Journal of Algebra 574 (2021) 278–291

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The mapping class group is generated by two commutators

R. İnanç Baykur [a,*], Mustafa Korkmaz [b,*]

[a] *Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003-9305, USA*
[b] *Department of Mathematics, Middle East Technical University, 06800 Ankara, Turkey*

A R T I C L E I N F O

Article history:
Received 15 June 2020
Available online 30 January 2021
Communicated by Martin Liebeck

Keywords:
Mapping class group
Commutators
Generators of a group
Perfect groups
Symplectic group

A B S T R A C T

We show that the mapping class group of any closed connected orientable surface of genus at least five is generated by only two commutators, and if the genus is three or four, by three commutators.

1. Introduction

The mapping class group $\mathrm{Mod}(\Sigma_g)$ of a closed connected orientable surface of genus g is known to be perfect, i.e. equal to its commutator subgroup, when $g \geq 3$ [11]. We prove the following peculiar result:

Theorem 1. *The mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by two commutators if* $g \geq 5$*, and by three commutators if* $g \geq 3$*.*

* Corresponding authors.
E-mail addresses: baykur@math.umass.edu (R.İ. Baykur), korkmaz@metu.edu.tr (M. Korkmaz).

https://doi.org/10.1016/j.jalgebra.2021.01.021

Our result is clearly sharp when $g \geq 5$, for $\mathrm{Mod}(\Sigma_g)$ is not cyclic. When $g = 1$ and 2, the abelianization of $\mathrm{Mod}(\Sigma_g)$ is $\mathbb{Z}_{12}$ and $\mathbb{Z}_{10}$, respectively, so the mapping class group cannot be generated by commutators in these low genera cases.

A geometric implication of Theorem 1 is that any pair of genus $g \geq 5$ surface bundle over the circle are cobordant through a finite sequence of basic building blocks, which are fibrations over two-holed tori, prescribed by the two commutator generators and their inverses.

Since the action of mapping classes on the integral first homology group $H_1(\Sigma_g)$ of Σ_g induces an epimorphism from $\mathrm{Mod}(\Sigma_g)$ onto the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$, another immediate implication is the following:

Corollary 2. *The symplectic group* $\mathrm{Sp}(2g, \mathbb{Z})$ *is generated by two commutators if* $g \geq 5$*, and by three commutators if* $g \geq 3$*.*

Theorem 1 adds to the ever-growing literature on minimal generating sets for $\mathrm{Mod}(\Sigma_g)$; e.g. by $2g+1$ Dehn twists [3,9,6], by three involutions [1,8], or by two general elements [13,7].

It is interesting to know for which perfect groups the minimal number of generators is equal to the minimal number of commutator generators. There are numerous other groups satisfying this property. For example; any finite non-abelian simple group, such as the alternating group A_n for $n \geq 5$, is a perfect group generated by two elements, whereas by the resolution of Ore's conjecture [10], any element in such a group is a commutator. The same holds for the special linear group $SL(n, R)$ for various $n \geq 3$ and coefficient rings R, which goes back to the classical works of Thompson [12]. However, the situation is much more subtle for the mapping class group, since $\mathrm{Mod}(\Sigma_g)$, for $g \geq 3$, is not even uniformly perfect [4], i.e. there is not even a fixed positive number that any element in $\mathrm{Mod}(\Sigma_g)$ can be expressed as a product of that many commutators.

The explicit set of generators we provide for Theorem 1 consists of a finite order mapping class and an infinite order one (or two) that is a product of disjoint Dehn twists. In Section 2, we review the basic results regarding Dehn twists. The torsion elements, and their expressions as commutators, come from the symmetries of the surface, and are discussed in Section 3. Various new generating sets for $\mathrm{Mod}(\Sigma_g)$ featuring the above mapping classes are obtained in Section 4, and the proof of Theorem 1 is given in Section 5.

Acknowledgments. The first author was partially supported by The National Science Foundation Grants DMS-1510395, DMS-2005327, and The Simons Foundation Grant 634309. The second author thanks UMass Amherst for its generous support and wonderful research environment during this project.

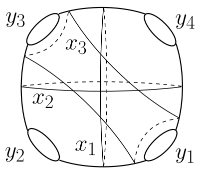

Fig. 1. The curves of the lantern relation.

2. Preliminaries

The *mapping class group* $\mathrm{Mod}(\Sigma)$ of a compact connected oriented surface Σ is the group of orientation–preserving diffeomorphisms of $\Sigma \to \Sigma$ which restrict to the identity near the boundary $\partial\Sigma$ modulo isotopies of the same type. We are primarily interested in the case when $\Sigma = \Sigma_g$, the closed surface of genus g.

We denote simple closed curves on Σ by lowercase letters such as a, b, c, d, and denote *positive (right-handed) Dehn twists* t_a, t_b, t_c, t_d about them by the corresponding capital letters A, B, C, D, all with indices. In our notation, both the curves on Σ and self-diffeomorphisms of Σ should be understood up to isotopy. We use the functional notation for the composition of diffeomorphisms (i.e. for $\phi\psi$, ψ acts on Σ first), yet we still express the *commutator of* ϕ *and* ψ as $[\phi, \psi] = \phi\psi\phi^{-1}\psi^{-1}$.

We make repeated use of the following basic relations in $\mathrm{Mod}(\Sigma)$, without referring to them explicitly: for two simple closed curves a and b on Σ, and for any $f \in \mathrm{Mod}(\Sigma)$,

- (*Conjugation*) $f t_a f^{-1} = t_{f(a)}$,
- (*Commutativity*) $AB = BA$ if a and b are disjoint,
- (*Braid relation*) $ABA = BAB$ if a and b intersect transversely at one point.

We also need

- (*Lantern relation*) for x_i, y_j the simple closed curves on the four-holed sphere in Fig. 1 (embedded in Σ), $X_1X_2X_3 = Y_1Y_2Y_3Y_4$.

All the relations above appeared in the pioneering work of Dehn [3], who then proved that $\mathrm{Mod}(\Sigma_g)$ is generated by finitely many Dehn twists. Later works of Lickorish [9] and Humphries [6] led to the following minimal collection of Dehn twist generators $\{A_i, B_j, C_k\}$ along the curves $\{a_i, b_j, c_k\}$ on Σ_g in Fig. 2.

Theorem 3. (Dehn-Lickorish-Humphries) *The mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by* $\{A_1, A_2, B_1, B_2, \ldots, B_g, C_1, C_2, \ldots, C_{g-1}\}$.

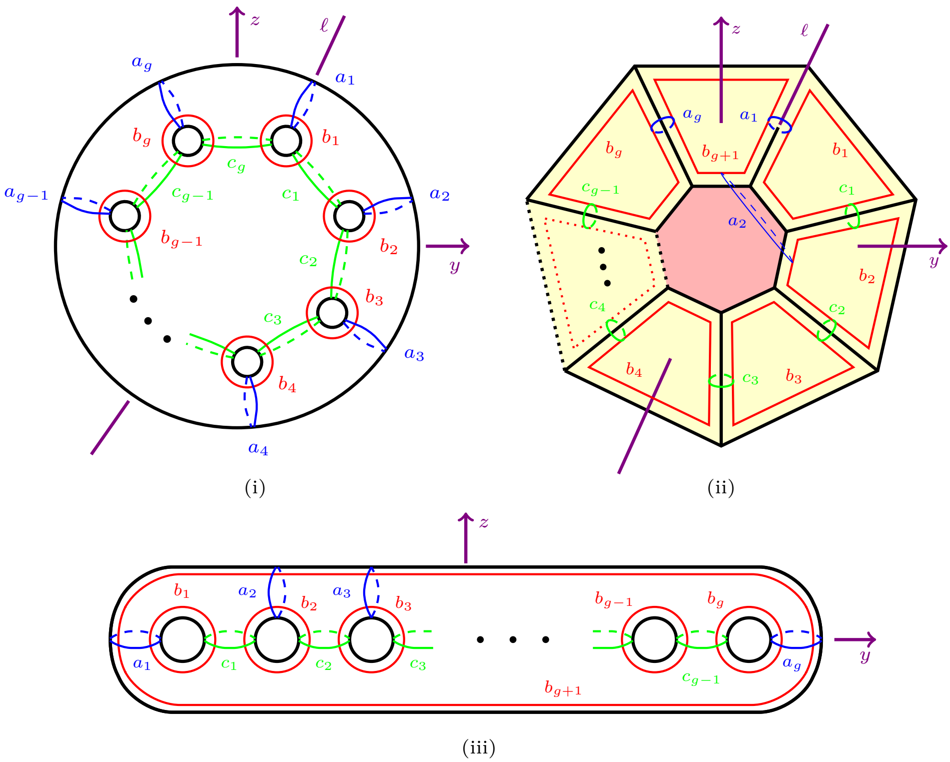

Fig. 2. The three models for the surface Σ_g. The x–axis is perpendicular to the page in all.

3. Finite order mapping classes as commutators

Consider the three different embeddings of the closed surface Σ_g in $\mathbb{R}^3$ as depicted in Fig. 2. The surface in (ii) is the boundary of the solid handlebody which consists of two thickened $(g+1)$–gons, stacked on top of each other, and $(g+1)$ solid handles joining their corresponding vertices. There are orientation-preserving diffeomorphisms between the three models which identify the curves labeled as a_i, b_j, c_k in each one. These models allow us to easily introduce and study certain torsion elements in $\mathrm{Mod}(\Sigma_g)$ coming from the symmetries of the surface. The surface Σ_g is invariant under the following maps:

- the clockwise $\frac{2\pi}{g}$–rotation R about the x–axis in Fig. 2(i),
- the clockwise $\frac{2\pi}{g+1}$–rotation S about the x–axis in Fig. 2(ii),
- the rotations ρ_1 and ρ_2 by π about the z–axis and the line ℓ, respectively, in Fig. 2(i),
- the rotations σ_1 and σ_2 by π about the z–axis and the line ℓ, respectively, in Fig. 2(ii),
- the rotations σ and h by π about the z–axis and the y–axis, respectively, in Fig. 2(iii).

Clearly, in $\mathrm{Mod}(\Sigma_g)$, the rotations R and S yield torsion elements of orders g and $g+1$, respectively, and $\rho_i, \sigma_i, \sigma, h$ yield involutions (elements of order 2), where h is a *hyperelliptic involution.* It is easy to check (say by the Alexander's method applied to the maximal chain $a_1, b_1, c_1, b_2, c_2, \ldots, c_{g-1}, b_g$) that the involutions σ and h we described on the model (iii) correspond on the model (ii) to σ_1 and the involution h_1 which interchanges the top and bottom thickened $(g+1)$–gons. Under these identifications, we define one more torsion element:

- T is the "alternating rotation" of the surface in Fig. 2(ii), prescribed as $T = Sh_1$.

Here S and h_1 commute, so $T = h_1 S$ as well. Note that T is of order $g+1$ if g is odd, and $2(g+1)$ if g is even.

Proposition 4. *In* $\mathrm{Mod}(\Sigma_g)$*, the mapping classes* R *when* $g = 2k+1$, $k \geq 1$*, and* S, h, T *when* $g = 2k$, $k \geq 1$*, are all commutators, which can be expressed as*

(1) $R = [R^{k+1}, \rho_1]$,
(2) $S = [S^{k+1}, \sigma_1]$,
(3) $h = [\sigma, P^{-(2k+1)}]$,
(4) $T = [S^{k+1}P^{2k+1}, P^{-(2k+1)}\sigma_1 P^{2k+1}]$,

where $P = A_1B_1(C_1B_2)\cdots(C_{k-1}B_k)$.

Proof. From the dihedral symmetries of the models (i) and (ii) in Fig. 2, we easily deduce that

$$R = \rho_2\rho_1 \text{ and } \rho_2 = R^{k+1}\rho_1 R^{-(k+1)}, \text{ when } g = 2k+1 \geq 3, \text{ and}$$
$$S = \sigma_2\sigma_1 \text{ and } \sigma_2 = S^{k+1}\sigma_1 S^{-(k+1)}, \text{ when } g = 2k \geq 2.$$

Therefore $R = \rho_2\rho_1 = (R^{k+1}\rho_1 R^{-(k+1)})\rho_1 = [R^{k+1}, \rho_1]$, as $\rho_1^{-1} = \rho_1$, and similarly $S = [S^{k+1}, \sigma_1]$. This proves (1) and (2).

For (3), let $g = 2k$ and let δ be the separating curve that is the intersection of Σ_g with the xz–plane in Fig. 2(iii), so that δ is the common boundary of two compact genus–k subsurfaces Σ and Σ'. The surfaces Σ and Σ' are tubular neighborhoods of the $(2k)$–chains $a_1, b_1, c_1, b_2, \ldots, c_{k-1}, b_k$ and $a_g, b_g, c_{g-1}, b_{g-1}, \ldots c_{k+1}, b_{k+1}$.

We first show that, by letting

$$P = A_1B_1C_1B_2\cdots C_{k-1}B_k$$

and

$$P' = A_gB_gC_{g-1}B_{g-1}\cdots C_{k+1}B_{k+1},$$

the hyperelliptic involution h can be expressed as

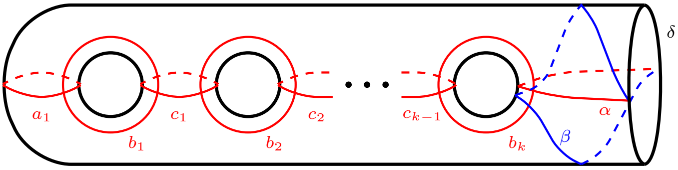

Fig. 3. The genus–k subsurface Σ with boundary δ.

$$h = P^{2k+1}(P')^{-(2k+1)} = (P')^{-(2k+1)}P^{2k+1}.$$

Note that the diffeomorphism P^{2k+1} is a π–rotation of the subsurface Σ along the y–axis, followed by isotoping the boundary back to its original position, so that its square $P^{4k+2} = t_\delta$. (The latter equality is known as the $(2k)$–*chain relation*; see e.g. [5].) This can be easily checked by the Alexander's method: P^{2k+1} maps each one of the curves $a_1, b_1, c_1, \ldots, c_{k-1}, b_k$ to itself, but with reversed orientation, whereas it maps the arc α to the arc β in Fig. 3. By the same token, $(P')^{2k+1}$ is a similar π–rotation of the subsurface Σ' along the y–axis, albeit in the opposite direction. So taking the inverse of one, as we did above, we get a π–rotation of the whole surface $\Sigma_g = \Sigma \cup_\delta \Sigma'$, which is the hyperelliptic involution h.

Next, we observe that the involution σ on Σ_g interchanges these two $2k$–chains. It follows that $P' = \sigma P \sigma^{-1}$, and therefore

$$h = (\sigma P \sigma^{-1})^{-(2k+1)}\, P^{2k+1} = \sigma P^{-(2k+1)}\sigma^{-1}P^{2k+1} = [\sigma, P^{-(2k+1)}].$$

Finally, since σ and h correspond to σ_1 and h_1 in model (ii), we have

$$\begin{aligned}
T &= Sh_1 \\
&= [S^{k+1}, \sigma_1][\sigma_1,\ P^{-(2k+1)}] \\
&= (S^{k+1}\sigma_1 S^{-(k+1)}\sigma_1^{-1})(\sigma_1 P^{-(2k+1)}\sigma_1^{-1}P^{2k+1}) \\
&= S^{k+1}\sigma_1 S^{-(k+1)}P^{-(2k+1)}\sigma_1^{-1}P^{2k+1} \\
&= S^{k+1}(P^{2k+1}P^{-(2k+1)})\sigma_1(P^{2k+1}P^{-(2k+1)})S^{-(k+1)}P^{-(2k+1)}\sigma_1^{-1}P^{2k+1} \\
&= (S^{k+1}P^{2k+1})(P^{-(2k+1)}\sigma_1 P^{2k+1})(S^{k+1}P^{2k+1})^{-1}(P^{-(2k+1)}\sigma_1 P^{2k+1})^{-1} \\
&= [S^{k+1}P^{2k+1}, P^{-(2k+1)}\sigma_1 P^{2k+1}],
\end{aligned}$$

which[1] concludes (4). □

4. New generating sets for the mapping class group

Here we obtain several new generating sets for $\mathrm{Mod}(\Sigma_g)$, focusing on generators that can be expressed as commutators (for suitable g).

[1] In general, $[x, y][y, z] = [xz^{-1}, zyz^{-1}]$ for *any* three elements x, y, z in a group (cf. [2]).

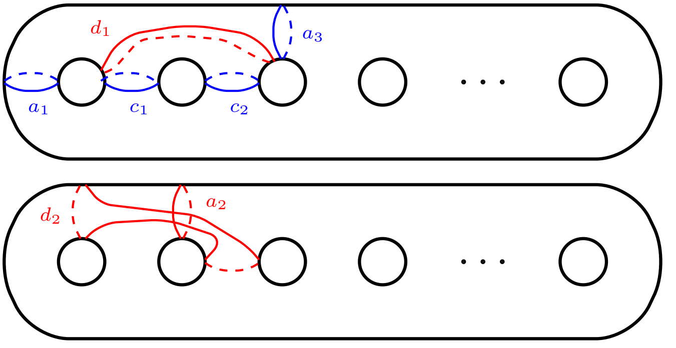

Fig. 4. The curves of the embedded lantern relation $A_1C_1C_2A_3 = A_2D_1D_2$ in Σ_g.

Lemma 5. *For $g \geq 3$, the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by* $A_1B_1^{-1}, A_2B_2^{-1}$, $B_1C_1^{-1}$, $B_iB_{i+1}^{-1}$ *and* $C_jC_{j+1}^{-1}$, *where* $1 \leq i \leq g-1$ *and* $1 \leq j \leq g-2$.

Proof. Let Γ be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set

$$\{A_1B_1^{-1}, A_2B_2^{-1}, B_1C_1^{-1}, B_iB_{i+1}^{-1}, C_jC_{j+1}^{-1}\}_{\forall i,j}.$$

Then

$$\{A_2A_1^{-1}, B_iB_j^{-1}, C_iC_j^{-1}, B_iC_j^{-1}, A_1B_i^{-1}, A_2B_i^{-1}, A_1C_j^{-1}, A_2C_j^{-1}\}_{\forall i,j} \subset \Gamma.$$

For example, we have $A_2C_1^{-1} = (A_2B_2^{-1})(B_1B_2^{-1})^{-1}(B_1C_1^{-1}) \in \Gamma$. Since $C_jC_{j+1}^{-1} \in \Gamma$, multiplying these elements in increasing index, we get $C_1C_j^{-1} \in \Gamma$. So $A_2C_j^{-1} \in \Gamma$. The others can be easily verified in a similar fashion.

By the lantern relation, the following holds in $\mathrm{Mod}(\Sigma_g)$:

$$A_1C_1C_2A_3 = A_2D_1D_2,$$

where the curves are as in Fig. 4. We can rewrite this relation as

$$A_3 = (A_2C_1^{-1})(D_1C_2^{-1})(D_2A_1^{-1}), \tag{1}$$

as C_1, C_2 and A_3 commute with all the other Dehn twists here. Note that $A_2C_1^{-1} \in \Gamma$. One can check that the diffeomorphism

$$F = (A_1B_2^{-1})(A_1C_1^{-1})(A_1C_2^{-1})(A_1B_2^{-1})$$

maps the pair (a_2, a_1) of simple closed curves to (d_2, a_1). Since $F \in \Gamma$, we have $F(A_2A_1^{-1})F^{-1} = D_2A_1^{-1} \in \Gamma$ by the conjugation relation. We also have $D_2C_2^{-1} = (D_2A_1^{-1})(A_1C_2^{-1}) \in \Gamma$.

Likewise, the diffeomorphism

$$G = (C_2B_1^{-1})(C_2A_1^{-1})(C_2C_1^{-1})(C_2B_1^{-1})$$

lies in Γ and maps the pair (d_2, c_2) of simple closed curves to (d_1, c_2). Therefore, the element $G(D_2C_2^{-1})G^{-1} = D_1C_2^{-1}$ is also in Γ, once again by the conjugation relation.

Now the equality (1) implies that the Dehn twist A_3 is in Γ. Also, $A_3(B_3B_1^{-1})A_3(B_1B_3^{-1})A_3^{-1} = B_3$ is in Γ. It now follows easily that $A_1, A_2, B_1, B_2, \dots, B_g$ and $C_1, C_2, \dots, C_{g-1}$ are all contained in Γ. We conclude from Theorem 3 that $\Gamma = \text{Mod}(\Sigma_g)$. □

We now present various new generators for the mapping class group $\text{Mod}(\Sigma_g)$ we need for our main theorem.

Theorem 6. *For $g \geq 5$, the mapping class group $\text{Mod}(\Sigma_g)$ is generated by R and $A_1A_2C_2^{-1}B_4^{-1}$.*

Proof. First note that the rotation R on Σ_g maps each a_i, b_i, c_i to $a_{i+1}, b_{i+1}, c_{i+1}$. Here, the indices are considered modulo g. The Dehn twist curves of mapping classes F_i defined below are illustrated in Fig. 5 for the case $g = 5$.

Let $F_1 := A_1A_2C_2^{-1}B_4^{-1}$ and let Γ be the subgroup of $\text{Mod}(\Sigma_g)$ generated by R and F_1. We make the following series of observations:

$$F_2 := RF_1R^{-1} = A_2A_3C_3^{-1}B_5^{-1} \in \Gamma.$$
$$F_3 := (F_2F_1)F_2(F_2F_1)^{-1} = A_2A_3B_4^{-1}B_5^{-1} \in \Gamma.$$

Let us spell out the details of this calculation, as we have several others akin to this one. It is easy to see that the diffeomorphism F_2F_1 maps the curves a_2, a_3, c_3, b_5 to a_2, a_3, b_4, b_5, respectively, so that

$$\begin{aligned} F_3 &:= (F_2F_1)F_2(F_2F_1)^{-1} \\ &= (F_2F_1)(A_2A_3C_3^{-1}B_5^{-1})(F_2F_1)^{-1} \\ &= A_2A_3B_4^{-1}B_5^{-1}. \end{aligned}$$

We then have $F_3^{-1}F_2 = B_4C_3^{-1} \in \Gamma$ and $F_3F_2^{-1} = B_4^{-1}C_3 \in \Gamma$. By conjugating these elements with powers of R, we see that

$$B_iC_{i-1}^{-1} \in \Gamma \quad \text{and} \quad B_i^{-1}C_{i-1} \in \Gamma$$

for all i.

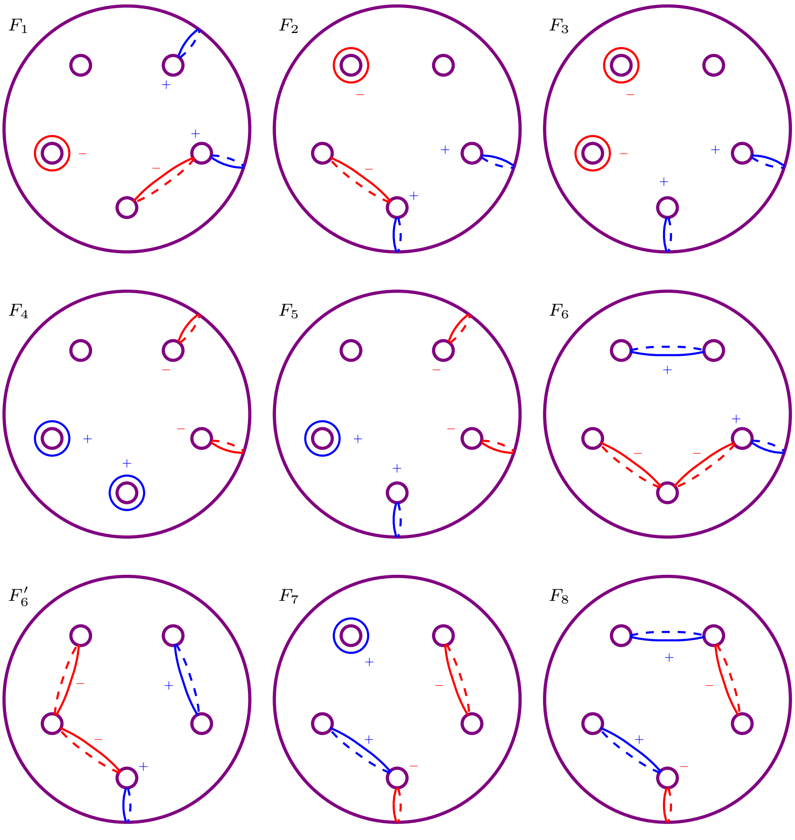

Fig. 5. The Dehn twist curves of $F_1, \ldots, F_8$ and F_6' in the proof of Theorem 6 drawn on Σ_5.

We also have

$$F_4 := R^{-1}F_3^{-1}R = A_1^{-1}A_2^{-1}B_3B_4 \in \Gamma,$$
$$F_5 := (F_4F_3)F_4(F_4F_3)^{-1} = A_1^{-1}A_2^{-1}A_3B_4 \in \Gamma.$$

Thus, $F_5F_4^{-1} = A_3B_3^{-1} \in \Gamma$ and $F_5^{-1}F_4 = A_3^{-1}B_3 \in \Gamma$. Again, by conjugating with powers of R, we conclude that

$$A_iB_i^{-1}, A_i^{-1}B_i \in \Gamma, \tag{2}$$

and in turn,

$$A_iC_{i-1}^{-1} \in \Gamma$$

as well, as we already have $B_iC_{i-1}^{-1} \in \Gamma$.

Furthermore,

$$F_6 := (C_g A_1^{-1}) F_1 (B_4 C_3^{-1}) = C_g A_2 C_2^{-1} C_3^{-1} \in \Gamma,$$
$$F_6' := R^{-4} F_6 R^4 = C_{g-4} A_{g-2} C_{g-2}^{-1} C_{g-1}^{-1} \in \Gamma,$$
$$F_7 := (F_6')^{-1} (C_{g-1}^{-1} B_g) = C_{g-4}^{-1} A_{g-2}^{-1} C_{g-2} B_g \in \Gamma$$

and

$$F_8 := (F_7 F_6) F_7 (F_7 F_6)^{-1} = C_{g-4}^{-1} A_{g-2}^{-1} C_{g-2} C_g \in \Gamma,$$

by a similar calculation to the one we had for F_3 above. From these, we get $F_7 F_8^{-1} = B_g C_g^{-1} \in \Gamma$, so that

$$B_i C_i^{-1} \in \Gamma \tag{3}$$

by the action of R.

Hence we have all of the following elements in Γ:

$$B_i B_{i+1}^{-1} = (B_i C_i^{-1})(C_i B_{i+1}^{-1}), \tag{4}$$
$$C_i C_{i+1}^{-1} = (C_i B_{i+1}^{-1})(B_{i+1} C_{i+1}^{-1}). \tag{5}$$

It follows from (2)–(5) and Lemma 5 that $\Gamma = \mathrm{Mod}(\Sigma_g)$. □

Theorem 7. *For $g \geq 3$, the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by $T, A_1 A_2^{-1}$ and $A_1 B_1 C_1 C_2^{-1} B_3^{-1} A_3^{-1}$.*

Proof. Let Γ be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by $T, A_1 A_2^{-1}$ and $F := A_1 B_1 C_1 C_2^{-1} B_3^{-1} A_3^{-1}$.

Below we repeatedly use the conjugation relation, both when conjugating with F and with powers of T. The action of T on Σ_g maps a_1 to c_1, c_i to c_{i+1} for each $i = 1, \ldots, g-2$, c_{g-1} to a_g, and a_g back to a_1, whereas it maps b_i to b_{i+1} for each $i = 1, \ldots, g$, and b_{g+1} back to b_1.

Note that $F(a_2) = a_2$. Since $F(a_1) = b_1$ and $F(b_1) = c_1$,

$$F(A_1 A_2^{-1}) F^{-1} = B_1 A_2^{-1} \in \Gamma$$

and

$$F(B_1 A_2^{-1}) F^{-1} = C_1 A_2^{-1} \in \Gamma.$$

It follows that

$$A_1 B_1^{-1}, B_1 C_1^{-1} \in \Gamma.$$

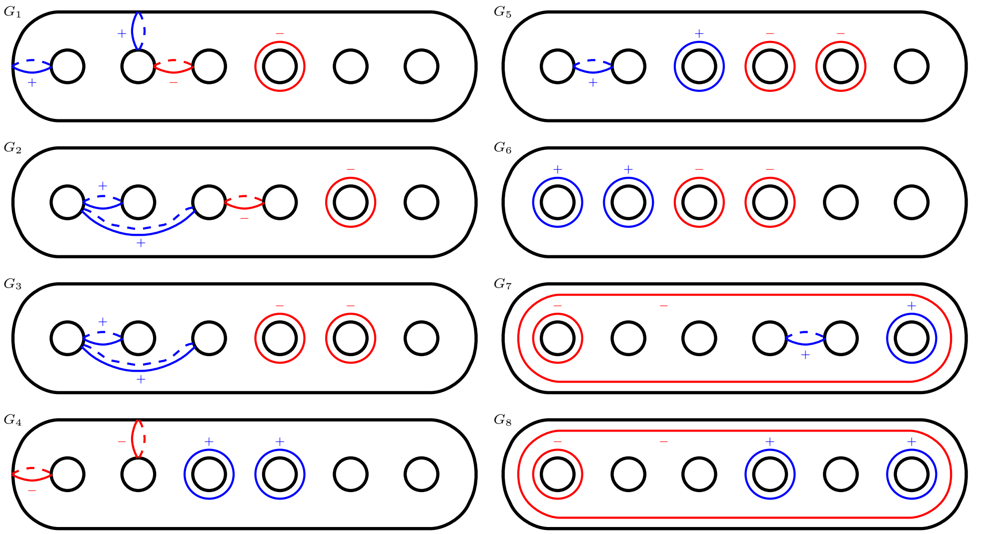

Fig. 6. The Dehn twist curves of $G_1, \ldots, G_8$ in the proof of Theorem 8 drawn on Σ_6.

Hence, the elements

$$\begin{aligned}
C_1B_2^{-1} &= T(A_1B_1^{-1})T^{-1},\\
A_2B_2^{-1} &= (A_2C_1^{-1})(C_1B_2^{-1}),\\
B_2C_2^{-1} &= T(B_1C_1^{-1})T^{-1},\\
C_1C_2^{-1} &= (C_1B_2^{-1})(B_2C_2^{-1}),\\
B_1B_2^{-1} &= (B_1C_1^{-1})(C_1B_2^{-1}),\\
C_jC_{j+1}^{-1} &= T^{j-1}(C_1C_2^{-1})T^{-(j-1)} \text{ for } 1 \leq j \leq g-2,\\
B_iB_{i+1}^{-1} &= T^{i-1}(B_1B_2^{-1})T^{-(i-1)} \text{ for } 1 \leq i \leq g-1
\end{aligned}$$

are all in Γ. It follows now from Lemma 5 that $\Gamma = \mathrm{Mod}(\Sigma_g)$. □

Theorem 8. *For $g \geq 6$, the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by T and $A_1A_2C_2^{-1}B_4^{-1}$.*

Proof. Let Γ be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by T and $G_1 := A_1A_2C_2^{-1}B_4^{-1}$. The Dehn twist curves of mapping classes G_i defined in this proof are illustrated in Fig. 6 for the case $g = 6$.

Let $d = T(a_2)$.[2] We then have

$$G_2 := TG_1T^{-1} = C_1DC_3^{-1}B_5^{-1} \in \Gamma.$$

[2] To view the action of T (specifically on the curve a_2) in the "standard" model as in Fig. 6, one can observe that $S = (A_1B_1C_1B_2C_2 \cdots B_{g-1}C_{g-1}B_gA_g)^2$.

Since G_2G_1 maps the curve c_3 to b_4 and fixes c_1, d, b_5, the conjugation of G_2 with G_2G_1 gives[3]

$$\begin{aligned} G_3 &:= (G_2G_1)G_2(G_2G_1)^{-1} \\ &= C_1DB_4^{-1}B_5^{-1} \in \Gamma. \end{aligned}$$

Hence, the subgroup Γ contains the elements $G_2G_3^{-1} = C_3^{-1}B_4$ and $G_2^{-1}G_3 = C_3B_4^{-1}$. Thus, by conjugating by powers of T, we see that

$$A_1B_1^{-1} \in \Gamma \text{ and } C_iB_{i+1}^{-1} \in \Gamma \tag{6}$$

for $1 \le i \le g-1$.

We also have

$$G_4 := G_1^{-1}(C_2^{-1}B_3) = A_1^{-1}A_2^{-1}B_3B_4 \in \Gamma$$

and

$$G_5 := (G_3G_4)G_3(G_3G_4)^{-1} = C_1B_3B_4^{-1}B_5^{-1} \in \Gamma$$

by a similar calculation to that of G_3 above. From this we get

$$G_3G_5^{-1} = DB_3^{-1} \in \Gamma$$

and hence

$$T^{-1}(DB_3^{-1})T = A_2B_2^{-1} \in \Gamma. \tag{7}$$

Let

$$G_6 := T^{-1}(B_2C_1^{-1})G_5T = B_1B_2B_3^{-1}B_4^{-1} \in \Gamma$$

and

$$G_7 := (T^3G_5T^{-3}) = C_4B_6B_7^{-1}B_8^{-1} \in \Gamma.$$

Here, we take $B_8 = B_1$ if $g = 6$. It is then easy to see that

$$G_8 := (G_7G_6^{-1})G_7(G_7G_6^{-1})^{-1} = B_4B_6B_7^{-1}B_8^{-1} \in \Gamma.$$

[3] Here we use the fact that a_2 is disjoint from $d = T(a_2)$, which would *not* be the case for $S(a_2)$. This is essentially the reason why we preferred to work with the slightly more complicated torsion element T rather than S.

We then have

$$G_8G_7^{-1} = B_4C_4^{-1} \in \Gamma.$$

Thus, by the action of T, we get

$$B_iC_i^{-1} \in \Gamma \tag{8}$$

for all i. As in the proof of Theorem 6, we obtain that

$$B_iB_{i+1}^{-1}, C_iC_{i+1}^{-1} \in \Gamma. \tag{9}$$

Once again from (6)–(9) and Lemma 5, we conclude $\Gamma = \mathrm{Mod}(\Sigma_g)$. □

5. Proof of the main theorem

The proof of our main theorem now follows easily from the array of results we have obtained thus far.

Proof of Theorem 1. We obtain our sharpest results in four cases:

$\underline{g \geq 5 \text{ is odd}}$: By Theorem 6, the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by R and $A_1A_2C_2^{-1}B_4^{-1}$. By Proposition 4, R is a single commutator. On the other hand, there is clearly a diffeomorphism ϕ of Σ_g mapping the pair (a_1, a_2) to (b_4, c_2), so that

$$\begin{aligned} A_1A_2C_2^{-1}B_4^{-1} &= A_1A_2(B_4C_2)^{-1} \\ &= A_1A_2(\phi A_1A_2\phi^{-1})^{-1} \\ &= A_1A_2\phi(A_1A_2)^{-1}\phi^{-1} \\ &= [A_1A_2, \phi]. \end{aligned}$$

$\underline{g \geq 6 \text{ is even}}$: By Theorem 8, the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by T and $A_1A_2C_2^{-1}B_4^{-1}$. Again T is a commutator by Proposition 4, and $A_1A_2C_2^{-1}B_4^{-1} = [A_1A_2, \phi]$ as above.

$\underline{g = 3}$: By similar arguments we had in Section 4, one can show that when $g \geq 3$, $\mathrm{Mod}(\Sigma_g)$ is generated by the elements $R, A_1A_2^{-1}$, and $A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1}$; see Theorem 6 in [8]. Once again R is a commutator by Proposition 4. Clearly there is a diffeomorphism ψ of Σ_3 mapping a_1 to a_2 and a diffeomorphism φ mapping (a_1, b_1, c_1) to (a_3, b_3, c_2). It follows that $A_1A_2^{-1} = [A_1, \psi]$ and $A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1} = [A_1B_1C_1, \varphi]$.

$\underline{g = 4}$: By Theorem 7, $\mathrm{Mod}(\Sigma_4)$ is generated by the three elements $T, A_1A_2^{-1}$ and $A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1}$. Once again T is a commutator by Proposition 4 and so are the other two elements, as we have argued above.

This completes the proof of Theorem 1. □

References

[1] T.E. Brendle, B. Farb, Every mapping class group is generated by 6 involutions, J. Algebra 278 (2004) 187–198.
[2] R.D. Carmichael, Introduction to the Theory of Groups of Finite Order, Dover Publications, New York, 1956.
[3] M. Dehn, The group of mapping classes, in: Papers on Group Theory and Topology, Springer-Verlag, 1987;
Translated from the German by J. Stillwell, Die Gruppe der Abbildungsklassen, Acta Math. 69 (1938) 135–206.
[4] H. Endo, D. Kotschick, Bounded cohomology and non-uniform perfection of mapping class groups, Invent. Math. 144 (1) (2001) 169–175.
[5] B. Farb, D. Margalit, A Primer on Mapping Class Groups, Princeton University Press, 2011.
[6] S. Humphries, Generators for the mapping class group, in: Topology of Low-Dimensional Manifolds, Proc. Second Sussex Conf., Chelwood Gate, 1977, in: Lecture Notes in Math., vol. 722, Springer-Verlag, 1979, pp. 44–47.
[7] M. Korkmaz, Generating the surface mapping class group by two elements, Trans. Am. Math. Soc. 357 (2005) 3299–3310.
[8] M. Korkmaz, Mapping class group is generated by three involutions, Math. Res. Lett. 27 (4) (2020) 1095–1108.
[9] W.B.R. Lickorish, A finite set of generators for the homeotopy group of a 2–manifold, Proc. Camb. Philos. Soc. 60 (1964) 769–778.
[10] M.W. Liebeck, E.A. O'Brien, A. Shalev, P.H. Tiep, The Ore conjecture, J. Eur. Math. Soc. 12 (4) (2010) 939–1008.
[11] J. Powell, Two theorems on the mapping class group of a surface, Proc. Am. Math. Soc. 68 (3) (1978) 347–350.
[12] R.C. Thompson, Commutators in the special and general linear groups, Trans. Am. Math. Soc. 101 (1961) 16–33.
[13] B. Wajnryb, Mapping class group of a surface is generated by two elements, Topology 35 (1996) 377–383.