# On the Power of Randomization for Scheduling Real-Time Traffic in Wireless Networks

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***Abstract*— In this paper, we consider the problem of scheduling real-time traffic in wireless networks under a conflict-graph interference model and single-hop traffic. The objective is to guarantee that at least a certain fraction of packets of each link are delivered within their deadlines, which is referred to as *delivery ratio*. This problem has been studied before under restrictive frame-based traffic models, or greedy maximal scheduling schemes like LDF (Largest-Deficit First) that can lead to poor delivery ratio for general traffic patterns. In this paper, we pursue a different approach through randomization over the choice of maximal links that can transmit at each time. We design randomized policies in collocated networks, multi-partite networks, and general networks, that can achieve delivery ratios much higher than what is achievable by LDF. Further, our results apply to any traffic (arrival and deadline) process that evolves as an *unknown* positive recurrent Markov chain. Hence, this work is an improvement with respect to both efficiency and traffic assumptions compared to the past work. We further present extensive simulation results over various traffic patterns and interference graphs to illustrate the gains of our randomized policies over LDF variants.**

***Index Terms*— Scheduling, real-time traffic, Markov processes, stability, wireless networks.**

## I. Introduction

MUCH of the prior work on scheduling algorithms for wireless networks focus on maximizing throughput. However, for many real-time applications, e.g., in Internet of Things (IoT), vehicular networks, and other cyber-physical systems, delays and deadline guarantees on packet delivery are more important than long-term throughput [2]–[4]. Recently, there has been an interest in developing scheduling algorithms specifically targeted towards handling deadline-constrained traffic [5]–[10], when each packet has to be delivered within a *strict deadline*, otherwise it is of no use. The key objective in these works is to guarantee that at least a fraction of the packets will be delivered to their destinations within their deadlines, which is referred to as *delivery ratio* (QoS). Providing such guarantees is very challenging as it crucially depends on the temporal pattern of packet arrivals and their deadlines, as opposed to long-term averages in traditional throughput maximization. One can construct adversarial traffic patterns that all have the same long-term average but their achievable delivery ratio is vastly different [9], [11].

Recently, there have been two approaches for providing QoS guarantees for real-time traffic in wireless networks. One is the frame-based approach [5]–[8], and the other is a greedy scheduling approach like the largest-deficit-first policy (LDF) [9], [10]. In the frame-based approach, it is assumed that each frame is a number of consecutive time slots, and packets arriving in each frame have to be scheduled before the end of the frame. They crucially rely on the assumption that all packets of all users arrive at the beginning of frames [5]–[7], or the complete knowledge of future packet arrivals and their deadlines in each frame is available at the beginning of the frame [8]. This restricts the application of such policies to specific traffic patterns with periodic arrivals and synchronized users. Partial generalizations of the frame-based traffic are considered in [12], [13] without performance guarantees. The results for general traffic patterns without such frame assumptions are very limited, as in such settings, the real-time rate region is difficult to characterize and the optimal policy is unknown. A popular algorithm for providing QoS guarantees for real-time traffic is the largest-deficit-first (LDF) policy [5], [9], [10], [14], which is the real-time variation of the longest-queue-first (LQF) policy (see, e.g., [15], [16]). It is known that LDF is optimal in collocated networks *under the frame-based model* [5], [14]. The performance of LDF in the non-frame-based setting has been studied in [9] in terms of the *efficiency ratio*, which is the fraction of the real-time throughput region guaranteed by LDF. It is shown that LDF achieves an efficiency ratio of at least $\frac{1}{1+\beta}$ for a network with interference degree[1] $\beta$, under *i.i.d.* (independent and identically distributed) packet arrivals and deadlines. Further, when traffic is not i.i.d., the efficiency ratio of LDF is as low as $\frac{1}{1+\sqrt{\beta}}$ [9]. In particular, for collocated networks, the efficiency ratio of LDF under Markovian traffic is $1/2$, and in a simple star topology with one center link and $K$ neighboring links, it scales down as low as $O(\frac{1}{\sqrt{K}})$. This shows that LDF might not be suitable for high throughput real-time applications, especially with non-i.i.d. traffic, which is the case if packet drops due to deadline expiry trigger re-transmissions.

Besides the works above on providing QoS guarantees for wireless networks, there is literature on approximation

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[1] The interference degree is the maximum number of links that can be scheduled simultaneously out of a link and its neighboring links.

algorithms for *single-link* buffer management problem [17], [18]. In this problem, packets arrive to a single link, each with a non-negative constant weight and a deadline. The goal is to maximize the total weight of transmitted packets for the worst input sequence. The approximation algorithms include the maximum-weight greedy algorithm [17], [18], EDF$_\alpha$ [19] which schedules the earliest-deadline packet with weight at least $\alpha \leq 1$ of the maximum-weight packet, or randomized algorithms such as [19]–[22] where the scheduling decision is randomized over pending packets in the link's buffer. Some of these randomized algorithms have used a novel amortized analysis technique initially introduced in [23]. Inspired by such randomization techniques, we design randomized algorithms for wireless networks under a general interference model and given the delivery ratio requirements for the links in the network.

### A. Contributions

Contributions of this paper can be summarized as follows.

*Markovian Traffic Model:* Our traffic model allows traffic (*arrival* and *deadline*) processes that evolve as an *unknown* irreducible Markov chain over a finite state space. This model is a significant extension from i.i.d. or frame-based traffic models in [5]–[9]. A key technique in analyzing the achievable efficiency ratio in our model is to look at the return times of the traffic Markov chain and analyze the performance of scheduling algorithms over long enough cycles consisting of multiple return times.

*Randomized Algorithms with Improved Efficiency:* We propose randomized scheduling algorithms that can significantly outperform deterministic greedy algorithms like LDF. The key idea is to identify a structure for the optimal policy and randomize over the possible scheduling choices of the optimal policy, rather than solely relying on the deficit queues. For *collocated networks* and *complete bipartite graphs* our randomized algorithms achieve an efficiency ratio of at least $0.63$ and $2/3$, respectively, and in *general graphs*, achieve an efficiency ratio greater than $1/2$, all *independent of the network size* and *without the knowledge of the traffic model*.

## II. Model and Definitions

*Wireless Network Model.* We consider a set of $K$ links (or users) denoted by the set $\mathcal{K}$, where $K = |\mathcal{K}|$. Time is slotted, and at each time slot $t \in \mathbb{N}_0$, each link can transmit one packet successfully, if there are no interfering links transmitting at the same time. As in [9], it is standard to represent the interference relationships between links by an *interference graph* $G_I = (\mathcal{K}, E_I)$. Each vertex of $G_I$ is a link, and an edge $(l_1, l_2) \in E_I$ indicates links $l_1$ and $l_2$ interfere with each other. Let $I_l(t) = 1$ if link $l$ is transmitting a packet at time $t$, and $I_l(t) = 0$ otherwise. Hence, any *feasible schedule* $M(t) := \{l \in \mathcal{K} : I_l(t) = 1\}$ at time $t$ has to form an independent set of $G_I$ over links that have packets, i.e., no two transmitting links can share an edge in $G_I$. We say a feasible schedule $M(t)$ is maximal if no more links can be added to the schedule without interfering with some other active link in $M(t)$. Let $\mathcal{B}(t)$ be the set of links that have packets available to transmit at time $t$. Let $\mathcal{I}$ denote the set of all maximal independent sets of $G_I$. Then, at any time $t$,

$$M(t) \subseteq (\mathcal{B}(t) \cap D), \quad \text{for some } D \in \mathcal{I},$$

where '$\subseteq$' holds with '$=$' if $M(t)$ is a maximal schedule.

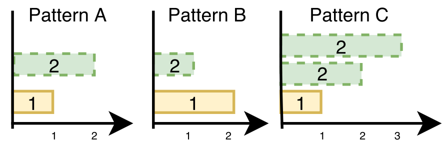



Fig. 1. An example of a Markovian traffic process with three traffic patterns repeating as $A \to B \to C \to A \cdots$. Each rectangle indicates a packet for a link indicated by its number. The left side of the rectangle corresponds to its arrival time, and its length corresponds to its deadline. For example on pattern $A$, we have 2 packets, 1 from link 2, with deadline 2 slots after the arrival, and 1 from link 1, with deadline in the same slot.

*Traffic Model.* We consider a single-hop traffic with deadlines for each link. Let $a_l(t)$ denote the number of packets arriving on link $l$ at time $t$, with $a_l(t) \leq a_{\max}$, for some $a_{\max} < \infty$. Each packet upon arrival has a deadline which is the maximum delay that the packet can tolerate. We define a vector $\tau_l(t) = (\tau_{l,d}(t); d = 1, \cdots, d_{\max})$, where $\tau_{l,d}(t)$ is the number of packets with deadline $d$ arriving to link $l$ at time $t$. A packet arriving with deadline $d$ at time $t$ has to be transmitted before the end of time slot $t + d - 1$, otherwise it will be dropped. The maximum deadline is bounded by a constant $d_{\max}$. Hence, the network traffic (arrival, deadline) process is described by $\tau(t) = (\tau_l(t); l \in \mathcal{K}),\ t \geq 0$. We also use $u(t)$ to denote any unobservable (hidden) information of the traffic process, so that the complete traffic process $\mathbf{x}(t) = (\tau(t), u(t))$ evolves as an irreducible Markov chain over a finite state space $\mathcal{X} = \Gamma \times \mathcal{U}$, where $\Gamma = \{0, \cdots, a_{\max}\}^{d_{\max} \times K}$ and $\mathcal{U} := \{1, \cdots, U_{\max}\}$ for a finite $U_{\max}$.[2]

Note that the arrival and deadline processes do not need to be i.i.d. across times or users. Since the state space $\mathcal{X}$ is finite, $\mathbf{x}(t)$ is a positive recurrent Markov chain [24] and the time-average of any bounded function of $\mathbf{x}(t)$ is well-defined, in particular, the packet arrival rate $\overline{a}_l$, $l \in \mathcal{K}$,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=1}^{t} a_l(s) = \overline{a}_l. \tag{1}$$

See Figure 1 for an example of a Markovian traffic process.

*Buffer Dynamics.* The buffer of link $l$ at time $t$ contains the existing packets at link $l$ which have not expired yet and also the newly arrived packets $\tau_l(t)$. Formally, we define the buffer of link $l$ by a vector $\Psi_l(t) = (\Psi_{l,d}(t); d = 1, \cdots, d_{\max})$, where $\Psi_{l,d}(t)$ is the number of packets in the buffer with *remaining* deadline $d$ at time $t$. The remaining deadline of each packet in the buffer decreases by one at every time slot, until the packet is successfully transmitted or reaches the deadline 0, which in either case the packet is removed from the buffer, i.e., the buffer at the beginning of slot $t + 1$ is

$$\Psi_{l,d}(t+1) = \Psi_{l,d+1}(t) + \tau_{l,d}(t+1) - I_{l,d+1}(t), \tag{2}$$

where $I_l(t) = \sum_{d=1}^{d_{\max}} I_{l,d}(t) \leq 1$, and $I_{l,d}(t) = 1$ if the scheduler selects a packet with deadline $d$ to transmit at time $t$

[2] Essentially, $u(t)$ assigns labels to $\tau(t)$ to allow more complicated dependencies in $\tau(t)$. If $\mathcal{U} = \{1\}$, then $\tau(t)$ itself evolves as a Markov chain.

on link $l$. By convention, we set $\Psi_{l,d_{\max}+1}(t) = 0$, $\Psi_{l,0}(t) = 0$. We define the network buffer state as $\Psi(t) = (\Psi_l(t); l \in \mathcal{K})$.

*Delivery Requirement and Deficit.* As in [5]–[9], we assume that there is a minimum delivery ratio $p_l$ (QoS requirement) for each link $l$, $l \in \mathcal{K}$. This means the scheduling algorithm must successfully deliver at least $p_l$ fraction of the incoming packets on each link $l$ in long term. Formally,

$$\liminf_{t\to\infty} \frac{\sum_{s=1}^{t} I_l(s)}{\sum_{s=1}^{t} a_l(s)} \geq p_l. \tag{3}$$

We define a deficit $w_l(t)$ which measures the amount of service owed to link $l$ up to time $t$ to fulfill its minimum delivery rate. As in [8], [9], the deficit evolves as

$$w_l(t+1) = \Big[w_l(t) + \widetilde{a}_l(t) - I_l(t)\Big]^+, \tag{4}$$

where $[\cdot]^+ = \max\{\cdot, 0\}$, and $\widetilde{a}_l(t)$ indicates the amount of deficit increase due to packet arrivals. To determine $\widetilde{a}_l(t)$, for the $n$-th arriving packet on link $l$, we increase the deficit of link $l$ by $X_l(n) \geq 0$, where $X_l(\cdot)$ is i.i.d. with $\mathbb{E}[X_l(\cdot)] = p_l$, i.e., we increase the deficit on average by $p_l$. For example, we can increase the deficit by exactly $p_l$ for each packet arrival to link $l$, or use a coin tossing process as in [8], [9], i.e., each packet arrival at link $l$ increases the deficit by one with the probability $p_l$, and zero otherwise. We refer to $\tilde{a}_l(t)$ as the *deficit arrival* process for link $l$. Note that it holds that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=1}^{t} \widetilde{a}_l(s) = \overline{a}_l p_l := \lambda_l, \quad l \in \mathcal{K}. \tag{5}$$

We refer to $\lambda_l$ as the deficit arrival rate for link $l$. We would like to emphasize that the arriving packet is always added to the link's buffer, regardless of whether and how much deficit is added for that packet. Also note that in (4), each time a packet is scheduled from the link, $I_l(t) = 1$, the deficit is reduced by one. The dynamics in (4) define a deficit queueing system, with bounded increments/decrements, whose stability, e.g.,

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{s=1}^{t} \mathbb{E}[w_l(s)] < \infty, \tag{6}$$

implies that (3) holds.[3] Define the vector of deficits as $w(t) = (w_l(t), l \in \mathcal{K})$. The system state at time $t$ is then defined as

$$\mathcal{S}(t) = (\Psi(t), w(t), \mathbf{x}(t)). \tag{7}$$

*Objective.* Define $\mathcal{P}_C$ to be the set of all causal policies, i.e. policies that do not know the information of future arrivals and deadlines in order to make scheduling decisions. We assume that policies in $\mathcal{P}_C$ can potentially utilize the information of the hidden state $u(t)$ of the traffic process $\mathbf{x}(t)$, however, we emphasize that the policies designed in this paper *do not* need to know this information when making decisions. For a given traffic process $\mathbf{x}(t)$, $t \geq 0$, with fixed $\overline{a}_l$, defined in (1), we are interested in causal policies that can stabilize the deficit queues for the largest set of delivery rate vectors $\mathbf{p} = (p_l, l \in \mathcal{K})$, or equivalently largest set of $\boldsymbol{\lambda} = (\lambda_l := \overline{a}_l p_l, l \in \mathcal{K})$ possible. For a given traffic process, we say the rate vector $\boldsymbol{\lambda} = (\lambda_l, l \in \mathcal{K})$ is supportable under some policy $\mu \in \mathcal{P}_C$ if all the deficit queues remain stable. Then one can define the supportable (real-time) rate region of the policy $\mu$ as

$$\Lambda_\mu = \{\boldsymbol{\lambda} \geq 0 : \boldsymbol{\lambda} \text{ is supportable by } \mu\}. \tag{8}$$

Note that for a given traffic distribution, a vector $\boldsymbol{\lambda}$ corresponds to a single vector of delivery rate requirements $\mathbf{p}$ exactly. The supportable rate region under all the causal policies is defined as $\Lambda = \bigcup_{\mu \in \mathcal{P}_C} \Lambda_\mu$. The overall performance of a policy $\mu$ is evaluated by the efficiency ratio $\gamma^\star_\mu$ which is defined as

$$\gamma^\star_\mu = \sup\{\gamma : \gamma\Lambda \subseteq \Lambda_\mu\}. \tag{9}$$

For a casual policy $\mu$, we aim to provide a *universal lower bound* on the efficiency ratio that holds for "all" Markovian traffic processes (without knowing the transition probability matrix).

[3]Actually only the rate stability is enough to establish (3) [25], however we consider this stronger notion of stability.

## III. Randomized Scheduling Algorithms

In this section, we present our randomized scheduling algorithms. We start with the collocated networks, and then proceed to general networks.

### A. Collocated Networks

In a collocated network, only one of the links can transmit a packet at any time. Hence the interference graph $G_I$ is a complete graph.

Define $e_l(t) = \min\{d : \Psi_{l,d}(t) > 0\}$ to be the deadline of the earliest-deadline packet available at link $l$ at time $t$. By convention, the minimum of an empty set is considered infinity. We use a tuple $(w_l(t), e_l(t))_l$ to denote the earliest-deadline packet of link $l$ with deadline $e_l(t)$ and link deficit $w_l(t)$. We make the following dominance definition.

*Definition 1: We say that a link $l_1$ dominates a link $l_2$ at time t if $w_{l_1}(t) \geq w_{l_2}(t)$ and $e_{l_1}(t) \leq e_{l_2}(t)$. If one of the two inequalities is strict, we call it a strict dominance. A non-dominated link is a nonempty link that is not dominated strictly by any other link at that time.*

Recall that $\mathcal{B}(t)$ is the set of links with nonempty buffers. At every time slot, we first find the set of non-dominated links $\mathcal{B}_{\text{ND}}(t)$. One way to do that is as follows:

**Algorithm 1** Finding Set of Non-Dominated Links

1: $H \leftarrow \mathcal{B}(t)$, $\mathcal{B}_{\text{ND}}(t) \leftarrow \varnothing$, $i \leftarrow 0$
2: **while** $H \neq \varnothing$ **do**
3: $\quad i \leftarrow i + 1$
4: $\quad$ Find the largest-deficit non-dominated link $h_i \in H$.
5: $\quad$ Add $h_i$ to $\mathcal{B}_{\text{ND}}(t)$
6: $\quad$ Remove $h_i$ and all the links dominated by it, i.e.
$$H \leftarrow H \setminus \{l \in H : e_l(t) \geq e_{h_i}(t)\}.$$
7: **end while**

Algorithm 1 returns a set $\mathcal{B}_{\text{ND}}(t) = \{h_1, .., h_k\}$, where $h_i$ is the link selected in the $i$-th iteration, and the links are ordered in the order of their deficits, i.e., $w_{h_1}(t) > w_{h_2}(t) > \cdots > w_{h_k}(t)$. See Figure 2 for an illustrative example of the

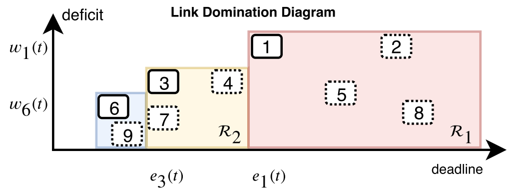



Fig. 2. An example for non-dominated links. Each numbered rectangle denotes the earliest-deadline packet of a link. A solid rectangle indicates that the link is non-dominated. Dashed rectangles (links) that fall in regions $\mathcal{R}_i$ will be dominated.

non-dominated links. Our scheduling algorithm transmits the earliest-deadline packet of one of the links $h_i \in \mathcal{B}_{\text{ND}}(t)$ randomly, where the probabilities $p_{h_i}(t)$ are computed recursively as in Algorithm 2. We refer to Algorithm 2 as AMIX-ND

**Algorithm 2** AMIX-ND: Randomized Scheduling in Collocated Networks

Use Algorithm 1 to find $\mathcal{B}_{\text{ND}}(t) = \{h_1, .., h_k\}$.
$r \leftarrow 1$
**for** $i = 1$ to $k-1$ **do**
  $p_{h_i}(t) = \min\left(1 - \frac{w_{h_{i+1}}(t)}{w_{h_i}(t)}, r\right)$
  $r \leftarrow r - p_{h_i}(t)$
**end for**
$p_{h_k}(t) = r$
Send the earliest-deadline packet from link $h_i$ with probability $p_{h_i}(t)$.

which stands for *Adaptive Mixing over Non-Dominated links*.

*Theorem 1: In a collocated wireless network with $K$ links,* AMIX-ND *achieves an efficiency ratio of at least*

$$\gamma^{\star}_{\text{AMIX-ND}} \geq 1 - \left(1 - \frac{1}{K}\right)^{K} > \frac{e-1}{e}. \tag{10}$$

*Remark 1:* Note that AMIX-ND has an efficiency ratio which is bounded below by $0.63$, regardless of the number of links. In contrast, we can construct Markovian traffic processes where the efficiency ratio of LDF is less than $1/2 + \epsilon$ [9]. For example, for the traffic patterns of Figure 1 in the model section, we will see in simulations in Section VI that, while AMIX-ND can achieve delivery ratios close to $0.99$, LDF cannot do better than $0.5 + \epsilon$. Note that our traffic model does allow traffic patterns as in Figure 1, since we do not need the traffic Markov chain to be aperiodic.

*Remark 2:* Assuming access to the earliest deadline packet of every link, the computational complexity of AMIX-ND is $O(K \log K)$ for assigning probabilities and choosing a packet for transmission. We describe one such implementation in Appendix A.

### *B. Multipartite Networks and General Networks*

Recall that $\mathcal{B}(t)$ is the set of links with nonempty buffers, and $\mathcal{I}$ is the set of maximal independent sets of the interference graph $G_I$. The set of maximal schedules is defined as $\mathcal{M}(t) = \{D \cap \mathcal{B}(t), D \in \mathcal{I}\}$. Our randomized algorithm selects a maximal schedule (*MS*) $M \in \mathcal{M}(t)$ probabilistically and schedules the earliest-deadline packets of the links of $M$. We refer to this algorithm as AMIX-MS which stands for *Adaptive Mixing over Maximal Schedules*. Before presenting the algorithm, we make a few definitions.

*Definition 2: The weight of a MS $M \in \mathcal{M}(t)$ at time $t$ is*

$$W_M(t) = \sum_{l \in M} w_l(t). \tag{11}$$

*Let $R = |\{M \in \mathcal{M}(t), W_M(t) > 0\}|$. We index and order $M \in \mathcal{M}(t)$ such that $M_i$ has the $i$-th largest weight at time $t$, i.e.,*

$$W_{M_1}(t) \geq W_{M_2}(t) \cdots \geq W_{M_R}(t).$$

*Definition 3: Define the subharmonic average of weights of the first $n$ MS, $n \leq R$, at time $t$ to be*

$$C_n(t) = \frac{n-1}{\sum_{i=1}^{n}(W_{M_i}(t))^{-1}}. \tag{12}$$

The probabilities used by AMIX-MS to select MS $M_i$, at time $t$, are as follows

$$p^{\bar{n}}_{M_i}(t) \equiv p^{\bar{n}}_i(t) = \begin{cases} 1 - \dfrac{C_{\bar{n}}(t)}{W_{M_i}(t)}, & 1 \leq i \leq \bar{n} \\ 0, & \bar{n} < i \leq |\mathcal{M}(t)| \end{cases} \tag{13}$$

where $\bar{n}$ is the largest $n \leq R$ such that $\{p^n_i(t), 1 \leq i \leq n\}$ defines a valid probability distribution over $1 \leq i \leq n$. Noting that $p^n_i(t) \geq p^n_{i+1}(t)$ for $i < n$, and $\sum_{i \leq n} p^n_i(t) = 1$, $\bar{n}$ is therefore given by

$$\bar{n} := \bar{n}(t) = \max\{n : p^n_n(t) \geq 0\}. \tag{14}$$

We drop the dependence on $t$ for $\bar{n}(t)$ when there is no ambiguity. Algorithm 3 gives a description of AMIX-MS where $\bar{n}$ is found using a binary search. Then AMIX-MS selects a MS $M_i$ with probability $p^{\bar{n}}_i(t)$ as in (13).

**Algorithm 3** AMIX-MS: Randomized Scheduling in General Interference Graphs

$n_1 \leftarrow 1, n_2 \leftarrow R$
**while** $n_1 \neq n_2$ **do**
  $n \leftarrow \lceil \frac{n_1+n_2}{2} \rceil$
  **if** $p^n_n(t) \geq 0$ **then**
    $n_1 \leftarrow n$
  **else**
    $n_2 \leftarrow n-1$
  **end if**
**end while**
$\bar{n} \leftarrow n_1$
Select MS $M_i$ with probability $p^{\bar{n}}_{M_i}(t)$ as in (13) and transmit the earliest-deadline packet of each link in $M_i$.

The following theorem states the main result regarding the efficiency ratio of AMIX-MS.

*Theorem 2: In a wireless network with interference graph $G_I$ and maximal independent sets $\mathcal{I}$, the efficiency ratio of* AMIX-MS *is at least*

$$\gamma^{\star}_{\text{AMIX-MS}} \geq \frac{|\mathcal{I}|}{2|\mathcal{I}| - 1} > \frac{1}{2}.$$

A special case of this theorem is for networks with a complete $n$-partite interference graph, $n \geq 2$. In a complete $n$-partite graph, with $n$ components, $V_1, \cdots, V_n$, links in each component do not share any edge but there is an edge between any two links in different components. Hence, each component $V_i$, $1 \leq i \leq n$ induces a MS. We state the result as the following corollary which immediately follows from Theorem 2.

*Corollary 2.1: For a wireless network with a complete $n$-partite interference graph, under* AMIX-MS*,*

$$\gamma^\star_{\text{AMIX-MS}} \geq \frac{n}{2\,n-1}.$$

*Remark 3:* We emphasize on the importance of Theorem 2 using a simple interference graph with 'star' topology. This is a special case of a bipartite graph with only two components, $V_1$ is the center node, and $V_2$ are the leaf nodes. Notice that the guarantee of AMIX-MS in this case is at least $\frac{2}{3}$, regardless of the number of nodes $K$. This is a significant improvement over LDF, whose efficiency ratio is at least $\frac{1}{K}$ under i.i.d. traffic but not better than $\frac{1}{\sqrt{K-1}+1}$ under Markovian traffics [9].

*Remark 4:* We note that the computational complexity of AMIX-MS could be high for general graphs as it requires finding an ordering of maximal schedules, the number of which can be exponential in the number of vertices of the graph [26]. However, it is easily applicable for small graphs or graphs with limited number of independent sets. Moreover, we can further approximate the algorithm by only ordering a subset of maximal schedules as opposed to finding all of them. The randomization in AMIX-MS could be potentially implemented in a distributed manner by using CSMA-like schemes such as [27]–[29].

## IV. Analysis Technique

We provide an overview of the techniques in our proofs. We first mention a lemma below which should be intuitive and will allow us to restrict our attention to *natural* policies.

*Lemma 1: Without loss of generality, we consider natural policies that use a maximal schedule to transmit at each time. Further, if a link is included in the schedule, its earliest-deadline packet will be selected for transmission.*

*Proof:* The proof is through a standard exchange argument and can be found in Appendix B. □

*Frame Construction:* A key step in the analysis of our scheduling algorithms is a careful frame construction. We emphasize that the frame construction is only for the purpose of analysis and is *not* part of our algorithms. The F-framed construction in [9] only works for i.i.d. arrivals and deadlines. Here, we need a construction that can handle our Markovian traffic model. We present this construction below where frames have random length as opposed to fixed length in [9].

*Definition 4 (Frames and Cycles): Starting from an initial traffic state $\mathbf{x}(0) = \mathbf{x} \in \mathcal{X}$, let $t_i$ denote the $i$-th return time of traffic Markov chain $\mathbf{x}(t)$ to $\mathbf{x}$, $i = 1, \cdots$. By convention, define $t_0 = 0$. The $i$-th cycle $\mathcal{C}_i$ is defined from the beginning of time slot $t_{i-1}+1$ until the end of time slot $t_i$, with cycle length $C_i = t_i - t_{i-1}$. Given a fixed $k \in \mathbb{N}$, we define the $i$-th frame $\mathcal{F}_i^{(k)}$ as $k$ consecutive cycles $\mathcal{C}_{(i-1)k+1}, \cdots, \mathcal{C}_{ik}$, i.e., from the beginning of slot $t_{(i-1)k}+1$ until the end of slot $t_{ik}$. The length of the $i$-th frame is denoted by $F_i^{(k)} = \sum_{j=(i-1)k+1}^{ik} C_j$. Define $\mathcal{J}(\mathcal{F}^{(k)})$ to be the space of all possible traffic patterns $(\tau(t), t \in \mathcal{F}^{(k)})$ during a frame $\mathcal{F}^{(k)}$. Note that these patterns start after $\mathbf{x}$ and end with $\mathbf{x}$.*

By the strong Markov property and the positive recurrence of traffic Markov chain, frame lengths $F_i^{(k)}$ are i.i.d with mean $\mathbb{E}[F^{(k)}] = k\mathbb{E}[C]$, where $\mathbb{E}[C]$ is the mean cycle length which is a bounded constant [24]. In fact, since state space $\mathcal{X}$ is finite, all the moments of $C$ (and $F^{(k)}$) are finite. We choose a fixed $k$, and, when the context is clear, drop the dependence on $k$ in the notation.

Define the class of *non-causal $\mathcal{F}$-framed* policies $\mathcal{P}_{NC}(\mathcal{F})$ to be the policies that, at the beginning of each frame $\mathcal{F}_i$, have complete information about the traffic pattern in that frame, but have a restriction that they drop the packets that are still in the buffer at the end of the frame. Note that the number of such packets is at most $d_{\max} a_{\max} K$, which is negligible compared to the average number of packets in the frame, $\overline{a}_l \mathbb{E}[F] = \overline{a}_l k \mathbb{E}[C]$, as $k \to \infty$. Define the rate region

$$\Lambda_{NC}(\mathcal{F}) = \bigcup_{\mu \in \mathcal{P}_{NC}(\mathcal{F})} \Lambda_\mu. \tag{15}$$

Given a policy $\mu \in \mathcal{P}_{NC}(\mathcal{F})$, the time-average service rate $\bar{I}_l$ of link $l$ is well defined. In fact, by the renewal reward theorem (e.g. [30], Theorem 5.10), and boundedness of $\mathbb{E}[F]$,

$$\lim_{t\to\infty} \frac{\sum_{s=1}^t I_l(s)}{t} = \frac{\mathbb{E}\left[\sum_{t\in\mathcal{F}} I_l(t)\right]}{\mathbb{E}[F]} = \bar{I}_l. \tag{16}$$

Similarly for the deficit arrival rate $\lambda_l$, defined in (5),

$$\frac{\mathbb{E}[\sum_{t\in\mathcal{F}} \widetilde{a}_l(t)]}{\mathbb{E}[F]} = \lambda_l, \quad l \in \mathcal{K}. \tag{17}$$

In Definition 4, each frame consists of $k$ cycles. Using similar arguments as in [9], it is easy to see (and it is intuitive) that

$$\liminf_{k\to\infty} \Lambda_{NC}(\mathcal{F}^{(k)}) \supseteq \operatorname{int}(\Lambda).$$

where $\operatorname{int}(\cdot)$ is the interior. Hence, if we prove that for a causal policy ALG, there exists a constant $\rho$, and a large $k_0$, such that for all $k \geq k_0$,

$$\rho \operatorname{int}(\Lambda_{NC}(\mathcal{F}^{(k)})) \subseteq \Lambda_{ALG}, \tag{18}$$

then it follows that $\Lambda_{ALG} \supseteq \rho \operatorname{int}(\Lambda)$. For our algorithms, we find a $\rho$ such that (18) holds for *any traffic process* under our model. Then it follows that $\gamma^\star_{ALG} \geq \rho$.

We define the *gain* of a policy $\mu$ at time $t$ as

$$\mathcal{G}_\mu(t) = \sum\nolimits_{l\in\mathcal{K}} w_l^\mu(t) I_l^\mu(t), \tag{19}$$

and the gain over a frame is $\sum_{t\in\mathcal{F}} \mathcal{G}_\mu(t)$. To prove (18), we rely on comparing the gain (total deficit of packets transmitted) by ALG and an optimal max-gain non-causal policy over a frame. The following proposition states the result for any general interference graph.

*Proposition 1: Consider a frame $\mathcal{F} \equiv \mathcal{F}^{(k)}$, for some fixed $k$ based on returns of traffic process $\mathbf{x}(t)$ to a state $\mathbf{x}$. Let $\|w(t_0)\| = \sum_{l\in\mathcal{K}} w_l(t_0)$ be the norm of the initial* *deficit vector at the start of the frame. Suppose for a causal policy ALG, given any $\epsilon > 0$, there is a $W'$ such that when $\|w(t_0)\| > W'$,*

$$\frac{\mathbb{E}\left[\sum_{t\in\mathcal{F}}\mathcal{G}_{ALG}(t)|\mathcal{S}(t_0)\right]}{\mathbb{E}\left[\sum_{t\in\mathcal{F}}\mathcal{G}_{\mu^\star}(t)|\mathcal{S}(t_0)\right]} \geq \rho - \epsilon, \tag{20}$$

*where $\mathcal{S}(t_0) = (\Psi(t_0), w(t_0), \mathbf{x}(t_0))$, and $\mu^\star$ is the non-causal policy that maximizes the gain over the frame. Then for any $\lambda \in \rho\,\mathrm{int}(\Lambda_{NC}(\mathcal{F}))$, the deficit queues are bounded in the sense of (6).*

*Proof:* The proof of Proposition 1 is through a Lyapunov argument. It is provided in Appendix C. □

*Gain Analysis.* With Proposition 1 in hand, we analyze the achievable gain of our algorithm over a frame, compared with that of the optimal non-causal policy $\mu^\star$. Since characterizing $\mu^\star$ is hard, we extend a gain comparison technique from [19]–[21], [31] (developed for constant-weight single buffer analysis) to stochastic process $(\Psi(t), w(t), \mathbf{x}(t))$ in a general network.

Consider a state $(\Psi(t), w(t), \mathbf{x}(t))$ under our randomized algorithms at time $t \in \mathcal{F}$, and the state $(\Psi^{\mu^\star}(t), w^{\mu^\star}(t), \mathbf{x}(t))$ under the max-gain policy $\mu^\star$. Note that the traffic process $\mathbf{x}(t)$ is the same during the frame for both algorithms since we do not assume dependence between the policy decisions and the traffic process. We change the state of $\mu^\star$ (by modifying its buffers and deficits) to make it identical to $(\Psi(t), w(t), \mathbf{x}(t))$, and give $\mu^\star$ appropriate additional compensation that guarantees that, alongside the state modification, we have $\sum_{t\in\mathcal{F}}\mathcal{G}'_{\mu^\star}(t) > \sum_{t\in\mathcal{F}}\mathcal{G}_{\mu^\star}(t)$, where $\mathcal{G}'_{\mu^\star}(t)$ is the modified gain, i.e., the changes are advantageous for $\mu^\star$ considering the rest of the frame. Then, taking the expectation $\mathbb{E}[\mathcal{G}'(t)]$ with respect to the random decisions of our algorithm, AMIX-ND or AMIX-MS, and traffic patterns in a frame, we can bound the optimal gain of $\mu^\star$. Then we can prove the main results in view of Proposition 1.

The gain analysis of AMIX-ND in collocated networks and AMIX-MS in general networks is presented in Sections V-A and V-B, respectively.

## V. Proofs of Main Results

In view of Proposition 1, we provide the gain analysis of our algorithms. In what follows, we define

$$w_{max}(t) = \max_{l\in\mathcal{K}} w_l(t)\mathbb{1}(\Psi_l \neq 0), \tag{21}$$

to be the maximum deficit of a nonempty link at time $t$. Also define $[N] := \{1, 2, \ldots, N\}$. We use $\mathbb{E}_X[\cdot]$ to denote conditional expectation $\mathbb{E}[\cdot|X]$. $\mathbb{E}^Y[\cdot]$ is used to explicitly indicate that expectation is taken with respect to some random variable $Y$. $|A|$ is used to denote the cardinality of set $A$.

### A. Gain Analysis of AMIX-ND in Collocated Networks

Consider a subclass $\mathcal{P}_{ND}$ of all the policies that schedule Non-Dominated (ND) links at each slot (recall Definition 1). We refer to policies in $\mathcal{P}_{ND}$ as *ND-policies*. The roadmap for proving Theorem 1 through Proposition 1 is as follows. We first show that the optimal ND-policy is close to the optimal non-restricted policy (Lemma 2). This allows us to focus on comparing the gain of our policy with ND-policies. The gain comparison is initially performed through the gain analysis technique described in Section IV on a per-time slot basis (Lemma 3) and then extended to the whole frame (Lemma 4). As Proposition 1 is with regard to general policies, we convert the comparison with ND-policies to that with general policies (Theorem 3) to conclude the proof of Theorem 1.

The formal statement of Lemma 2 is as follows.

*Lemma 2: Consider any policy $\mu$ for scheduling packets in a frame $\mathcal{F}$. Then there is an ND-policy $\hat{\mu} \in \mathcal{P}_{ND}$ such that, under the same pattern $J \in \mathcal{J}(\mathcal{F})$ and initial state $\mathcal{S}(t_0)$,*

$$\sum_{t\in\mathcal{F}}\mathcal{G}_{\hat{\mu}}(t) \geq \sum_{t\in\mathcal{F}}\mathcal{G}_{\mu}(t) - a_{max}F^2,$$

*where $F$ is the length of the frame.*

*Proof:* Suppose the first time $\mu$ does not schedule a non-dominated link is $t_0$. Suppose $\mu$ sends earliest-deadline packet $(w_y(t_0), d_y)$ from link $y$ and $(w_x(t_0), d_x)$ be the earliest-deadline packet at a link $x$ ($x \neq y$) that strictly dominates $y$, i.e. $w_x(t_0) \geq w_y(t_0)$, $d_x \leq d_y$. Consider some alternative policy $\mu'$ which has the same transmissions as $\mu$ up to time $t_0$ but transmits the packet of $x$ at time $t_0$ instead. Let $w'_l(t)$, $l \in \mathcal{K}$ denote the link deficits under $\mu'$. Note that $w'_l(t) = w_l(t),\ \forall t \leq t_0$. We differentiate between 2 cases:

1) $\mu$ does not transmit packet $x$ in the remaining time slots. In this case, let $\mu'$ transmit the same packets as $\mu$ in the remaining slots (after $t_0$). Let $I_l(t_1, t_2) = \sum_{t=t_1}^{t_2} I_l(t)$ be the number of packets transmitted between $t_1$ and $t_2$ at link $l$ under $\mu$ (and subsequently under $\mu'$). And let $\Delta\mathcal{G} := \sum_{t\in\mathcal{F}}\mathcal{G}_{\mu'}(t) - \sum_{t\in\mathcal{F}}\mathcal{G}_{\mu}(t)$. Then we have
$$\begin{aligned}\Delta\mathcal{G} &\overset{(a)}{=} w_x(t_0) + I_y(t_0+1, F)\\ &\quad - (w_y(t_0) + I_x(t_0+1, F))\\ &\overset{(b)}{\geq} w_x(t_0) - w_y(t_0) - F \geq -F.\end{aligned}$$
To see $(a)$, notice that as a result of transmitting from link $x$ instead of link $y$, the deficit of link $y$ under $\mu'$ will be one more than that under $\mu$ at any time $t > t_0$. Similarly, the deficit of link $x$ under $\mu'$ will be one less than that under $\mu$ at any time $t > t_0$. In $(b)$, we have used the fact that $I_l(t) \in \{0, 1\}$ and $w_x(t_0) \geq w_y(t_0)$.
2) $\mu$ transmits packet $x$ at some time slot $t_a$ where $t_0 < t_a < t_0 + d_x$. In this case we let $\mu'$ transmit the same packets as $\mu$ for all $t > t_0$ except for time slot $t_a$ in which it transmits packet $y$ instead, which still has not expired yet by the domination inequality $d_y \geq d_x$. It is easy to check that
$$\begin{aligned}&\sum\nolimits_{t\in\mathcal{F}}\mathcal{G}_{\mu'}(t) - \sum\nolimits_{t\in\mathcal{F}}\mathcal{G}_{\mu}(t)\\ &\quad = w_x(t_0) + w'_y(t_a) + I_y(t_0+1, t_a-1)\\ &\qquad - w_y(t_0) - w_x(t_a) - I_x(t_0+1, t_a-1).\end{aligned} \tag{22}$$
The total deficit arrival to a link in the frame cannot be more than $a_{\max}F$. Hence,
$$\begin{aligned}&w_x(t_a) \leq w_x(t_0) + a_{\max}F - I_x(t_0, t_a-1),\\ &w'_y(t_a) \geq w_y(t_0) - I_y(t_0, t_a-1).\end{aligned}$$

Using these two inequalities in (22) yields

$$\sum_{t\in\mathcal{F}}\mathcal{G}_{\mu'}(t)-\sum_{t\in\mathcal{F}}\mathcal{G}_{\mu}(t)\geq -a_{max}F. \quad (23)$$

By repeating this process (at most $F$ times), we can transform $\mu$ to $\hat{\mu}$. From this, the final result follows. □

In what follows, let $\rho_1$ denote the efficiency ratio bound stated in Theorem 1, i.e.,

$$\rho_1 := 1-(1-1/K)^K. \quad (24)$$

Lemma 3 below relates the per time-slot gain of AMIX-ND to the amortized gain of any other ND-policy.

*Lemma 3: Under any pattern $J\in\mathcal{J}(\mathcal{F})$ of length F, for each slot $t\in\mathcal{F}$, the gain obtained by* AMIX-ND*, and the amortized gain by any ND-policy $\hat{\mu}$, starting from some state $\mathcal{S}(t)$ satisfy:*

$$\mathbb{E}^R[\mathcal{G}'_{\hat{\mu}}(t)|\mathcal{S}(t),J]\leq w_{max}(t)+\mathcal{E}_0 \quad (25)$$

$$\mathbb{E}^R[\mathcal{G}_{\text{AMIX-ND}}(t)|\mathcal{S}(t)]\geq w_{max}(t)\rho_1 \quad (26)$$

*where $\mathcal{E}_0=(a_{max}+1)d_{\max}+F$, $w_{max}(t)$ is defined in (21), $\rho_1$ is defined in (24), and $\mathbb{E}^R[\cdot]$ denotes expectation with respect to the random decisions of* AMIX-ND.

*Proof:* At time $t$, after the new arrivals have happened, we have state $\mathcal{S}(t)$. AMIX-ND decides probabilistically to transmit a packet $\mathsf{p}_f=(w_f,e_f)$ from a non-dominated link $f\in\mathcal{B}_{\text{ND}}(t)$, and the ND-policy $\hat{\mu}$ transmits a packet $\mathsf{p}_z=(w_z,e_z)$ from some other non-dominated link $z$. We distinguish two cases following the same method as in [21] but for time-varying weights.

1) $e_f\leq e_z, w_f\leq w_z$: In this case, to maintain the same buffers for both algorithms, we remove the packet $\mathsf{p}_f$ from the buffer of link $f$ under $\hat{\mu}$ and inject the packet $\mathsf{p}_z$ to link $z$ so that $\hat{\mu}$ gets a packet with higher deadline and higher weight at the time $t$. Since both packets will expire in at most $d_{\max}$ slots, the deficit of $f$ can only increase by at most $d_{\max}a_{\max}$ before packet $e_f$ expires, whereas the deficit of $z$ can decrease by at most $d_{\max}$. Therefore giving $\hat{\mu}$ an additional compensation of $d_{\max}(a_{\max}+1)$ will guarantee that the modification is advantageous. Further, we decrease the deficit of link $f$ by one ($w_f-1$ in $\hat{\mu}$) and we increase the deficit of link $z$ by one ($w_z+1$ in $\hat{\mu}$). Then $\hat{\mu}$ and AMIX-ND have the same exact state. Making this change in the deficit will reduce the gain for each packet transmitted from link $f$ in the future by one. To compensate for this, we give $\hat{\mu}$ extra gain which is the number of packets transmitted from link $f$ for the rest of the frame, which is less than $F$. Hence, the total compensation is bounded by $F+(a_{max}+1)d_{\max}$.
2) $e_z\leq e_f, w_z\leq w_f$: In this case, we allow $\hat{\mu}$ to additionally transmit the packet $\mathsf{p}_f$ at time $t$, and inject a copy of packet $\mathsf{p}_z$ to the buffer of link $z$. Allowing $\hat{\mu}$ to transmit packet $\mathsf{p}_f$ at time $t$ instead of a later time can only be disadvantageous from the total-gain perspective in the case where the deficit of $f$ increases due to other arrivals in subsequent times, but such increase of deficit can be at most $d_{\max}a_{\max}$, hence giving this compensation guarantees that this modification is advantageous for $\hat{\mu}$. Further we decrease the deficit counter of link $f$ in $\hat{\mu}$ by one, which might not be advantageous for $\hat{\mu}$ for future times. Similarly to the other case, to guarantee that the change is advantageous for $\hat{\mu}$, we give it one extra reward for each possible transmission from link $f$ in the rest of the frame, which is less than $F$.

Note that the additional compensation in both cases is bounded by $\mathcal{E}_0 := F+(a_{\max}+1)d_{\max}$. Let $\mathcal{G}'^{(h_i)}_{\hat{\mu}}(t)$ denote the reward (including the compensation) gained by $\hat{\mu}$ when it transmits a non-dominated packet $h_i$ (recall $h_i$ from Algorithm 1). In each case, $\hat{\mu}$ collects the gain of the transmitted packet $w_{h_i}(t)$, and further when AMIX-ND transmits a packet $h_j$ such that case 2 applies (i.e. when $j<i$), $\hat{\mu}$ collects the gain from the additional transmission. As a result, we have,

$$\mathbb{E}^R[\mathcal{G}'^{(h_i)}_{\hat{\mu}}(t)|\mathcal{S}(t)]\leq w_{h_i}(t)+\sum_{h_j:j<i}p_{h_j}(t)w_{h_j}(t)+\mathcal{E}_0. \quad (27)$$

Note that the right-hand side of (27) is maximized over $i$ for $i=1$. This can be seen by showing that the difference of the values for two successive indexes, $i$ and $i+1$, is non-negative:

$$\begin{aligned}&w_{h_i}(t)+\sum_{h_j:j<i}p_{h_j}(t)w_{h_j}(t)+\mathcal{E}_0\\&\quad-(w_{h_{i+1}}(t)+\sum_{h_j:j<i+1}p_{h_j}(t)w_{h_j}(t))+\mathcal{E}_0)\\&=w_{h_i}(t)-w_{h_{i+1}}(t)+p_{h_i}(t)w_{h_i}(t)\overset{(a)}{\geq}0, \end{aligned} \quad (28)$$

where $(a)$ follows from the assigned probabilities (line 4 in Algorithm 2). Further note that for $i=1$ the right-hand side of (27) is equal to $w_{h_1}(t)+\mathcal{E}_0=w_{max}(t)+\mathcal{E}_0$. Hence, (25) indeed holds.

Now regarding AMIX-ND, similar derivation applies as in [22] to get the final bound. To see that, first let the number of links with positive probability be $B\leq K$. Then

$$\begin{aligned}&\mathbb{E}^R[\mathcal{G}_{\text{AMIX-ND}}(t)|\mathcal{S}(t)]\\&\quad=\sum_{i\in[B]}w_{h_i}(t)p_{h_i}(t)\\&\quad=\sum_{i\in[B-1]}w_{h_i}(t)p_{h_i}(t)+p_{h_B}(t)w_{h_B}(t)\\&\quad\overset{(a)}{=}w_{h_1}(t)-w_{h_B}(t)(1-p_{h_B}(t))\\&\quad\overset{(b)}{=}w_{h_1}(t)\Big(1-\prod_{i=1}^{B}(1-p_{h_i}(t))\Big)\\&\quad\overset{(c)}{\geq}w_{h_1}(t)\Big(1-(\frac{B-1}{B})^B\Big),\end{aligned}$$

where $(a)$ follows since $w_{h_i}(t)p_{h_i}(t)=w_{h_i}(t)-w_{h_{i+1}}(t)$ for $i<B$ (by line 4 in Algorithm 2 where the minimum is equal to the first of the two terms for $i<B$), $(b)$ follows by $w_{h_{i+1}}=w_{h_i}(t)(1-p_{h_i}(t))$ for $i<B$, and $(c)$ follows by applying the inequality between arithmetic and geometric means for the product of the $B$ terms: $1-p_{h_i}(t),\ i\in[B]$. □

Next in Lemma 4, we convert the per time-slot gain ratio of Lemma 3 to a ratio between the total gain of AMIX-ND and any ND-policy over a frame.

*Lemma 4: Over any frame $\mathcal{F}$, with initial state $\mathcal{S}(t_0) = (\Psi(t_0), w(t_0), \mathbf{x}(t_0))$, and any ND-policy $\hat{\mu}$.*

$$\lim_{\|w(t_0)\|\to\infty} \frac{\mathbb{E}^{R,J}[\sum_{t\in\mathcal{F}}\mathcal{G}_{\text{AMIX-ND}}(t)|\mathcal{S}(t_0)]}{\mathbb{E}^{J}[\sum_{t\in\mathcal{F}}\mathcal{G}_{\hat{\mu}}(t)|\mathcal{S}(t_0)]} \geq \rho_1 \quad (29)$$

*Proof:* Consider the initial state $\mathcal{S}(t_0)$ and a pattern $J$ of size $F$. Taking expectations of the result of Lemma 3, conditioned on traffic pattern $J$ of length $F$, we get

$$\mathbb{E}^R[\mathbb{E}^R[\mathcal{G}'_{\hat{\mu}}(t)|\mathcal{S}(t), J]|\mathcal{S}(t_0), J] \leq \mathbb{E}^R[w_{max}(t)|\mathcal{S}(t_0), J] + \mathcal{E}_0$$
$$\mathbb{E}^R[\mathbb{E}^R[\mathcal{G}_{ALG}(t)|\mathcal{S}(t)]|\mathcal{S}(t_0), J] \geq \mathbb{E}^R[w_{max}(t)|\mathcal{S}(t_0), J]\rho_1,$$

where ALG = AMIX-ND. Now notice that

$$\begin{aligned}&\mathbb{E}^R[\mathbb{E}^R[\mathcal{G}'_{\hat{\mu}}(t)|\mathcal{S}(t)]|\mathcal{S}(t_0), J]\\ &= \mathbb{E}^R[\mathbb{E}^R[\mathcal{G}'_{\hat{\mu}}(t)|\mathcal{S}(t), \mathcal{S}(t_0), J]|\mathcal{S}(t_0), J]\\ &= \mathbb{E}^R[\mathcal{G}'_{\hat{\mu}}(t)|\mathcal{S}(t_0), J]\end{aligned}$$

where the first equality is due to the fact that, given $\mathcal{S}(t)$, the gain of $\hat{\mu}$ at time $t$ does not depend on $J$ and $\mathcal{S}(t_0)$. The second equality is by the tower property of conditional expectation. Therefore, we get

$$\mathbb{E}^R[\mathcal{G}'_{\hat{\mu}}(t)|\mathcal{S}(t_0), J] \leq \mathbb{E}^R[w_{max}(t)|\mathcal{S}(t_0), J] + \mathcal{E}_0 \quad (30)$$

Using similar arguments for the expected gain of AMIX-ND,

$$\mathbb{E}^R[\mathcal{G}_{ALG}(t)|\mathcal{S}(t_0), J] \geq \mathbb{E}^R[w_{max}(t)|\mathcal{S}(t_0), J]\rho_1. \quad (31)$$

Summing the gains over time slots in the frame, we have

$$\begin{aligned}&\mathbb{E}\Big[\sum_{t=t_0}^{t_0+F}\mathcal{G}_{\hat{\mu}}(t)|\mathcal{S}(t_0), J\Big]\\ &\leq \mathbb{E}^R\Big[\sum_{t=t_0}^{t_0+F}\mathcal{G}'_{\hat{\mu}}(t)|\mathcal{S}(t_0), J\Big]\\ &\leq \mathbb{E}^R\Big[\sum_{t=t_0}^{t_0+F}w_{max}(t)|\mathcal{S}(t_0), J\Big] + \mathcal{E}_0\ F\end{aligned}$$

and taking the expectation with respect to the pattern $J$,

$$\mathbb{E}^J\Big[\sum_{t\in\mathcal{F}}\mathcal{G}_{\hat{\mu}}(t)|\mathcal{S}(t_0)\Big] \leq \mathbb{E}^{R,J}[\sum_{t\in\mathcal{F}}w_{max}(t)|\mathcal{S}(t_0)] + \bar{\mathcal{E}} \quad (32)$$

where $\bar{\mathcal{E}} = (a_{max}+1)\mathbb{E}[F]d_{\max} + 2\mathbb{E}[F^2]$. Similarly,

$$\mathbb{E}^{R,J}\Big[\sum_{t\in\mathcal{F}}\mathcal{G}_{\text{AMIX-ND}}(t)|\mathcal{S}(t_0)\Big] \geq \rho_1\mathbb{E}^{R,J}\Big[\sum_{t\in\mathcal{F}}w_{max}(t)|\mathcal{S}(t_0)\Big] \quad (33)$$

Now consider link $l_1$ that has the maximum deficit at time $t_0$. At any time $t\in\mathcal{F}$,

$$w_{l_1}(t_0) + a_{max}F \geq w_{l_1}(t) \geq w_{l_1}(t_0) - F.$$

Recall that $w_{max}(t)$ denotes the maximum deficit among the nonempty links, and $a_{l_1(t)} > 0$ implies that the link $l_1$'s buffer is nonempty at time $t$. Therefore

$$w_{max}(t) \geq w_{l_1}(t)\mathbb{1}(a_{l_1}(t) > 0) \geq w_{l_1}(t)\frac{a_{l_1}(t)}{a_{\max}}. \quad (34)$$

Hence,

$$\begin{aligned}&\mathbb{E}^{R,J}[\sum_{t\in\mathcal{F}}w_{max}(t)|\mathcal{S}(t_0)] \geq \mathbb{E}^{R,J}\Big[\sum_{t\in\mathcal{F}}w_{l_1}(t)\frac{a_{l_1}(t)}{a_{\max}}\big|\mathcal{S}(t_0)\Big]\\ &\geq \frac{1}{a_{\max}}\mathbb{E}^{R,J}\Big[(w_{l_1}(t_0)-F)\sum_{t\in\mathcal{F}}a_{l_1}(t)\big|\mathcal{S}(t_0)\Big]\\ &\geq \frac{\|w(t_0)\|}{K}\mathbb{E}[F]\frac{\overline{a}_{l_1}}{a_{\max}} - \mathbb{E}[F^2] \quad (35)\end{aligned}$$

and therefore

$$\lim_{\|w(t_0)\|\to\infty}\mathbb{E}^{R,J}\Big[\sum_{t\in\mathcal{F}}w_{max}(t)|\mathcal{S}(t_0)\Big] = \infty.$$

Using this and (32) and (33), the result follows. From which it follows that

$$\frac{\mathbb{E}^{R,J}[\sum_{t\in\mathcal{F}}\mathcal{G}_{\text{AMIX-ND}}(t)|\mathcal{S}(t_0)]}{\mathbb{E}^{J}[\sum_{t\in\mathcal{F}}\mathcal{G}_{\hat{\mu}}(t)|\mathcal{S}(t_0)]} \geq \rho_1 - \epsilon$$

as $\|w(t_0)\|\to\infty$. □

Theorem 3 below states the relationship between the gain of AMIX-ND and that of any policy (not necessarily an ND-policy), using Lemma 4 and Lemma 2.

*Theorem 3: For any policy $\mu$, and* AMIX-ND*, given any $\epsilon > 0$, there is $W'$ such that when $\|w(t_0)\| \geq W'$:*

$$\mathbb{E}_{t_0}\Big[\sum_{t\in\mathcal{F}}\mathcal{G}_{\text{AMIX-ND}}(t)\Big] \geq (\rho_1-\epsilon)\mathbb{E}_{t_0}\Big[\sum_{t\in\mathcal{F}}\mathcal{G}_{\mu}(t)\Big].$$

*Proof:* Using Lemma 2 for the optimal $\mu$ over a frame $\mathcal{F}$, and the fact that $\mu$ is at least as effective as $\hat{\mu}$

$$\mathbb{E}_{t_0}[\sum_{t\in\mathcal{F}}\mathcal{G}_{\mu}(t)] \geq \mathbb{E}_{t_0}[\sum_{t\in\mathcal{F}}\mathcal{G}_{\hat{\mu}}(t)] \geq \mathbb{E}_{t_0}[\sum_{t\in\mathcal{F}}\mathcal{G}_{\mu}(t)] - a_{max}\mathbb{E}[F^2]$$

Dividing by $\mathbb{E}_{t_0}[\sum_{t\in\mathcal{F}}\mathcal{G}_{\mu}(t)]$ and taking limits as $\|w(t_0)\|\to\infty$, the squeeze limits theorem yields:

$$\frac{\mathbb{E}_{t_0}[\sum_{t\in\mathcal{F}}\mathcal{G}_{\hat{\mu}}(t)]}{\mathbb{E}_{t_0}[\sum_{t\in\mathcal{F}}\mathcal{G}_{\mu}(t)]} \to 1 \quad (36)$$

since, as we showed in the proof of Lemma 4, $\mathbb{E}_{t_0}[\sum_{t\in\mathcal{F}}\mathcal{G}_{\mu}(t)] \to \infty$, as $\|w(t_0)\|\to\infty$. Using (36) and Lemma 4, the result follows. □

Using Theorem 3 and Proposition 1 concludes the proof of Theorem 1.

### *B. Gain Analysis of* AMIX-MS *in General Networks*

First we show that binary search in Algorithm 3 suffices for computing $\bar{n}$ defined in (14).

*Proposition 2: The binary search in Algorithm 3 computes $\bar{n}$ as defined in (14).*

*Proof:* The proof of Proposition 2 is straightforward and provided in Appendix D. □

To prove Theorem 2, similarly to the proof of Theorem 1, we rely on the amortized gain analysis technique for a single slot (Lemma 7) which we then extend to the entire frame (Theorem 4) and use Proposition 1. We introduce a few auxiliary Lemmas below to simplify these main steps.

First, we state Lemmas 5 and 6 below regarding the properties of the probabilities used by AMIX-MS, which are used in the gain analysis. Their proofs follow directly from the probabilities used by AMIX-MS.

*Lemma 5: $C_n(t)$ (defined in (12)) is strictly decreasing as a function of $n$, for $\bar{n} \le n \le R$.*

*Proof:* The proof is through algebraic manipulations and provided in Appendix E. □

*Lemma 6: If $i \notin [\bar{n}]$ and $j \in [\bar{n}]$, for the choice of probabilities $p_k^{\bar{n}}(t)$ in (13) selected by* AMIX-MS*, we have*

$$W_{M_i}(t) + \sum\nolimits_{k\in[\bar{n}]} p_k^{\bar{n}}(t) W_{M_k}(t) \le W_{M_j}(t) + \sum\nolimits_{k\in[\bar{n}]\setminus\{j\}} p_k^{\bar{n}}(t) W_{M_k}(t)$$

*Proof:* Equivalently after simplifying the inequality, we need to prove:

$$W_{M_i}(t) \le W_{M_j}(t)(1 - p_j^{\bar{n}}(t)) = C_{\bar{n}}(t).$$

This is trivially true for $i > R$ since $W_{M_i}(t) = 0$ by the definition of $R$ (Definition 2). Since $i \notin [\bar{n}]$, for the case of $i \le R$ we have $W_{M_i}(t) < C_i(t)$, and from the monotonicity of $C_n(t)$ for $R \ge n \ge \bar{n}$ (Lemma 5), since $i > \bar{n}$, we have $C_i(t) < C_{\bar{n}}(t)$. Therefore, $W_{M_i}(t) < C_{\bar{n}}(t)$. □

Lemma 7 below relates the per time-slot gain of AMIX-MS with the amortized gain of the Max-Gain policy, similarly to Lemma 3 in the collocated case.

*Lemma 7: For any pattern $J \in \mathcal{J}(\mathcal{F})$, for each time $t \in \mathcal{F}$, the gain obtained by* AMIX-MS*, and the amortized gain obtained by the Max-Gain policy $\mu$, starting from some state $\mathcal{S}(t)$, satisfy:*

$$\mathbb{E}^R[\mathcal{G}'_\mu(t)|\mathcal{S}(t), J] \le \sum_{i\in[\bar{n}]} W_{M_i}(t) - (\bar{n}-1)C_{\bar{n}}(t) + \mathcal{E}_m \tag{37}$$

$$\mathbb{E}^R[\mathcal{G}_{\text{AMIX-MS}}(t)|\mathcal{S}(t)] = \sum_{i\in[\bar{n}]} W_{M_i}(t) - \bar{n}C_{\bar{n}}(t) \tag{38}$$

*where $\mathcal{E}_m = K(F + a_{max}d_{\max})$ and $\mathbb{E}^R$ is with respect to decisions of* AMIX-MS.

*Proof:* Using the probabilities computed by AMIX-MS, the expected gain of AMIX-MS at time $t$ is

$$\begin{aligned}\mathbb{E}[\mathcal{G}_{\text{AMIX-MS}}(t)|\mathcal{S}(t)] &= \sum_{i\in[\bar{n}]} p_i^{\bar{n}}(t) W_{M_i}(t)\\ &= \sum_{i\in[\bar{n}]} W_{M_i}(t) - \bar{n}C_{\bar{n}}(t).\end{aligned}$$

Next for the amortized gain of the Max-Gain Policy $\mu$, we will apply the same technique as in the collocated networks case, where we modify the buffers and give $\mu$ additional reward. Suppose $\mu$ transmits $M_i$, and AMIX-MS transmits some $M_j$. We make the buffers the same by allowing $\mu$ to additionally transmit all the packets that are transmitted by AMIX-MS but not by $\mu$ (i.e., in links $M_j \setminus M_i$). As the deficit of these packets can increase by at most $d_{\max}a_{max}$ before they expire, we give a compensation of $Kd_{\max}a_{max}$ to $\mu$. Since transmitting these additional packet will result in a decrease of the deficit by one for each link in $M_j \setminus M_i$ for $\mu$ in the remaining slots, we give $\mu$ an additional reward $KF$ which is an upper bound on the number of packets transmitted by $\mu$ from links $M_j \setminus M_i$ in the remaining slots. Thus the total compensation is $\mathcal{E}_m = K(F + a_{max}d_{\max})$.To compute the expected gain, we differentiate between two cases:

*Case 1:* $i \in [\bar{n}]$. In this case, we can write

$$\begin{aligned}&\mathbb{E}_t[\mathcal{G}'^{M_i}_{\hat{\mu}}(t)|J]\\ &= W_{M_i}(t) + \sum_{j\in[\bar{n}]\setminus\{i\}} p_j^{\bar{n}}(t)\left(W_{M_j\setminus M_i}(t) + \mathcal{E}_m\right)\\ &\le W_{M_i}(t) + \sum_{j\in[\bar{n}]\setminus\{i\}} p_j^{\bar{n}}(t)\left(W_{M_j}(t) + \mathcal{E}_m\right) \quad (39)\\ &= W_{M_i}(t)(1 - p_i^{\bar{n}}(t)) + \sum_{j\in[\bar{n}]} p_j^{\bar{n}}(t) W_{M_j}(t)\\ &\quad + \sum_{j\in[\bar{n}]\setminus\{i\}} p_j^{\bar{n}}(t)\mathcal{E}_m\\ &= C_{\bar{n}}(t) + \sum_{i\in[\bar{n}]} W_{M_i}(t) - \bar{n}C_{\bar{n}}(t) + \sum_{j\in[\bar{n}]\setminus\{i\}} p_j^{\bar{n}}(t)\mathcal{E}_m \quad (40)\\ &\le C_{\bar{n}}(t) + \sum_{i\in[\bar{n}]} W_{M_i}(t) - \bar{n}C_{\bar{n}}(t) + \mathcal{E}_m. \quad (41)\end{aligned}$$

*Case 2:* $i \notin [\bar{n}]$. In this case, we have

$$\begin{aligned}\mathbb{E}_t[\mathcal{G}'^{M_i}_{\hat{\mu}}(t)|J] &\le W_{M_i}(t) + \sum_{k\in[\bar{n}]} p_k^{\bar{n}}(t)(W_{M_k}(t) + \mathcal{E}_m)\\ &\overset{(a)}{\le} W_{M_j}(t) + \sum_{k\in[\bar{n}]\setminus\{j\}} p_k^{\bar{n}}(t) W_{M_k}(t) + \mathcal{E}_m\\ &= C_{\bar{n}}(t) + \sum_{i\in[\bar{n}]} W_{M_i}(t) - \bar{n}C_{\bar{n}}(t) + \mathcal{E}_m,\end{aligned}$$

where in $(a)$ we applied Lemma 6 for $i, j$. Note that in both cases, the upper bound is the same and does not depend on the particular choice of $M_i$. □

Lemma 8 below provides a bound on the ratio between the bounds in Lemma 7 that will be helpful in our subsequent analysis.

*Lemma 8: For $C_{\bar{n}}(t)$ in (12), We have*

$$\frac{\sum_{i\in[\bar{n}]} W_{M_i}(t) - \bar{n}C_{\bar{n}}(t)}{\sum_{i\in[\bar{n}]} W_{M_i}(t) - (\bar{n}-1)C_{\bar{n}}(t)} \ge \frac{|\mathcal{I}|}{2|\mathcal{I}| - 1}.$$

*Proof:* Suffices to show that

$$\frac{\sum_{i\in[\bar{n}]} W_{M_i}(t) - \bar{n}C_{\bar{n}}(t)}{\sum_{i\in[\bar{n}]} W_{M_i}(t) - (\bar{n}-1)C_{\bar{n}}(t)} \ge \frac{\bar{n}}{2\bar{n} - 1}, \tag{42}$$

since $|\mathcal{I}| \ge |\mathcal{M}(t)| \ge \bar{n}$. For the non-trivial case, we have $\bar{n} - 1 > 0$, and therefore inequality (42) can equivalently be written as $(\bar{n} - 1)\sum_{i\in[\bar{n}]} W_{M_i}(t) \ge \bar{n}^2(t)C_{\bar{n}}(t)$. This inequality holds since it follows by applying the inequality between arithmetic and harmonic means:

$$\frac{1}{\bar{n}}\sum_{i\in[\bar{n}]} W_{M_i}(t) \ge \frac{\bar{n}}{\sum_{i\in[\bar{n}]} W_{M_i}(t)^{-1}},$$

and the fact that $\bar{n} - 1 \ge 1$. □

Combining previous Lemmas, we relate the total gain in a frame of AMIX-MS and any non-causal policy, given that both are initialized with the same state, in Theorem 4.

*Theorem 4: Under* AMIX-MS*, given any $\epsilon > 0$ there is $W'$ such that for all $\|w_0\| = \sum_{l\in\mathcal{K}} w_L(t_0) \geq W'$,*

$$\mathbb{E}_{t_0}^{R,J}\Big[\sum_{t\in\mathcal{F}}\mathcal{G}_{\text{AMIX-MS}}(t)\Big] \geq (\rho_2-\epsilon)\mathbb{E}_{t_0}^{J}\Big[\sum_{t\in\mathcal{F}}\mathcal{G}_{\mu}(t)\Big],$$

*where $\mu$ is any non-causal policy, and $\rho_2 = \frac{|\mathcal{I}|}{2|\mathcal{I}|-1}$.*

*Proof:* By using Lemma 7, summing and taking expectation similar to the proof of Lemma 4, it follows that

$$\mathbb{E}\Big[\sum_{t\in\mathcal{F}}\mathcal{G}_{\text{AMIX-MS}}(t)|\mathcal{S}(t_0)\Big] = \mathbb{E}\Big[\sum_{t\in\mathcal{F}} x(t)|\mathcal{S}(t_0)\Big],$$
$$\mathbb{E}\Big[\sum_{t\in\mathcal{F}}\mathcal{G}_{\hat{\mu}}(t)|\mathcal{S}(t_0)\Big] \leq \bar{\mathcal{E}}_m + \mathbb{E}\Big[\sum_{t\in\mathcal{F}} y(t)|\mathcal{S}(t_0)\Big],$$

where $\bar{\mathcal{E}}_m = K(\mathbb{E}[F^2] + \mathbb{E}[F]a_{max}d_{\max})$, and $x(t) = y(t) - C_{\bar{n}}(t)$, where

$$y(t) := \sum_{i\in[\bar{n}]} W_{M_i}(t) - (\bar{n}(t)-1)C_{\bar{n}}(t).$$

Now notice that

$$\begin{aligned} y(t) &= C_{\bar{n}}(t) + \sum_{i\in[\bar{n}]} W_{M_i}(t) - \bar{n}C_{\bar{n}}(t) \\ &= W_{M_1}(t)(1-p_1^{\bar{n}}(t)) + \sum_{i\in[\bar{n}]} p_i^{\bar{n}}(t) W_{M_i}(t) \\ &= W_{M_1}(t) + \sum_{i\in[\bar{n}]\setminus\{1\}} p_i^{\bar{n}}(t) W_{M_i}(t) \\ &\geq W_{M_1}(t) \geq w_{max}(t). \end{aligned} \tag{43}$$

It then follows

$$\begin{aligned} &\lim_{\|w_0\|\to\infty} \frac{\mathbb{E}[\sum_{t\in\mathcal{F}} x(t)|\mathcal{S}(t_0)]}{\mathbb{E}[\sum_{t\in\mathcal{F}} y(t)|\mathcal{S}(t_0)] + \bar{\mathcal{E}}_m} \\ &\overset{(a)}{=} \lim_{\|w_0\|\to\infty} \frac{\mathbb{E}[\sum_{t\in\mathcal{F}} x(t)|\mathcal{S}(t_0)]}{\mathbb{E}[\sum_{t\in\mathcal{F}} y(t)|\mathcal{S}(t_0)]} \overset{(b)}{\geq} \frac{|\mathcal{I}|}{2|\mathcal{I}|-1}, \end{aligned}$$

where in $(a)$ we used the fact that $\bar{\mathcal{E}}_m < \infty$, and that the remaining expression in the denominator goes to infinity using the inequality derived in (43) alongside the argument in (35). In $(b)$ we used Lemma 8. □

Using Theorem 4 and Proposition 1 concludes the proof of Theorem 2.

*Remark 5:* The design of the probabilities in AMIX-ND and AMIX-MS were such that all the scheduling choices by the Max-Gain policy $\mu$ lead to an equal amortized gain or an equal bound on it. In particular, for the collocated case this was done by choosing probabilities appropriately in the bound (27), whereas for the general case, this was obtained by choosing the probabilities appropriately in (39). This ensures that an optimal policy does not have an option that provides a big advantage over our policy.

### C. Tightness of Gain Analysis

We construct adversarial examples for each algorithm in order to find upper bounds on their performance.

*AMIX-ND:* For AMIX-ND, we can construct an example such that its expected gain over a frame of size $F$ is approximately $\rho_1 = 1-(1-1/K)^K$ fraction of the optimal gain. Given $K$ links, suppose the $i$-th non-dominated link $h_i$ has deficit $L(1-1/K)^i$ for a large enough constant $L$, so that during the frame the deficits are effectively constant. Then assume that each link $h_i$ has a large number $L' > F$ of packets with equal deadline $F-i$, so that it always has packets for transmission during the frame. Further assume $K \ll F$. In this case, every link has an equal probability $1/K$ of being scheduled and it can be shown that the majority of the time, i.e., for any $t \leq F-K$, the expected gain of AMIX-ND is $\approx L(1-1/K)(1-(1-1/K)^K)$ whereas the optimal policy can obtain $\approx L(1-1/K)$, i.e., the ratio of the two is $\rho_1$. This shows the tightness of Lemma 4.

*AMIX-MS:* Note that under AMIX-MS, maximal schedules with equal weights have equal probabilities. Consider a collocated network with $K$ links and apply AMIX-MS, in which case $\rho_2 = \frac{K}{2K-1}$. Suppose link $l_i$, for $i \in [K]$, has deadline $i$ and deficit $L$ (equal across links). The optimal policy in this example can transmit all the packets using an EDF rule, whereas the expected gain of AMIX-MS can be found through a recursive program. For example for $K=2$, we have two links with packets of deadlines 1 and 2, and AMIX-MS yields an expected gain of $1.5L$ and the optimal yields $2L$, thus the ratio is $0.75$. For $K = 3,\ldots,6$, the ratio is $0.722, 0.698, 0.685, 0.676$, respectively. As $K\to\infty$, we can show that the ratio converges to $\frac{e-1}{e}$. We provide an informal proof based on fluid limits below.

Let $|\Psi^K(t)|$ denote the number of remaining packets at time $t$ in the system, if we start with $K$ packets. Hence $|\Psi^K(0)| = K$. Let $\psi^K(t) = \frac{|\Psi^K(\lfloor Kt\rfloor)|}{K}$ denote the fraction of remaining packets at time $\lfloor Kt\rfloor$, and consider the fluid limit

$$\psi(t) := \lim_{K\to\infty}\psi^K(t) = \lim_{K\to\infty}\frac{|\Psi^K(\lfloor Kt\rfloor)|}{K}.$$

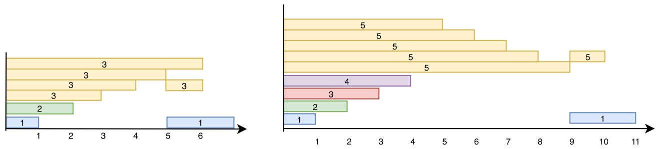
In this limiting regime, time $t$ in the scaled system changes in interval $[0,1]$, and similarly deadlines are in the range $[0,1]$. During $[0,t)$, the scheduler has transmitted packets from those with deadlines in the ranges $[0,t)$ and $[t,1]$. The packets scheduled from those with deadline in $[t,1]$ were chosen *uniformly at random*, and the number of existing packets at time $t$ with deadline in $[t,1]$ is $\psi(t)$ at the fluid limit. Hence, at time $t$, the density of existing packets with deadline $t' > t$ is $\frac{\psi(t)}{1-t}$, for any $t' > t$. Hence, the remaining number of packets with deadline in interval $(t, t+dt)$ at the fluid limit is $\frac{\psi(t)}{1-t}dt$, which will all expire by $t+dt$. Also note that if $\psi(t) > 0$, buffer is nonempty and we always have a packet transmission, thus the rate of packet transmission is one. Hence, the evolution of $\psi(t)$ can be described as

$$\frac{d}{dt}\psi(t) = -\frac{\psi(t)}{1-t} - 1,$$

with $\psi(0) = 1$. It then follows that

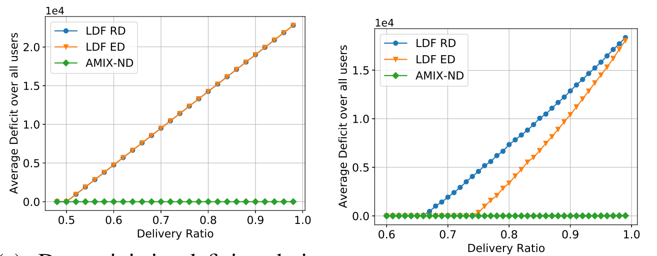
$$\psi(t) = 1 - t + (1-t)\ln(1-t).$$

It can be seen that $\psi(t) > 0$ for $t \in [0, \frac{e-1}{e})$, and $\psi(\frac{e-1}{e}) = 0$. This implies that AMIX-MS transmits $\frac{e-1}{e}K + o(K)$ packets. Recall that the optimal policy can transmit all $K$ packets. Hence, as $K\to\infty$, the approximation ratio is $\frac{e-1}{e}$. Therefore, the approximation ratio derived for AMIX-MS cannot be greater than $\frac{e-1}{e}$. We leave closing the gap between the upper



(a) Traffic pattern F (b) Traffic pattern E

Fig. 3. Two of the traffic patterns used in simulations.




(a) Deterministic deficit admission (b) Coin-tossing deficit admission

Fig. 4. Comparison between AMIX-ND and LDF policies in a two-link network.

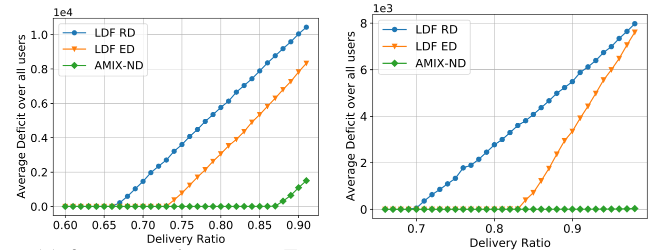
bound and lower bound on the performance of AMIX-MS as an open problem.

## VI. Simulation Results

If the packet arrival rate becomes very large, any policy inevitably will be restricted to a small delivery ratio $\mathbf{p}$. But then due to high availability of packets in the buffers, the policy can always schedule packets, thus making the deadlines irrelevant. Similarly, if the packet deadlines become very large, the problem is reduced to the regular non real-time scheduling and deadline-oblivious algorithms like LDF should perform reasonably well. Hence, we focus on the interesting scenario when packet arrival rates or deadlines are not excessively large. In our simulations, we also consider two cases for the deficit admission (see the model section): one is based on coin tossing where each arrival on a link $l$ is counted as deficit with probability $p_l$, and the other is deterministic, where each arrival increases the deficit by exactly $p_l$.

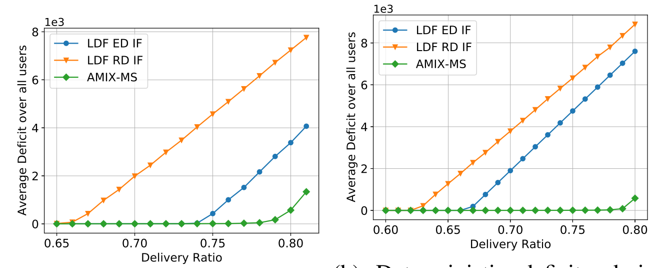
We compare the performance of our randomized algorithms, AMIX-ND and AMIX-MS with LDF. Recall that LDF chooses the longest-deficit link, then removes the interfering links with this link, and repeat the procedure. We further consider two versions of LDF: One is LDF that does a random tie breaking when presented with a deficit tie (LDF-RD), and the other version tries to schedule the non-dominated link and its earliest-deadline packet (LDF-ED) in such tie situations. In the plots, we compare the average deficit (over all links) as we vary the value of the delivery ratio.

*Collocated Networks:* We first consider two interfering links with deterministic deficit admission. The traffic is periodic and consists of alternating Pattern A and Pattern B of Figure 1, with the delivery ratios satisfying $p_2 = p_1 + 0.001$. Figure 4a shows the result. As we can see, AMIX-ND is able to achieve roughly $p_1 = 0.996$, whereas both versions of LDF become unstable for $p_1 = 0.5 + \epsilon$. In Figure 4b, again for two users, we used a traffic that consists of Pattern C followed by Pattern B, repeatedly. This time we keep $p_1 = p_2$.




(a) 3 users under pattern F. (b) 5 users under pattern E.

Fig. 5. Comparison between AMIX-ND and LDF policies in collocated networks with coin-tossing deficit admission.




(a) Coin-tossing deficit admission (b) Deterministic deficit admission

Fig. 6. Comparison between AMIX-MS and LDF policies in a lightly connected interference graph with 5 links.

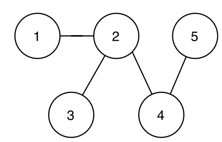
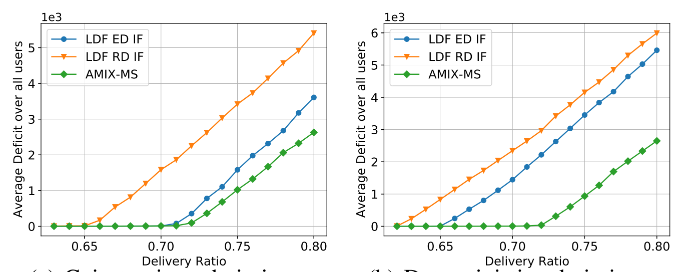


Fig. 7. Interference graph $\mathcal{G}_1$ used for simulations in Figure 6.




(a) Coin-tossing admission (b) Deterministic admission

Fig. 8. Comparison between policies on a complete bipartite graph with 8 links, and i.i.d. and Markovian arrivals.

AMIX-ND achieved near $p_1 = 1.0$, whereas the better version of LDF achieved roughly 0.75, resulting in a gap of around 0.25.

Figure 5a and Figure 5b show the results for collocated networks with various number of users, when traffic F and traffic E from Figure 3 are used, respectively. In traffic F, when $p_1 = p_2 = p_3 = p$, the optimal policy can support at most $p = 7/8 = 0.875$. In this case AMIX-ND achieves at least $p = 0.87$, whereas LDF-ED achieves roughly $p = 0.73$. Traffic E is similar in nature, but with more users and AMIX-ND is able to achieve near optimal behavior; the result is shown in Figure 5b.

*General Networks:* We first consider the interference graph $\mathcal{G}_1$ in Figure 7 involving 5 links, and interference edges

$E_l = \{(l_1, l_2), (l_2, l_3), (l_2, l_4), (l_4, l_5)\}$. For links $l_2$ and $l_5$, we have a periodic traffic with period $t = 5$, where in slot 1 there are 2 packets arriving with deadline 2 and 3 and in slot 4 a packet arrives with deadline 1, and for links $l_1, l_3, l_5$, we have 1 packet arriving with deadline 1 at slot 1, and 1 packet arriving with deadline 2 at slot 4. The result for this graph is shown in Figure 6.

Next, we consider a complete bipartite graph $\mathcal{G}_2$ with two components, $V_1 = \{l_1, l_2, l_3, l_4\}$ and $V_2 = \{l_5, l_6, l_7, l_8\}$. The traffic used for links $l_1, l_2$ is the same as that of link $l_1$ in Graph $\mathcal{G}_1$ above. For links $l_3, l_4$ we used i.i.d. Bernulli with 1 arrival having deadline 1 with probability 0.25. For links $l_5, l_6$ we used the traffic used for link $l_2$ in Graph $\mathcal{G}_1$. For links $l_7, l_8$ we used i.i.d. traffic with 7 arrivals with probability 0.05, and 0 arrivals otherwise, and deadline 10. The results are depicted in Figures 8a and 8b.

As we see, simulation results indicate that there are many scenarios that result in significant gap between our algorithms and LDF variants. This gap is especially pronounced when deterministic deficit admission is used, which is preferable as it provides a short-term guarantee on the deficit of a user.

## VII. Conclusion

In this paper, we studied real-time traffic scheduling in wireless networks under an interference-graph model. Our results indicated the power of randomization over the prior deterministic greedy algorithms for scheduling real-time packets. In particular, our proposed randomized algorithms significantly outperform the well-known LDF policy in terms of efficiency ratio. As future work, we will investigate efficient and distributed implementation of AMIX-MS for general graphs, and incorporating fading channels in the wireless network model.

## Appendix A
## Computational Complexity of AMIX-ND

We present an implementation of AMIX-ND. We assume we have access to the earliest deadline packet of every link.

At every time $t$, we can sort the active links according to their deficits (and break ties in favor of the earliest-deadline link) in an array $A_1$ and according to their deadlines in an array $A_2$ using $O(K \log K)$ operations. While sorting, we can connect the sorted elements in $A_1$ to their corresponding positions in $A_2$. Finding the set of non-dominated links can then be obtained as follows: Start with the largest deficit link $l$ (obtained as the first element of $A_1$), add it to the set of non-dominated links $\mathcal{B}_{\text{ND}}$. Find its associated position $i$ in array $A_2$ in constant time (as these arrays are linked). Every element before index $i$ in $A_2$ corresponds to a dominated link, hence for each of those links we mark them as dominated. Then continue in array $A_1$ to the next non-dominated link, add it in the list of non-dominated links, find its index in $A_2$ and mark all its preceding elements in $A_2$ as dominated if not marked already. This way, we can find the non-dominated links in $O(K \log K)$. We can then compute the probabilities and schedule a packet according to these probabilities.

## Appendix B
## Proof of Lemma 1

For the first part, assume that a policy $\mu$ at time $t_0$ chooses a non-maximal schedule, hence a packet $x$ from link $l$ could have been included in the schedule. Consider an alternative policy $\mu'$ that does schedule any link that could have been included at time $t_0$ so that the schedule becomes maximal, and for the rest of the time, it transmits exactly the same packets as the initial policy $\mu$, except for the transmission of any packet $x$, if $\mu$ schedules it at a later point. This results in $\sum_{s=1}^{t} I_l^{\mu'}(s) \geq \sum_{s=1}^{t} I_l^{\mu}(s), \forall t \geq 1$, and at the same time every schedule transmitted by $\mu'$ for $t \leq t_0$ is maximal. We can repeat this argument for times $t > t_0$ to convert $\mu$ to a policy $\widetilde{\mu}$ that transmits maximal schedules. We then have $\sum_{s=1}^{t} I_l^{\widetilde{\mu}}(s) \geq \sum_{s=1}^{t} I_l^{\mu}(s), \forall t \geq 1$ and from (3) we see that any delivery ratio supported by $\mu$ is also supported by $\widetilde{\mu}$.

For the second part, consider a policy $\mu$ that at some time $t_0$ transmits a packet that is not the earliest-deadline packet $x_1 = (w_1(t), d_1)_l$ in link $l$. Then there is some other packet $x_2 = (w_1(t), d_2)_l$ in link $l$ with $d_2 < d_1$. If we let $\mu$ transmit $x_2$ instead of $x_1$, the buffer state will be improved since we will have the same set of packets in link $l$ except for one packet with a longer deadline now. Further, the link's deficit will not change.

## Appendix C
## Proof of Proposition 1

We look at the state process $\{\mathcal{S}(t)\}$ at times $t_i$ when frames start. We show that the deficits of this sampled chain are stable in the sense of (6). From this it follows that the deficits of the original process $\{\mathcal{S}(t)\}$ are also stable as the mean frame size $\mathbb{E}[F]$ is bounded and the mean deficits within a frame can change at most by $a_{max} K \mathbb{E}[F]$.

Since $\lambda \in \rho \operatorname{int}(\Lambda_{NC})$, we have for some $\epsilon > 0$, and some policy $\mu \in \mathcal{P}_{NC}(\mathcal{F})$,

$$\lambda \mathbb{E}[F](1 + 2\epsilon) \preceq \rho \mathbb{E}[\sum_{t \in \mathcal{F}} I^{\mu}(t)], \tag{44}$$

where $\preceq$ is the component-wise inequality between vectors. this is simply due to the fact that in each frame, the number of deficit arrivals $\sum_{t \in \mathcal{F}} \widetilde{a}(t)$ and the number of departures under the policy $\mu$ are i.i.d across the frames, with means $\mathbb{E}[F]\lambda$ and $\mathbb{E}[\sum_{t \in \mathcal{F}} I^{\mu}(t)]$, respectively, by the renewal reward theorem. hence, to ensure stability, (44) must hold. Next, consider the lyapunov function

$$V(t) := V(\mathcal{S}(t)) = \frac{1}{2} \sum_{l \in \mathcal{K}} w_l^2(t).$$

Let $\{I(t), t \in \mathcal{F}\}$ denote the scheduling decisions by ALG within the frame. Using (4), we get

$$\begin{aligned} w_l^2(t+1) - w_l^2(t) &\leq (w_l(t) + \widetilde{a}_l(t) - I_l(t))^2 - w_l^2(t) \\ &= 2\, w_l(t)(\widetilde{a}_l(t) - I_l(t)) + (\widetilde{a}_l(t) - I_l(t))^2 \\ &\leq 2\, w_l(t)(\widetilde{a}_l(t) - I_l(t)) + a_{max}^2. \end{aligned}$$

Then we compute the drift over $F$ slots

$$V(t_0+F)-V(t_0)=\frac{1}{2}\sum_{l\in\mathcal{K}}\left(w_l^2(t_0+F)-w_l^2(t_0)\right)$$
$$=\frac{1}{2}\sum_{t\in\mathcal{F}}\sum_{l\in\mathcal{K}}\left(w_l^2(t+1)-w_l^2(t)\right)$$
$$\leq Ka_{max}^2F/2+\sum_{t\in\mathcal{F}}\sum_{l\in\mathcal{K}}w_l(t)\left(\widetilde{a}_l(t)-I_l(t)\right). \quad (45)$$

Let $\mathbb{E}_{t_0}[\cdot]=\mathbb{E}[\cdot|\mathcal{S}(t_0)]$. then, over a frame,

$$\mathbb{E}_{t_0}\left[V(t_0+f)-V(t_0)\right]$$
$$\leq \mathbb{E}_{t_0}[\sum_{t\in\mathcal{F}}\sum_{l\in\mathcal{K}}w_l(t)\widetilde{a}_l(t)]-\mathbb{E}_{t_0}[\sum_{t\in\mathcal{F}}\sum_{l\in\mathcal{K}}w_l(t)I_l(t)]+\mathsf{C}_1, \quad (46)$$

where $\mathsf{C}_1=Ka_{max}^2\mathbb{E}[F]/2$. noting that

$$w_l(t_0)-F\leq w_l(t)\leq w_l(t_0)+a_{max}F, \quad (47)$$

at any $t\in\mathcal{F}$, we can bound

$$\mathbb{E}_{t_0}[\sum_{t\in\mathcal{F}}\sum_{l\in\mathcal{K}}w_l(t)\widetilde{a}_l(t)]\leq\sum_{l\in\mathcal{K}}(w_l(t_0)\lambda_l\mathbb{E}[F])+\mathsf{C}_2, \quad (48)$$

where we have used (17) and (47), and $\mathsf{C}_2=a_{max}^2\mathbb{E}[F^2]K<\infty$.

Let $I^\star(t)$ be the scheduling decisions by the policy $\mu^\star$, and $I^\mu(t)$ be the scheduling decisions by the policy $\mu\in\mathcal{P}_{NC}(\mathcal{F})$ in (44). note that $\mu^\star$ is the non-causal policy that maximizes the gain over the frame and can transmit packets from a previous frame (included in the initial buffer $\psi(t_0)$). this only improves the performance of $\mu^\star$, compared to starting with empty buffers, hence,

$$\mathbb{E}_{t_0}\big[\sum_{t\in\mathcal{F}}\sum_{l\in\mathcal{K}}w_l^\star(t)I_l^\star(t)\big]\geq\mathbb{E}_{t_0}\big[\sum_{l\in\mathcal{K}}\sum_{t\in\mathcal{F}}w_l^\mu(t)I_l^\mu(t)\big]. \quad (49)$$

using (49) and the proposition assumption, given $\epsilon>0$, there is a $w'$ such that, if $\|w(t_0)\|>w'$,

$$\mathbb{E}_{t_0}\big[\sum_{t\in\mathcal{F}}\sum_{l\in\mathcal{K}}w_l(t)I_l(t)\big]$$
$$\geq(\rho-\epsilon)\mathbb{E}_{t_0}\big[\sum_{t\in\mathcal{F}}\sum_{l\in\mathcal{K}}w_l^\star(t)I_l^\star(t)\big]$$
$$\geq(\rho-\epsilon)\mathbb{E}_{t_0}\big[\sum_{l\in\mathcal{K}}\sum_{t\in\mathcal{F}}w_l^\mu(t)I_l^\mu(t)\big]$$
$$\geq(\rho-\epsilon)\mathbb{E}_{t_0}\big[\sum_{l\in\mathcal{K}}\sum_{t\in\mathcal{F}}(w_l(t_0)-f)I_l^\mu(t)\big]$$
$$\geq(\rho-\epsilon)\mathbb{E}_{t_0}\big[\sum_{l\in\mathcal{K}}\sum_{t\in\mathcal{F}}w_l(t_0)I_l^\mu(t)\big]-\mathsf{C}_3, \quad (50)$$

where $\mathsf{C}_3=K\mathbb{E}[F^2]$ is a constant. Using (50), (48), (46),

$$\mathbb{E}_{t_0}\big[V(t_0+F)-V(t_0)\big]$$
$$\leq\mathsf{C}_4+\sum_{l\in\mathcal{K}}\mathbb{E}[F]w_l(t_0)\lambda_l-(\rho-\epsilon)\sum_{l\in\mathcal{K}}w_l(t_0)\mathbb{E}_{t_0}\big[\sum_{t\in\mathcal{F}}I_l^\mu(t)\big]$$
$$\leq\mathsf{C}_4+\sum_{l\in\mathcal{K}}w_l(t_0)\left(\lambda_l\mathbb{E}[F]-(\rho-\epsilon)\mathbb{E}_{t_0}\big[\sum_{t\in\mathcal{F}}I_l^\mu(t)\big]\right)$$
$$\leq\mathsf{C}_4-\epsilon\mathbb{E}[F]\sum_{l\in\mathcal{K}}\lambda_lw_l(t_0) \quad (51)$$
$$\leq\mathsf{C}_4-\epsilon\mathbb{E}[F]K\lambda_{min}\sum_{l\in\mathcal{K}}w_l(t_0), \quad (52)$$

where $\mathsf{C}_4=\mathsf{C}_1+\mathsf{C}_2+\mathsf{C}_3$, $\lambda_{min}=\min_l\lambda_l$, and in (51) we have used (44). From this inequality, the stability in the mean sense (6) follows for the sampled Markov chain by classical Lyapunov arguments (for example see Section 3.1 in [32]) and hence stability of the deficits for the original chain follows as $\mathbb{E}[F]<\infty$.

## Appendix D
## Proof of Proposition 2

Assume that for some $n$, $p_n^n(t)\geq0$. In this case we know that $\bar{n}\geq n$ since $n$ satisfies (14). Now assume that $p_n^n(t)<0$. Then we claim that we can conclude $\bar{n}<n$, or equivalently $p_{n'}^{n'}(t)<0$ for any $R\geq n'>n$. It suffices to prove that $p_n^n(t)<0$ implies $p_{n+1}^{n+1}(t)<0$, from which inductively the claim follows. To arrive at a contradiction, assume $p_n^n(t)<0$, $p_{n+1}^{n+1}(t)\geq0$, or equivalently $(a)$: $C_n(t)>W_{M_n}(t)$ and $(b)$: $C_{n+1}(t)\leq W_{M_{n+1}}(t)$. Then

$$\frac{1}{W_{M_{n+1}}(t)}-\frac{1}{nW_{M_{n+1}}(t)}\overset{(b')}{\leq}\frac{1}{C_{n+1}(t)}-\frac{1}{nW_{M_{n+1}}(t)}$$
$$=\frac{\sum_{i\in[n+1]}W_{M_i}(t)^{-1}}{n}-\frac{1}{nW_{M_{n+1}}(t)}=\frac{\sum_{i\in[n]}W_{M_i}(t)^{-1}}{n}$$
$$=\frac{n-1}{n}\frac{\sum_{i\in[n]}W_{M_i}(t)^{-1}}{n-1}\overset{(a')}{<}\frac{n-1}{n}\frac{1}{W_{M_n}(t)},$$

where in $(a')$ we used $(a)$ and in $(b')$ we used $(b)$. This shows $\frac{1}{W_{M_{n+1}}(t)}<\frac{1}{W_{M_n}(t)}$ or $W_{M_{n+1}}(t)>W_{M_n}(t)$, which is a contradiction with the ordering of $M_i$. Hence $p_n^n(t)<0$ implies $p_{n+1}^{n+1}(t)<0$.

## Appendix E
## Proof of Lemma 5

Take any $n$, $\bar{n}<n\leq R$. By the definition of $\bar{n}$ it must be the case that $p_n^n(t)<0$, which implies $W_{M_n}(t)<C_n(t)$. From this, and by using (12),

$$W_{M_n}(t)^{-1}(n-1)>\sum_{i\in[n]}W_{M_i}(t)^{-1}. \quad (53)$$

We then have

$$\sum_{i\in[n]}W_{M_i}(t)^{-1}$$
$$=\sum_{i\in[n-1]}W_{M_i}(t)^{-1}+W_{M_n}(t)^{-1}$$
$$=\sum_{i\in[n-1]}W_{M_i}(t)^{-1}+\frac{n-1}{n-2}W_{M_n}(t)^{-1}-\frac{W_{M_n}(t)^{-1}}{n-2}$$
$$\overset{(a)}{>}\sum_{i\in[n-1]}W_{M_i}(t)^{-1}+\frac{1}{n-2}\sum_{i\in[n]}W_{M_i}(t)^{-1}-\frac{W_{M_n}(t)^{-1}}{n-2}$$
$$=\sum_{i\in[n-1]}W_{M_i}(t)^{-1}+\frac{1}{n-2}\sum_{i\in[n-1]}W_{M_i}(t)^{-1}$$
$$=\frac{n-1}{n-2}\sum_{i\in[n-1]}W_{M_i}(t)^{-1},$$

where in $(a)$ we used (53). Dividing both sides by $n-1$, we get $C_n(t)^{-1} > C_{n-1}(t)^{-1}$.

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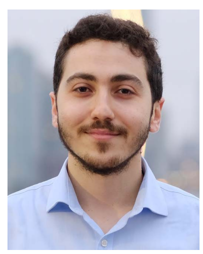

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