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ABSTRACT
A 3D numerical investigation on the magnetization of dilute magnetic emulsions subjected to shear flows and external magnetic fields is carried out. The present study is an extension of the previous work of Cunha et al. [“Effects of external magnetic fields on the rheology and magnetization of dilute emulsions of ferrofluid droplets in shear flows,” Phys. Fluids 32, 073306 (2020)] for the 2D analogous system. The ferrofluid is assumed superparamagnetic such as the bulk magnetization depends on droplet shape and orientation. The magnetic field is applied in the main flow, main velocity gradient, and main vorticity directions. For the two former cases, the emulsion magnetization does not perfectly align with the external field, such stronger shear rates leading to larger misalignment angles. For fields parallel with flow direction, stronger fields lead to a decrease in this misalignment angle, while for external magnetic fields in the main velocity gradient direction, stronger magnetic fields lead to an increase in misalignment angle. Although these results are qualitatively similar to those presented by Cunha et al. [“Effects of external magnetic fields on the rheology and magnetization of dilute emulsions of ferrofluid droplets in shear flows,” Phys. Fluids 32, 073306 (2020)], the misalignment angles observed are significantly smaller than those of the analogous 2D. The magnetic forces at the droplet interface create a magnetic torque in the emulsion, resulting in asymmetries of the bulk stress tensor. For external magnetic fields in the main velocity gradient direction, magnetic torques increase monotonically with increasing field intensities, while for external fields in the main vorticity direction, magnetic torques are smaller and remain roughly constant for $\Gamma_{mag} > 8$. We observed that the magnetic field applied in the main vorticity direction may lead to oblate droplets, rather than the more conventional prolate one.

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I. INTRODUCTION
When an emulsion of magnetic droplets is subjected to an external magnetic field, an additional force is induced in the flow field.2 This phenomenon is due to a coupling between matter and magnetic fields, in which different magnetic permeability leads to changes in the underlying magnetic field and modifications in the structure of the mixture.3,4

Due to this coupling between magnetic fields and fluid properties, magnetic fluids can be classified as a special category of smart nanomaterials covering a wide range of applications, including optical, biomedical, and microfluidic systems.5

In the biomedical field, magnetic fluids are widely studied6,7 and play a significant role in prognostic, diagnostic, and treatment of serious health issues.8 Recently, significant improvements were achieved in targeted drug delivery and active isolation of circulating tumor cells using magnetic fields.9 Magnetic fluids are also used for cancer treatments via hyperthermia, in which oscillating magnetic fields are used to heat a targeted location due to viscous dissipation of energy in ferrofluids.8,10

Some potential applications for magnetic fluids in other fields also include the prevention of vapor explosion in fusion reactors by the transition from explosive to stable nucleation,11 the formation of
Pickering emulsions, in which solid particles are used to stabilize and tune surface properties,\textsuperscript{12} accurate control of bubbles to prevent liquid metal columns from clogging,\textsuperscript{7} enhance convection in heat transfer systems,\textsuperscript{7} and applications of ferrofluids to dampers.\textsuperscript{7}

In recent years, more research attention has been directed toward the rheology and dynamics of magnetic colloidal particles and magnetizable droplets in the presence of external magnetic fields.\textsuperscript{13–22} Cunha and Couto\textsuperscript{13} extended the well-developed boundary integral formulation for viscous droplets\textsuperscript{23–25} to describe the dynamics of ferrofluid droplets under the action of external magnetic fields. Jesus et al.\textsuperscript{17} using the 3D immersed boundary methodology studied the effects of magnetic fields on the geometry of ferrofluid droplets in shear flows. Using the Lattice Boltzmann method, Ghaderi et al.\textsuperscript{18} investigate the effects of magnetic fields on the formation of ferrofluid droplets in coflowing configurations. The authors show that the use magnetic field allows controlling the size of the droplets and frequency of formation. In another study, Hassan et al.\textsuperscript{19} demonstrated experimentally how a uniform magnetic field can affect lateral migration in microchannels.

Cunha et al.\textsuperscript{13} analyzed the effects of external magnetic fields on the rheological properties of magnetic emulsions using 2D level set simulations. Using a modified expression for the bulk stress accounting for the magnetic effects, they revealed a complex coupling between the local induced magnetic field and the flow near the droplet. This coupling, in turn, leads to a strong dependence between the bulk viscosity of the emulsion and the external magnetic field direction and intensity. For magnetic fields perpendicular to the main flow direction, the magnetic field promotes a dramatic increase in reduced viscosity as the droplet is deformed alongside the magnetic field direction. With the magnetic field parallel to the main flow direction, there is a delay in the droplet breakup process since the magnetic forces confine the droplet in regions of weaker effective shear. The authors also showed how bulk magnetization produces internal torques and breaks down the symmetry of the stress tensor.

Ishida and Matsunaga\textsuperscript{20} extended the results of Cunha et al.\textsuperscript{13} to the three-dimensional case, using a lattice-Boltzmann model, providing important contributions to the study of the rheology of ferrofluid emulsions. This allowed the authors to investigate the effects of an external magnetic field on the droplet dynamics when applied either in the main flow direction, in the main velocity gradient direction, and in the main vorticity direction. Moreover, the authors were able to study the response of the second normal stress difference to the magnetic field.

All these recent advances in the rheological characterization of magnetic emulsions reveal an even greater potential since they can also be an effective form to take advantage of the torque created in the magnetically polarized matter due to the induced force by a magnetic field.\textsuperscript{26}

In this work, we present a numerical analysis of the magnetic behavior of a ferrofluid emulsion under the combined action of an external magnetic field and a simple shear flow. For this, we consider a three-dimensional domain, with a single ferrofluid droplet immersed in a Newtonian, nonmagnetic matrix fluid. The numerical method is based on the solution of the Navier–Stokes equations with a projection method, alongside the level set method for interface capturing, and Maxwell’s equations to model the magnetic problem. In our results, we find that, despite assuming a superparamagnetic ferrofluid droplet, the bulk magnetization of the emulsion does not perfectly align with the external magnetic field. We then investigate how the bulk magnetization is affected by the shape and inclination of the droplet and, in turn, by different shear rates and magnetic field intensities. We also found that the misalignment between bulk magnetization and external magnetic field gives rise to a magnetic torque acting on the emulsion, which is then counteracted by an equal and opposite hydrodynamic torque.

The remainder of this work is organized as follows: in Sec. II, we present the problem formulation, including the governing equations and the nondimensional parameters that govern the model problem. Section III presents the numerical methodology used to solve the governing equations. Results and discussions are presented in Sec. IV, with concluding remarks presented in Sec. V.

II. PROBLEM STATEMENT

We consider a single ferrofluid droplet suspended in an immiscible, nonmagnetizable liquid. The system is confined in a three-dimensional channel bounded by two parallel walls normal to the $y$-direction, and periodic in the $x$- and $z$-directions. Initially, the droplet is circular with a radius $a$ and located at the center of the domain. Both phases are assumed to be incompressible Newtonian fluids, of the same density $\rho$ and viscosity $\eta$. The continuous phase has magnetic permeability $\mu_0$, assumed to be the same as free space. The ferro-fluid has magnetic permeability $\mu_i$, such that $\xi$ represents the continuum-to-dispersed phase magnetic permeability ratio. The droplet interface is assumed to be free of tensioactive substances, such that it has a constant surface tension coefficient $\sigma$. The movement of the channel walls, in opposite directions, imposes a shear flow of shear rate $\dot{\gamma}$, with vorticity in the $z$-direction. Moreover, the system is subjected to an external uniform magnetic field $\mathbf{H}_0$. This external magnetic field can be parallel to either the main flow direction ($x$), the main velocity gradient direction ($y$), or the main vorticity direction ($z$).

A schematic illustration of the problem is presented in Fig. 1, for the external magnetic field in the $y$-direction.

Considering the absence of electric fields and currents, Maxwell’s equations reduce to $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{H} = 0$, where $\mathbf{B}$ is the magnetic induction and $\mathbf{H}$ is the magnetic field. Due to the usual composition of ferrofluids, we assume that it behaves as a superparamagnetic material.\textsuperscript{27} Hence, we assume that the magnetization in the droplet follows the linear relation $\mathbf{M} = \chi \mathbf{H}$, where $\chi$ is the magnetic susceptibility. The magnetic induction is composed of a contribution from the magnetic field and the material magnetization, so that $\mathbf{B} = \mu_0 \mathbf{M} + \mathbf{H}$. To accommodate this relation for the entire domain, we define the continuous function $\zeta_\phi(\mathbf{x})$ such that $\zeta_\phi(\mathbf{x}) = 1$ for the matrix fluid and $\zeta_\phi(\mathbf{x}) = 1 + \gamma$ inside the droplet. In this sense, we have $\mathbf{B} = \mu_0 \zeta_\phi \mathbf{H}$. Since the magnetic field is irrotational, it can be written as the gradient the magnetic potential field $\psi$, such that $\mathbf{H} = -\nabla \psi$. Thus, given that $\mathbf{B}$ is solenoidal, the magnetic potential field is governed by

$$\nabla \cdot (\zeta_\phi(\mathbf{x}) \nabla \psi) = 0.$$  

Following a continuum perspective, we have that the hydrodynamic problem for the two-phase flow is governed by the incompressible Navier–Stokes equations, accounting for the magnetic and capillary forces as the body forces per unit of volume $\mathbf{F}_c$ and $\mathbf{F}_m$, respectively. Therefore,
where 

and 

where \( u \) is the velocity field, \( t \) is time, and \( P \) is the pressure. The use of the level set function \( \phi \) to capture the interface located at \( \phi = 0 \) as further discussed in Sec. III B—leads to a smooth transition between both phases, which permits the computation of interfacial forces using the Young-Laplace equation as

\[
F_l = -\sigma \kappa \delta(\phi) |\nabla \phi| \hat{n},
\]

where \( \kappa \) is the mean curvature, \( \delta \) is the Dirac delta function, and \( \hat{n} \) is the unit normal vector outward the droplet surface. The magnetic force term is accounted as

\[
F_m = \mu_0 (\zeta \phi(x) - 1) \mathbf{H} \cdot \nabla \mathbf{H}.
\]

The nondimensional form of the problem is obtained using the following characteristic scales: \( a \) for length, \( 1/\gamma \) for time, \( \gamma a^2 \) for velocity, \( \rho \gamma^2 a^2 / \eta \) for pressure, and \( H_0 = |\mathbf{H}_0| \) for magnetic field. The mass balance in Eq. (3) remains the same in the nondimensional form, while the momentum balance in Eq. (2) is rewritten as

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \frac{1}{Re} \nabla^2 \mathbf{u} - \frac{1}{Re Ca} \kappa \delta(\phi) |\nabla \phi| \mathbf{n} + \frac{Ca_{mag}}{Re Ca} (\zeta \phi(x) - 1) \mathbf{H} \cdot \nabla \mathbf{H}.
\]
Note that in this scheme, the incompressibility constraint is imposed at every time step.

The spatial discretization is performed using a regular, staggered (Marker and Cell) grid, with centered second-order finite differences, except for the advective term, which uses a second-order essentially non-oscillatory (ENO) scheme with upwinding. The magnetic potential equation (Eq. (1)) is discretized with centered second-order finite differences. For an efficient algorithm, suitable for three-dimensional simulations, the three Helmholtz equations for are solved with a fast Poisson solver based on Fourier analysis and Gaussian elimination, and the Laplace equation for is solved using a conjugate gradient algorithm with multigrid preconditioning.

### B. Level set method

To capture the droplet dynamics, we use the well-established level set method. This method consists of evolving a signed distance function , which by construction defines the droplet interface position at the collection of points where is equal to zero. In this work, is initialized as a signed distance function, considering a sphere of the same radius as the droplet, and defining positive values outside the droplet and negative values inside of it. Note that any nonzero gradient of a function will always be orthogonal to the droplet surface. Hence, we can define the unit normal vector as , and the local mean curvature as .

The motion of the interface is accurately captured by the advection of the level set function, which conserves the values of on fluid particles.

The level set method assumes smooth transitions of quantities through the interface, which is characterized by a finite thickness . Such a smoothing process is essential for avoiding numerical instabilities and ensuring that the interface is accurately captured in a discrete setting. In this sense, we adopt the smoothed Heaviside function defined as

\[
H_s(\phi) = \begin{cases} 
0, & \text{if } \phi < -\epsilon, \\
\frac{1}{2} \left[ 1 + \frac{\phi}{\epsilon} - \frac{1}{\pi} \sin \left( \pi \phi / \epsilon \right) \right], & \text{if } |\phi| \leq \epsilon, \\
1, & \text{if } \phi > \epsilon,
\end{cases}
\]

with a corresponding smoothed Dirac delta function \( \delta_s(\phi) = \frac{dH_s(\phi)}{d\phi} \).

We define the interface thickness as \( \epsilon = 1.5 \Delta x \), where \( \Delta x \) is the size of a grid cell. This formulation allows us to rewrite the continuous function for the magnetic permeability as \( \kappa(\phi) = \zeta + (1 - \zeta)H_s(\phi) \).

Although the evolution of Eq. (11) ensures that the zero level set of is accurately transported, it also causes it to diverge from a signed distance function. Since the stability and accuracy of the level set method are strongly dependent on the level set function remaining close to signed distance function near the interface, it must be periodically reinitialized, reverting it back to a signed distance function while preserving the position of the interface. In this work, reinitialization is performed in each time step by solving the equation,

\[
\frac{\partial \phi}{\partial t} + S(\phi)(|\nabla \phi| - 1) - \lambda \delta f(\phi) = 0,
\]

where \( \tau \) is an artificial time, \( \delta f(\phi) \) is volume-preserving correction parameters, and \( S(\phi) \) is the signal function, defined as

\[
S(\phi) = \frac{\phi}{\sqrt{\phi^2 + |\nabla \phi|^2 \Delta x^2}}.
\]

Due to numerical errors, the reinitialization procedure causes slight changes in the position of the interface. If not addressed, these changes can accumulate and, over time, lead to significant changes in droplet volume. To ensure local conservation of volume, the parameter \( \delta f(\phi) \) is introduced, with

\[
\lambda = \frac{\int_{\Omega} \delta (\phi)(S(\phi)(|\nabla \phi| - 1)) \, dV}{\int_{\Omega} \delta (\phi) f(\phi) \, dV},
\]

where \( \Omega \) is an arbitrary, fixed domain. In the discrete implementation, \( \Omega \) is defined as each individual grid cell, in order to ensure local conservation of volume.

To solve Eqs. (11) and (13), we use a fifth-order weighted essentially non-oscillatory (WENO) scheme for spatial discretization, and a third-order strong stability preserving (SSP) Runge–Kutta scheme for temporal evolution. This discretization is based on a basic upwinding scheme for Eq. (11), with a conservative discrete form based on Godunov’s method used for Eq. (13).

In the level set framework, surface integrals of an arbitrary function \( \mathcal{R} \) over the interface \( \Gamma \) can be evaluated as volume integrals over the entire domain \( V \) using the Dirac delta function. This is performed using the relationship,

\[
\int_{\Gamma} \mathcal{R} dS = \int_{V} \nabla \delta(\phi) |\nabla \phi| \, dV.
\]

In this work, this volume integral is approximated with a second-order quadrature, using a 27 point stencil.

### IV. RESULTS AND DISCUSSION

In this section, we present a study on the bulk magnetization of ferrofluid emulsions in shear flows, and on the magnetic torque created in the systems due to the misalignment between the bulk magnetization and the external magnetic field. Given that these phenomena are in part determined by the droplet dynamics, we start by investigating the effects of the external magnetic fields and the shear stresses on the shape and inclination of ferrofluid droplets, extending the results of Ishida and Matsunaga for different values of \( \mu \) and a broader range of \( \alpha_{mag} \).

For all cases presented in this work, we use a domain size of \( 10 \times 10 \times 7.5 \) and a respective mesh discretization of \( 128 \times 128 \times 96 \). We also fix the parameters \( Re = 0.01 \) and \( \zeta = 2 \). This domain size corresponds to a dispersed phase volume fraction \( \beta \approx 0.56 \%), characterizing a dilute emulsion with negligible magnetic and hydrodynamic interaction between the droplets. Tests considering one case of large droplet deformation \( \text{Ca} = 0.15, \text{Ca}_{mag} = 12 \) were performed to verify the absence of confinement and mesh refinement effects. Comparing

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the results obtained with our standard mesh to those obtained using a more refined mesh of $192 \times 192 \times 144$ cells, we found that discrepancies in droplet deformation, droplet inclination angle, and reduced viscosity were all smaller than 3%. Additionally, comparisons with a domain size $15 \times 15 \times 11.25$, with the same mesh density, resulted in discrepancies smaller than 1% for the aforementioned parameters. We also found that the influence of $\Delta t$ is negligible for steady-state results, provided that it is small enough to ensure stability. It worth mentioning that the droplet deformation for the case in this analysis corresponds to one of the largest reported in this work. Therefore, we are confident that the results presented in this work are free of confinement and mesh refinement effects.

### A. Droplet shape and inclination

Since the droplet shape and inclination have significant effects on both the rheology and magnetization of the system, understanding how they vary for different shear rates and applied magnetic fields is fundamental for the development of this work. In this context, two parameters of interest are the droplet deformation and inclination. Figures 2(a) and 2(b) present the droplet deformation and inclination, respectively, when it is subjected to an external magnetic field. One can see that, up to moderate magnetic field intensities ($C_{\text{mag}} \approx 10$), the droplet deformation varies significantly across the different capillary numbers. As the values of $C_{\text{mag}}$ increase past this value, the curves for different $C_a$ start to converge, indicating that the magnetic stresses have a dominant effect on the droplet shape. For the droplet inclination, one observes that increasing values of $C_{\text{mag}}$ lead to the alignment of the droplet with the applied magnetic field, and, consequently, to the main flow direction. Additionally, the inclination angle displays a nonmonotonic behavior. In the absence of magnetic fields ($C_{\text{mag}} = 0$), higher values of $C_a$ lead to smaller inclination angles, while the opposite behavior is observed for $C_{\text{mag}} \geq 4$. Intermediate values of $C_{\text{mag}}$ correspond to a transition region, where such behaviors are not as clearly defined.

Figure 2 presents the droplet deformation and inclination angle as a function of the magnetic capillary number, for different external magnetic field directions and capillary numbers. Data from Ishida and Matsunaga$^{32}$ are presented alongside our results and in a close agreement. Since the two works use different numerical approaches, such an agreement suggests that both methodologies are accurately capturing the magnetic and hydrodynamic aspects of the problem.

When subjected to an external magnetic field, the magnetic forces at the interface stretch the droplet in the external magnetic field direction, allowing for an active control of both the droplet deformation and inclination. Figures 2(a) and 2(b) present the droplet deformation and inclination, respectively, when it is subjected to an external magnetic field in the $x$-direction. One can see that, up to moderate magnetic field intensities ($C_{\text{mag}} \approx 10$), the droplet deformation varies significantly across the different capillary numbers. As the values of $C_{\text{mag}}$ increase past this value, the curves for different $C_a$ start to converge, indicating that the magnetic stresses have a dominant effect on the droplet shape. For the droplet inclination, one observes that increasing values of $C_{\text{mag}}$ lead to the alignment of the droplet with the applied magnetic field, and, consequently, to the main flow direction. Additionally, the inclination angle displays a nonmonotonic behavior. In the absence of magnetic fields ($C_{\text{mag}} = 0$), higher values of $C_a$ lead to smaller inclination angles, while the opposite behavior is observed for $C_{\text{mag}} \geq 4$. Intermediate values of $C_{\text{mag}}$ correspond to a transition region, where such behaviors are not as clearly defined. The less deformed geometries, characteristic of smaller values of $C_a$, are more susceptible to variations in inclination due to elongations in the magnetic field direction, explaining the steeper slope of $\theta$ for smaller $C_a$. Also, as suggested by Cunha et al.$^1$ at high values of $C_{\text{mag}}$, the stronger shear effects, characteristic of larger values of $C_a$, are more likely to align the droplet at angles closer to those with no external fields.

For the external magnetic field applied in the $y$-direction, however, shear-induced effects remain relevant across the entire range of $C_{\text{mag}}$. Such an effect is evidenced in Figs. 2(c) and 2(d), where the

![Figure 2](image-url)
droplet deformation and inclination vary significantly with $Ca$. The main reason for this behavior is that, as the droplet deforms alongside the magnetic field direction, it becomes subjected to stronger effective shears, so that the magnetic effects do not become overly dominant. For the droplet deformation, shear-induced and magnetic-induced elongations are additive. Hence, it increases with both $Ca$ and $Camag$ up to the point of breakup. The inclination angle $\theta$ decreases with $Ca$ and increases with $Camag$. This behavior is due to the balance between shear stresses, attempting to rotate the droplet with the vorticity, and the magnetic forces, attempting to align the droplet with the external magnetic field. It is interesting to note that, for $Camag > 6$, further increases in $Camag$ only have minor effects on inclination angle.

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For the external magnetic fields in the $z$-direction, Figs. 2(e) and 2(f) indicate that the intensity of the external magnetic field has a minor influence on the droplet deformation and inclination angle, as previously discussed by Ishida and Matsunaga. As the magnetic effects stretch the droplet in the $z$-direction, the mass conservation causes the droplet cross section in the shear plane to reduce. As a consequence, the droplet becomes subject to a weaker effective shear, and the increased curvature of the interface leads to stronger interfacial forces. The result is that increasing values of $Camag$ have similar effects to decreasing values of $Ca$, represented by reduced droplet deformations and inclination angles closer to $45^\circ$. It should be noticed that we observe slight deviations from the results of Ishida and Matsunaga for $\theta$ when $Camag > 5$ with $Ca = 0.20$, and when $Camag > 2$ with $Ca = 0.15$. In our analysis, the curves for $\theta$ as a function of $Camag$ are smooth and continuous, while the results from Ishida and Matsunaga present an apparent discontinuity. As a matter of fact, such discontinuities appear in the point of $Camag$ where the droplet’s length in the $z$-direction becomes greater than the one corresponding to $B$ in Fig. 1.

Given that the magnetic fields in the main vorticity direction can introduce a droplet deformation normal to the shear plane, in contrast to magnetic fields in the other two directions, it can lead to unusual droplet shapes. If the shear-induced deformations are more pronounced than the magnetic-induced ones, such as for $Ca = 0.3$ and $Camag = 1$, for example, the droplet presents a prolate geometry similar to that of a nonmagnetic droplet. If the magnetic-induced deformations are more pronounced than the shear-induced ones, such as for $Ca = 0.1$ and $Camag = 12$, for example, the droplet once again presents a prolate shape, although elongated in the main vorticity direction, as previously stated by Ishida and Matsunaga. However, if both deforming mechanisms are similarly pronounced, such as for $Ca = 0.6$ and $Camag = 16$, the droplet is deformed to a distinct disc-like shape, as shown in Fig. 3, as it is stretched in two orthogonal directions. Such a phenomenon might be of great interest for the fabrication of micromaterials with oblate shapes.

![FIG. 3. Three-dimensional view of a ferrofluid droplet subjected to an external magnetic field parallel to the main vorticity direction, with slices crossing the droplet center projected to the domain boundaries. $Ca = 0.6$, $Camag = 16$.](image-url)
Moving our attention to the viscosity of the emulsion, we again compare our results to those of Ishida and Matsunaga.\(^3\) In order to eliminate the influence of the dispersed phase volume fraction $\beta$, our results are presented for the reduced viscosity of the emulsion, defined as

$$\eta = \frac{S_y}{\beta},$$

(18)

where $S_y$ is the shear-stress component of the particle stress scaled by $\eta_j$. The particle stress is a tensor that gives the contribution of the dispersed phase to the bulk stress tensor of the emulsion.\(^5\) In the case of an emulsion of ferrofluid droplets with equal viscosities, the particle stress can be calculated by the surface integral,

$$S = \frac{1}{V} \int_{\Gamma} \left[ \left( \frac{\kappa}{Ca} - \frac{Ca_{mag}}{2Ca} (\zeta - 1)H^2 \right) \mathbf{x} \right] dS,$$

(19)

where $\Gamma$ is the droplet surface and $V$ is the total volume of both phases.

Figure 4 presents the reduced viscosity as a function of $Ca_{mag}$ for the three external magnetic field directions, and $Ca = 0.2$. We verify a close agreement between our results and those of Ishida and Matsunaga,\(^3\) further suggesting the accuracy of both methodologies, not only for the droplet geometry but also for the rheology of the emulsion.

Distinct behaviors for the reduced viscosity are observed for each of the external magnetic field directions. For external magnetic fields in the $y$-direction, stronger fields lead to drastic increases of the reduced viscosity. As the droplet is deformed alongside the magnetic field direction, it assumes a shape that imposes more resistance to the flow, by interacting with regions of higher velocity flow and presenting a larger cross-sectional area relative to the flow direction. An opposite behavior is observed for external magnetic fields in the $x$-direction. In this case, the droplet is deformed alongside the magnetic field direction, it assumes a shape that imposes less resistance to the flow, both by confining it to regions of lower velocity flow and reducing its cross-sectional area relative to the flow direction.\(^6\) For external magnetic fields applied in $z$-direction, different magnetic field intensities have minor influences on the reduced viscosity of the emulsion, with higher values of $Ca_{mag}$ leading to slight decreases in reduced viscosity.\(^3\) Such a phenomenon is most likely due to the reduction in the cross-sectional area of the droplet in the shear plane, although part of this effect is probably counteracted by the stronger capillary forces and its restorative effects on the shape of the droplet, similarly to that of the shear-thinning behavior of nonmagnetic emulsions.

### B. Bulk magnetization of the emulsion

In this section, we investigate the magnetization of the dilute emulsion under the combined action of a simple shear flow and an external magnetic field. Due to the superparamagnetic property of the ferrofluid phase, the droplet magnetization is a function of its geometry and the external magnetic field. However, as discussed in Sec. IV A, the droplet geometry is dependent on $Ca$, $Ca_{mag}$ and the direction of the external magnetic field.\(^1,2,9,3,13,32\) In this sense, we have that the droplet magnetization must also be a function of these three parameters, assuming constant $\zeta$. The bulk magnetization of the system can be calculated from the magnetization of a single droplet as

$$\langle M \rangle = \frac{1}{V} \int_\Gamma (\zeta (\phi) - 1) H dV,$$

(20)

Notably, the nonmagnetizable phase does not contribute to the mean magnetization since $\zeta(\phi) = 1$ outside the droplet.

Figure 5 presents the magnetic field lines and the mean magnetization direction (red arrow) for an external magnetic field parallel to the main velocity gradient direction, with $Ca = 0.15$ and $Ca_{mag} = 12$. We observe that the magnetic field and, consequently, the magnetization inside the droplet, are mostly uniform. This result agrees with the classical theory of Maxwell (1873) which postulates that ellipsoids are the only finite bodies that can be uniformly magnetized in the presence of a uniform inducing magnetic field.\(^5,35\) Hence, as long as the droplet assumes an ellipsoidal shape, either prolate or oblate, the magnetization should be uniform. Additionally, we observe a slight misalignment angle, $\theta_{mag}$, between the bulk magnetization direction and the external magnetic field $H_0$, contrary to what a naive expectation would indicate given the superparamagnetic assumption. Such a misalignment is a consequence of the droplet elongation and inclination with respect to $H_0$. Moreover, the regions with the highest jump of magnetic field intensity across the interface (adjacent to the regions of highest field intensity), and, consequently, the strongest magnetic forces, are located where the surface is perpendicular to the external magnetic field direction. Note that such regions are close to the tips of the droplet but offset in the field direction, then creating an asymmetry of the magnetic forces with respect to the tip-to-tip direction. Given that the magnetic forces point outward the droplet in the direction normal to the surface, this asymmetry gives rise to magnetic torques in the system.

As elucidated by Cunha et al.,\(^4\) the misalignment angle $\theta_{mag}$ is associated with the droplet asymmetry relative to the direction of the external magnetic field. If the droplet is nearly spherical, or nearly aligned to the external magnetic field, the resulting asymmetry, and thus misalignment angle, is small. On the other hand, if the droplet is largely deformed and not aligned to the external magnetic field, the resulting misalignment angles are significantly larger. Figure 6 presents $\theta_{mag}$ as a function of $Ca_{mag}$ for different values of $Ca$ and for external magnetic fields applied in three different directions.
For the magnetic field in the $y$-direction, one would expect a decrease in $\theta_{\text{mag}}$ with decreasing $Ca$ and increasing $Ca_{\text{mag}}$ due to weaker shear effects and stronger magnetic alignment effects, respectively. Indeed, $\theta_{\text{mag}}$ does decrease with decreasing $Ca$. However, we observe that higher values of $Ca_{\text{mag}}$ lead to larger values of $\theta_{\text{mag}}$. This phenomenon happens because, as discussed in Sec. IV A, the droplet inclination angle does not change significantly with variations in magnetic field intensity if $Ca_{\text{mag}} > 6$. Thus, as the droplet deformation increases for higher values of $Ca_{\text{mag}}$, this leads to larger asymmetries relative to the magnetic field direction, which in turn leads to larger misalignment angles.

For external magnetic fields in the $x$-direction, the misalignment angles for $Ca_{\text{mag}} = 0$ show similar results to those with the field in the $y$-direction, especially for $Ca = 0.05$, where the droplet inclination in the absence of magnetic forces is the closest to $45^\circ$. For all cases, $\theta_{\text{mag}}$ decreases with larger values of $Ca_{\text{mag}}$ as the droplet tends to align with the external magnetic field. The misalignment angle $\theta_{\text{mag}}$ also increases with larger values of $Ca$, since shear-induced deformations are not perfectly aligned with the magnetic field direction (recall that non-magnetic droplets assume an inclination angle $\theta \approx 45^\circ$). Thus, higher values of $Ca_{\text{mag}}$ lead to larger droplet asymmetries relative to the magnetic field direction and, in turn, to larger misalignment angles. Figure 7 presents the droplet shape for the case of $Ca = 0.15$ and $Ca_{\text{mag}} = 12$. As can be seen, the droplet is, indeed, nearly (although not perfectly) parallel to the external magnetic field, resulting in a small, but present, magnetization misalignment.

The trends of the curves for $\theta_{\text{mag}}$ as a function of $Ca_{\text{mag}}$ for external magnetic fields applied in the x- and y-directions are similar to those reported by Cunha et al.\textsuperscript{1} for the two-dimensional analogous cases. However, we notice that for three-dimensional cases the values of $\theta_{\text{mag}}$ are considerably smaller. As shown in Fig. 6, the bulk magnetization perfectly aligns with the external magnetic field when it is in the...
z-direction, regardless of capillary or magnetic capillary numbers. For this particular case, the external magnetic field is normal to the shear plane, and so the droplet deformation caused by the shear flow does not generate asymmetries relative to any of the planes tangent to the external magnetic field direction.

In addition to the magnetization misalignment, the shape of the droplet also has a significant influence on the magnitude of the magnetization, $|\mathbf{M}|$, shown in Fig. 8. Recall that, in Eq. (20), both $\mathbf{M}$ and $\langle \mathbf{M} \rangle$ are scaled by the external magnetic field intensity $H_0$. Thus, variations in $|\mathbf{M}|$ are caused solely by changes in the droplet shape, since the permeability ratio is held constant throughout this work, with larger droplet elongations in the magnetic field direction leading to larger magnitudes of the magnetization.

The overall behavior of $|\mathbf{M}|$ has a very similar trend regardless of the external magnetic field direction, increasing with $\text{Ca}_{\text{mag}}$ regardless of magnetic field direction, as the magnetic forces stretch the droplet in the magnetic field direction. On the other hand, variations of $\text{Ca}$ have different effects on $|\mathbf{M}|$ depending on the magnetic field direction, similar to the two-dimensional analogous system. For magnetic fields in the x-direction, $|\mathbf{M}|$ increases with larger values of $\text{Ca}$, although such dependence gets weaker with larger values of $\text{Ca}_{\text{mag}}$. Given that the shear-induced droplet deformation is mostly aligned to the magnetic field direction, increasing $\text{Ca}$ leads to larger droplet elongations in the magnetic field direction, and thus to an increase in $|\mathbf{M}|$. Contrarily, for magnetic fields in the y-direction, $|\mathbf{M}|$ decreases with larger values of $\text{Ca}$, as the stronger shear stresses rotate the droplet further away from the magnetic field direction, reducing the droplet elongation in the magnetic field direction. For magnetic fields in the z-direction, the influence of different values of $\text{Ca}$ is mostly negligible, since the shear-induced deformation happens on a plane orthogonal to the magnetic field direction, and thus has only marginal effects on the droplet elongation in the field direction.
C. Magnetic torque

As discussed in Sec. IV B, the emulsion magnetization does not, in general, perfectly align to the external magnetic field, due to shear-induced effects on the droplet geometry. In addition to the purely magnetic effects previously discussed, such a misalignment creates a magnetic torque in the emulsion, given by

$$\tau_{\text{mag}} = \frac{C_{\text{mag}}}{C_0} (M) \times H_0,$$

where $\tau_{\text{mag}}$ is the magnetic torque normalized by $\eta^{\ast}$. Equation (21) shows that a misalignment between the bulk magnetization and the external magnetic field will lead to the existence of a magnetic torque, which rotates the droplet toward the external magnetic field direction. Hence, this magnetic torque is proportional to both $|M|$ and $\sin(\theta_{\text{mag}})$.

Since the magnetic torque acts in the emulsion through the ferrofluid droplets, an opposite hydrodynamic torque of the same magnitude, manifesting itself as an asymmetry in the particle stress, must act to satisfy the local conservation of angular momentum. To isolate this asymmetry, the particle stress can be split into a symmetric and an antisymmetric part, respectively, stresslet and couplet,

$$S = \text{sym}(S) + \text{asym}(S) = \text{sym}(S) + \frac{1}{2} \varepsilon \mathcal{U},$$

where the antisymmetric part can be written in terms of the Levi-Civita permutation tensor $\varepsilon$ and the dual vector, $\mathcal{U}$, with the latter defined as

$$\mathcal{U} = -\varepsilon : S.$$

The hydrodynamic torque can then be calculated as $\tau_{\text{hyd}} = \mathcal{U}$, and, ensuring the local conservation of angular momentum, we have that $\tau_{\text{mag}} + \tau_{\text{hyd}} = 0$.

Figure 9 presents the nondimensional torques, normalized by the dispersed phase volume fraction $\beta$, as a function of the magnetic capillary number and for varying capillary numbers and magnetic field directions. Black curves represent the opposite values of the magnetic torque, $-\tau_{\text{mag}}$, and blue markers represent the hydrodynamic torque $\tau_{\text{hyd}}$. Similar to what was reported by Cunha et al., we also observed a close agreement between $-\tau_{\text{mag}}$ and $\tau_{\text{hyd}}$, indicating that both our magnetization and hydrodynamic measurements are accurate.

For external magnetic fields in the $x$-direction, the magnitude of the normalized torque remains always below 1.20 for all studied cases, significantly lower than that of magnetic fields in the $y$-direction, due to the combination of geometric and magnetic aspects. As discussed in Sec. IV B, misalignment angles decrease with increasing $C_{\text{mag}}$, while the magnitude of the magnetization increases. Such characteristics, alongside the influence of $C_{\text{mag}}$ on the normalized torque definition, balance themselves out, resulting in near-constant magnetic torques for $C_{\text{mag}} > 8$, a trend that was suggested but not confirmed by Cunha et al.

To better understand the influence of $Ca$ on the normalized torque acting in the emulsion, Fig. 10 reproduces the curve in Fig. 9 for magnetic fields in the $x$-direction with an adjusted scale. The normalized torque is shown to decrease with larger values of $Ca$, due to the reduction in misalignment angle associated with stronger magnetic fields.

For external magnetic fields applied in the $y$-direction, the magnitude of the normalized torque increases continuously with larger magnetic capillary numbers, across the entire $C_{\text{mag}}$ range, similar to the behavior observed by Cunha et al. for the analogous two-dimensional system. Such a behavior is due to larger values of $C_{\text{mag}}$ leading to increases in both misalignment angle and bulk magnetization magnitude, as discussed in Sec. IV B. For $C_{\text{mag}} \leq 8$, varying capillary numbers have little influence in the normalized torque. However, variations in $Ca$ start to have an influence of normalized torque for $C_{\text{mag}} > 8$, with larger capillary numbers leading to larger normalized torques, due to the significantly larger misalignment angles. Note that, for external magnetic fields in the $x$- and $y$-directions, the magnetic torque does indeed act in the direction of aligning the droplet to the external magnetic field. This corresponds to a counterclockwise rotation in Fig. 5 and a clockwise rotation in Fig. 7.
In the case of external magnetic fields in the z-direction, there is no misalignment between the external magnetic field and the bulk magnetization, and so the resulting magnetic torques are always zero.

V. CONCLUDING REMARKS

In this work, we presented a computational analysis of the magnetic behavior of a dilute ferrofluid emulsion, consisting of a ferrofluid droplet immersed in a nonmagnetizable, Newtonian fluid under simple shear flows, with external, uniform magnetic fields applied in different directions. Such study is an extension of the previous work of Cunha et al. for the analogous two-dimensional case. Our simulations considered a single ferrofluid droplet contained in a three-dimensional domain, with finite-difference discretization and a level set method for interface capturing. The Navier–Stokes equations were solved with a projection method based on the Crank–Nicolson scheme.

We considered external magnetic fields parallel to either the main flow direction, the main velocity gradient direction, or the main vorticity direction, and analyzed the resulting magnetic behavior as a function of varying capillary and magnetic capillary numbers, as well as its dependencies on the droplet geometry. A general behavior is that the magnitude of the bulk magnetization in the emulsion increases with higher magnetic capillary numbers, due to the droplet elongation in the magnetic field direction. Although effects associated with the shear flow, represented by different capillary numbers, did influence the magnetization magnitude, such an influence was very minor.

The most interesting aspect, however, is that, in most cases, the bulk magnetization does not perfectly align to the external magnetic field, even though the ferrofluid is assumed to be superparamagnetic. Such effects are due to asymmetries in the droplet shape relative to the magnetic field direction, caused by the deformations associated with the shear flow, and thus dependent on complex interactions between viscous, capillary, and magnetic forces. This misalignment between the emulsion magnetization and the external magnetic field was found to be most prominent in the case of external magnetic fields in the main velocity gradient direction, with magnetic fields in the main flow direction resulting in significantly smaller misalignment, and magnetic fields in the main vorticity direction resulting in no misalignment at all. The fundamental reasoning for this is that, in the first case, the shear flow has very significant effects of rotating the droplet away from the magnetic field direction, while such effects are significantly smaller in the second case, and nonexistent in the third case, since the magnetic field is normal to the shear plane.

The misalignment angles for external magnetic fields in the main flow direction and main velocity gradient direction were found to be qualitatively similar to the results of Cunha et al. for the analogous two-dimensional system, although our three-dimensional results display significantly smaller misalignment angles.

Such a misalignment between the bulk magnetization and the external magnetic field, in turn, gives rise to a magnetic torque acting on the system. This then gives rise to an equal and opposite hydrodynamic torque, manifesting itself as an asymmetric stress tensor, ensuring conservation of angular momentum. We then compared torque measurements based on both magnetization and hydrodynamic stress tensor and found an excellent agreement between them. The magnetic torques were found to increase monotonically with increasing magnetic field intensities in the case of external magnetic fields in the main velocity gradient direction, with external magnetic fields in the main flow direction leading to magnetic torques that are smaller in magnitude and remain roughly constant for $C_{\text{mag}} > 8$. Since there are no misalignment angles for external magnetic fields in the main vorticity direction, no magnetic torques arise in this case.

We also found that for external magnetic fields in the main vorticity direction, the droplet can assume a distinct oblate, or disc-like shape, in contrast to the traditional prolate shape. This behavior was observed when both shear and magnetic stresses were large, such as for $Ca = 0.6$ and $C_{\text{mag}} = 16$.

In summary, we found that the magnetic behavior of a ferrofluid emulsion is far from trivial, as it is significantly dependent on the underlying droplet geometry, which is in turn dependent on complex interactions between viscous, capillary, and magnetic forces. We then analyzed the reasons behind such behaviors, generating insights that should prove useful for further studies related to ferrofluid emulsions and their eventual practical applications.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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