of the Clebsch-Gordan decomposition applied to each factor corresponding to a cluster, the full Hilbert space $\mathcal{H}$ in (8) decomposes into the direct sum of invariant spaces of the form (9). We can then take a basis of the full Hilbert space $\mathcal{H}$ by putting together the (orthogonal) bases of the subspaces of the type (9). In this basis the dynamical Lie algebra $\mathcal{G}$ takes a block diagonal form.

The dimension of each subspace S in (9) is

$$D^S := \dim(F_1) \dim(F_2) \cdots \dim(F_N). \tag{10}$$

Subspace controllability is a feature of each invariant subspace in (9).

Definition 1.3. An invariant subspace (9) is said to be **subspace controllable** if and only if, for every M in $su(D^S)$, there exists a matrix in $\mathcal{G}$ such that its restriction to $S = F_1 \otimes F_2 \otimes \cdots \otimes F_N$ in (9) is equal to M. The full system is called subspace controllable if every invariant subspace is subspace controllable. More generally we define a **subspace dynamical Lie algebra** $\mathcal{G}_S$ for the subspace (9) as the largest Lie subalgebra of $su(D^S)$ such that for every matrix $M \in \mathcal{G}_S$ there exists an element in $\mathcal{G}$ whose restriction to $S = F_1 \otimes F_2 \otimes \cdots \otimes F_N$ is equal to M. Subspace controllability is verified when $\mathcal{G}_S = su(D^S)$.

1.4 Statement of the main result

The subspace dynamical Lie algebra, and therefore subspace controllability, can be assessed using a graph associated with the invariant subspace (9) which we shall call the **associated graph**. Such a graph is obtained from the connectivity graph of the spin network by removing the nodes corresponding to values of j such that $F_j = V^0$ in (9) and all the edges having such nodes as endpoint. Even if the original connectivity graph was connected (as we have assumed) the resulting associated graph for a subspace (9) might not be be connected, and, in general, it will have a number m_c of connected components $\mathcal{C}_1, \mathcal{C}_2,...,\mathcal{C}_{m_c}$. We define the dimension associated with h-th connected component, as (cf., (10))

$$D_h^S := \prod_{j \in \mathcal{C}_h} \dim(F_j). \tag{11}$$

In the special case where $m_c = 1$, we have only D_1^S which coincides with D^S in (10).

Example 1.4. Reconsider the network of Example 1.2 and Figures 1, 2, for which we have calculated the decompositions for any cluster as $V^2 \oplus V^0$, $V^2 \oplus V^0$, $V^3 \oplus V^1 \oplus V^1$, V^1. The possible invariant subspaces (9) are $T_{2,2,3,1} := V^2 \otimes V^2 \otimes V^3 \otimes V^1$, $T_{2,2,1,1} := V^2 \otimes V^2 \otimes V^1 \otimes V^1$, $T_{2,0,3,1} := V^2 \otimes V^0 \otimes V^3 \otimes V^1$, $T_{2,0,1,1} := V^2 \otimes V^0 \otimes V^1 \otimes V^1$, $T_{0,2,3,1} := V^0 \otimes V^2 \otimes V^3 \otimes V^1$, $T_{0,2,3,1} := V^0 \otimes V^2 \otimes V^3 \otimes V^1$, $T_{0,2,1,1} := V^0 \otimes V^2 \otimes V^1 \otimes V^1$, $T_{0,0,3,1} := V^0 \otimes V^0 \otimes V^3 \otimes V^1$, $T_{0,0,1,1} := V^0 \otimes V^0 \otimes V^1 \otimes V^1$. In Figure 3 we report the associated graphs for $T_{2,2,3,1}$ (which coincides with the connectivity graph), $T_{2,0,3,1}$, and $T_{0,2,1,1}$.
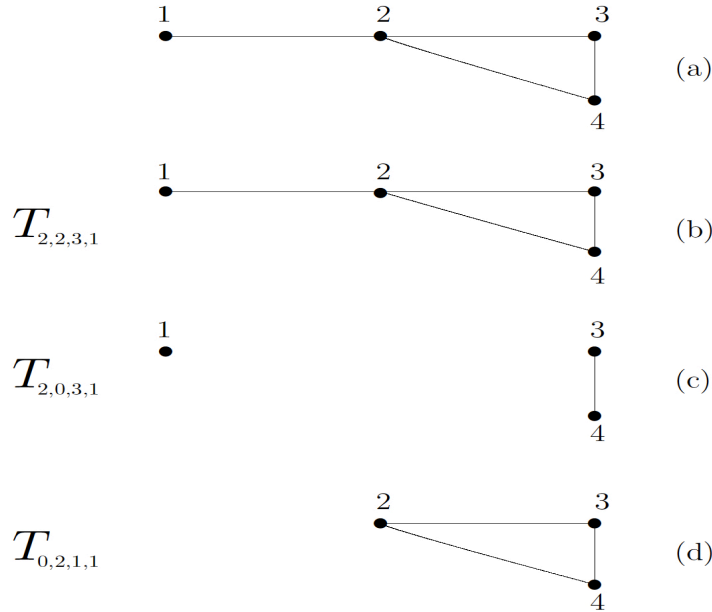

Figure 3: Associated graphs for invariant subaspaces $T_{2,2,3,1}$ (b), $T_{2,0,3,1}$ (c), $T_{0,2,1,1}$ (d), as compared with the connectivity graph of the network in part (a).

The following result is the main theorem of this paper. It allows to characterize the subspace dynamical Lie algebra and therefore subspace controllability in every case.

Theorem 1. *Consider an invariant subspace of the form (9) and its associated graph with m_c connected components. Then, the subspace dynamical Lie algebra $\mathcal{G}_S$ has the form of a direct sum*

$$\mathcal{G}_S = \mathcal{G}_1 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{G}_2 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \mathcal{G}_{m_c}, \tag{12}$$

where $\mathcal{G}_h$ is a Lie algebra acting on the space given by $\otimes_{j \in \mathcal{C}_h} F_j$. This space has dimension D_h^S in (11) and it corresponds to the h-th connected component in the associated graph to (9). In (12) $\mathcal{G}_h$, $h = 1, ..., m_c$ is (modulo multiples of the identity)

1. *Equal to the* $\dim(F_j)$-*irreducible representation of* $su(2)$ *if* $\mathcal{C}_h$ *only contains one node, the node* j.

2. *Equal to* $su(D_h^S)$ *if* $\mathcal{C}_h$ *contains more than one node.*

From the above theorem the following exact characterization of subspace controllability follows.

Corollary 1.5. A subspace (9) is subspace controllable if and only if the associated graph is connected and contains at least two nodes.

Example 1.6. Consider the subspaces of Example 1.4 with the associated graphs reported in Figure 3. According to Corollary 1.5 subspace controllability is verified in the cases $T_{2,2,3,1}$ and $T_{0,2,1,1}$. It is not verified in the case of $T_{2,0,3,1}$. In this case, on the given subspace, the subspace dynamical Lie algebra is the direct sum of two subalgebras, one subalgebra given by the irreducible representation of $su(2)$ on V^2, i.e., a representation of dimension 3, and a subalgebra given by $su(D)$. Here $D = \dim(V^3) \dim(V^1) = 4 \times 2 = 8$ acting on invariant spaces associated with the clusters 3 and 4.

Remark 1.7. In Theorem 1, every invariant subspace $S = F_1 \otimes F_2 \otimes \cdots \otimes F_N$ in (9), is written in the form $E_1 \otimes E_2 \otimes \cdots \otimes E_{m_c}$ where each subspace $E_h = \otimes_{j \in \mathcal{C}_h} F_j$ refers to one connected component of the associated graph. On $E_1 \otimes E_2 \otimes \cdots \otimes E_{m_c}$ the dynamical Lie algebra (and of the associated group of possible evolutions which is a subgroup of the unitary group) acts as a tensor product (that is, in the case of the group, as $X_1 \otimes X_2 \otimes \cdots \otimes X_{m_c}$, for unitary matrices $X_1, ..., X_{m_c}$). Moreover, such a decomposition into invariant subspaces is *minimal* in the following sense: Given $E_1 \otimes E_2 \otimes \cdots \otimes E_{m_c}$ there is no other invariant subspace $E_1' \otimes E_2' \otimes \cdots \otimes E_{m_c}'$, with $E_h' \subseteq E_h$, $h = 1, ..., m_c$, where the strict inclusion holds for at least one h. This is due to the fact that every Lie algebra $\mathcal{G}_h$ in (12) is an *irreducible representation*, either of $su(2)$ or of $su(D_h^S)$ being the standard representation for the given dimension D_h^S, which is also irreducible.

Remark 1.8. Sometimes (cf. the on line version of [15] and also [11]) a distinction is made between *external* and *internal* symmetries. External symmetries are operators commuting with all the Hamiltonians of a quantum system (the matrices A and B_j in (1)). In our model they can be taken in an appropriate basis to be matrices with identity submatrices on the diagonal (and zero every where else) so as to commute with the dynamical Lie algebra which (in the given basis) takes a block diagonal form. Internal symmetries are, by definition, present when the dynamical Lie algebra acts on a vector space that cannot be decomposed (irreducible representation) but the corresponding Lie group preserves a symmetric or anti-symmetric inner product. In our case, each invariant subspace (9) corresponds to an irreducible representation (cf. the previous remark), but then, each tensor product (9) has to be decomposed according to the connected components of the associated graph. On each such factor the dynamical Lie algebra acts either as $su(D)$ (for appropriate D) or as an irreducible representation of $su(2)$ on a subspace V^f (of dimension $f + 1$). Such representations are (modulo a change of coordinates) subalgebras of $so(f + 1)$, for $f + 1$ odd or $sp\left(\frac{f+1}{2}\right)$ (for $f + 1$ even) (cf. Theorem 1.5. in [6]) and therefore preserve asymmetric or anti-symmetric inner product, giving in this case, internal symmetries.

2 Proof of Theorem 1

2.1 Casimir operators

An important operator for what will follow will be the **Casimir operator** C^j (on the j-th space $(V^1)^{\otimes n_j}$ in (8)) defined as

$$C^j := (S_x^j)^2 + (S_y^j)^2 + (S_y^j)^2, \tag{13}$$

which is scalar on each irreducible representation V^f with value on V^f given by $\frac{f}{2}(\frac{f}{2} + 1)$ [17]. In particular, it is zero on (and only on) V^0. In the following, operators will appear which are products of certain powers of the Casimir operator at certain locations in $\{1, ..., N\}$ and other operators at other locations. For example $C^j S_x^k$, is the product of the Casimir operator at location j with S_x at location k, with $j \neq k$. Another example would be $(C^j)^2 C^l S_x^k$ with all different j, k, l, which is a square of C^j together with C^l and S_x^k. Another example is A^j itself for an operator A where all the powers of the Casimir operators are zero. Linear combinations of powers of Casimir operators form a (unital) commutative algebra. Therefore, their behavior in Lie brackets calculations when

generating a given Lie algebra is easy to control. We shall denote by Υ a general operator which is the product of Casimir operators. If we write $\Upsilon A^{j_1} B^{j_2} \cdots L^{j_r}$, we mean an operator which is A in location j_1, B in location j_2,...,L in location j_r and unspecified powers of Casimir operators in the remaining locations. If we want to point out the fact that these latest factors might be different from one operator to the other we use $\Upsilon_1 A^{j_1} B^{j_2} \cdots L^{j_r}$ and $\Upsilon_2 A^{j_1} B^{j_2} \cdots L^{j_r}$, for example.

2.2 Reduction of the problem

We first prove that we can reduce ourselves to the following special case.

Proposition 2.1. Assume that no subspace F_j in (9) is equal to V^0 and that the connectivity graph of the network is connected. Then, if $N = 1$, $\mathcal{G}_S$ is the representation of $su(2)$ associated with F_1. If $N \geq 2$ then $\mathcal{G}_S = su(D^S)$, with D^S in (10).

Notice that if $F_j \neq V^0$ for all $j = 1, \ldots, N$, then the connectivity graph of the network coincides with the associated graph relative to the invariant subspace.

To see that the general case can be reduced to the special case of Proposition 2.1, write the tensor product S in (9) by placing the V^0 spaces in the first $\bar{N}$ positions, i.e., like

$$S = V^0 \otimes V^0 \otimes \cdots \otimes V^0 \otimes F_{\bar{N}+1} \otimes F_{\bar{N}+2} \otimes \cdots \otimes F_N, \tag{14}$$

where $F_j = V^{r_j}$ with $r_j \geq 1$, for $j = \bar{N} + 1, \ldots, N$. The dynamical Lie algebra $\mathcal{G}$ is generated by all the $S^j_{x,y,z}$ and by all the $S^j_z S^k_z$ for which the coupling constant $A_{j,k}$ are different from zero (Lemma 1.1). However, on the subspace (14) $S^j_{x,y,z}$, $j = 1, ..., \bar{N}$ are all zero, since, as we have mentioned when we introduced the Clebsch-Gordan decomposition, $S^j_{x,y,z}$ is zero on the V^0 representation of $su(2)$. For the same reason, $S^j_z S^k_z$, with $j < k$, and with $j = 1, ..., \bar{N}$ are also zero. Moreover zeros are also all their (repeated) Lie brackets. As a consequence, on these spaces, the dynamical Lie algebra is the one generated by $iS^j_{x,y,z}$ and $iS^j_z S^k_z$, $j < k$, for all pairs j and k such that $A_{j,k} \neq 0$ and $j = \bar{N} + 1, ..., N$.

The connectivity graph of the network of $N - \bar{N}$ clusters of spins is not necessarily connected and coincides with the graph associated with the subspace (14), i.e., the one obtained by removing the first $\bar{N}$ nodes and corresponding edges. Now by collecting in $F_{\bar{N}+1} \otimes F_{\bar{N}+2} \otimes \cdots \otimes F_N$ elements corresponding to the same connected components in order, we notice that the element $S^j_z S^k_z$ and $S^j_{x,y,z}$ corresponding to pairs (j, k) in the same connected component generate a subalgebra which commutes with the ones correspoonding to the other connected components. Therefore the whole subspace dynamical Lie algebra $\mathcal{G}_S$ takes the form in (12).

Each term corresponds to one connected component of the associated graph and if we reduce ourselves to only one connected component the proof is reduced to the case of Proposition 2.1.

The case $N = 1$ of Proposition 2.1 follows immediately because if $N = 1$ there is no interaction matrix of the form $S^j_z S^k_z$ but only the matrices $iS^1_{x,y,z}$ form the Lie algebra, which form indeed a representation of $su(2)$. The type of representation depends on the nature of the space F_1.

The next subsections are devoted to prove the case $N \geq 2$ of Proposition 2.1.

2.3 Generation of terms $S^j_z S^k_z$

Lemma 1.1 shows that the matrices $iS^j_z S^k_z$ belong to the dynamical Lie algebra $\mathcal{G}$ for every pair of clusters j, k with nonzero coupling. The following Lemma shows that for a connected connectivity graph, $\mathcal{G}$ contains matrices of the form $i\Upsilon S^j_z S^k_z$, for *any* pair of clusters j, k (recall that Υ indicates a general operator which is the product of Casimir operators)

Lemma 2.2. Assume the connectivity graph of the network is connected. Then, for every pair $j < k \in \{1, 2, ..., N\}$, there exists in the dynamical Lie algebra $\mathcal{G}$ a matrix

$$i\Upsilon S^j_z S^k_z. \tag{15}$$

Proof. Fix two nodes $1 \leq j < k \leq N$. Given the connectedness assumption for the graph, we know that there exists a path of length $r \geq 1$ of nodes $\hat{n}_i$, $i = 0, \ldots, r$, with $\hat{n}_0 = j$ and $\hat{n}_r = k$ such that $A_{\hat{n}_i, \hat{n}_{i+1}} \neq 0$. The claim will be proved by induction on the length r of the path joining the two nodes.

If $r = 1$, the claim follows from Lemma 1.1. Assume $r > 1$. Since the nodes $\hat{n}_0 = j$ and $\hat{n}_{r-1}$ are connected by a path of length $r - 1$, by the inductive assumption, we know that the dynamical Lie algebra $\mathcal{G}$ contains a matrix of the type:

$$i\Upsilon S_z^j S_z^{\hat{n}_{r-1}}, \tag{16}$$

Moreover since $A_{\hat{n}_{r-1},k} \neq 0$, we know from Lemma 1.1 that the matrix

$$iS_z^{\hat{n}_{r-1}} S_z^k, \tag{17}$$

is in the dynamical Lie algebra $\mathcal{G}$ as well. Since all the matrices of the type $iS_{x,y,z}^l$ are in $\mathcal{G}$, for any $l = 1, ..., N$, by taking Lie brackets of the matrices in (16) and (17), with these matrices we get that $\mathcal{G}$ contains all matrices of the type:

$$i\Upsilon S_{x,y,z}^j S_{x,y,z}^{\hat{n}_{r-1}}, \tag{18}$$

and

$$S_{x,y,z}^{\hat{n}_{r-1}} S_{x,y,z}^k, \tag{19}$$

respectively. Notice that all Υ operators appearing in (18) are the same. Now, we calculate (using (7))

$$\left[i\Upsilon S_z^j S_x^{\hat{n}_{r-1}}, iS_y^{\hat{n}_{r-1}} S_z^k \right] = i\Upsilon S_z^j S_z^{\hat{n}_{r-1}} S_z^k, \tag{20}$$

which belongs to $\mathcal{G}$ as well. Again, since all the matrices of the type $iS_{x,y,z}^l$ are in $\mathcal{G}$, by taking Lie brackets of these matrices with the one in (20) we get that:

$$i\Upsilon S_{x,y,z}^j S_{x,y,z}^{\hat{n}_{r-1}} S_{x,y,z}^k \in \mathcal{G}, \tag{21}$$

for all possible choices of x, y and z. Now, we use matrices of type (18) and (21), and we get:

$$\left[i\Upsilon S_x^j S_x^{\hat{n}_{r-1}}, i\Upsilon S_y^j S_x^{\hat{n}_{r-1}} S_z^k \right] = i\Upsilon_1 S_z^j (S_x^2)^{\hat{n}_{r-1}} S_z^k \in \mathcal{G}$$

By using $S_{y,z}^{\hat{n}_{r-1}}$ instead of $S_x^{\hat{n}_{r-1}}$ in the previous computation, we get that the three matrices

$$i\Upsilon_1 S_z^j (S_{x,y,z}^2)^{\hat{n}_{r-1}} S_z^k$$

are all in $\mathcal{G}$, with the sames value for Υ_1 for x, y, and z. By summing these matrices, using the definition of the Casimir operator (13), we get

$$i\Upsilon_2 S_z^j S_z^k \in \mathcal{G},$$

which is the claim of the Lemma.

$\square$

2.4   Generation of terms $I_{zz}^j - I_{yy}^j$ and $I_{yy}^j - I_{xx}^j$

Lemma 2.3. For every cluster $j = 1, ..., N$, there exists a matrix $i\Upsilon(I_{zz}^j - I_{yy}^j)$ and a matrix $i\Upsilon(I_{yy}^j - I_{xx}^j)$ in the dynamical Lie algebra $\mathcal{G}$.

In the case where the j-th cluster contains only one spin $I_{(x,y,z)(x,y,z)}^j$ are taken equal to zero. So the statement is trivially true.

Proof. Let us set $j = 1$ (without loss of generality) and $k = 2$. We have that taking the Lie brackets between $i\Upsilon S_z^1 S_z^2$ (from Lemma 2.2) and $S_{x,y}^1$ and $S_{x,y}^2$, we obtain all possible $i\Upsilon S_{x,y}^1 S_{x,y}^2$, and in fact, taking, possibly one extra Lie bracket with $S_{x,y}^1$ or $S_{x,y}^2$, we obtain all possible matrices

$$i\Upsilon S_{x,y,z}^1 S_{x,y,z}^2 \in \mathcal{G}. \tag{22}$$

Also observe from the calculation that the unspecified powers of Casimir operators in (22), which are collected in the term Υ, are the same for all the matrices in (22). Now consider

$$\left[i\Upsilon S_z^1 S_x^2, i\Upsilon S_z^1 S_y^2 \right] = i\Upsilon_1 (S_z^1)^2 S_z^2 = i\Upsilon_1 \left(\frac{n_1}{4} \mathbf{1}^1 + 2I_{zz}^1 \right) S_z^2, \tag{23}$$

since, as it is easily seen by induction, on a space of n_1 spin $\frac{1}{2}$ of dimension 2^{n_1},

$$(S_g)^2 = \frac{n_1 \mathbf{1}}{4}\mathbf{1} + 2I_{gg}, \qquad \text{for} \qquad g = x, y, z. \tag{24}$$

Now, by using S_y^1 instead of S_z^1 in (23) we obtain the matrix $i\Upsilon_1(S_y^1)^2 S_z^2 = i\Upsilon_1(\frac{n_1}{4}\mathbf{1}^1 + 2I_{yy}^1)S_z^2$. Taking the difference between this matrix and the one in (23) we obtain that $i\Upsilon_2(I_{zz}^1 - I_{yy}^1)(S_z^2)^2$ belongs to $\mathcal{G}$. With analogous calculations, replacing S_z^2 with S_x^2 or S_y^2 we obtain also $i\Upsilon_2(I_{zz}^1 - I_{yy}^1)(S_x^2)^2$ and $i\Upsilon_2(I_{zz}^1 - I_{yy}^1)(S_y^2)^2$. It is important to notice at this point that since the omitted Casimir operators in (22) are all equal and the sequence of calculation is the same in all three cases (with $x, y,$ or z on the right hand side), the omitted Casimir operators (in the operator Υ_2) are the same in all three cases. We can therefore sum these three matrices and obtain using the definition of the Casimir operator (13) that $i\Upsilon_3(I_{zz}^1 - I_{yy}^1)$ belongs to $\mathcal{G}$, for some Υ_3 operator. A completely analogous calculation gives that $i\Upsilon(I^1{}_{yy} - I_{xx}^1)$ also belongs to $\mathcal{G}$, for some Υ operator. $\qquad\square$

2.5 Lie subalgebra of $u(2^n)$ commuting with the symmetric group

We now need to recall some general facts on the Lie subalgebra of $u(2^n)$ of matrices commuting with the permutation group P_n. Denote this subalgebra as $u^{P_n}(2^n)$. Its dimension is given by (cf. [3])

$$\dim\left(u^{P_n}(2^n)\right) = \binom{n+3}{n}. \tag{25}$$

One of the main results of [3] is the following

Theorem 2. $\{iI_{zz}, iS_{x,y,z}, i\mathbf{1}\}$ *generate* $u^{P_n}(2^n)$, *and* $\{iI_{zz}, iS_{x,y,z}\}$ *generate* $u^{P_n}(2^n) \cap su(2^n)$.

As we already recalled, the space $(V^1)^{\otimes n}$ decomposes according to (iterated) Clebsch-Gordan decomposition of a tensor product representation in the direct sum of (possibly repeated) V^n, V^{n-2},..., irreducible representations of $su(2)$. Since $S_{x,y,z}$ and I_{zz} leave such subspaces invariant,[1] these spaces are invariant for $u^{P_n}(2^n)$ as well, because of Theorem 2. Therefore, in coordinates given by the bases of these spaces, the matrices of $u^{P_n}(2^n)$ take a block diagonal form.[2] Consider two subspaces in the decomposition of the form V^f for some f, i.e., two subspaces of the same dimension, say V_1^f and V_2^f. A basis for these spaces can be obtained starting with the *highest weight vector* and then successively applying the lowering operator as described for example in [17]. The operators $S_{x,y,z}$ and therefore $I_{zz} = \frac{1}{2}(S_z^2 - \frac{n}{4}\mathbf{1})$ as well as the identity $i\mathbf{1}$ act in the same way on these bases, and therefore (by induction), each repeated Lie bracket of them. Therefore we can take a basis so that the blocks of $u^{P_n}(2^n)$ of the same dimension are equal to each other. Furthermore, each block of dimension $f + 1$ can take any value in $u(f + 1)$ independently of the other blocks of different dimensions, that is, for each block of dimension $f + 1$ there are $(f + 1)^2$ degrees of freedom. If this was not the case for one block, we would have a total number of degrees of freedom, *which is the dimension of* $u^{P_n}(2^n)$, strictly less than T_n, where T_n is defined, for n odd, as

$$T_n = 2^2 + 4^2 + \cdots + (n+1)^2, \tag{26}$$

and, for n even, as

$$T_n = 1^2 + 3^2 + \cdots + (n+1)^2. \tag{27}$$

However in both cases, n odd in (26) and n even in (27), an induction argument shows that

$$T_n = \binom{n+3}{n},$$

which is from (25) the dimension of $u^{P_n}(2^n)$. So we obtain a contradiction. Therefore, we have the following consequence of Theorem 2, which will be useful for us

Corollary 2.4. *The restrictions of* $\{iI_{zz}, iS_{x,y,z}, i\mathbf{1}\}$ *to every irreducible representation* V^f *of* $su(2)$ *generate* $u(f + 1)$.

---

[1] To see this for I_{zz} recall that $S_z^2 = \frac{n}{4}\mathbf{1} + 2I_{zz}$ (from (24)) so that $I_{zz} = \frac{1}{2}(S_z^2 - \frac{n}{4}\mathbf{1})$.

[2] In [10] such a block diagonal form was described using a different approach based on Young symmetrizers.

2.6 Controllability on a single factor in (9)

We now show a notion of controllability on each factor F_j in (9). Recall that each of these factors is assumed of the form V^f, with $f \geq 1$ in Proposition 2.1, although the next lemma can be stated without restrictions on f.

Lemma 2.5. Fix any $j \in \{1, ..., N\}$ with F_j in (9) equal to $F_j = V^f$ so that $f + 1 = \dim(F_j)$. Then for every $M \in su(f + 1)$ the dynamical Lie algebra $\mathcal{G}$ contains a matrix $i\Upsilon A^j$ such that the restriction of iA^j to F_j is equal to M.

Proof. As we have done above, to simplify notations, we set, without loss of generality $j = 1$. The statement is trivially true (and not useful for us because we are assuming in Proposition 2.1 that all F_j have dimensions strictly larger than 1) if $\dim(F_1) = 1$ and it is also true in the case $\dim(F_1) = 2$ since $iS^j_{x,y,z}$ belong to $\mathcal{G}$.

It is useful to adopt the notation $\langle B_1, ..., B_s \rangle$ for the Lie algebra generated by certain matrices $\{B_1, ..., B_s\}$ so that, for instance, the first statement of Theorem 2 reads as $\langle iI_{zz}, iS_{x,y,z}, i\mathbf{1} \rangle = u^{P_n}(2^n)$. Denote by n_1 the number of spin $\frac{1}{2}$ particles in the first cluster. Consider the matrix $Q^1 := I^1_{xx} + I^1_{yy} + I^1_{zz} = \frac{1}{2}(C^1 - 3\frac{n_1}{4}\mathbf{1}^1)$, with the Casimir operator (13) on the first set, which commutes with every matrix in $\{i(I^1_{zz} - I^1_{yy}), i(I^1_{yy} - I^1_{xx}), iS^1_{x,y,z}\}$ (and therefore with each repeated Lie bracket of them). Then we have by Theorem 2

$$\left(su(2^{n_1}) \cap u^{P_{n_1}}(2^{n_1})\right) \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} = \langle iI^1_{zz}, iS^1_{x,y,z} \rangle \subseteq \langle i(I^1_{zz} - I^1_{yy}), i(I^1_{yy} - I^1_{xx}), iS^1_{x,y,z}, iQ^1 \rangle = \quad (28)$$

$$\langle i(I^1_{zz} - I^1_{yy}), i(I^1_{yy} - I^1_{xx}), iS^1_{x,y,z} \rangle + \texttt{span}(iQ^1) \subseteq u^{P_{n_1}}(2^{n_1}) \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}.$$

In the first equality, we used Theorem 2 and in the second equality we used the commutativity of Q^1. Now, consider relation (28) in the basis where matrices are block diagonal and in particular on the block corresponding to $F_1 \otimes F_2 \otimes \cdots \otimes F_N$ in (9). Restricting to this block we notice that $\texttt{span}(Q^1)$ is included in the span of the identity on it (it commutes with an irreducible representation of $su(2)$ given by the restriction of $\texttt{span}\{iS^1_x, iS^1_y, iS^1_z)\}$ and therefore it must be a multiple of the identity according to Schur's lemma (see, e.g., [17])). Consider now the block diagonal form of the relation (28), and its form on the block corresponding to $F_1 \otimes F_2 \otimes \cdots \otimes F_N$. The first Lie algebra on the left is $su(f + 1) \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$, the second to last Lie algebra is the restriction of $\langle i(I^1_{zz} - I^1_{yy}), i(I^1_{yy} - I^1_{xx}), i(S^1_{x,y,z}) \rangle$ to F_1 plus the span of the identity, everything tensored by the identity $N - 1$ times. The last Lie algebra is $u(f + 1) \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$. Now, using the fact from Lemma 2.3 that $\mathcal{G}$ contains $\{i\Upsilon(I^1_{zz} - I^1_{yy}), i\Upsilon(I^1_{yy} - I^1_{xx}), iS^1_{x,y,z}\}$ and that Casimir operators are *all non zero* on the subspaces $F_2, F_3, ..., F_N$ because of our assumption on the dimension, it follows that we can generate every element of the restriction of $\langle i(I^1_{zz} - I^1_{yy}), i(I^1_{yy} - I^1_{xx}), iS^1_x, iS^1_y, iS^1_z \rangle = su(2^{n_1}) \cap u^{P_{n_1}}(2^{n_1})$ to F_1. This concludes the proof. $\square$

2.7 Maximal subalgebras in $su(rs)$

Now that we know that $\mathcal{G}$ acts as any desired element of $su(f + 1)$ on any factor in (9) we need to show that from these elements we can generate all of $su(D^S)$ with D^S in (10). Recall that $\mathcal{G}$ also contains $i\Upsilon S^j_z S^k_z$ for every pair j, k according to Lemma 2.2. Denote by $f_j + 1$, $j = 1, ..., N$ the dimension of F_j. According to Lemma 2.5 we have on $F_1 \otimes F_2 \otimes \cdots \otimes F_N$, $su(f_1 + 1) \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$, $\mathbf{1} \otimes su(f_2 + 1) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$,..., $\mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes su(f_N + 1)$, besides the restriction of $i\Upsilon S^j_z S^k_z$. We will apply iteratively the following result

Theorem 3. *For each pair $r, s \geq 2$, the Lie algebra which is a direct sum of $su(r) \otimes \mathbf{1}$ and $\mathbf{1} \otimes su(s)$ is a maximal Lie algebra of $su(rs)$.*

A maximal Lie algebra $\mathcal{L} \subseteq su(rs)$ is by definition such that for every element $A \in su(rs)$ with $A \notin \mathcal{L}$, $\langle A, \mathcal{L} \rangle = su(rs)$. Theorem 3 was proved by E.B. Dynkin in [6] (Theorem 1.3 in that paper). We only need a simpler version of it, which says that for each $iA \otimes B$, $\notin su(r) \otimes \mathbf{1}$ and $\notin \mathbf{1} \otimes su(s)$, $\langle iA \otimes B, su(r) \otimes \mathbf{1}, \mathbf{1} \otimes su(s) \rangle \rangle = su(rs)$. In order to see this, consider

$$+^\infty_{m=0} ad^m_{\mathbf{1} \otimes su(s)} iA \otimes B = iA \otimes \left(+^\infty_{m=0} ad^m_{su(s)} B\right).$$

Since $+^\infty_{m=0} ad^m_{su(s)} B$ is a nonzero ideal in $su(s)$ and $su(s)$ is simple, it must be equal to $su(s)$. Therefore for every matrix $C \in su(s)$ we have that $iA \otimes C$ belongs to the generated Lie algebra. Fixing C, and doing the same thing on the left, we have that for every $E \in su(r)$, $iE \otimes C$ also belongs to the generated Lie algebra. Therefore, in conclusion, such a Lie algebra contains all the matrices of the form $iE \otimes C$ with $E \in su(r)$ and $C \in su(s)$ beside $su(r) \otimes \mathbf{1}$ and $\mathbf{1} \otimes su(s)$. Putting these together, they span all of $su(rs)$.

2.8 Conclusion of the proof

The proof of the Proposition 2.1 and therefore of the theorem is completed as follows. On the space $F_1 \otimes F_2$, we have $su(f_1 + 1) \otimes \mathbf{1}$ and $\mathbf{1} \otimes su(f_2 + 1)$ along with the restriction of $i\Upsilon S_z^1 S_z^2$ which is nonzero because all the restriction of all the Casimir operators are nonzero multiples of the identity, and it is not in $su(f_1 + 1) \otimes \mathbf{1}$ nor in $\mathbf{1} \otimes su(f_2 + 1)$. Therefore, using Theorem 3, we have that $\mathcal{G}$ contains matrices that are equal to M for any $M \in su((f_1 + 1)(f_2 + 1))$ on $F_1 \otimes F_2$ and equal to the identity on the other factors in (9). Then we iterate this argument by using $i\Upsilon S_z^2 S_z^3$ to show this fact for $M \in su((f_1 + 1)(f_2 + 1)(f_3 + 1))$, $i\Upsilon S_z^3 S_z^4$ and so on up to $i\Upsilon S_z^{N-1} S_z^N$ for $M \in su(D^S)$.

3 Discussion and Extensions

We now discuss several possible extensions of the result of Theorem 1 to networks different from the *multipartite* case with Ising coupling above treated.

3.1 Networks with different type of coupling between spins

The Ising coupling between spins in two different clusters, $A_{j,k} S_z^j S_z^k$, can be replaced by a more general two body coupling so that A in (5) is replaced by $\hat{A}$ with

$$i\hat{A} = \sum_{1 \leq j < k \leq N} A_{j,k} S_z^j S_z^k + B_{j,k} S_x^j S_x^k + C_{j,k} S_y^j S_y^k. \tag{29}$$

The result of Theorem 1 is still valid as long as we consider as a non-zero interaction between the j-th and the k-th cluster if $(A_{j,k}, B_{j,k}, C_{j,k}) \neq (0, 0, 0)$. In order to see this, notice that the subspaces (9) are still invariant for the dynamics if the interaction takes the more general form (29) and that the reduction to the case of Proposition 2.1 still holds. If there is only one cluster in the network, there is no term of the two-body form (29) and so the result of the proposition holds. If there is more than one cluster in a connected network we have proven in Proposition 2.1 subspace controllability in the Ising $Z - Z$ case. Let us see why this is true in the general case of interaction (29). By taking repeated Lie brackets of the interaction (29) with matrices of the form $iS_{x,y,z}^j$ we can obtain (as long as the coupling is nonzero) the Ising terms $iS_z^j S_z^k$. Therefore, the dynamical Lie algebra generated by replacing the Ising interaction (5) with the more general (29) two body interaction is larger than or equal to the one obtained with Ising interaction. Since in the latter case we have subspace controllability, the same is true for the more general interaction (29).

3.2 Coupling between spins in the same cluster

If we add to the interaction A in (5) a term modeling interaction between spins in the same cluster, the coupling takes the more general form

$$iA_{gen} = iA + \sum_{j=1}^{N} H_0^j, \tag{30}$$

where A is the same as in (5) (or (29)) and H_0^j models these 'internal' interactions. By using the form of the interaction A in (5) and taking repeated Lie brackets of (30) with $S_{x,y,z}^j$, $j = 1, ..., N$, alternating $S_{x,y,z}^j$ with $S_{x,y,z}^k$ ($j \neq k$) similarly with what was done in Lemma 1.1, we can detach iA from iA_{gen} in (30). Therefore the dynamical Lie algebra in this case is generated by the same dynamical Lie algebra calculated above for the case without internal interactions, and A_{gen}. Therefore the dynamical Lie algebra will be in general larger and the spaces (9) will in general not be invariant anymore.

3.3 Different coupling strength for spins in the same cluster

As it is intuitive, if we allow spins of the same cluster to interact differently with the same spin in another cluster we increase the controllability of the system in that some of the subspaces (9) will not be invariant anymore and larger invariant subspaces have to be considered. We illustrate this fact with a simple example.

Example 3.1. Consider first two spin $\frac{1}{2}$ particles with the same gyromagnetic ratio interacting in the same way with one spin $\frac{1}{2}$ particle with a different gyromagnetic ratio. We have two clusters with two and one spin respectively. On the first cluster, the Hilbert space $V^1 \otimes V^1$ splits according to Clebsch-Gordan decomposition as $V^1 \otimes V^1 = V^2 \oplus V^0$ so that the full space $(V^1 \otimes V^1) \otimes V^1$ splits as $(V^2 \otimes V^1) \oplus (V^0 \otimes V^1)$. Therefore the spaces $V^2 \otimes V^1$ and $V^0 \otimes V^1 \simeq V^1$ are the ones to be considered in (9). In the first case the associated graph coincides with the connectivity graph of the network, the dimension D^S in (10) is equal to $D^S = 6$, and the dynamical Lie algebra acting on this invariant space coincides with $su(6)$. In the second case the associated graph only has the node corresponding to the second cluster. The dynamical Lie algebra on the given subspace coincides with $su(2)$ (its irreducible standard representation). Therefore in the appropriate basis, the dynamical full Lie algebra $\mathcal{G}$ can be written in block diagonal form, with blocks of dimension 6 and 2. However, if the coupling constants are different in absolute value, a direct calculation of the dynamical Lie algebra shows that it is equal to $su(8)$. Therefore, there is no nontrivial invariant subspace and the system is controllable as a whole. The two subspaces above are included in a single invariant subspace equal to the whole space.

We now want to obtain some insight into the mechanism of increase in controllability and enlargement of the invariant subspaces when the coupling constants differ which we have seen in the previous example. We start with the basic situation of the type of networks considered in the previous sections and then perturb some coupling constants. Consider, in particular, a network with N clusters as in the previous sections, each cluster with uniform coupling with any other cluster. Consider then an associate invariant subspace as in (9). Assume now that the coupling constants of one of the cluster, say the cluster $N-1$, with another cluster, say the N-th cluster, split. A subcluster of the $(N-1)$-th cluster has coupling constant with the N-th cluster equal to W and another subcluster has coupling constant Y (we assume for simplicity that there are only two values of coupling constants and furthermore we assume the stronger condition $|Y| \neq |W|$). The matrix A in (5) can then be written as

$$iA := \left(\sum_{1 \leq j < k \leq N, \, (j,k) \neq (N-1,N)} A_{j,k} S_z^j S_z^k \right) + W S_{z,1}^{N-1} S_z^N + Y S_{z,2}^{N-1} S_z^N, \tag{31}$$

where we have split S_z^{N-1} in two parts, $S_{z,1}^{N-1}$ and $S_{z,2}^{N-1}$, according to their interaction with the N-th cluster. Now, if $F_N = V^0$ in (9), the last two terms in (31) as well as all the coupling $S_z^j S_z^N$ and also $S_{x,y,z}^N$ give zero, the associated graph to the subspace (9) only contains the first $N-1$ nodes. The splitting of the coupling constants in the cluster $N-1$ plays no role and the situation is equivalent to the one we considered in the previous sections but with the first $N-1$ clusters only. If however, $F_N \neq V^0$, by taking (repeated) Lie brackets of A in (31) with $iS_{x,y,z}^N$ and $iS_{x,y,z}^{N-1}$ we obtain all matrices of the form $iW S_{x,y,z,1}^{N-1} S_{x,y,z}^N + iY S_{x,y,z,2}^{N-1} S_{x,y,z}^N$ where we have split $S_{x,y,z}^{N-1}$ as $S_{x,y,z}^{N-1} = S_{x,y,z,1}^{N-1} + S_{x,y,z,2}^{N-1}$, generalizing what we have done above. Taking the Lie brackets of $iW S_{x,1}^{N-1} S_x^N + iY S_{x,2}^{N-1} S_x^N$ with $iW S_{y,1}^{N-1} S_x^N + iY S_{y,2}^{N-1} S_x^N$, we obtain $\left(iW^2 S_{z,1}^{N-1} + iY^2 S_{z,2}^{N-1} \right) (S_x^N)^2$. Analogously we obtain $\left(iW^2 S_{z,1}^{N-1} + iY^2 S_{z,2}^{N-1} \right) (S_y^N)^2$ and $\left(iW^2 S_{z,1}^{N-1} + iY^2 S_{z,2}^{N-1} \right) (S_z^N)^2$, and summing all we obtain

$$\left(iW^2 S_{z,1}^{N-1} + iY^2 S_{z,2}^{N-1} \right) C^N, \tag{32}$$

where C^N is the Casimir operator. Analogously, we can obtain (32) with z replaced by x and y, respectively. Since C^N is a multiple of the identity on F_N, we effectively obtain $W^2 S_{x,y,z,1} + Y^2 S_{x,y,z,2}$ and since we already had $S_{x,y,z} = S_{x,y,z,1} + S_{x,y,z,2}$ we obtain the two matrices $S_{x,y,z,1}$ and $S_{x,y,z,2}$. We have effectively split the cluster $N-1$ into two subclusters. The subspace F_{N-1} is not invariant anymore. If we reconsider the separation of the $(N-1)-$th cluster into the two subclusters as above we can apply the Clebsch-Gordan decomposition to each subcluster. Assuming that the first subcluster has m_1 spins and the second m_2 (thus $n_{N-1} = m_1 + m_2$), we will have a decomposition of $(V^1)^{\otimes m_1}$ for the first subcluster and a decomposition of $(V^1)^{\otimes m_2}$ for the second subcluster. Pick a space in the first decomposition, say V^{f_1} and a space in the second decomposition, say V^{f_2}, which carry respectively an irreducible representation corresponding to f_1 and f_2 of $su(2)$. To $V^{f_1} \otimes V^{f_2}$ we can apply the Clebsch-Gordan decomposition into the direct sum of invariant subspaces. The original invariant subspace F_{N-1} was selected among such spaces. However, with the division into two subclusters above, the tensor product $V^{f_1} \otimes V^{f_2}$ has to be considered as a whole, giving therefore a larger invariant space.

3.4   Controls acting on certain spins only

A topic of current interest (see, e.g., [15], [16]) is subspace controllability of a network when the control is only localized on certain spins. In the context of this paper, this corresponds to setting some of the gyromagnetic ratios γ equal to zero for the spins which are not directly coupled to the controls. In this case, the result given in Theorem 1 is no longer valid because we have assumed (cf. the discussion preceding Lemma 1.1) that all the gyromagnetic ratios are different from zero. However, some of the techniques used in this paper can be applied to analyze these cases as well. If the gyromagnetic ratio of one cluster (say cluster number 1) is zero, then the argument preceding Lemma 1.1 shows that the dynamical Lie algebra is generated by $iS_{x,y,z}^j$, for $j = 2, 3, ..., N$ and A in (5). One can perform again the Clebsch-Gordan decomposition for each cluster so that the combined subspaces of the type (9) are still invariant under the dynamics. However, they are not in general minimal invariant. We illustrate this point with a simple example.

Assume to have a chain of $n = 3$ spins, $\circ^1 \longleftrightarrow \circ^3 \longleftrightarrow \circ^2$, where the interaction between spin $\circ^1$ and $\circ^3$ and the interaction between spin $\circ^2$ and $\circ^3$ are equal. Assume that the gyromagnetic ratio is nonzero only for spin $\circ^3$. This chain can be seen as a bipartite network with $N = 2$ clusters: the first cluster containing spin $\circ^1$ and $\circ^2$ and the second cluster with the middle spin $\circ^3$ only. The dynamical Lie algebra is generated by $A := i(\sigma_z \otimes 1 + 1 \otimes \sigma_z) \otimes \sigma_z$ and $B_{x,y,z} = i 1 \otimes 1 \otimes \sigma_{x,y,z}$, and it is computed to be spanned by $\{(\sigma_z \otimes 1 + 1 \otimes \sigma_z) \otimes \sigma, 1 \otimes 1 \otimes \sigma\}$, for general $\sigma \in su(2)$. The overall space with spaces V^1 grouped according to clusters is $(V^1 \otimes V^1) \otimes V^1$. Now, the first factor $(V^1 \otimes V^1)$ can be decomposed according to the Clebsch-Gordan decomposition as $(V^1 \otimes V^1) = V^2 \oplus V^0$ giving for the possible subspaces (9) $V^2 \otimes V^1$ and $V^0 \otimes V^1 \simeq V^1$. These are invariant under the dynamical Lie algebra. However it is also possible to decompose the factor $(V^1 \otimes V^1)$ as $(V^1 \otimes V^1) = S_1 \oplus S_2 \oplus S_3 \oplus S_4$ where S_j, $j = 1, ..., 4$, are one dimensional subspaces spanned by the common eigenvectors of $\mathbf{1} \otimes \mathbf{1}$ and $\sigma_z \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_z = S_z^1$. This decomposition is *finer* than the Clebsch-Gordan one and results in smaller invariant subspaces for the dynamics, $S_j \otimes V^1$, $j = 1, ..., 4$. This example deals with *chains* of spin. Chains rarely fall in the type of multipartite models considered in this paper. If two spins have the same gyromagnetic ratio they have to belong to the same cluster and not been connected to each other. In order to keep the same interaction with a second cluster and to keep the chain structure such a second cluster must contain one spin. This gives the case of three spins considered in the example. Another possibility is a chain with n spin each of them giving a cluster. In this case (under our assumption that all the component of the electro-magnetic field are available for control) the system is controllable and no invariant subspace exists. More possibilities exist in the case of more general branched networks.

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