J. Math. Anal. Appl. 492 (2020) 124457

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Effective noncommutative Nevanlinna-Pick interpolation in the row ball, and applications

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ARTICLE INFO

Article history:
Received 20 May 2020
Available online 4 August 2020
Submitted by J.A. Ball

Keywords:
Nevanlinna-Pick interpolation
Free semigroup algebra
Noncommutative function theory

ABSTRACT

We provide an effective single-matrix criterion, in terms of what we call the *elementary Pick matrix*, for the solvability of the noncommutative Nevanlinna-Pick interpolation problem in the row ball, and provide some applications. In particular we show that the so-called “column-row property” fails for the free semigroup algebras, in stark contrast to the analogous commutative case. Additional applications of the elementary Pick matrix include a local dilation theorem for matrix row contractions and interpolating sequences in the noncommutative setting. Finally we present some numerical results related to the failure of the column-row property.

1. Introduction

1.1. The purpose of this paper is to give an effective solution of the so-called “noncommutative Nevanlinna-Pick interpolation problem” in the row ball, which is an analog, in the modern setting of noncommutative function theory, of the classical Nevanlinna-Pick interpolation problem. The main result is the construction of a single matrix, in closed form, such that the problem has a solution if and only if this matrix is positive semidefinite. In this introductory section we pose the problem and describe some of the applications of our solution.

1.2. Noncommutative Pick interpolation in the row ball

We work in the general setting of noncommutative function theory, as laid out e.g. in [10]. Fix an integer $d \geq 1$. For each $n = 1, 2, 3, \ldots$, let $\mathcal{M}_n^d$ denote the set of d-tuples of $n \times n$ matrices with complex entries:

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1 Research Supported by NSF grant DMS-1900364.

https://doi.org/10.1016/j.jmaa.2020.124457

$$\mathcal{M}_n^d = \{X = (X_1, \dots, X_d) : X_i \in \mathcal{M}_n\}$$

and let $\mathcal{M}^d$ be the disjoint union of the $\mathcal{M}_n^d$ over all n (When $d = 1$ we drop the superscripts and just write $\mathcal{M}_n, \mathcal{M}$). Let $\mathcal{M}_{s\times t}$ denote the set of $s \times t$ matrices with complex entries. By the *row ball* $\mathcal{B}^d$ we mean the graded subset of $\mathcal{M}_d$, defined at each "level" n by

$$\mathcal{B}_n^d = \{X = (X_1, \dots, X_d) \in \mathcal{M}_n^d : \|X_1X_1^* + \dots + X_dX_d^*\| < 1\} \subset \mathcal{M}_n^d.$$

The row ball $\mathcal{B}^d$ is a prototypical example of an *nc domain*; this means that (1) at each level, the set $\mathcal{B}_n^d \subset \mathcal{M}_n^d$ is open, (2) $\mathcal{B}^d$ respects direct sums, i.e. if $X \in \mathcal{B}_n^d$ and $Y \in \mathcal{B}_m^d$ then $X \oplus Y \in \mathcal{B}_{m+n}^d$ (here the direct sum means coordinatewise direct sum: $X \oplus Y = (X_1 \oplus Y_1, \dots, X_d \oplus Y_d)$; and (3) $\mathcal{B}^d$ respects unitary equivalence, i.e. if $U \in \mathcal{M}$ is a unitary matrix and $X = (X_1, \dots, X_d) \in \mathcal{B}_n^d$ then $U^*XU = (U^*X_1U, \dots, U^*X_dU) \in \mathcal{B}_n^d$.

The nc-domain $\mathcal{B}^d$ then supports *nc-functions*, which are graded functions $f : \mathcal{B}^d \to \mathcal{M}$ (that is, a family of functions $f_n : \mathcal{B}_n^d \to \mathcal{M}_n$, $n = 1, 2, 3, \dots$ which (1) respect direct sums: for $X \in \mathcal{B}_n^d$, $Y \in \mathcal{B}_m^d$, we have $f_{m+n}(X \oplus Y) = f_n(X) \oplus f_m(Y)$; and (2) respect similarities, in the sense that if $X \in \mathcal{B}^d$ and S is a similarity such that $S^{-1}XS$ is also in $\mathcal{B}^d$, then $f(S^{-1}XS) = S^{-1}f(X)S$. Let

$$H^\infty(\mathcal{B}^d) = \{f : \mathcal{B}^d \to \mathcal{M} : f \text{ is an nc function and } \sup_{X \in \mathcal{B}^d} \|f(X)\| < \infty\}.$$

We refer to the supremum in this definition as the H^∞ norm of the nc function f, denoted $\|f\|_\infty$.

The *noncommutative Nevanlinna-Pick interpolation problem* in the row ball is the following (see [4] and the references therein): given a finite set of points ("nodes") $X^1, \dots, X^m$ in $\mathcal{B}^d$, with $X^j \in \mathcal{B}_{n_j}^d$, and matrices $Y^1, \dots Y^m$, with $Y^j \in \mathcal{M}_{n_j}$, find an interpolating function $f \in H^\infty(\mathcal{B}^d)$ (if it exists)

$$f(X^j) = Y^j \quad j = 1, \dots m \tag{1.1}$$

of minimal H^∞ norm. The fact that the domain $\mathcal{B}^d$ and the nc functions f respect direct sums means that every such problem can be immediately reduced to a "one-point problem": putting $X = \oplus X^j$ and $Y = \oplus Y^j$, the problem (1.1) has a solution if and only if the one-point problem

$$f(X) = Y, \tag{1.2}$$

has a solution, and the minimal norms are the same. Instead of asking for the minimal norm, one could pose the essentially equivalent problem of asking whether or not there exists a solution of norm $\|f\|_\infty \le 1$. It is also possible to consider a generalized problem in which the single $n \times n$ matrix Y is replaced by an $s \times t$ block matrix $(Y_{ij}), i = 1, \dots, s; j = 1, \dots, t$, where each Y_{ij} is an $n \times n$ matrix. We then seek an $s \times t$ matrix of nc functions $F = (f_{ij})$ so that

$$f_{ij}(X) = Y_{ij} \quad i = 1, \dots, s; j = 1, \dots, t$$

and the H^∞ norm of the $s \times t$ matrix nc function F is the evident supremum norm.

When $d = 1$ and all the X^j, Y^j are 1×1 matrices this reduces to the classical Nevanlinna-Pick interpolation problem in the unit disk. In that case, interpolating functions always exists (e.g. one can take a Lagrange interpolating polynomial), so the problem is just one of finding the minimal H^∞ norm. However in the noncommutative setting solutions need not always exist; a necessary and sufficient condition for a solution of the one-point problem (1.2) is that the matrix Y belong to the subalgebra of $\mathcal{M}_n$ generated by the coordinates $X_1, \dots, X_d$ of the point X.

Consider for a moment the classical Nevanlinna-Pick interpolation problem: given points $x^1, \ldots, x^m$ in the open unit disk, and complex numbers $y^1, \ldots, y^m$, does there exist an analytic function f, bounded by 1 in the disk, with

$$f(x^j) = y^j, \quad j = 1, \ldots, m? \tag{1.3}$$

The problem has a solution if and only if the *Pick matrix*

$$P = \left(\frac{1 - y^i \overline{y^j}}{1 - x^i \overline{x^j}} \right)_{i,j=1}^m$$

is positive semidefinite. It turns out that it is also possible to give a necessary and sufficient condition for the existence of a norm-one solution of the noncommutative problem (1.2) in terms of a single matrix involving the data X, Y, this was given by Ball, Marx, and Vinnikov in [4]; however the single matrix in question is expressed as an infinite sum and does not have a readily apparent closed form. The main result of the present paper is to present a closed-from expression for this "noncommutative Pick matrix," which is amenable at least in some cases to machine computation, thus providing an effective solution to the problem which is numerically stable for suitably conditioned data. We construct this closed form expression in Section 3, the key idea is a matrix involution introduced previously in [12] in connection with the problem of determining the algebra generated by a family of matrices $X_1, \ldots, X_d$ (which is connected to the interpolation problem, as remarked above). Finally, we note that the interpolation problem considered here can also be understood as a special case of an interpolation problem in Hardy algebras over C^*-correspondences as considered in the work of Muhly and Solel (in particular the interpolation theorem [11, Theorem 5.3]), though we have not attempted to interpret the objects of the present paper in this formalism.

1.3. Failure of the column-row property in $\mathcal{L}^d$

Let $\mathcal{H}$ be a Hilbert space, $B(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$, and fix a subset $\mathcal{A} \subseteq B(\mathcal{H})$. For each fixed $n \geq 1$, we define C_n to be the least number C_n such that the inequality

$$\Big\| \sum_{i=1}^n A_i A_i^* \Big\|^{1/2} \leq C_n \Big\| \sum_{i=1}^n A_i^* A_i \Big\|^{1/2}$$

holds for all n-tuples $A_1, \ldots, A_n$ of elements from $\mathcal{A}$. The **column-row constant** of $\mathcal{A}$ is the least number C such that

$$\Big\| \sum_{i=1}^\infty A_i A_i^* \Big\|^{1/2} \leq C \Big\| \sum_{i=1}^\infty A_i^* A_i \Big\|^{1/2}$$

for all sequences $(A_i)_{i=1}^\infty$ from $\mathcal{A}$ for which the sums are SOT-convergent. Evidently the C_n form an increasing sequence with $\lim C_n = C$; it is possible that $C = \infty$. If C is finite, we say that $\mathcal{A}$ has the **column-row property**. (One could analogously define a row-column property but this will not concern us here.) For example, $\mathcal{A} = M_n(\mathbb{C})$ has column-row constant at least equal to $\sqrt{n}$. (To see this, let E_{ij} denote the standard $n \times n$ matrix units; putting $A_i = E_{1i}$ for $i = 1, \ldots, n$ one checks easily that $\| \sum_{i=1}^n A_i^* A_i \| = 1$ while $\| \sum_{i=1}^n A_i A_i^* \| = n$.) It is also easy to verify that for any set of operators $\mathcal{A}$, we have $C_n \leq \sqrt{n}$ for every n.

Of particular interest is the case when $\mathcal{A}$ is the algebra of bounded multiplication operators on a reproducing kernel Hilbert space. In this setting a number of important spaces are known to have this property.

Trivially, the algebra $\mathcal{A} = H^\infty(\mathbb{D})$ (the algebra of bounded analytic functions in the unit disk $\mathbb{D}$, equipped with the supremum norm) has the column-row property. Beyond this, the multiplier algebra of the Dirichlet space $\mathcal{D}$ over the unit disk has the column-row property with constant $C \le \sqrt{18}$ [18], and the multiplier algebras of the *Drury-Arveson spaces* H^2_d over the unit ball $\mathbb{B}^d \subset \mathbb{C}^d$ (denoted $Mult(H^2_d)$) have the column-row property with constants $C = C(d)$; [3], in the proof given in [3] the obtained estimates on the constants $C(d)$ grow to infinity with the dimension d. The Dirichlet space and the H^2_d spaces are particular examples of spaces with a *complete Nevanlinna-Pick (CNP) kernel*, the column-row property (when it holds) turns out to have important consequences in such spaces, e.g. in applications to interpolating sequences [2] and in factorization of weak products [9], [3]. Very recently, M. Hartz has shown that the column-row property holds in all CNP spaces, with constant 1 [8].

The connection with the present paper is as follows: it turns out that the multiplier algebras of H^2_d can be viewed as the "commutative collapse" of the so-called *free semigroup algebras* $\mathcal{L}_d$, $d \ge 2$. (We refer to the survey [6] for the basic facts about the free semigroup algebras.) One may then ask if an analog of the column-row property holds for these algebras. In detail, if we let $\mathbb{F}^+_d$ denote the free semigroup of all noncommuting words in d letters $\{1, 2, \dots, d\}$, (including the "empty word" $\varnothing$), then we can form a Hilbert space $\mathcal{F}^2_d$ with orthonormal basis $\{\xi_w\}_{w \in \mathbb{F}^+_d}$. For each letter i we define an operator

$$L_i \xi_w = \xi_{iw}, \quad w \in \mathbb{F}^+_d.$$

The operators L_i are isometries with orthogonal ranges, i.e. we have $L_i^* L_j = \delta_{ij} I$ for $i, j = 1, \dots, d$. The free semigroup algebra is the WOT-closed algebra generated by the L_i, $i = 1, \dots, d$.

By a result of Salomon, Shalit, and Shamovich [16, Theorem 3.1] the free semigroup algebra $\mathcal{L}_d$ may be completely isometrically identified with the algebra $H^\infty(\mathcal{B}^d)$ of bounded nc functions in the row ball. Moreover the map $f \to f(z)$ obtained by restricting an nc function to level 1 (the scalar unit ball $\mathbb{B}^d \subset \mathbb{C}^d$) is a completely contractive homomorphism from $H^\infty(\mathcal{B}^d)$ onto the multiplier algebra $Mult(H^2_d)$, (see [17, Theorem 4.4.1, Subsection 4.9] or [5, Section 2]; this latter reference makes clear the connection with Nevanlinna-Pick interpolation).

In particular, we observe that for each d, and n, the column-row constants C_n for $H^\infty(\mathcal{B}^d)$ dominate the corresponding constants for $Mult(H^2_d)$. The question naturally arises of whether or not the free semigroup algebras $H^\infty(\mathcal{B}^d)$ have the column-row property. It turns out they do not; in fact we will prove the constant is infinity for $H^\infty(\mathcal{B}^d)$, and the constant $C_n = \sqrt{n}$.

Theorem 1.1. *For the algebra of bounded nc functions in the row ball, $H^\infty(\mathcal{B}^d)$, $d \ge 2$, we have $C_n = \sqrt{n}$ for all $n = 1, 2, \dots$.*

Thus, in contrast to $Mult(H^2_d)$, the column-row property fails in $H^\infty(\mathcal{B}^d)$ in the strongest possible way, establishing a stark contrast between the commutative multiplier algebras $Mult(H^2_d)$ and their noncommutative "parents." Theorem 1.1 is proved in Section 6.

1.4. Readers' guide

Section 2 gives a definition of the ψ-involution first introduced in [12]. We use the ψ-involution liberally throughout Section 3 to first construct for a (contractive) matrix tuple $X = (X_1, \dots, X_d) \subset \mathcal{M}^d_n$ its elementary Pick matrix: a matrix P_X whose range encodes the unital subalgebra of $\mathcal{M}_n$ generated by $X_1, \dots, X_d$. This in turn is used to establish two of the main results of the paper: Theorem 3.5 and Theorem 3.9.

Section 4 consists of several technical results leading up to the construction of an isometry in Section 5 and its immediate use in Theorem 5.2, a so-called "mini-dilation."

In Section 6 we apply Theorem 3.9 to prove Theorem 1.1: the column-row property fails for the Fock space on two or more generators.

Section 7 gives a more concrete approach to the results in Section 6. Section 8 introduces a condition number for a matrix tuple $X = (X_1, \dots, X_d)$ and explores its properties and interpolating sequences. Finally, Section 9 discusses computational consequences of effective NP-interpolation.

2. Preliminaries. The ψ involution and its properties

If $A \in \mathcal{M}_{n\times m}$ and $B \in \mathcal{M}_{r\times s}$ then their **Kronecker product** $A \otimes B \in M_{nr\times ms}$ is the block matrix given by

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nm}B \end{pmatrix}. \tag{2.1}$$

Or, in other words, $(A \otimes B)_{n(i-1)+k,n(j-1)+\ell} = a_{i,j}b_{k,\ell}$.

Let $\boldsymbol{\tau} : \mathcal{M} \to \mathcal{M}$ be the transpose operator and let $\mathbf{vec} : \mathcal{M}_n \to \mathcal{M}_{n^2\times 1}$ be the linear map taking the columns of a matrix and stacking them to get a column vector:

$$\mathbf{vec}\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \\ a_{12} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

We have the classical identity

$$\mathbf{vec}(AXB) = (B^T \otimes A)\,\mathbf{vec}(X). \tag{2.2}$$

Typically we treat $\mathbf{vec}$ as a graded function on $\mathcal{M}$. That is, $\mathbf{vec} = (\mathbf{vec}[n])_{n=1}^{\infty}$, where each $\mathbf{vec}[n] : \mathcal{M}_n \to \mathcal{M}_{n^2\times 1}$, and if $A \in \mathcal{M}_n$, then $\mathbf{vec}(A) = \mathbf{vec}[n](A)$. This greatly simplifies notation.

Definition 2.1. Define $\psi : \mathcal{M}_{n^2} \to \mathcal{M}_{n^2}$ to be $\psi = (\boldsymbol{\tau} \circ \mathbf{vec}) \otimes \mathbf{vec}$. If $A \in \mathcal{M}_{n^2}$ then we write the evaluation of ψ on A as

$$A^\psi = \big[(\boldsymbol{\tau} \circ \mathbf{vec}) \otimes \mathbf{vec}\big](A).$$

Or, more explicitly, if $C, D \in \mathcal{M}_n$ then

$$[C \otimes D]^\psi = \mathbf{vec}(C)^T \otimes \mathbf{vec}(D) = \mathbf{vec}(D)\,\mathbf{vec}(C)^T. \tag{2.3}$$

Indeed, observe

$$[A\otimes B]^\psi = [(\boldsymbol{\tau} \circ \mathbf{vec})\otimes \mathbf{vec}](A\otimes B) = \mathbf{vec}(A)^T \otimes \mathbf{vec}(B) = \mathbf{vec}(B)\,\mathbf{vec}(A)^T.$$

Our motivation for writing ψ as a superscript is that ψ is an involution on $\mathcal{M}_{n^2}$:

Lemma 2.2. *For any E_{ij} and $E_{k\ell}$ we have*

$$[E_{ij} \otimes E_{k\ell}]^\psi = E_{\ell j} \otimes E_{ki}. \tag{2.4}$$

Consequently, ψ is an involution.

Proof. We have the following equalities:

$$\begin{aligned}[E_{ij} \otimes E_{k\ell}]^{\psi} &= \mathbf{vec}(E_{k\ell})\,\mathbf{vec}(E_{ij})^T \\ &= E_{n(\ell-1)+k,n(j-1)+i} \\ &= E_{\ell j} \otimes E_{ki}.\end{aligned}$$

Evidently applying ψ again gives us back $E_{ij} \otimes E_{k\ell}$. Therefore ψ is an involution. □

Proposition 2.3 *(ψ modularity). If $U \in \mathcal{M}_{n^2}$ and $A, B, C, D \in \mathcal{M}_n$ then*

$$\Big[(A \otimes B)U(C \otimes D)\Big]^{\psi} = (D^T \otimes B)U^{\psi}(C \otimes A^T).$$

Proof. First we recall that if u, v are column vectors, then $u^T \otimes v = vu^T$. We first prove the result for $U = E_{ij} \otimes E_{k\ell}$. Using (2.2) and (2.3), we have

$$\begin{aligned}\Big[(A \otimes B)(E_{ij} \otimes E_{k\ell})(C \otimes D)\Big]^{\psi} &= \Big[(AE_{ij}C) \otimes (BE_{k\ell}D)\Big]^{\psi} \\ &= \mathbf{vec}(BE_{k\ell}D)(\mathbf{vec}(AE_{ij}C))^T \\ &= \big[(D^T \otimes B)\,\mathbf{vec}(E_{k\ell})\big]\,\big[(C^T \otimes A)\,\mathbf{vec}(E_{ij})\big]^T \\ &= (D^T \otimes B)\,\mathbf{vec}(E_{k\ell})\,\mathbf{vec}(E_{ij})^T(C \otimes A^T) \\ &= [D^T \otimes B][E_{ij} \otimes E_{k\ell}]^{\psi}[C \otimes A^T].\end{aligned}$$

Since ψ is linear and the $E_{ij} \otimes E_{k\ell}$ form a basis for $\mathcal{M}_{n^2}$, we are done. □

Remark 2.4. We could just as easily use Equation (2.4) as the definition of the ψ-involution. The ψ-involution was introduced by the third named author in [12], where its key properties (including the modularity property) were described; we have included proofs here for the sake of convenience. What we will call the *elementary kernel matrix*, defined in the next section, also appears in [12].

3. Noncommutative Pick interpolation and the matrix P_X

Recall $X = (X_1, \ldots, X_d) \in \mathcal{M}_n^d$ is a **row contraction** if $[X_1 \quad \ldots \quad X_d]$ has norm strictly less than 1:

$$\Big\| \sum_{i=1}^{d} X_i X_i^* \Big\| < 1.$$

Definition 3.1. For $X = (X_1, \ldots, X_d) \in \mathcal{M}_n^d$, we put

$$P_X := \Big[(I_n \otimes I_n - \sum_{i=1}^{d} \overline{X_i} \otimes X_i)^{-1}\Big]^{\psi}$$

(when it is defined). If X is a row contraction, then it follows from [13, Proposition 3.1] that the spectral radius of $\sum_{i=1}^{d} \overline{X_i} \otimes X_i$ is strictly less than 1, so P_X exists. In this case we call P_X the **elementary kernel matrix**. (It may be thought of as an analog, in our setting, of the ordinary Szegő kernel function $k(x, x) = (1 - \overline{x}x)^{-1}$ at a single point x in the unit disk.)

Definition 3.2. For $\mathbb{x} = \{x_1, \ldots, x_d\}$, a set of freely noncommuting indeterminates, let $\langle \mathbb{x} \rangle = \langle x_1, \ldots, x_d \rangle$ denote the unital free semigroup generated by $x_1, \ldots, x_d$ with empty product $\varnothing$ acting as the identity. If $w = i_1 i_2 \ldots i_n$ is a word in the letters $\{1, 2, \ldots, d\}$ we write

$$x^w := x_{i_1} x_{i_2} \cdots x_{i_n}.$$

In particular, for a system of matrices $X = (X_1, \ldots, X_d)$ and a word w we write

$$X^w := X_{i_1} X_{i_2} \cdots X_{i_n}.$$

Thus, when X is a row contraction we can express P_X as a norm-convergent power series

$$P_X = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{d} \overline{X_i} \otimes X_i \right)^n = \sum_{w \in \langle \mathbb{x} \rangle} \overline{X}^w \otimes X^w.$$

Suppose now X is a row contraction. We recall the one-point nc Pick interpolation problem from the introduction: given $Y = (Y_{i,j}) \in \mathcal{M}_{s \times t} \otimes \mathcal{M}_n$, does there exist an nc function $f \in \mathcal{M}_{s \times t} \otimes H^\infty(\mathcal{B}^d)$ such that $\|f\|_\infty \leq 1$ and $f(X) = Y$?

From [4, Theorem 6.5], this problem has a solution if and only if the map $\Phi : \mathcal{M}_n \to \mathcal{M}_{ns}$

$$\Phi(H) = \sum_{w \in \langle \mathbb{x} \rangle} (X^w H X^{w*}) \otimes I_s - Y \left(\sum_{w \in \langle \mathbb{x} \rangle} (X^w H X^{w*}) \otimes I_t \right) Y^*$$

is completely positive. Our goal is to recast this condition in terms of the elementary kernel matrix P_X introduced above. To do this we first apply Choi's criterion to reduce the problem of checking the complete positivity of Φ to checking the positivity of a single matrix. We then use the ψ involution to express this single matrix in closed form.

Definition 3.3. For each n, the **Choi Matrix** is the matrix

$$\mathfrak{C}_n = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij} \in \mathcal{M}_{n^2} \tag{3.1}$$

By Choi's Theorem (see e.g. [15, Theorem 3.14]), a map $\Phi : \mathcal{M}_n \to \mathcal{M}_m$ is completely positive if and only if the single $nm \times nm$ matrix

$$(I_n \otimes \Phi)(\mathfrak{C}_n) = \sum_{i,j} E_{ij} \otimes \Phi(E_{ij})$$

is positive semidefinite.

The Choi Matrix also has the following important relation with ψ:

$$[I_{n^2}]^\psi = \mathfrak{C}_n, \tag{3.2}$$

as is trivially verified using (2.4) and (3.1).

Lemma 3.4. *If $X = (X_1, \ldots, X_d) \in \mathcal{M}_n^d$ is a row contraction then*

$$\sum_{w \in \langle \mathbb{x} \rangle} (I \otimes X)^w \mathfrak{C}_n (I \otimes X)^{w*} = P_X.$$

Proof. Since X is a row contraction, the series is norm convergent. Using the fact that ψ is an involution, the modularity property (Proposition 2.3), and the action of ψ on the Choi matrix (3.2), we have

$$\begin{aligned}\sum_{w\in\langle \mathbb{x}\rangle}(I\otimes X)^w\mathfrak{C}_n(I\otimes X)^{w*} &= \left[\textstyle\sum_{w\in\langle \mathbb{x}\rangle}[(I\otimes X^w)\mathfrak{C}_n(I\otimes X^{w*})]^\psi\right]^\psi\\ &= \left[\textstyle\sum_{w\in\langle \mathbb{x}\rangle}(\overline{X}^w\otimes X^w)I_{n^2}(I\otimes I)\right]^\psi\\ &= \left[\textstyle\sum_{w\in\langle \mathbb{x}\rangle}(\overline{X}\otimes X)^w\right]^\psi\\ &= \left[\left(I\otimes I-\textstyle\sum_i\overline{X_i}\otimes X_i\right)^{-1}\right]^\psi\\ &= P_X \quad \square\end{aligned}$$

Theorem 3.5. *Suppose $X=(X_1,\dots,X_d)\in\mathcal{M}_n^d$ is a row contraction and $Y=(Y_{i,j})_{i,j=1}^{s,t}\in\mathcal{M}_{s\times t}\otimes\mathcal{M}_n$ is an $s\times t$ block matrix with $n\times n$ blocks. There exists an nc function $f\in\mathcal{M}_{s\times t}\otimes H^\infty(\mathcal{B}^d)$ such that $\|f\|\le 1$ and $f(X)=Y$ if and only if*

$$P_X\otimes I_s-(I_n\otimes Y)(P_X\otimes I_t)(I_n\otimes Y^*)\succeq 0.$$

Proof. Let $\Phi:\mathcal{M}_n\to\mathcal{M}_{ns}$ be the operator defined by

$$\Phi(H)=\sum_{w\in\langle \mathbb{x}\rangle}(X^wHX^{w*})\otimes I_s-Y\left(\sum_{w\in\langle \mathbb{x}\rangle}(X^wHX^{w*})\otimes I_t\right)Y^*.$$

Next observe

$$\begin{aligned}\sum_{i,j}E_{ij}\otimes\sum_w X^wE_{ij}X^{w*}\otimes I_s &= \sum_w\sum_{i,j}(E_{ij}\otimes I)(I\otimes X^wE_{ij}X^{w*})\otimes I_s\\ &= \sum_w\sum_{i,j}(I\otimes X^w)(E_{ij}\otimes E_{ij})(I\otimes X^{w*})\otimes I_s\\ &= \sum_w(I\otimes X)^w\mathfrak{C}_n(I\otimes X)^{w*}\otimes I_s\\ &= P_X\otimes I_s,\end{aligned}$$

where the last equality uses Lemma 3.4. Hence,

$$(I_n\otimes\Phi)(\mathfrak{C}_n)=P_X\otimes I_s-(I_n\otimes Y)(P_X\otimes I_t)(I_n\otimes Y^*).$$

Thus, Choi's Theorem tells us Φ is completely positive if and only if $P_X\otimes I_s-(I_n\otimes Y)(P_X\otimes I_t)(I_n\otimes Y^*)\succeq 0$. Finally, as already noted, [4, Theorem 6.5] says that Φ is completely positive if and only if there is a solution to the interpolation problem. This completes the proof. $\square$

We now turn an essentially equivalent version of the interpolation problem: if X is a row contraction and Y is given, find the minimal norm of a solution f to the interpolation problem $f(X)=Y$. First of all, we must note that there may not be any f with $f(X)=Y$; this will happen if and only if the blocks of Y belong to the subalgebra generated by $X_1,\dots,X_d$. (The sufficiency of this condition is trivial; necessity follows e.g. from a result of Agler and McCarthy [1, Lemma 3.2] which says that if there is some bounded f

with $f(X) = Y$, then there is a polynomial with $p(X) = Y$. In this respect see also [16, Lemma 4.4], which connects the condition $Y \in \text{alg}_X$ to the "full envelope" condition in [4].) By the main Theorem of [12], we know that a matrix Z is in the algebra generated by X if and only if $\mathbf{vec}(Z) \in \text{ran}(P_X)$. When each block of Y belongs to this algebra, then there will exist an nc polynomial matrix f with $f(X) = Y$.

We will define the NP-norm of Y to be the minimal norm of a solution to $f(X) = Y$, and show how to compute this minimal norm using P_X.

Definition 3.6. Suppose $X = (X_1, \dots, X_d) \in \mathcal{B}^d \subset \mathcal{M}_n^d$ and $Y \in \mathcal{M}_{s\times t} \otimes \mathcal{M}_n$. We define the NP($X$) norm of Y to be

$$\|Y\|_{\text{NP}(X)} := \inf_{\substack{f \in M_{s\times t}\otimes H^\infty(\mathcal{B}^d)\\ f(X)=Y}} \|f\|_{H^\infty} \tag{3.3}$$

and note that implicitly we consider only nc functions f. Moreover, if $\| \sum_{i=1}^d X_i X_i^* \| = 1$, so that X lies in the boundary of $\mathcal{B}^d$ at level n, we define the ANP(X) norm of Y (the asymptotic NP(X) norm at the boundary point X) as

$$\|Y\|_{\text{ANP}(X)} := \lim_{t \nearrow 1} \|Y\|_{\text{NP}(tX)}. \tag{3.4}$$

We note that the NP(X) norm could be equivalently defined by taking the infimum just over nc polynomial matrices f in the expression (3.3).

We will compute $\|Y\|_{\text{NP}(X)}$ for all Y whose blocks are in the algebra generated by X, and $\|Y\|_{\text{ANP}(X)}$ for a special subclass of boundary points X, which will be useful in applications. We first introduce some notation and make some elementary observations.

Definition 3.7. If $X = (X_1, \dots, X_d) \in \mathcal{M}_n^d$ then let alg_X denote the unital subalgebra of $\mathcal{M}_n$ generated by $X_1, \dots, X_d$.

Note:

- If $X = (X_1, \dots, X_d) \in \mathcal{M}_n^d$ is a row contraction, then P_X is self-adjoint and positive semi-definite, so $P_X^{1/2}$ exists, and we let ${P_X}^{\dagger/2}$ denote the Moore-Penrose pseudoinverse of $P_X^{1/2}$.
- Note ${P_X}^{\dagger/2} P_X^{1/2} = P_X^{1/2} {P_X}^{\dagger/2} = Q_X$, where Q_X is the projection onto $\text{ran}(P_X) = \mathbf{vec}(\text{alg}_X)$.
- The matrices P_X and $P_X^{1/2}$ are invertible if and only if $\text{alg}_X = \mathcal{M}_n$. In this case, ${P_X}^{\dagger/2} = P_X^{-1/2}$.

Corollary 3.8. *Suppose $Y \in \mathcal{M}_{s\times t} \otimes \text{alg}_X$. If*

$$^P Y^P = ({P_X}^{\dagger/2} \otimes I_s)(I_n \otimes Y)(P_X^{1/2} \otimes I_t)$$

then

$$\|Y\|_{\text{NP}(X)} = \|{}^P Y^P\|.$$

Proof. We begin by multiplying the main equation in Theorem 3.5 on the left and right by $({P_X}^{\dagger/2} \otimes I_s)$:

$$\begin{aligned} Q_X \otimes I_s &\succeq ({P_X}^{\dagger/2} \otimes I_s)(I_n \otimes Y)(P_X \otimes I_t)(I_n \otimes Y^*)({P_X}^{\dagger/2} \otimes I_s) \\ &= {}^P Y^P ({}^P Y^P)^* \end{aligned}$$

Suppose $c > 0$. By considering the interpolation problem for $c^{-1}Y$ instead of Y, it follows from above that there exists $f \in \mathcal{M}_{s\times t}\otimes H^\infty(\mathcal{B}^d)$ such that $\|f\| = c$ and $f(X) = Y$ if and only if ${}^PY^P({}^PY^P)^* \preceq c^2 Q_X \otimes I_s$ if and only if $\|{}^PY^P\| \leq c$. If there exists $f \in \mathcal{M}_{s\times t}\otimes H^\infty(\mathcal{B}^d)$ such that $\|f\| = c$ and $f(X) = Y$, then $\|Y\|_{\mathrm{NP}(X)} \leq c$. Hence, $\|{}^PY^P\| \leq c$ implies $\|Y\|_{\mathrm{NP}(X)} \leq c$.

On the other hand, if $\|Y\|_{\mathrm{NP}(X)} \leq c$ then for each $\varepsilon > 0$ there exists $f_\varepsilon \in \mathcal{M}_{s\times t}\otimes H^\infty(\mathcal{B}^d)$ such that $\|f_\varepsilon\| = c + \varepsilon$ and $f_\varepsilon(X) = Y$. However, this implies $\|{}^PY^P\| \leq c + \varepsilon$ for all $\varepsilon > 0$, hence $\|{}^PY^P\| \leq c$. Thus, $\|{}^PY^P\| \leq c$ if and only if $\|Y\|_{\mathrm{NP}(X)} \leq c$. Therefore, $\|Y\|_{\mathrm{NP}(X)} = \|{}^PY^P\|$. □

Theorem 3.9. *Let $X = (X_1, \ldots, X_d) \in M_n(\mathbb{C})^d$ be a row co-isometry (that is, $\sum_{i=1}^d X_i X_i^* = I$). If the algebra generated by $X_1, \ldots, X_d$ is all of $M_n(\mathbb{C})$ then*

$$\lim_{t\to 1}(P_{tX}^{-1/2}\otimes I_s)(I_n\otimes Y)(P_{tX}^{1/2}\otimes I_t) = I_n\otimes Y$$

and consequently

$$\|Y\|_{\mathrm{ANP}(X)} = \lim_{t\to 1}\|Y\|_{\mathrm{NP}(tX)} = \|Y\|.$$

Proof. We claim $\frac{1-t^2}{t^2}P_{tX} \to \overline{W}\otimes I_n$ as $t \to 1$, for some positive definite matrix $W \in M_n(\mathbb{C})$. If the claim is true, then

$$\begin{aligned}\lim_{t\to 1}(P_{tX}^{-1/2}\otimes I_s)(I_n\otimes Y)(P_{tX}^{1/2}\otimes I_t) &= (\overline{W}^{-1/2}\otimes I_n\otimes I_s)(I_n\otimes Y)(\overline{W}^{1/2}\otimes I_n\otimes I_t)\\ &= I_n\otimes Y\end{aligned}$$

from which we conclude, by Corollary 3.8, that $\|Y\|_{\mathrm{ANP}(X)} = \|Y\|$. Thus, it is sufficient to prove the claim.

Consider $T = \sum_{i=1}^d \overline{X_i}\otimes X_i$. Using the identity (2.2) we see

$$T\,\mathbf{vec}(I) = \mathbf{vec}\left(\sum_{i=1}^d X_i X_i^*\right) = \mathbf{vec}(I),$$

that is, $\mathbf{vec}(I)$ is an eigenvector for T with eigenvalue 1. Since X is a row contraction and the algebra generated by $X_1, \ldots, X_d$ is all of $M_n(\mathbb{C})$, it follows from the quantum Perron-Frobenius theorem of Evans and Høegh-Krohn [7] that the spectral radius of T is equal to 1, and the (generalized) eigenspace corresponding to 1 is one dimensional. (For a treatment of this result more tailored to the present application, see [13, Theorem 5.4].) Let $\mathbf{vec}(W)$ be the corresponding left eigenvector to 1 (that is, $\mathbf{vec}(W)^*T = \mathbf{vec}(W)^*$), normalized so that $\mathbf{vec}(W)^*\,\mathbf{vec}(I) = 1$. From [13, Theorem 5.4] and the remarks following it, the matrix W must be positive definite. (This conclusion again relies on the fact that $X_1, \ldots, X_d$ generate all of $\mathcal{M}_n$.) Thus, we may decompose T as

$$T = G + B$$

where $G = \mathbf{vec}(I)\,\mathbf{vec}(W)^*$, and B is the remainder $T - G$. It follows that B has spectral radius less than or equal to 1, and that 1 is not an eigenvalue of B. Next, we see that

$$G^2 = \mathbf{vec}(I)\,\mathbf{vec}(W)^*\,\mathbf{vec}(I)\,\mathbf{vec}(W)^* = \mathbf{vec}(I)\,\mathbf{vec}(W)^* = G.$$

Moreover, since $\mathbf{vec}(I)$ is a right eigenvector to 1 and $\mathbf{vec}(W)$ is a left eigenvector to 1, we also have

$$TG = G = GT$$

and consequently $GB = BG = 0$. In particular, $T^n = (G + B)^n = G^n + B^n = G + B^n$ and

$$(I - t^2T)^{-1} = \sum_{n=0}^{\infty}(t^2T)^n = I + \sum_{n=1}^{\infty}(t^{2n}G + t^{2n}B^n)$$
$$= I + \frac{t^2}{1-t^2}G + t^2B(I - t^2B)^{-1}.$$

Thus,

$$P_{tX} = [(1 - t^2T)^{-1}]^{\psi} = \left[1 + \frac{t^2}{1-t^2}G + t^2B(1 - t^2B)^{-1}\right]^{\psi}.$$

Since, as noted above, $r(B) \leq 1$ and 1 is not an eigenvalue of B, we have

$$\lim_{t\to 1}(1 - t^2)B(1 - t^2B)^{-1} = 0.$$

Taking the limit of $\frac{1-t^2}{t^2}P_{tX}$ as $t \to 1$, we obtain

$$\lim_{t\to 1}\frac{1-t^2}{t^2}P_{tX} = G^{\psi} = [\mathbf{vec}\, I(\mathbf{vec}\, W)^*]^{\psi} = \overline{W} \otimes I_n,$$

which finishes the proof of the claim, and hence the theorem. □

We restate the above theorem in terms of the condition of the interpolation problem in Section 8.

4. The Boomerang matrix

In this section we collect some calculations which will be useful in the next section.

Definition 4.1. Define $\breve{B} \in \mathcal{M}_{n^3\times n}$ to be

$$\breve{B} = \sum_{i,j=1}^{n}\mathbf{vec}(E_{ij}) \otimes E_{ij} = \sum_{i,j=1}^{n} e_i \otimes e_j \otimes E_{ij}$$

The matrix $\breve{B}$ is known as the **Boomerang matrix**.

Lemma 4.2. *If $C \in \mathcal{M}_n$, then*

$$(C \otimes I_n \otimes I_n)\breve{B} = (I_n \otimes I_n \otimes C^T)\breve{B}. \tag{4.1}$$

Moreover, if $A \in \mathcal{M}_{n^2}$ and $D \in \mathcal{M}_n$ then

$$\breve{B}^T(A \otimes CD)\breve{B} = \breve{B}^T([(C^T \otimes I)A(D^T \otimes I)] \otimes I)\breve{B}. \tag{4.2}$$

Proof. We prove the first item for $C = E_{k\ell}$ and then extend linearly to all of $\mathcal{M}_n$. Note

$$(E_{k\ell} \otimes I \otimes I)\breve{B} = \sum_{i,j}^{n}(E_{k\ell} \otimes I \otimes I)(e_i \otimes e_j \otimes E_{ij}) = \sum_{j}^{n} e_k \otimes e_j \otimes E_{\ell j}$$

$$
\begin{aligned}
&= \sum_{i,j}^{n} e_i \otimes e_j \otimes E_{\ell k} E_{ij} = \sum_{i,j}^{n} (I \otimes I \otimes E_{\ell k})(e_i \otimes e_j \otimes E_{ij}) \\
&= (I \otimes I \otimes E_{\ell k})\breve{B}.
\end{aligned}
$$

Extending linearly we have Equation (4.1).

Now suppose $A \in \mathcal{M}_{n^2}$ and $D \in \mathcal{M}_n$. Observe

$$
\begin{aligned}
(A \otimes D)\breve{B} = (A \otimes I)(I \otimes I \otimes D)\breve{B} &= (A \otimes I)(D^T \otimes I \otimes I)\breve{B} \\
&= ([A(D^T \otimes I)] \otimes I)\breve{B},
\end{aligned}
$$

and by taking transposes we have

$$
\breve{B}^T(A \otimes C) = \breve{B}^T([(C^T \otimes I)A] \otimes I).
$$

Using (4.1) finally yields

$$
\begin{aligned}
\breve{B}^T(A \otimes CD)\breve{B} &= \breve{B}^T(A \otimes C)(I \otimes I \otimes D)\breve{B} \\
&= \breve{B}^T([(C^T \otimes I)A(D^T \otimes I)] \otimes I)\breve{B}. \quad \square
\end{aligned}
$$

Lemma 4.3. *For any row contraction $X \in \mathcal{M}_n^d$,*

$$
P_X - \sum_{i=1}^{d} (X_i^T \otimes I_n) P_X (\bar{X}_i \otimes I_n) = [I_{n^2}]^\psi = \mathfrak{C}_n,
$$

where $\mathfrak{C}_n = \sum_{i,j=1}^n E_{ij} \otimes E_{ij}$ is the Choi matrix.

Proof. Observe by Proposition 2.3,

$$
\begin{aligned}
P_X - \textstyle\sum_{i=1}^d (X_i^T \otimes I_n) P_X (\bar{X}_i \otimes I_n) &= P_X - \left[\textstyle\sum_{i=1}^d (I \otimes I) P_X^\psi (\bar{X}_i \otimes X_i)\right]^\psi \\
&= \left[(P_X)^\psi\right]^\psi - \left[P_X^\psi \textstyle\sum_{i=1}^d (\bar{X}_i \otimes X_i)\right]^\psi \\
&= \left[[P_X]^\psi (I - \textstyle\sum_{i=1}^d \bar{X}_i \otimes X_i)\right]^\psi \\
&= \left[(I - \textstyle\sum_{i=1}^d \bar{X}_i \otimes X_i)^{-1} (I - \textstyle\sum_{i=1}^d \bar{X}_i \otimes X_i)\right]^\psi \\
&= [I_{n^2}]^\psi = \mathfrak{C}_n. \quad \square
\end{aligned}
$$

Lemma 4.4. *Suppose $X \in \mathcal{M}_n^d$ is a row contraction. If $H \in \mathcal{M}_n$ then*

$$
\breve{B}^T \left(P_X \otimes \left(H - \textstyle\sum_{i=1}^d X_i H X_i^*\right)\right) \breve{B} = H.
$$

Proof. We begin by computing the left hand side for a fixed i:

$$
\begin{aligned}
\breve{B}^T(P_X \otimes (H - X_i H X_i^*))\breve{B} &= \breve{B}^T(P_X \otimes H)\breve{B} - \breve{B}^T(P_X \otimes X_i H X_i^*)\breve{B} \\
&= \breve{B}^T(P_X \otimes H)\breve{B} - \breve{B}^T([(X_i^T \otimes I)P_X(\bar{X}_i \otimes I)] \otimes H)\breve{B} \\
&= \breve{B}^T([P_X - (X_i^T \otimes I)P_X(\bar{X}_i \otimes I)] \otimes H)\breve{B}.
\end{aligned}
$$

Summing over i and applying Lemma 4.3 implies

$$\breve{B}^T \left(P_X \otimes \left(H - \sum_{i=1}^d X_i H X_i^*\right)\right) \breve{B} = \breve{B}^T (\mathfrak{C}_n \otimes H)\breve{B}.$$

Using the definition of the Choi matrix we finish the computation:

$$\begin{aligned}
\breve{B}^T(\mathfrak{C}_n \otimes H)\breve{B} &= \sum (e_{i_1}^T \otimes e_{j_1}^T \otimes E_{j_1 i_1})(\mathfrak{C}_n \otimes H)(e_{i_2}T \otimes e_{j_2} \otimes E_{i_2 j_2}) \\
&= \sum (e_{i_1}^T \otimes e_{j_1}^T)\mathfrak{C}_n(e_{i_2} \otimes e_{j_2}) \otimes E_{j_1 i_1} H E_{i_2 j_2} \\
&= \sum e_{i_1}^T E_{k\ell} e_{i_2} \otimes e_{j_1}^T E_{k\ell} e_{j_2} \otimes E_{j_1 i_1} H E_{i_2 j_2} \\
&= \sum_{k,\ell} 1 \otimes 1 \otimes E_{kk} H E_{\ell\ell} \\
&= H. \quad \square
\end{aligned}$$

Definition 4.5. Let $s, t \in \mathbb{Z}_+$ and let $\mathcal{Q}_{n,s} \in \mathcal{M}_{ns}$ denote the permutation matrix such that

$$\mathcal{Q}_{n,s}(U \otimes V)\mathcal{Q}_{n,s}^T = V \otimes U$$

for all $U \in \mathcal{M}_n$ and $V \in \mathcal{M}_s$. In particular, if $W \in \mathcal{M}_{n\times m}$ and $Z \in \mathcal{M}_{t\times s}$ then

$$\mathcal{Q}_{n,t}(W \otimes Z)\mathcal{Q}_{m,s}^T = Z \otimes W.$$

For each $r \geq 1$, we define the $n^3 r \times nr$ matrix

$$\breve{B}_r = \left(\sum_{i,j} e_i \otimes e_j \otimes I_r \otimes E_{ij}\right) \mathcal{Q}_{n,r}$$

to be the **ampliated boomerang matrix**.

Proposition 4.6. *Suppose $A \in \mathcal{M}_{n^2}$, $C \in \mathcal{M}_n$, $Z \in \mathcal{M}_{nt\times ns}$ and $W \in \mathcal{M}_{ns\times nt}$. We have the following identities;*

$$\big[(A \otimes I_t)(I_n \otimes Z) \otimes C\big]\breve{B}_s = \big[A \otimes I_t \otimes C\big]\breve{B}_t Z \tag{4.3}$$

and

$$\breve{B}_s^T \big[(I_n \otimes W)(A \otimes I_t) \otimes C\big] = W\breve{B}_t^T \big[A \otimes I_t \otimes C\big]. \tag{4.4}$$

If, in addition, $J, K \in \mathcal{M}_{ns}$, then

$$\breve{B}_s^T \big[(I_n \otimes J)(A \otimes I_s)(I_n \otimes K) \otimes C\big]\breve{B}_s = J\breve{B}_s^T \big[A \otimes I_s \otimes C\big]\breve{B}_s K. \tag{4.5}$$

Furthermore, the ampliated boomerang matrix satisfies the ampliated versions of Equations (4.1) and (4.2).

Proof. Suppose $A \in \mathcal{M}_{n^2}$ and $C \in \mathcal{M}_n$. Let $\mathcal{E}_{k\ell}$ be the $t \times s$ matrix with a 1 in the k,ℓ-entry and zeros elsewhere. We prove Equation (4.3) with $E_{pq} \otimes \mathcal{E}_{k\ell}$ first:

$$\begin{aligned}[(A\otimes I_t)(I_n\otimes E_{pq}\otimes\mathcal{E}_{k\ell})\otimes C]\breve{B}_s &= \textstyle\sum_{ij}\left[A(I_n\otimes E_{pq})\otimes\mathcal{E}_{k\ell}\otimes C\right]\left[e_i\otimes e_j\otimes I_s\otimes E_{ij}\right]\mathcal{Q}_{n,s}\\ &= \textstyle\sum_{ij}[A(e_i\otimes E_{pq}e_j)\otimes\mathcal{E}_{k\ell}\otimes CE_{ij}]\mathcal{Q}_{n,s}\\ &= \textstyle\sum_{i}[A(e_i\otimes e_p)\otimes\mathcal{E}_{k\ell}\otimes CE_{iq}]\mathcal{Q}_{n,s}\\ &= \textstyle\sum_{ij}[A(e_i\otimes e_j)\otimes\mathcal{E}_{k\ell}\otimes CE_{ij}E_{pq}]\mathcal{Q}_{n,s}\\ &= \textstyle\sum_{ij}(A\otimes I_t\otimes C)\left(e_i\otimes e_j\otimes I_t\otimes E_{ij}\right)(\mathcal{E}_{k\ell}\otimes E_{pq})\mathcal{Q}_{n,s}\\ &= (A\otimes I_t\otimes C)(\breve{B}_t\mathcal{Q}_{n,t}^T)(\mathcal{E}_{k\ell}\otimes E_{pq})\mathcal{Q}_{n,s}\\ &= (A\otimes I_t\otimes C)(\breve{B}_t)(E_{pq}\otimes\mathcal{E}_{k\ell}).\end{aligned}$$

Thus, with linearity and by taking adjoints we have Equations (4.3) and (4.4).

For Equation (4.5), set $t = s$ and combine Equations (4.3) and (4.4):

$$\breve{B}_s^T\left[A\otimes JK\otimes C\right]\breve{B}_s = J\breve{B}_s^T[A\otimes I_s\otimes C]\breve{B}_sK.$$

Finally, the ampliated versions of Equations (4.1) and (4.2) follow readily from adapting their proofs. □

Proposition 4.7. *Suppose $X \in \mathcal{M}_n^d$ is a row contraction and $H \in \mathcal{M}_n$. An ampliated version of Lemma 4.4 is satisfied:*

$$\breve{B}_s^T\left[P_X\otimes I_s\otimes\left(H-\textstyle\sum_{i=1}^d X_iHX_i^*\right)\right]\breve{B}_s = H\otimes I_s.$$

Proof. This follows from adapting the proof of Lemma 4.4. □

5. Popescu mini-dilations

Once more, we suppose $X = (X_1,\dots,X_d) \in \mathcal{M}_n^d$ is a row-contraction. Set $\Delta_X = I_n - \sum_{i=1}^d X_iX_i^*$ and observe that both P_X and Δ_X are self-adjoint and positive semi-definite.

Define $\mathcal{V}_X = (P_X^{1/2}\otimes I_n\otimes\Delta_X^{1/2})\breve{B}_n \in \mathcal{M}_{n^4\times n^2}$ and remark that Proposition 4.7 with $H = I_n$ implies $\mathcal{V}_X$ is an isometry: $\mathcal{V}_X^*\mathcal{V}_X = I_{n^2}$.

Recall ${P_X}^{\dagger/2}$ is the pseudoinverse of $P_X^{1/2}$ and Q_X is the projection onto $\mathbf{vec}(\mathrm{alg}_X)$, where alg_X is the unital algebra generated by $X_1,\dots,X_d$.

Lemma 5.1. *If $W \in \mathrm{alg}_X$ then*

$$Q_X(I_n\otimes W)P_X = (I_n\otimes W)P_X \quad \textit{and} \quad P_X(I_n\otimes W^*)Q_X = P_X(I_n\otimes W^*).$$

Proof. We begin by taking $v \in \mathcal{M}_{n^2}$ and recalling that $\mathrm{ran}(P_X) = \mathbf{vec}(\mathrm{alg}_X)$, hence $P_Xv = \mathbf{vec}(V)$, for some $V \in \mathrm{alg}_X$. Moreover, since W is also in alg_X, it follows that $WV \in \mathrm{alg}_X$ and $Q_X\,\mathbf{vec}(WV) = \mathbf{vec}(WV)$. Thus,

$$\begin{aligned}Q_X(I_n\otimes W)P_Xv &= Q_X(I_n\otimes W)\,\mathbf{vec}(V) = Q_X\,\mathbf{vec}(WV)\\ &= \mathbf{vec}(WV) = (I_n\otimes W)\,\mathbf{vec}(V)\\ &= (I_n\otimes W)P_Xv,\end{aligned}$$

allowing us to conclude that $Q_X(I_n\otimes W)P_X = (I_n\otimes W)P_X$. Taking adjoint shows

$$P_X(I_n\otimes W^*)Q_X = P_X(I_n\otimes W^*). \quad \square$$

Theorem 5.2. *Suppose* $\alpha, \beta \in \mathbb{C}\langle x\rangle$. *If* $\tilde{X} = {P_X}^{\dagger/2}(I_n \otimes X)P_X^{1/2}$ *then*

$$\mathcal{V}_X^*\big(\alpha(\tilde{X})\beta(\tilde{X})^* \otimes I_{n^2}\big)\mathcal{V}_X = \alpha(X)\beta(X)^* \otimes I_n.$$

Proof. Observe that Lemma 5.1 implies that $\alpha(\tilde{X}) = {P_X}^{\dagger/2}(I_n \otimes \alpha(X))P_X^{1/2}$. Take $W, Z \in \text{alg}_X$ and set

$$\tilde{W} = {P_X}^{\dagger/2}(I_n \otimes W)P_X^{1/2} \quad \text{and} \quad \tilde{Z} = {P_X}^{\dagger/2}(I_n \otimes Z)P_X^{1/2}.$$

Note $\tilde{Z}^* = P_X^{1/2}(I_n{\otimes}Z^*){P_X}^{\dagger/2}$ and applying Lemma 5.1 once more implies

$$\begin{aligned} P_X^{1/2}\tilde{W}\tilde{Z}^*P_X^{1/2} &= P_X^{1/2}{P_X}^{\dagger/2}(I_n{\otimes}W)P_X^{1/2}P_X^{1/2}(I_n{\otimes}Z^*){P_X}^{\dagger/2}P_X^{1/2} \\ &= Q_X(I_n{\otimes}W)P_X(I_n{\otimes}Z^*)Q_X \\ &= (I_n{\otimes}W)P_X(I_n{\otimes}Z^*). \end{aligned}$$

Using Proposition 4.6 we have the following chain of equalities:

$$\begin{aligned} \mathcal{V}_X^*(\tilde{W}\tilde{Z}^*{\otimes}I_{n^2})\mathcal{V}_X &= \left(\left(P_X^{1/2}{\otimes}I_n{\otimes}\Delta_X^{1/2}\right)\breve{B}_n\right)^*(\tilde{W}\tilde{Z}^*{\otimes}I_{n^2})\left(\left(P_X^{1/2}{\otimes}I_n{\otimes}\Delta_X^{1/2}\right)\breve{B}_n\right) \\ &= \breve{B}_n^T\left(P_X^{1/2}\tilde{W}\tilde{Z}^*P_X^{1/2}{\otimes}I_n{\otimes}\Delta_X\right)\breve{B}_n \\ &= \breve{B}_n^T\big((I_n{\otimes}W)P_X(I_n{\otimes}Z^*){\otimes}I_n{\otimes}\Delta_X\big)\breve{B}_n \\ &= \breve{B}_n^T\big((I_n{\otimes}W{\otimes}I_n)(P_X{\otimes}I_n)(I_n{\otimes}Z^*{\otimes}I_n){\otimes}\Delta_X\big)\breve{B}_n \\ &= (W{\otimes}I_n)\breve{B}_n^T\left(P_X{\otimes}I_n{\otimes}\Delta_X\right)\breve{B}_n(Z^*{\otimes}I_n) \\ &= WZ^*{\otimes}I_n. \end{aligned}$$

Since $\alpha(X), \beta(X) \in \text{alg}_X$, setting $W = \alpha(X)$ and $Z = \beta(X)$ finishes the proof. □

6. Proof of Theorem 1.1

Proof. It suffices to prove the theorem in the case $d = 2$. Fix n and choose a pair of $n \times n$ matrices $X = (X_1, X_2)$ such that $X_1X_1^* + X_2X_2^* = I_n$ and X_1, X_2 generate all of $\mathcal{M}_n$ as a unital algebra. (A construction of such a pair is given in the next section, see Example 7.1(a).) Consider the $n \times n$ matrices

$$Y_1 = E_{11}, \quad Y_2 = E_{12}, \quad \dots, \quad Y_n = E_{1n}.$$

Put

$$Y_{col} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}, \quad Y_{row} = [Y_1 \quad Y_2 \quad \cdots \quad Y_n],$$

then $\|Y_{col}\| = \|\sum_{i=1}^n Y_i^*Y_i\|^{1/2} = 1$ and $\|Y_{row}\| = \|\sum_{i=1}^n Y_iY_i^*\|^{1/2} = \sqrt{n}$.

Let $0 < \epsilon < 1$. By Theorem 3.9, for all t sufficiently close to 1 we have both

$$\|Y_{col}\|_{\text{NP}(tX)} < (1+\epsilon) \quad \text{and} \quad \|Y_{row}\|_{\text{NP}(tX)} > (1-\epsilon)\sqrt{n}.$$

Fix such a t. By the definition of the $NP(tX)$ norm, there exists an $n \times 1$ column of elements of $\mathcal{L}_2$

$$F_{col} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

such that $\|F_{col}\|_\infty < 1 + \epsilon$ and $F_{col}(tX) = Y_{col}$, that is, $f_i(tX) = Y_i$ for each $i = 1, \dots, n$, and

$$\|\sum_{i=1}^{n} f_i^* f_i\|^{1/2} < 1 + \epsilon. \tag{6.1}$$

If we take these $f_1, \dots, f_n$ and form the row

$$F_{row} = [f_1 \quad f_2 \quad \cdots \quad f_n]$$

then F_{row} solves the interpolation problem $F_{row}(tX) = Y_{row}$, and hence, again by the definition of the $NP(tX)$ norm, we must have

$$\|\sum_{i=1}^{n} f_i f_i^*\|^{1/2} = \|F_{row}\|_\infty \geq \|Y_{row}\|_{\mathrm{NP}(tX)} > (1 - \epsilon)\sqrt{n}. \tag{6.2}$$

Comparing (6.1) and (6.2), and keeping in mind that ϵ was arbitrary, we conclude that $C_n \geq \sqrt{n}$. As noted earlier, the reverse inequality always holds, so the theorem is proved. □

7. Examples

As we have seen, a central role is played by d-tuples of $n \times n$ matrices $X = (X_1, \dots, X_d)$ with the following two properties:

- the row X is a *co-isometry*, i.e. $\sum_{i=1}^{d} X_i X_i^* = I_n$, and
- X is *irreducible* in the sense that $\mathrm{alg}_X = \mathcal{M}_n$.

We now give several examples of such systems X; the first is important for the proof of Theorem 3.9 in the sense that it shows that such systems exist for $d = 2$ and all n (hence for all d and n).

7.1. Irreducible representations of groups

a) Let $d = 2$ and let $X = (\frac{1}{\sqrt{2}}S, \frac{1}{\sqrt{2}}M)$ where S is the cyclic permutation matrix and M is the discrete Fourier transform of S. That is,

$$Se_i = e_{i+1 \pmod{n}}, Me_i = \omega^i e_i$$

where ω is an n-th root of unity. Since S and M are unitary, it is trivial that X is a row co-isometry. Again since S and M are unitary, it follows that the algebra they generate is a $*$-algebra, and it is straightforward to check that the only matrices commuting with both S and M are scalar multiples of the identity. Thus, $\mathrm{alg}_X = \mathcal{M}_n$.

b) More generally, given any group G with generators $g_1, \dots g_d$, we can consider any irreducible unitary representation $\pi : G \to \mathcal{M}_n$ and let $X_i = w_i \pi(g_i)$, $i = 1, \dots, d$ where the w_i are nonzero and $\sum |w_i|^2 = 1$. (Note that, in the previous example, M and S generate a group of cardinality n^3.) As before the algebra generated by the X_i is a $*$-algebra, and hence the irreducibility of the representation implies that $\mathrm{alg}_X = \mathcal{M}_n$.

7.2. Many variable example: the Choi point

When $d = n^2$, we can construct a special d-tuple of $n \times n$ matrices $X_1, \ldots, X_d$, for which it is easy to verify the conclusion of Theorem 3.9 directly, without appeal to the machinery of the quantum Perron-Frobenius theorem. (In fact this is the context in which the failure of the column-row property for $\mathcal{L}_d$ was originally discovered. In particular, using the following lemma and imitating the proof of Theorem 3.9, one can show that for $\mathcal{L}_{n^2}$ the column-row constant C_n is $\sqrt{n}$. Since all the $\mathcal{L}_d$ embed completely isometrically in $\mathcal{L}_2$, one concludes $C_n = \sqrt{n}$ in $\mathcal{L}_2$, for all n.)

Lemma 7.1. *Fix $n > 1$ and let $d = n^2$. We consider the n^2 matrices $X_{i,j}$, each of size $n \times n$,*

$$X_{i,j} = \frac{1}{\sqrt{n}} E_{ij}, \quad 1 \leq i, j \leq n$$

arranged into a row (say, by listing the subscripts (ij) in lexicographic order). Then X is a row co-isometry and

$$\lim_{t \nearrow 1} \frac{1-t^2}{t^2} P_{tX} = \frac{1}{n} I_{n^2}.$$

Hence for any $Y \in \mathcal{M}_{s\times t} \otimes \mathcal{M}_{n^2}$,

$$\|Y\|_{\mathrm{ANP}(X)} = \|Y\|.$$

Proof. It is straightforward to verify that X is a row co-isometry. Now, let

$$T = \sum_{i,j=1}^{n} \overline{X_{i,j}} \otimes X_{i,j} = \frac{1}{n} \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij} = \frac{1}{n} \mathfrak{C}_n.$$

Note $T^2 = T$. Moreover, $T^{\psi} = \frac{1}{n} I_{n^2}$. So, computing P_{tX}, we see that

$$\begin{aligned} P_{tX} &= [(I - t^2 T)^{-1}]^{\psi} \\ &= [I + \frac{t^2}{1-t^2} T]^{\psi} \\ &= nT + \frac{t^2}{1-t^2} \frac{1}{n} I. \end{aligned}$$

This proves the first claim of the lemma, and the second claim follows exactly as in the proof of Theorem 3.9. □

8. Further remarks on the NP(X) norm and interpolating sequences

The Nevanlinna-Pick norm of a block matrix Y at X, denoted $\|Y\|_{\mathrm{NP}(X)}$, is the minimum block H^∞ norm of a function f satisfying the equation $f(X) = Y$. We define the **condition number** of X, denoted $\kappa(X)$, by the formula

$$\kappa(X) = \inf_{Y \neq 0, Y \in \mathcal{M}_{s\times t} \otimes \mathrm{alg}_X} \frac{\|Y\|_{\mathrm{NP}(X)}}{\|Y\|}.$$

Note that, by definition, for any $Y \in \mathrm{alg}_X$,

$$\|Y\| \leq \|Y\|_{\mathrm{NP}(X)} \leq \kappa(X)\|Y\|.$$

The two following fairly harmless assertions, which will be established momentarily, have somewhat explosive consequences:

(1) If X is an irreducible row co-isometry, then $\lim_{t\to 1} \kappa(tX) = 1$.
(2) If X_1 is a row contraction and X_2 is an irreducible row co-isometry, then

$$\lim_{t\to 1} \kappa(X_1 \oplus tX_2) = \kappa(X_1).$$

Together, they will be used to establish the following fact, which is somewhat surprising in light of the failure of the column-row property for the free semigroup algebras: *For any sequence of contractive target data, there is an interpolating sequence for that data such that the interpolating function can be chosen with norm less than or equal to* 1*.*

In fact, in this case, one can actually choose the interpolating sequence based only on the sequence of norms of the target data, and their sizes.

Recall the **elementary kernel matrix**:

$$P_X = [(I - \sum \overline{X_i}\otimes X_i)^{-1}]^{\psi}.$$

Given a set $S \subset \mathcal{M}_n$, we denote its **commutant** by S', and we denote the set of **invertible elements** in S by $S^{\times}$.

Corollary 8.1. *Suppose* $Y \in \mathcal{M}_{s\times t} \otimes \mathrm{alg}_X$*. Suppose* $D \in \{I \otimes X\}'^{\times}$*. Let* $Q_{X,D} = (DP_X D)^{1/2}$ *Then,*

$$\|Y\|_{\mathrm{NP}(X)} = \left\|(Q^{\dagger}_{X,D} \otimes I_s)(I_n \otimes Y)(Q_{X,D} \otimes I_t)\right\|.$$

(Here, † denotes the Moore-Penrose pseudoinverse.)

The corollary follows essentially trivially from Corollary 3.8 (by which it is sufficient to take $D = I$). However, especially in the case of multi-point Pick problems, D can act as a pre-conditioner. Note the necessity that $Y \in \mathcal{M}_{s\times t} \otimes \mathrm{alg}_X$, rather than merely $Y \in \mathcal{M}_{s\times t} \otimes \mathcal{M}_n$.

We define the **effective condition number** of X, denoted $\gamma(X)$, to be defined via the following formula,

$$\gamma(X) = \inf_{D\in(\{I\otimes X\}')^{\times}} \sqrt{\|DP_X D\|\|(DP_X D)^{-1}\|}.$$

The effective condition number gives a bound on the condition number.

Corollary 8.2. *For all* $X \in \mathcal{B}^d$*,*

$$\kappa(X) \leq \gamma(X).$$

In particular, for all $Y \in \mathrm{alg}_X$ *we have*

$$\|Y\|_{\mathrm{NP}(X)} \leq \gamma(X)\|Y\|.$$

We can also restate Theorem 3.9 in terms of condition numbers, which will be useful in the construction of interpolating sequences.

Theorem 8.3. *Let* $X = (X_1, \ldots, X_d) \in M_n(\mathbb{C})^d$ *be a row co-isometry. If the algebra generated by* $X_1, \ldots, X_d$ *is all of* $M_n(\mathbb{C})$ *then* $\lim \kappa(tX) = 1$*. That is, for* $Y \in \mathcal{M}_{s\times t} \otimes \mathcal{M}_n$ $\lim \|Y\|_{\mathrm{NP}(tX)} = \|Y\|$*.*

8.1. Interpolating sequences

We now do some basic constructions of interpolating sequences.

Lemma 8.4. *If X_1 is a row contraction and X_2 is an irreducible row co-isometry, then the spectral radius of $T = \sum \overline{(X_1)_i} \otimes (X_2)_i$ is less than 1.*

Proof. Note $T^n = \sum_{|w|=n} \overline{X_1}^w \otimes X_2^w$. Therefore, $(T^n)^\psi = \sum_{|w|=n} \mathbf{vec}\, X_1^w \otimes (\mathbf{vec}\, X_2^w)^*$. So,

$$\|(T^n)^\psi\| \leq \| \sup_{\sum_{|w|=n} |a_w|^2=1} a_w X_1^w \| \| \sup_{\sum_{|w|=n} |a_w|^2=1} a_w X_2^w \|.$$

So, by the Gelfand formula for outer spectral radius, we see that the spectral radius of T [13] is less than the geometric mean of the outer spectral radii of X_1 and X_2. □

We now show that the condition number of a direct sum of some tuple with a scaled co-isometric tuple has the same condition number as the original in the limit.

Lemma 8.5. *If X_1 is a row contraction and X_2 is an irreducible row co-isometry, then*

$$\lim_{t \to 1} \kappa(X_1 \oplus tX_2) = \kappa(X_1).$$

Proof. The reader may verify that $P_{X_1 \oplus tX_2}$ has a block 4 by 4 structure with four non-zero block entries, let $\hat{P}_{X_1 \oplus tX_2}$ be the matrix with the zero columns and rows removed. Note,

$$\hat{P}_{X_1 \oplus tX_2} = \begin{bmatrix} [(I - \sum \overline{(X_1)_i} \otimes (X_1)_i)^{-1}]^\psi & [(I - t\sum \overline{(X_1)_i} \otimes (X_2)_i)^{-1}]^\psi \\ [(I - t\sum \overline{(X_2)_i} \otimes (X_1)_i)^{-1}]^\psi & [(I - t^2\sum \overline{(X_2)_i} \otimes (X_2)_i)^{-1}]^\psi \end{bmatrix}$$

Preconditioning by a block diagonal D with 1 and $\sqrt{n(1-t^2)}$ on the diagonal, we get that

$$\tilde{P}_{X_1 \oplus tX_2} = \begin{bmatrix} [(I - \sum \overline{(X_1)_i} \otimes (X_1)_i)^{-1}]^\psi & \sqrt{n(1-t^2)}[(I - t\sum \overline{(X_1)_i} \otimes (X_2)_i)^{-1}]^\psi \\ \sqrt{n(1-t^2)}[(I - t\sum \overline{(X_2)_i} \otimes (X_1)_i)^{-1}]^\psi & n(1-t^2)[(I - t^2\sum \overline{(X_2)_i} \otimes (X_2)_i)^{-1}]^\psi \end{bmatrix}$$

Therefore, taking $t \to 1$

$$\lim_{t \to 1} \tilde{P}_{X_1 \oplus tX_2} = \begin{bmatrix} P_{X_1} & 0 \\ 0 & I \end{bmatrix}$$

Therefore, applying Corollary 8.1,

$$\lim_{t \to 1} \kappa(X_1 \oplus tX_2) = \kappa(X_1). \quad \square$$

We now immediately see the following theorem.

Theorem 8.6. *Given $(\rho_i)_{i=1}^\infty$, a sequence of numbers in $[0, 1)$, and $(n_i)_{i=1}^\infty$, a sequence of natural numbers, there is an sequence $(X^{(i)})_{i=1}^\infty$ such that each $X^{(i)}$ has size n_i and for any sequence $(Y^{(i)})_{i=1}^\infty$ such that $\|Y^{(i)}\| \leq \rho_i$, there is a function in H^∞ of norm 1 such that $f(X^{(i)}) = Y^{(i)}$.*

9. Numerics and random examples

In this section we present a pseudocode version of what was used to initially find counter-examples to the column-row property for the Fock space.

9.1. Code

The following pseudocode gives an algorithm that attempts to randomly generate tuples of matrices $X = (X_1, X_2)$ and $Y = (Y_1, \dots, Y_m)$ that satisfy the argument in Theorem 1.1. Much like the argument in Theorem 1.1, the algorithm presented relies on Corollary 3.8 and Theorem 3.9.

Recall that given a row contraction $X = (X_1, \dots, X_d) \in \mathcal{B}^d \subset \mathcal{M}_n^d$ we can solve the interpolation to a block matrix $Y \in M_{s\times t} \otimes \mathcal{M}_n$ if and only if $Y \in \mathcal{M}_{s\times t} \otimes \mathrm{alg}_X$. Thus our numeric approach to Theorem 1.1 certainly requires at least that $X = (X_1, X_2)$ is a row contraction and $Y \in \mathcal{M}_{1\times m} \otimes \mathrm{alg}_X$. Recall that in this case Corollary 3.8 implies

$$\|{}^P Y^P\| = \|Y\|_{\mathrm{NP}(X)}.$$

Thus, we choose $Y_1, \dots, Y_m \in \mathrm{alg}_X$ and set

$$Y_{\mathrm{row}} = [Y_1 \quad \dots \quad Y_m] \quad \text{and} \quad Y_{\mathrm{col}} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix}.$$

The goal is to find choices of $Y_1, \dots, Y_m$ such that

$$\sqrt{m} \approx \frac{\|{}^P(Y_{\mathrm{row}})^P\|}{\|{}^P(Y_{\mathrm{col}})^P\|} = \frac{\|Y_{\mathrm{row}}\|_{\mathrm{NP}(X)}}{\|Y_{\mathrm{col}}\|_{\mathrm{NP}(X)}}.$$

As was seen in Theorem 1.1, since the NP(X) norm is an infimum, there must be an interpolating function $F_{\mathrm{col}} \in M_{m\times 1}(H^\infty(\mathcal{B}^d))$ such that $F_{\mathrm{col}}(X) = Y_{\mathrm{col}}$ and $\|F_{\mathrm{col}}\|_\infty \approx \|Y_{\mathrm{col}}\|_{\mathrm{NP}(X)}$. Choosing F_{row} to be the row vector version of F_{col}, we have that $F_{\mathrm{row}}(X) = Y_{\mathrm{row}}$. Since $\|Y_{\mathrm{row}}\|_{\mathrm{NP}(X)} \le \|F_{\mathrm{row}}\|_\infty$ and $\|Y_{\mathrm{col}}\|_{\mathrm{NP}(X)} \approx \|F_{\mathrm{col}}\|_\infty$ we have the following

$$\frac{\|Y_{\mathrm{row}}\|_{\mathrm{NP}(X)}}{\|Y_{\mathrm{col}}\|_{\mathrm{NP}(X)}} \approx \frac{\|Y_{\mathrm{row}}\|_{\mathrm{NP}(X)}}{\|F_{\mathrm{col}}\|_\infty} \le \frac{\|F_{\mathrm{row}}\|_\infty}{\|F_{\mathrm{col}}\|_\infty} \le C_m \le \sqrt{m}.$$

Thus, with a correct choice of Y, we have that $\sqrt{m} \lesssim C_n \le \sqrt{m}$.

Now, fix n and m and choose a cut-off value $\gamma < \sqrt{m}$. The following pseudo-code describes a loop to find $X = (X_1, X_2) \in \mathcal{M}_n^2$ and $Y_1, \dots, Y_m \in \mathcal{M}_n$ that witness the ratio $\|Y_{\mathrm{row}}\|_{\mathrm{NP}(X)} > \gamma \|Y_{\mathrm{col}}\|_{\mathrm{NP}(X)}$.

Set a cut off value $\gamma < \sqrt{m}$;
Set the maximum ratio $M_r = 0$;
Choose a sufficiently small $\varepsilon > 0$;
LOOP while $M_r < \gamma$;
Randomly generate $Z = (Z_1, Z_2) \in \mathcal{M}_n^2$ such that $Z_1 Z_1^* + Z_2 Z_2^*$ is invertible;
Set $X = (1-\varepsilon)(Z_1 Z_1^* + Z_2 Z_2^*)^{-1/2} Z$;
Compute $P_X = \left[(I_{n^2} - \overline{X_1} \otimes X_1 - \overline{X_2} \otimes X_2)^{-1}\right]^\psi$;
Compute $P_X^{1/2}$ and $P_X^{\dagger/2}$;
Select $v_1, \dots, v_m$ to be distinct eigenvectors of P_X with the smallest positive associated eigenvalues;
Set each $Y_i = \mathbf{vec}^{-1}(v_i)$;
Form $Y = (Y_1, \dots, Y_m)$;
Compute $\|{}^P(Y_{\mathrm{row}})^P\|$ and $\|{}^P(Y_{\mathrm{col}})^P\|$;
IF $\gamma \|{}^P(Y_{\mathrm{col}})^P\| > \|{}^P(Y_{\mathrm{row}})^P\|$;
THEN set $M_r = \|{}^P(Y_{\mathrm{row}})^P\| / \|{}^P(Y_{\mathrm{col}})^P\|$ and PRINT(X, Y, M_r);

15: ELSE set $M_r = \max\{M_r, \|^P(Y_{\text{row}})^P\|/\|^P(Y_{\text{col}})^P\|\}$;
16: END LOOP.

The nature of the above algorithm implies that if ε is not sufficiently close to 0, then typically the column-row ratio will not be close to $\sqrt{m}$ and the loop will never terminate. It is perhaps advisable to randomly generate $\varepsilon' \in (0, \varepsilon)$ at each iteration of the loop to give a better chance that the loop terminates. A functioning version of the above pseudo-code (including a method of computing P_X) can be found at the following url: https://github.com/mericaugat/EffectiveNP.

9.2. Committee spaces

We note that if we had not done the normalization to make X asymptotically unitary in the above code and instead chose random tuples with independent entries, in the limit we would not find examples with large column-row ratio, as was proven in [14]. That is, sequences of random multipliers usually satisfy the true column-row property. Originally, our group did not normalize this way, and only found examples with a ratio of about 1.0043 after millions of trials.

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