A Safety Factor Method for Reliability-Based Component Design

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Abstract

Reliability-based design (RBD) employs optimization to identify design variables that satisfy the reliability requirement. For many routine component design jobs that do not need optimization, however, RBD may not be applicable, especially for those design jobs which are performed manually or with a spreadsheet. This work develops a modified RBD approach to component design so that the reliability target can be achieved by conducting traditional component design repeatedly using a deterministic safety factor. The new component design is based on the First Order Reliability Method (FORM), which iteratively assigns the safety factor during the design process until the reliability requirement is satisfied. In addition to several iterations of deterministic component design, the other additional work is the calculation of the derivatives of the design margin with respect to the random input variables. The proposed method can be used for a wide range of component design applications. For example, if a deterministic component design is performed manually or with a spreadsheet, so is the reliability-based component design. Three examples are used to demonstrate the practicality of the new design method.

Keywords: Reliability in design, Design of machine elements, Design methodologies, Algorithms

1. Introduction

Safety factors are routinely used in mechanical design to account for uncertainty [1-6]. They are particularly useful when complete distributions of random variables are unknown. When such distributions are available, the safety factor based design can be replaced by the RBD [7-16]. RBD solves an optimization problem [17-21] by identifying optimal design variables that minimize a cost-type objective function while satisfying reliability constraints. The reliability in RBD is the probability that a design requirement is satisfied [22].

There are many RBD methodologies. The most common ones employ the First Order Reliability Method (FORM) [23-25] to evaluate reliability constraints during the optimization process. FORM can not only provide a good balance between accuracy and efficiency, but also make it possible to decouple deterministic optimization from reliability analysis, thereby further reducing the computational cost. RBD has been successfully used in many applications, for example, design of composite over-wrapped tanks [26], B-pillar design for side impact [27], crashworthiness of vehicle side impact [28], and engine piston design for reducing slap noise due to the reducing secondary motion [29].

The concept of the safety factor, with which engineers are familiar, can also be incorporated in RBD. The safety-factor based approach for RBD [2, 5] is such a method. This method employs nonlinear optimization and FORM, calling deterministic optimization and FORM sequentially until all the reliability constraints are satisfied. During this process, partial safety factors are applied to all the input random variables.

RBD methodologies [30-34] that involve optimization are usually performed for system design and design of key elements whose reliability is required. Optimization is a commonly used tool for system and component design. Many engineering analysis and design software tools have an optimization module, and it is quite easy to use it. Some components, however, are not designed by optimization, and they may be designed by a safety-factor approach or may be designed based on physical experiments. It is therefore desirable to derive an RBD approach for components that do not need optimization.

One approach, which satisfies this requirement is the mechanical design approach using the First Order Second Moment (FOSM) method [35-40]. This method can find design variables for a given reliability target with only the minimal extra work: the calculation of derivatives of a design margin function with respect to input random variables. It is therefore practical and can be used for routine component design. The accuracy of the reliability produced by the design variables, however, may be poor. This means that the designed reliability may be far away from the required reliability. The reason is that FOSM uses a first-order approximation around the means of input random variables and that only the first two moments (means and standard deviations) are used.

This work develops a modified approach to reliability-based component design without optimization and a cost-type objective. It uses FORM and produces higher accuracy than FOSM. During the design process, the method iteratively updates a safety factor for the deterministic component design until the reliability requirement is satisfied. In addition to a number of iterations of the deterministic component design, the only additional work is the calculation of the derivatives of the design margin with respect to the random input variables. The major advantage of this approach is that engineers can use it in the same way as they perform their deterministic routine component design, either manually or by other means.

Reliability-based design and the safety factor are reviewed in Section 2, and the new component design approach is presented in Section 3, followed by three examples in Section 4. Conclusions are given in Section 5.

2. Review of RBD and Safety Factor

RBD is a design methodology that minimizes a cost-type objective and maintains reliability requirements when uncertainty (randomness) is present. Uncertainty can also be accommodated by using a safety factor. Both the design methodologies are briefly reviewed here.

2.1 Reliability-Based Design

A typical RBD model is given by

$$\begin{cases} \underset{\boldsymbol{d}}{\mathrm{Min}} & f(\boldsymbol{d}) \\ \mathrm{s.t.} & \Pr\{G_i(\boldsymbol{d},\boldsymbol{X}) > 0\} \geq [R_i], i = 1,2,\ldots,n_g \\ & \boldsymbol{d}^L \leq \boldsymbol{d} \leq \boldsymbol{d}^U \end{cases} \tag{1}$$

In the above model, $\boldsymbol{d}$ is the vector of design variables with their lower and upper bounds $\boldsymbol{d}^L$ and $\boldsymbol{d}^U$, respectively. $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is the vector of random variables. $f(\cdot)$ is a cost-type objective function, and $G_i(\boldsymbol{d},\boldsymbol{X}) = 0$ is a limit-state function. The requirement is $G_i(\boldsymbol{d},\boldsymbol{X}) > 0$, and the probability of satisfying the requirement is called reliability, denoted by R_i; namely

$$R_i = \Pr\{G_i(\boldsymbol{d},\boldsymbol{X}) > 0\} \tag{2}$$

The constraint associated with $G_i(\boldsymbol{d},\boldsymbol{X})$ is that R_i should be greater than or equal to the desired reliability $[R_i]$ or $1 - [p_{f_i}]$, where $[p_{f_i}]$ is the allowable probability of failure.

The reliability R_i is obtained by

$$R_i = \Pr\{G_i(\boldsymbol{d}, \boldsymbol{X}) > 0\} = \int_{G_i(\boldsymbol{d}, \boldsymbol{X})>0} f_{\boldsymbol{x}}(\boldsymbol{x})\, d\boldsymbol{x} \tag{3}$$

where $f_{\boldsymbol{x}}(\boldsymbol{x})$ is the joint probability density function (PDF) of $\boldsymbol{X}$. FORM is commonly used to calculate R_i. FORM first transforms $\boldsymbol{X}$ into independent standard normal variables $\boldsymbol{U}$ with $\boldsymbol{X} = T(\boldsymbol{U})$ [41], where $T(\cdot)$ denotes the transformation operation. The limit-state function then becomes

$$G(\boldsymbol{d}, \boldsymbol{X}) = G\big(\boldsymbol{d}, T(\boldsymbol{U})\big) \tag{4}$$

Then R_i is approximated by

$$R_i = \Phi(\beta) \tag{5}$$

where β is the reliability index, which is the shortest distance from the origin of the U-space to the limit-state contour $G(\boldsymbol{d}, T(\boldsymbol{U})) = 0$, $\Phi(\cdot)$ is the cumulative density function (CDF) of a standard normal variable. The distance is obtained by solving the following optimization model:

$$\begin{cases} \text{Min } \|\boldsymbol{u}\| \\ \text{s.t. } G\big(\boldsymbol{d}, T(\boldsymbol{u})\big) = 0 \end{cases} \tag{6}$$

The solution $\boldsymbol{u}^*$ is called the most probable point (MPP), whose norm is the reliability index.

$$\beta = \|\boldsymbol{u}^*\| \tag{7}$$

where $\|\cdot\|$ stands for the norm of a vector.

Directly solving the RBD model involves an expensive double-loop procedure if the MPP is used for the reliability analysis. Many sequential single-loop methods [30-33] have been developed to reduce the computational time. One of them is the sequential optimization and reliability analysis (SORA) method [42]. SORA solves the model in Eq. (1) with a sequence of cycles of optimization and reliability analysis. In each cycle, the optimization is performed by fixing the random variables $\boldsymbol{X}$ at fixed values determined by the reliability analysis from the last cycle. Then the design variables $\boldsymbol{d}$ are passed to the reliability analysis that is then performed. This process repeats till convergence.

2.2 Safety Factor and Traditional Deterministic Design

A safety factor is the ratio of the strength (capacity or resistance) divided by the maximum stress (demand or load). It is given by

$$S_F = \frac{S}{L} \tag{8}$$

where S and L are the general strength and general stress, respectively. The strength and load used in this work are in a general sense. The strength could be anything that is related to the capacity of a component, for example, a yield strength, permitted deflection, or required fatigue life; a load could be anything that related to the demand of the component or the loading acting on or generated in the component, such as normal stress, force, deflection, and fatigue damage accumulation.

The component design task is to identify design variables $\boldsymbol{d}$ so that the safety factor is greater than 1, and this gives a design margin function

$$g(\boldsymbol{d}) = \frac{S}{S_F} - L(\boldsymbol{d}) > 0 \tag{9}$$

For example, a cantilever shaft is subjected to a force P as shown in Fig. 1. The design margin function is given by

$$g(d) = \frac{S_y}{S_F} - \sigma' \tag{10}$$

where design variable d is the diameter, and σ' is the von Mises stress calculated by

$$\sigma' = \sqrt{\sigma_x^2 + 3\tau_{zx}^2} \tag{11}$$

in which

$$\sigma_x = \frac{32P(a+b)}{\pi d^3} \tag{12}$$

$$\tau_{zx} = \frac{16Pe}{\pi d^3} \tag{13}$$

Solving $g(d) = 0$ yield the design variable d.

Fig. 1 here

Fig. 1 A cantilever shaft

3. A Modified Approach to Reliability-Based Component Design

In this section, we discuss the modified approach to reliability-based component design. It is for mechanical component designs that do not have a cost-type objective function and therefore do not require optimization. The approach is practical because the design margin function is exactly the same as the one used in the deterministic component design as shown in Eq. (9). It does not dramatically alter the way that designers perform the component design. The only additional work is to perform the deterministic component design multiple times with different safety factors which are updated during the iterative design process.

3.1 The Proposed Approach

The proposed approach is based on FORM. The random variables and their nominal values (means) are $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$ and $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$, respectively. Let the CDF of X_i be $F_i(X_i)$, $i = 1,2, \dots, n$, and assume all the variables in $\boldsymbol{X}$ are independent. The general strength of the component is S, which is the first element of $\boldsymbol{X}$, namely, $X_1 = S$. S could be a yield strength, permissible deflection, or capacity. Let the rest of $\boldsymbol{X}$ be $\boldsymbol{Y} = (X_2, X_3, \dots, X_n)$. The general load L of the component is determined by $\boldsymbol{d}$ and $\boldsymbol{Y}$ and is therefore given by $L(\boldsymbol{d}, \boldsymbol{Y})$, where $\boldsymbol{d}$ and $\boldsymbol{Y}$ are vectors to represent multiple design variables and parameters, respectively. The general load could be a force, moment, and stress. For the example in Sec. 2.2, the general strength is the yield strength; namely, $S = S_y$; and the general load is the von Mises stress σ', namely, $L = \sigma'$, which is a function of the design variable or the diameter d.

If we use the nominal values of general strength and general load to calculate the safety factor, we obtain a deterministic safety factor S_F.

$$S_F = \frac{s}{l} = \frac{s}{L(\boldsymbol{d}, \boldsymbol{y})} \tag{14}$$

where s and l are nominal values of the strength and load, respectively, and $\boldsymbol{y}$ is a vector of the nominal values of $\boldsymbol{Y}$. Note that the nominal value of a random variable is the median of a random variable or its mean value if its distribution is symmetric. The deterministic design margin function is $g(\boldsymbol{d}) = \frac{s}{S_F} - L(\boldsymbol{d}) > 0$ as already been given in Eq. (9).

The actual design margin, or the difference between the general strength and general load, is given by

$$G(\boldsymbol{d}, \boldsymbol{X}) = S - L(\boldsymbol{d}, \boldsymbol{Y}) > 0 \tag{15}$$

As we have discussed, the probability of satisfying the design margin $R=\Pr\{G(\boldsymbol{d},\boldsymbol{X})>0\}$ is the component reliability. If the required reliability is $[R]$, the reliability index [43] is

$$\beta=\Phi^{-1}([R]) \tag{16}$$

Many studies [2, 5, 8, 16] have shown that the reliability requirement $R=\Pr\{G(\boldsymbol{d},\boldsymbol{X})>0\}>[R]$ is equivalent to

$$G(\boldsymbol{d},\boldsymbol{x}^*)=S^*-L(\boldsymbol{d},\boldsymbol{y}^*)>0 \tag{17}$$

where $\boldsymbol{x}^*=(S^*,\boldsymbol{y}^*)$ is the MPP in the X-space, and it is transformed from the MPP $\boldsymbol{u}^*=(u_i^*)_{i=1,n}$ in the standard normal space U-space. We rewrite Eq. (17) by

$$\frac{S^*}{s}\frac{s}{L(\boldsymbol{d},\boldsymbol{y})}-\frac{L(\boldsymbol{d},\boldsymbol{y}^*)}{L(\boldsymbol{d},\boldsymbol{y})}>0 \tag{18}$$

The X- to U-space transformation is given by

$$F_i(x_i^*)=\Phi(u_i^*) \tag{19}$$

Then

$$x_i^*=F_i^{-1}[\Phi(u_i^*)]=T(u_i^*) \tag{20}$$

where $T(\cdot)$ represents the transformation function for simplicity. Since the MPP $\boldsymbol{u}^*$ is the shortest-distance point to the surface $G(\cdot)=0$, $\boldsymbol{u}^*$ is collinear with the gradient of $G(\cdot)$. This gives [44]

$$u_i^*=-\beta\alpha_i \tag{21}$$

where the reliability index β is the magnitude of $\boldsymbol{u}^*$; α_i is the element of the unit vector of the gradient ∇_G and is given by

$$\alpha_i = \frac{\frac{\partial G(\boldsymbol{d}, T(\boldsymbol{u}^*))}{\partial u_i^*}}{\|\nabla_G\|} \tag{22}$$

∇_G is computed by

$$\nabla_G = \left(\frac{\partial G(\boldsymbol{d}, T(\boldsymbol{u}^*))}{\partial u_i^*}\right)_{i=1,\dots,n} \tag{23}$$

More details about the above equations can be found in [44]. By the chain rule of partial derivative, we have

$$\frac{\partial G(\boldsymbol{d}, T(\boldsymbol{u}^*))}{\partial u_i^*} = \frac{\partial G(\boldsymbol{d}, \boldsymbol{x}^*))}{\partial x_i^*}\frac{dx_i^*}{du_i^*} \tag{24}$$

Define $w_i = \frac{dx_i^*}{du_i^*}$, from Eq. (20), we have

$$w_i = \frac{dx_i^*}{du_i^*} = \frac{\phi\left(\Phi^{-1}\left(F_i(x_i^*)\right)\right)}{f_i(x_i^*)} \tag{25}$$

where $\phi(\cdot)$ and $f_i(\cdot)$ are the probability density function (PDF) of a standard normal variable and X_i, respectively, $\Phi(\cdot)$ is the CDF of a standard normal variable. For commonly used distributions, w_i is listed in the appendix. Therefore, we can rewrite Eq. (24) as

$$\frac{\partial G(\boldsymbol{d}, T(\boldsymbol{u}^*))}{\partial u_i^*} = w_i\frac{\partial G(\boldsymbol{d}, \boldsymbol{x}^*)}{\partial x_i^*} \tag{26}$$

And Eq. (23) can be rewritten as

$$\nabla_G = \left(w_i\frac{\partial G(\boldsymbol{d}, \boldsymbol{x}^*)}{\partial x_i^*}\right)_{i=1,\dots,n} = \left(w_1\frac{\partial G(\boldsymbol{d}, \boldsymbol{x}^*)}{\partial x_1^*}, w_2\frac{\partial G(\boldsymbol{d}, \boldsymbol{x}^*)}{\partial x_2^*}, \cdots, w_n\frac{\partial G(\boldsymbol{d}, \boldsymbol{x}^*)}{\partial x_n^*}\right) \tag{27}$$

From Eqs. (22) and (26), at the MPP $\boldsymbol{x}^*$ in X-space, we have

$$\alpha_i = \frac{w_i \dfrac{\partial G(\boldsymbol{d}, \boldsymbol{x}^*)}{\partial x_i^*}}{\|\nabla_G\|} \tag{28}$$

Plugging Eq. (28) into Eq. (21), we obtain the value of u_i^*. Then, we can obtain x_i^* by substituting u_i^* into Eq. (20). Define

$$\lambda_S = \frac{S^*}{s} \tag{29}$$

and

$$\lambda_L = \frac{L(\boldsymbol{d}, \boldsymbol{y}^*)}{L(\boldsymbol{d}, \boldsymbol{y})} \tag{30}$$

Substituting Eqs. (14), (29) and (30) into Eq. (18), we have

$$\lambda_S S_F - \lambda_L > 0 \tag{31}$$

By solving the inequality equation, we have the range for design variables. Once we specify the design variables, the safety factor for the given design is

$$S_F = \frac{\lambda_L}{\lambda_S} \tag{32}$$

As shown in Eq. (32), λ_L and λ_S indicate the contributions of the general strength and general stress to the overall safety faction S_F, and they can be considered as partial safety factors.

To design the component with the reliability target, we can then use the deterministic design function, which is rewritten here.

$$g(\boldsymbol{d}) = \frac{1}{S_F} s - L(\boldsymbol{d}, \boldsymbol{y}) > 0 \tag{33}$$

The design margin function in Eq. (33) is the same function used in deterministic component design since only the nominal values $\boldsymbol{y}$ of $\boldsymbol{Y}$ are involved. No random variables appear in the function. If the safety factor S_F is given, we can solve for $\boldsymbol{d}$. To determine the safety factor S_F that satisfies the reliability requirement, we need to repeat the above process iteratively to find the MPP $\boldsymbol{x}^*$. Unlike the reliability-based design optimization, the proposed approach performs the MPP search implicitly to update the safety factor. It does not require an explicit optimization model and is therefore easy to implement. The proposed approach depends totally on how the deterministic design is performed or in other words, how Eq. (33) is solved, either manually or numerically.

3.2 The Procedure

The design margin function $G(\boldsymbol{d}, \boldsymbol{X}) = S - L(\boldsymbol{d}, \boldsymbol{Y})$ and deterministic design margin function $g(\boldsymbol{d}) = \frac{1}{S_F} s - L(\boldsymbol{d}, \boldsymbol{y})$ are usually nonlinear functions. As the safety factor S_F depends on $\boldsymbol{d}$, directly solving for $\boldsymbol{d}$ from $g(\boldsymbol{d}) > 0$ requires a numerical procedure, which diminishes the practicality of the design. We develop a straightforward procedure so that the design variables can be obtained iteratively by performing deterministic design a number of times. The procedure is discussed below.

Initial design

1) Perform the initial design by using $S_F = 1$ or other value of $S_F > 1$. From $g(\boldsymbol{d}) = \frac{1}{S_F} s - L(\boldsymbol{d}, \boldsymbol{y}) > 0$, initial design variables $\boldsymbol{d}$ are obtained. If there are multiple design functions due to multiple design requirements or multiple failure modes, there are two ways to obtain $\boldsymbol{d}$. The first way involves the approach that can be found in a mechanical design textbook. It solves the design functions one by one and results in multiple designs. Then it selects the

design that satisfies all the design functions. The second way is to solve all the design functions simultaneously. Which way is used is the choice of designers, and the proposed approach can work for either way. If the preferred value of a design variable should be determined, designers can make a decision based on their experience. Or they can simply find whether a preferred value should be greater or smaller than the calculated value by verifying if the preferred value results in a positive design function.

Since the safety factor used in the initial design may not satisfy the reliability requirement, it will be updated iteratively next. To prepare for the iterations, set $\boldsymbol{d}$ to be the current design, and set the MPP $\boldsymbol{x}^*$ to be the nominal values of all random input variables.

Iterative design

2) At the current design point $\boldsymbol{d}$ and $\boldsymbol{x}^*$, calculate the gradient of the design margin function $G(\boldsymbol{d}, \boldsymbol{X})$ and update the MPP following the procedure in Fig. 2. And the gradient can be calculated either analytically or numerically.

3) Update λ_S and λ_L using Eqs. (29) and (30), and solve for the safety factor S_F using Eq. (32).

4) Solve for new design point $\boldsymbol{d}$ by plugging the new S_F into the deterministic design function $g(\boldsymbol{d}) = \frac{1}{S_F} s - L(\boldsymbol{d})$. If there are multiple design functions and preferred values of design variables need to be determined, follow the same guideline discussed for the initial design stage.

5) Check convergence. The criterion is that the difference of the safety factor (SF) of current iteration and previous iteration. It converges when the difference is sufficiently small, which is given by

$$\frac{\left\|\mathrm{SF}_{current} - \mathrm{SF}_{previous}\right\|}{\mathrm{SF}_{current}} \leq \varepsilon \tag{34}$$

where ε is a small positive quantity. $\varepsilon = 0.1\%$, $\varepsilon = 0.01\%$, or other values could be used. If convergence is not achieved, go to step 2); otherwise, go to step 6).

Final design

6) Based on $\boldsymbol{d}$ obtained, choose appropriate final design variables.

The MPP is updated after a new design $\boldsymbol{d}$ is identified. $\boldsymbol{u}^*$ obtained during each iteration before convergence is not the true MPP for a given design $\boldsymbol{d}$. Upon convergence of the entire design process, $\boldsymbol{u}^*$ will be the true MPP for the final design. This will not only save design time but also guarantee the target reliability is achieved.

The flowcharts of the proposed approach are provided in Figs. 2.

Fig. 2 here

Fig. 2 Flowchart of reliability-based component design

4. Examples

In this section, we provide three examples. Example 1 is the shaft design problem discussed previously in Sec. 2.2. Since the design is performed manually, all details of using the proposed approach are given so that an interested reader could easily repeat the process and reproduce the result. Example 2 shows a case with more than one failure mode. Example 3 involves discrete design variables selected from a table, non-normally distributed random variables, more than one failure mode, and black-box design functions.

4.1 A Shaft Design

This example involves a design that is performed manually as discussed in Sec. 2.2 and is shown in Fig. 1. The design margin function is given in Eq. (10). The yield strength and the applied force follow normal distributions $S_y \sim N(530, 20^2)$ MPa and $P \sim N(1200, 100^2)$ N, respectively. S_y and P are independent. The random variables are therefore $\boldsymbol{X} = (S_y, P)$. Other parameters are $a = 300$ mm, $b = 50$ mm, and $e = 350$ mm. The design task is to determine the diameter of the shaft d so that the reliability of the shaft is no less than $[R] = 0.9999$. The design margin function is

$$G(\boldsymbol{d}, \boldsymbol{X}) = S_y - L(\boldsymbol{d}, \boldsymbol{Y}) = \sqrt{\left[\frac{32P(a+b)}{\pi d^3}\right]^2 + 3\left(\frac{16Pe}{\pi d^3}\right)^2} \tag{35}$$

And the deterministic design function in Eq. (10) is rewritten as

$$g(\boldsymbol{d}) = \frac{s_y}{s_F} - L(\boldsymbol{d}) = \sqrt{\left[\frac{32p(a+b)}{\pi d^3}\right]^2 + 3\left(\frac{16pe}{\pi d^3}\right)^2} \tag{36}$$

where p is the nominal value of P.

Design process

Determine the reliability index

$$\beta = \Phi^{-1}([\mathrm{R}]) = \Phi^{-1}([0.9999]) = 3.7190$$

Derive the gradient

$$\nabla_G = \left(w_i \frac{\partial G(\boldsymbol{d}, \boldsymbol{X})}{\partial X_i}\right)_{i=1,\dots,n} = \left(w_1 \frac{\partial G}{\partial X_1}, w_2 \frac{\partial G}{\partial X_2}\right)$$

$$\frac{\partial G}{\partial X_1} = \frac{\partial G}{\partial S_y} = 1$$

$$\frac{\partial G}{\partial X_2} = \frac{\partial G}{\partial P} = -\frac{16\sqrt{4(a+b)^2 + 3e^2}}{\pi d^3}$$

From Table A1, we have

$$w_1 = \sigma_1 = 20 \text{ MPa}, w_2 = \sigma_2 = 1.2 \text{ kN}$$

Iteration 1

Start from the deterministic design by setting $S_F = 1.0$. Then plug the nominal values of S_y and P, which are $s_y = 530$ MPa and $p = 1200$ N, respectively, into

$$g(\boldsymbol{d}) = \frac{s_y}{S_F} - \sqrt{\left[\frac{32p(a+b)}{\pi d^3}\right]^2 + 3\left(\frac{16pe}{\pi d^3}\right)^2} > 0$$

We have

$$530(10)^6 - \sqrt{A_1^2 + 3B_1^2} > 0$$

where

$$A_1 = \frac{32(1.2)(10)^3(300+50)(10)^{-3}}{\pi d^3}$$

$$B_1 = \frac{16(1.2)(10)^3(350)(10)^{-3}}{\pi d^3}$$

which yields the initial design $d > 22.02$ mm. Substituting d into A_1 and B_1, the general load (normal stress) at the design point $d = 22.02$ mm is

$$L(\boldsymbol{d}, \boldsymbol{y}) = \sqrt{A_1^2 + 3B_1^2} = 530.0 \text{ MPa}$$

Iteration 2

At $d = 22.02$ mm, using Eq. (27) we obtain the gradient

$$\nabla_G = \left(w_1 \frac{\partial G}{\partial x_1}, w_2 \frac{\partial G}{\partial x_2}\right) = (2.0 \times 10^7, -4.4167 \times 10^7)$$

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2) = \left(\frac{w_1 \frac{\partial G}{\partial x_1}}{\|\nabla_G\|}, \frac{w_2 \frac{\partial G}{\partial x_2}}{\|\nabla_G\|}\right) = (0.4125, -0.9110)$$

$$\boldsymbol{u}^* = (u_1^*, u_2^*) = (-\beta\alpha_1, -\beta\alpha_2) = (-1.5341, 3.3879)$$

$$\boldsymbol{x}^* = (x_1^*, x_2^*) = (F_1^{-1}[\Phi(u_1)],\ F_2^{-1}[\Phi(u_2)]) = (499.3176 \text{ MPa}, 1.5388 \text{ kN})$$

and the general strength $S^* = x_1^* = 499.3176$ MPa.

$$\lambda_S = \frac{S^*}{s} = \frac{499.3176}{530} = 0.9421$$

The general load at $\boldsymbol{y}^* = (x_2^*)$ is

$$L(\boldsymbol{d}, \boldsymbol{y}^*) = \sqrt{A_2^2 + 3B_2^2} = 679.6302 \text{ MPa}$$

where

$$A_2 = \frac{32(1.5388)(10)^3(300+50)(10)^{-3}}{\pi(22.02\times10^{-3})^3}$$

$$B_2 = \frac{16(1.5388)(10)^3(350)(10)^{-3}}{\pi(22.02\times10^{-3})^3}$$

$$\lambda_L = \frac{L(\boldsymbol{d},\boldsymbol{y}^*)}{L(\boldsymbol{d},\boldsymbol{y})} = \frac{679.6302}{530.0} = 1.2823$$

Then the updated safety factor is

$$S_F = \frac{\lambda_L}{\lambda_S} = \frac{1.2813}{0.9421} = 1.3611$$

Plugging the new S_F into the deterministic design function in Eq. (33), we have

$$\frac{530(10)^6}{1.3611} - \sqrt{C_2^2 + 3D_2^2} > 0$$

where

$$C_2 = \frac{32(1.2)(10)^3(300+50)(10)^{-3}}{\pi d^3}$$

$$D_2 = \frac{16(1.2)(10)^3(350)(10)^{-3}}{\pi d^3}$$

which yields

$$d > 24.40 \text{ mm}$$

At $d = 24.40$ mm, $L(\boldsymbol{d},\boldsymbol{y}) = 389.3857$ MPa. Check the convergence using Eq. (34) and we obtain

$$\varepsilon = \frac{\left|S_{F,current} - S_{F,previous}\right|}{S_{F,previous}} = \frac{|1.3611 - 1.0|}{1.0} = 36.11\%$$

It is greater than the tolerance 0.01%, and the process continues.

Iteration 3

At $d = 24.40$ mm, we have

$$\nabla_G = \left(w_1 \frac{\partial G}{\partial x_1^*}, w_2 \frac{\partial G}{\partial x_2^*}\right) = (2.0 \times 10^7, -3.2449 \times 10^7)$$

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2) = \left(\frac{w_1 \frac{\partial G}{\partial x_1^*}}{\|\nabla_G\|}, \frac{w_2 \frac{\partial G}{\partial x_2^*}}{\|\nabla_G\|}\right) = (0.5247, -0.8513)$$

$$\boldsymbol{u}^* = (u_1^*, u_2^*) = (-\beta\alpha_1, -\beta\alpha_2) = (-1.9514, 3.1660)$$

$$\boldsymbol{x}^* = (x_1^*, x_2^*) = (F_1^{-1}[\Phi(u_1)],\ F_2^{-1}[\Phi(u_2)]) = (490.9729 \text{ MPa}, 1.5166 \text{ kN})$$

and the general strength $S^* = x_1^* = 490.9729$ MPa.

$$\lambda_S = \frac{S^*}{s} = \frac{490.9729}{530} = 0.9264$$

The general load at $\boldsymbol{y}^* = (x_2^*)$ is

$$L(\boldsymbol{d}, \boldsymbol{y}^*) = \sqrt{A_3^2 + 3B_3^2} = 492.1173 \text{ MPa}$$

where

$$A_3 = \frac{32(1.5166)(10)^3(300 + 50)(10)^{-3}}{\pi(24.40 \times 10^{-3})^3}$$

$$B_3 = \frac{16(1.5166)(10)^3(350)(10)^{-3}}{\pi(24.40 \times 10^{-3})^3}$$

$$\lambda_L = \frac{L(\boldsymbol{d}, \boldsymbol{y}^*)}{L(\boldsymbol{d}, \boldsymbol{y})} = \frac{492.1173}{389.3857} = 1.2638$$

Then the updated safety factor is

$$S_F = \frac{\lambda_L}{\lambda_S} = \frac{1.2638}{0.9264} = 1.3643$$

Plugging the new S_F into the deterministic limit-state function in Eq. (33), we have

$$\frac{530(10)^6}{1.3643} - \sqrt{C_3^2 + 3D_3^2} > 0$$

where

$$C_3 = \frac{32(1.2)(10)^3(300 + 50)(10)^{-3}}{\pi d^3}$$

$$D_3 = \frac{16(1.2)(10)^3(350)(10)^{-3}}{\pi d^3}$$

which yields

$$d > 24.42 \text{ mm}$$

Check the convergence using Eq. (34) and we obtain

$$\varepsilon = \frac{\left|S_{F,current} - S_{F,previous}\right|}{S_{F,previous}} = \frac{|1.3643 - 1.3611|}{1.3611} = 0.22\%$$

which is greater than the convergence tolerance 0.01%. After one more iteration, the process converges and the final design variable is $d > 24.42$ mm. This design will meet the reliability target 0.9999, which is equivalent to a probability of failure 10^{-4}. To verify this, Monte Carlo simulation (MCS) is performed with a large sample size of 10^8. The probability of failure produced by MCS is 1.01×10^{-4}, very close to the required probability of failure. For a manufacturability consideration, we can set the final design $d = 24.5$ mm, which ensures higher reliability than the required one. The entire design process is summarized in Table 1.

Table 1 Design process of the shaft

Table 1 here

To confirm that the safety factor RBD produces the same result as an optimization based RBD method, SORA is used to solve the same problem. Since no objective function exists, we set the deterministic design margin as the objective function and minimize it. The constraint function is the reliability constraint. The same design variable d is obtained from SORA, and both approaches find the final design variable in four iterations.

4.2 A Key Design

The task is to design a key (Fig. 3) for a shaft with a diameter of 22 mm so that its hub can withstand compression and shearing stress induced by the transmission power P. The target reliability is $[R] = 0.999999$. The width and height are determined given by shaft diameter according ANSI Standard, which are 8 mm and 7 mm, respectively. The random variables are $\boldsymbol{x} = (S_y, S_{sy}, P, \omega)$, where S_y is the compression (crushing) yield strength of the material, $S_{sy} = 0.577S_y$ is the shearing strength of the material, P is the transmission power, and ω is the angular velocity of the shaft. All the random variables are independent, and their distributions are given in Table 2. The design variable E is the length of the key, namely, $\boldsymbol{d} = (E)$, which should be less than 30 mm because the diameter of the shaft is 22 mm.

Fig. 3 here

Fig. 3 A key of shaft-hub gear

Table 2 Distributions of the random variables in Example 2

Table 2 here

There are two failure modes existed because the key needs to withstand compression and shearing stress induced by the transmission power. Therefore, the design margin functions are defined by

$$G_1(E,\boldsymbol{X}) = S_y - L_1(E,\boldsymbol{Y}) = \frac{S_y}{S_{F_1}} - \frac{4P}{DHE\omega} > 0 \tag{37}$$

$$G_2(E,\boldsymbol{X}) = S_{sy} - L_2(E,\boldsymbol{Y}) = \frac{S_{sy}}{S_{F_2}} - \frac{2P}{DWE\omega} > 0 \tag{38}$$

and the corresponding deterministic design functions are defined by

$$g_1(E) = \frac{s_y}{S_F} - L_1(E) = \frac{s_y}{S_{F_1}} - \frac{4p}{DHE\omega_\mu} > 0 \tag{39}$$

$$g_2(E) = \frac{s_{sy}}{S_F} - L_2(E) = \frac{s_{sy}}{S_{F_2}} - \frac{2p}{DWE\omega_u} > 0 \tag{40}$$

where s_y, s_{sy}, p, ω_u are the means of S_y, S_{sy}, P, ω, respectively.

Following the procedures in Fig. 2, we have two designs that are $E_1 = 28.6$ mm and $E_2 = 21.7$ mm for the two design functions. To meet the reliability target 0.999999, which is equivalent to a probability of failure 10^{-6}, the design is set to be $E = 28.6$ mm. To verify this, Monte Carlo simulation (MCS) is performed with a large sample size of 10^8. The probability of failure produced by MCS is 1.11×10^{-6}, very close to the required probability of failure. By using the same SORA strategy in Example 1, we have the final design $E = 28.6$ mm which is the same as the proposed approach. The proposed method and SORA call the design margin functions 8 and 10 times, respectively. For a manufacturability and safety consideration, we can set the final design $E = 29$ mm, which ensure higher reliability than the required one. The entire design process is summarized in Table 3.

Table 3 Design process of the key

Table 3 here

4.3 A Cantilever Tube Design

The design task is to select a tube (Fig. 4) so that it can withstand random forces F, P, and T, with the reliability greater than or equal to $[R] = 0.99998$. The random variables are $X = (\delta, S_y, P, F, T, E)$, where E is the young's modulus of the tube, δ is the maximum displacement of the tube, S_y is the yield strength of the material. All the random variables are independent, and their distributions are given in Table 4. The design variables are $\boldsymbol{d} = (H, W, d)$, which can be chosen only from the following list of preferred sizes for (H, W, d) mm: (35, 20, 2.5), (40, 15, 2.5), (40, 20, 2.5), (40, 25, 3), (40, 30, 3), (50, 25, 3), (50, 30, 3.2), (50, 30, 4), (50, 30, 5).

Fig. 4 here

Fig. 4 A cantilever tube

Table 4 Distributions of the random variables in Example 3

Table 4 here

This problem is more general than the first two examples because it involves three non-normally distributed random variables and more than one design variable with multiple failure modes, and the design variables are discrete, also it is solved by finite element method (FEM) which proves that the proposed method is compatible with black-box simulation. MATLAB PDE toolbox which is a FEM package is used to solve this example.

There are two failure modes for this example. Once the defection exceeds the allowed maximum defection or the tension exceeds the strength of the material, failure occurs. The design margin functions are defined with

$$G_1(\boldsymbol{d}, \boldsymbol{X_1}) = \delta - L_1(\boldsymbol{d}, \boldsymbol{Y_1}) = \delta - L_1(\boldsymbol{d}, P, F, T, E) \tag{41}$$

$$G_2(\boldsymbol{d}, \boldsymbol{X_2}) = S_y - L_2(\boldsymbol{d}, \boldsymbol{Y_2}) = S_y - L_2(\boldsymbol{d}, P, F, T) \tag{42}$$

And the corresponding deterministic design functions are

$$g_1(\boldsymbol{d}) = \frac{\delta_0}{S_F} - L(\boldsymbol{d}, p, f, t, E_0) \tag{43}$$

$$g_2(\boldsymbol{d}) = \frac{s_y}{S_F} - L(\boldsymbol{d}, p, f, t) \tag{44}$$

where $\delta_0, s_y, p, f, t, l, E_0$ are the means of $\delta, S_y, P, F, T, L, E$, respectively, $L_1(\cdot)$ and $L_2(\cdot)$ are the general load solved by FEM in this example.

This design involves two requirements or two design functions. As discussed in Sec. 3.2, there are two ways to perform the design for multiple design functions. The first way is to use a decoupled approach that is commonly found in a mechanical element design book. This approach considers multiple requirements separately one by one so that different designs are produced, and the final design is selected among the designs generated, and it is the design that satisfies all the requirements. In this example, we have two design functions, and we can obtain two designs from the two design functions. We then pick the one that satisfies the two design functions. The second way solves all the design functions simultaneously. For this example, the simultaneous functions Eqs. (43) and (44) are solved, producing a single design. If the design is performed manually or if the design involves a small number of design variables and design functions, the first way is easier; otherwise, the second way is preferred. Theoretically, if a unique solution exists, the solutions from the two ways should be identical. The details of the first way are given in Example 1, we just need to repeat the same process twice.

Following the procedures in Fig. 2, the design process converges after 5 and 6 iterations and the final design variables are $\boldsymbol{d_1} = (40, 30, 3)$ mm and $\boldsymbol{d_2} = (50, 30, 5)$ mm for the two design margin functions. Since the reliability of final design should be greater than the requirement $[R] = 0.99998$ for both failure modes, we choose the conservative one that is $(50, 30, 5)$ mm for the design. The reliability is confirmed by FORM, which produces 4.2124×10^{-7}, less than 2×10^{-5}. The calculations are summarized in Table 5. By using the same SORA strategy in Example 1, we obtain the design $\boldsymbol{d} = (50, 30, 5)$ mm which is the same as the proposed method. For this example, the proposed approach and SORA call the design margin functions 243 and 260 times.

Table 5 Design process of the cantilever tube

Table 5 here

We also use the second way, which produces the same design with the convergence history: The designs in the four iterations are (40, 30, 3), (50, 30, 5), (50, 30, 5), (50, 30, 5), (50, 30, 5), and (50, 30, 5) mm, and the final design is (50, 30, 5) mm.

This example involves a black-box FEM model. For design problems that need computationally expensive models, we can at first create cheaper surrogate models [45-48] to replace the original models, and then use the proposed approach based on the surrogate models.

The three examples demonstrate that the deterministic design is performed several times with the additional computations for the derivatives of the design margin with respect to random variables. In the examples, the deterministic design is conducted manually, and so is the proposed reliability-based design method.

5. Conclusion

This work develops a modified approach to reliability-based component design for which optimization is not required. The approach is easy to implement because it is based on the traditional safety factor with which engineers are familiar. The safety factor is determined by the specified reliability of the component. FORM is used to link the safety factor and component reliability. Since the safety factor for the required reliability also depends on design variables, the design process is iterative, and the proposed efficient numerical procedure ensures that the design process can converge with a few iterations.

The prerequisites of the modified reliability-based component design approach are as follows: the availability of derivatives of the design margin function with respect to basic input variables and the availability of distributions of the basic input variables. In addition to the derivative calculation, the traditional safety factor design method is performed repeatedly several times. The new approach can be therefore conducted in the same manner as the traditional safety factor design method, manually, numerically, or with the help of computer software such as a spreadsheet. No optimization is needed.

Note that the proposed approach doesn't involve optimization and cannot make decisions (find design variables) automatically. It provides a safety factor for engineers to meet their reliability targets. How to get the design variables from the safety factor largely depends on how engineers perform their deterministic component design. If the deterministic component design can deal with black-box models, so can the proposed approach.

The proposed approach is intended for routine mechanical component design without high dimensional complex models, for which regular reliability-based design approaches should be used. The proposed approach is based on the first order reliability method (FORM), and it performs a complete MPP search. It is possible, however, the proposed method does not converge, especially when the design margin function is highly nonlinear in the transformed normal space. The approach may produce a large error if multiple MPPs exist. Our future research will investigate possible ways to avoid divergence and to deal with multiple MPPs.

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Appendix

Table A1 w for distributions

Distribution	PDF	w
Normal	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ μ: mean, σ: standard deviation	σ
Lognormal	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)$ μ: mean of $\ln x$, σ: standard deviation of $\ln x$	$\frac{\phi\left[\Phi^{-1}\left(\frac{1}{2}\left(1+\operatorname{erf}\left(\frac{\ln x-\mu}{\sqrt{2}\sigma}\right)\right)\right)\right]}{\frac{1}{x\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)}$
Gumbel	$f(x)=\frac{1}{\beta}\exp\left(-\left(\frac{x-\mu}{\beta}+\exp\left(-\frac{x-\mu}{\beta}\right)\right)\right)$ μ: location parameter, β: scale parameter	$\frac{\phi\left[\Phi^{-1}\left(\exp\left(-\exp\left(-\frac{x-\mu}{\beta}\right)\right)\right)\right]}{\frac{1}{\beta}\exp\left(-\left(\frac{x-\mu}{\beta}+\exp\left(-\frac{x-\mu}{\beta}\right)\right)\right)}$
Exponential	$f(x)=\begin{cases}\frac{1}{\beta}\exp\left(-\frac{1}{\beta}x\right) & x\geq 0,\\ 0 & x<0.\end{cases}$ β: mean, β^2: variance	$\frac{\phi\left[\Phi^{-1}\left(1-\exp\left(-\frac{1}{\beta}x\right)\right)\right]}{\frac{1}{\beta}\exp\left(-\frac{1}{\beta}x\right)}$
Weibull	$f(x)=\begin{cases}\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\exp\left(-\left(\frac{x}{\lambda}\right)^k\right) & x\geq 0,\\ 0 & x<0.\end{cases}$ λ: scale parameter, k: shape parameter	$\frac{\phi[\Phi^{-1}(1-\exp(-(x/\lambda)^k))]}{\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\exp(-(x/\lambda)^k)}$
Uniform	$f(x)=\begin{cases}\frac{1}{b-a} & a\leq x\leq b,\\ 0 & \text{otherwise.}\end{cases}$ $\frac{a+b}{2}$: mean, $\frac{1}{12}(b-a)^2$: variance	$\frac{\phi\left[\Phi^{-1}\left(\frac{x-a}{b-a}\right)\right]}{\frac{1}{b-a}}$

List of Table Captions

List of Figure Captions

Table 1 Design process of the shaft

Iteration	∇_G	S_F	d (mm)	ε (%)
1	–	1.0	22.02	–
2	$(2.0\times 10^7, -4.4167\times 10^7)$	1.3611	24.40	36.11
3	$(2.0\times 10^7, -3.2449\times 10^7)$	1.3643	24.42	0.22
4	$(2.0\times 10^7, -3.2373\times 10^7)$	1.3643	24.42	0.00

Table 2 Distributions of the random variables in Example 2

Random Variable	Distribution	Mean	Standard Deviation
S_y (MPa)	Normal	450	30
S_{sy} (MPa)	Normal	0.577×450	0.577×30
P (Watt)	Lognormal	20000	1200
ω (rpm)	Normal	650	32.5

Table 3 Design process of the key

Iteration	$\nabla_{G_1}, \nabla_{G_2}$	S_{F_1}, S_{F_2}	E_1, E_2 (mm)	$\varepsilon_1,\ \varepsilon_2$ (%)
1	$(3\times 10^7, -2.69\times 10^7, 2.25\times 10^7)$, $(1.73\times 10^7, -1.55\times 10^7, 1.30\times 10^7)$	1.0, 1.0	17.0, 12.9	–
2	$(3\times 10^7, -2.14\times 10^7, 2.02\times 10^7)$, $(1.73\times 10^7, -1.24\times 10^7, 1.17\times 10^7)$	1.6788, 1.6788	28.5, 21.6	67.88, 66.11
3	$(3\times 10^7, -2.09\times 10^7, 1.97\times 10^7)$, $(1.73\times 10^7, -1.20\times 10^7, 1.13\times 10^7)$	1.6839, 1.6839	28.6, 21.7	0.30, 0.30
4	$(3\times 10^7, -2.08\times 10^7, 1.97\times 10^7)$, $(1.73\times 10^7, -1.20\times 10^7, 1.13\times 10^7)$	1.6840, 1.6840	28,6, 21.7	5.9×10^{-3}, 5.9×10^{-3}

Table 4 Distributions of the random variables in Example 3

Random Variable	Distribution	Mean	Standard Deviation
δ (mm)	Normal	10	1
S_y (MPa)	Normal	450	45
P (N)	Lognormal	80000	9000
F (N)	Lognormal	1500	100
T (N)	Lognormal	4000	500
E (GPa)	Normal	20	0.2

Table 5 Design process of the cantilever tube

Iteration	$\nabla_{G_1}, \nabla_{G_2}$	S_F	$\boldsymbol{d}_1, \boldsymbol{d}_2$	ε (%)
1	$(1.00\times10^{-3}, -2.29\times10^{-5}, -2.06\times10^{-4}, -8.02\times10^{-4}, -9.80\times10^{-5})$, $(45\times10^{8}, -1.14\times10^{7}, -5.53\times10^{6}, -2.86\times10^{7})$	1, 1	$(35, 20, 2.5)$, $(40, 30, 3)$	-
2	$(1.00\times10^{-3}, -1.60\times10^{-5}, -1.46\times10^{-4}, -5.38\times10^{-4}, -6.67\times10^{-5})$, $(45\times10^{8}, -7.10\times10^{6}, -2.83\times10^{6}, -1.85\times10^{7})$	1.8365, 1.7993	$(40, 30, 3)$, $(50, 30, 5)$	83.65, 79.93
3	$(1.00\times10^{-3}, -1.58\times10^{-5}, -1.45\times10^{-4}, -4.98\times10^{-4}, -6.33\times10^{-5})$, $(45\times10^{8}, -6.90\times10^{6}, -2.80\times10^{6}, -1.71\times10^{7})$	1.8207, 1.8054	$(40, 30, 3)$, $(50, 30, 5)$	0.86, 0.34
4	$(1.00\times10^{-3}, -1.59\times10^{-5}, -1.45\times10^{-4}, -4.91\times10^{-4}, -6.27\times10^{-5})$, $(45\times10^{8}, -6.89\times10^{6}, -2.80\times10^{6}, -1.69\times10^{7})$	1.8211, 1.8040	$(40, 30, 3)$, $(50, 30, 5)$	0.03, 0.08
5	$(1.00\times10^{-3}, -1.59\times10^{-5}, -1.45\times10^{-4}, -4.90\times10^{-4}, -6.27\times10^{-5})$, $(45\times10^{8}, -6.89\times10^{6}, -2.80\times10^{6}, -1.69\times10^{7})$	1.8211, 1.8037	$(40, 30, 3)$, $(50, 30, 5)$	0.00, 0.01
6	-, $(45\times10^{8}, -6.89\times10^{6}, -2.80\times10^{6}, -1.69\times10^{7})$	-, 1.8037	-, $(50, 30, 5)$	-, 0.00