Noncommutative Partial Convexity Via Γ-Convexity

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Abstract

Motivated by classical notions of partial convexity, biconvexity, and bilinear matrix inequalities, we investigate the theory of free sets that are defined by (low degree) noncommutative matrix polynomials with constrained terms. Given a tuple of symmetric polynomials Γ, a free set $\mathcal{K}$ is called Γ-convex if for all $X \in \mathcal{K}$ and isometries V satisfying $V^*\Gamma(X)V = \Gamma(V^*XV)$, we have $V^*XV \in \mathcal{K}$. We establish an Effros–Winkler Hahn–Banach separation theorem for Γ-convex sets; they are delineated by linear pencils in the coordinates of Γ and the variables x.

Keywords Partial convexity · Biconvexity · Bilinear matrix inequalities · Noncommutative matrix polynomial · Matrix convexity · Free semialgebraic set · Linear pencil · Γ-convexity · Effros–Winkler theorem

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1 Introduction

Convexity is ubiquitous in quantitative sciences. A set $C \subset \mathbb{R}^g$ is convex if for any two points in C the line segment connecting them lies entirely in C. Such sets, whenever they are closed, are described by (possibly infinitely many) affine linear inequalities. Convexity is fundamental in many areas of mathematics, including functional analysis, optimization, and geometry [4]. The convex sets described by finitely many linear inequalities are precisely the polytopes, a very restrictive class. A much bigger but still very tractable class of convex sets C, which are the central objects in semidefinite programing [5], are described by linear matrix inequalities (LMIs), i.e.,

$$C = \left\{ x \in \mathbb{R}^g \mid A_0 + \sum_j A_j x_j \succeq 0 \right\},$$

where A_j are self-adjoint $d \times d$ matrices and $T \succeq 0$ means that the self-adjoint matrix T is positive semidefinite. Such sets are called spectrahedra. They appear in several branches of mathematics, e.g., optimization and algebraic geometry [5].

The **linear pencil** $L(x) = A_0 + \sum_j A_j x_j$ is naturally evaluated at tuples of self-adjoint $n \times n$ matrices X by

$$L(X) = A_0 \otimes I_n + \sum_j A_j \otimes X_j$$

leading to the notion of a **free spectrahedron**

$$\mathcal{D}_L = (\mathcal{D}_L(n))_n \quad \text{where} \quad \mathcal{D}_L(n) = \{ X \in \mathbb{S}_n(\mathbb{C})^g \mid L(X) \succeq 0 \},$$

where $\mathbb{S}_n(\mathbb{C})$ denotes the set of all $n \times n$ self-adjoint matrices. Free spectrahedra arise naturally in applications such as systems engineering and control theory [6]. They are matrix convex sets [7,9,11,14,15,18,19,32] and are dual to operator systems and thus intimately connected to the theory of completely positive maps [27]. Moreover, the Effros–Winkler Hahn–Banach separation theorem [11] says that matrix convex sets are determined by LMIs in much the same way that convex sets are determined by linear inequalities.

Sets and functions that have some partial convexity or other geometric features, say convex in some coordinates with the other held fixed, arise in applications. Free noncommutative polynomials, and more generally free rational functions, arise in engineering systems problems governed by a signal flow diagram. Typically, there are two classes of variables. The system variables depend on the choice of system parameters and produce polynomial, or more generally rational, inequalities (in the sense of positive semidefinite) in the state variables. The algebraic form of these inequalities involves (matrix-valued) free polynomials or rational functions and depends only upon the flow diagram, and not the particular choice of system variables. Convexity

in the state variables, for a given choice of system variables, is an important optimization consideration. One way to study partial convexity is through bilinear matrix inequalities (BMIs) [22][1]. A BMI is an expression of the form

$$A_0 + \sum A_j x_j + \sum_k B_k y_k + \sum_{p,q} C_{pq} x_p y_q \succeq 0 \qquad (1.1)$$

for self-adjoint matrices A_j, B_k, $C_{p,q}$ [31]. Domains defined by BMIs are convex in the x and y variables separately. The article [16] contains some noncommutative results on partial convexity.

In analogy with matrix convexity and BMIs, it makes sense to consider matrix polynomial inequalities built from a restricted set of predetermined polynomials giving rise to the notion of Γ-convexity. One type of inequality we consider is of the form

$$I - Ax - By - Cy^2 \succeq 0.$$

Sets describable in this form are "convex in x" and unconstrained in y. That is, by allowing extra nonlinear terms in the matrix inequality, we can isolate certain geometric features of the domain. Allowing xy, yx as in (1.1), obtains a class of biconvex sets.

Often times in this setting, the results, while finite-dimensional in nature, require working with operator inputs or coefficients for the inequalities, fitting in with larger trends in matrix convexity [8,12,13,25,26] and the emerging area of free analysis [1,3,21,28–30].

1.1 Free Polynomials and Their Evaluations

Let $x = (x_1, \ldots, x_{\mathsf{g}})$ denote a g-tuple of freely noncommuting variables. Let $\langle x \rangle$ denote the semigroup of words in x and we often use 1 to denote the unit $\varnothing$. Let $\mathbb{C}\langle x \rangle$ denote the algebra of finite $\mathbb{C}$-linear combinations of words in x. An element $p \in \mathbb{C}\langle x \rangle$ is a **free polynomial**, or just polynomial for short, and takes the form

$$p(x) = \sum_{w \in \langle x \rangle} p_w w, \qquad (1.2)$$

where the sum is finite and $p_w \in \mathbb{C}$. There is a natural involution $*$ on $\langle x \rangle$ determined by $x_j^* = x_j$ for $1 \leq j \leq \mathsf{g}$ and $(wu)^* = u^* w^*$ for $u, w \in \langle x \rangle$. This involution naturally extends to $\mathbb{C}\langle x \rangle$. For instance, for the polynomial p of Eq. (1.2),

$$p^* = \sum \overline{p_w} w^*.$$

Since $x_j^* = x_j$ the variables x are referred to as **symmetric variables**.

Let $\mathbb{S}(\mathbb{C})^{\mathsf{g}}$ denote the sequence, or graded set, $(\mathbb{S}_n(\mathbb{C})^{\mathsf{g}})_{n=1}^{\infty}$, where $\mathbb{S}_n(\mathbb{C})^{\mathsf{g}}$ is the set of g-tuples $(X_1, \ldots, X_{\mathsf{g}})$ of self-adjoint elements of $M_n(\mathbb{C})$. Elements of $\mathbb{C}\langle x \rangle$ are

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naturally **evaluated** at an $X \in \mathbb{S}(\mathbb{C})^g$. For a word

$$w = x_{j_1} x_{j_2} \cdots x_{j_N} \in \langle x \rangle,$$

and $X \in \mathbb{S}_n(\mathbb{C})^g$,

$$w(X) = X_{j_1} X_{j_2} \cdots X_{j_N} \in M_n(\mathbb{C}).$$

Given p as in Eq. (1.2),

$$p(X) = \sum_{w \in \langle x \rangle} p_w w(X) \in M_n(\mathbb{C}).$$

Thus p determines a (graded) function $p : \mathbb{S}(\mathbb{C})^g \to M(\mathbb{C})$, where $M(\mathbb{C})$ is the (graded) set $M(\mathbb{C}) = (M_n(\mathbb{C}))_n$. Likewise a tuple $p = (p_1, \ldots, p_r) \in \mathbb{C}\langle x \rangle^{1 \times r} = M_{1,r}(\mathbb{C}\langle x \rangle)$ determines a mapping $p : \mathbb{S}(\mathbb{C})^g \to M(\mathbb{C})^r$.

A polynomial is **symmetric** if it is invariant under the involution. As is well known, $p \in \mathbb{C}\langle x \rangle$ is symmetric if and only if $p(X)^* = p(X)$ for all $X \in \mathbb{S}(\mathbb{C})^g$. In this case p determines a mapping $p : \mathbb{S}(\mathbb{C})^g \to \mathbb{S}(\mathbb{C})^1$.

Certain matrix-valued free polynomials will play an important role in this article. A $\mu \times \mu$ matrix-valued free polynomial $p \in M_\mu(\mathbb{C}\langle x \rangle)$ takes the form of equation (1.2), but now the coefficients p_w lie in $M_\mu(\mathbb{C})$. Such a polynomial p is **evaluated** at a tuple $X \in \mathbb{S}_n(\mathbb{C})^g$ using the tensor (Kronecker) product as

$$p(X) = \sum p_w \otimes w(X) \in M_\mu(\mathbb{C}) \otimes M_n(\mathbb{C}),$$

and p is **symmetric** if $p(X)^* = p(X)$ for all $X \in \mathbb{S}(\mathbb{C})^g$. Equivalently, p is symmetric if $p_{w^*} = p_w^*$ for all words w.

1.2 Γ-Convex Sets

Let $\Gamma = (\gamma_1, \ldots, \gamma_r)$ denote a tuple of *symmetric* free polynomials with $\gamma_j = x_j$ for $1 \le j \le g \le r$. We also use $\Gamma : \mathbb{S}(\mathbb{C})^g \to \mathbb{S}(\mathbb{C})^r$ to denote the resulting mapping,

$$\Gamma(X) = (\gamma_1(X), \ldots, \gamma_r(X)).$$

A pair (X, V), where $X \in \mathbb{S}_n(\mathbb{C})^g$ and $V : \mathbb{C}^m \to \mathbb{C}^n$ is an isometry, is a Γ-**pair** provided

$$V^* \Gamma(X) V = \Gamma(V^* X V).$$

Let $\mathcal{C}_\Gamma$ denote the collection of Γ-pairs. For instance, if U is an $n \times n$ unitary matrix and $X \in \mathbb{S}_n(\mathbb{C})^g$, then (X, U) is a Γ-pair.

A subset $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^g$ is a sequence $(\mathcal{K}(n))_n$ where $\mathcal{K}(n) \subset \mathbb{S}_n(\mathbb{C})^g$ for each n. A subset $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^g$ is a **free set** if it is closed with respect to direct sums, simultaneous

unitary similarity, and restrictions to reducing subspaces.[2] A set $\mathcal{K}$ is a **Γ-convex set** if it is free and if

$$X \in \mathcal{K} \text{ and } (X, V) \in \mathcal{C}_\Gamma \implies V^*XV \in \mathcal{K}.$$

In the special case that $\mathtt{r} = \mathtt{g}$ (equivalently $\Gamma(x) = x$) Γ-convexity reduces to ordinary matrix convexity.

Example 1.1 Consider the case of two variables $(x, y) = (x_1, x_2)$ and $\Gamma = \{x, y, y^2\}$. For notational ease, we write y^2-convex instead of $\{x, y, y^2\}$-convex. A pair $((X, Y), V)$ is in $\mathcal{C}_\Gamma$ if and only if the range of V reduces Y and, as shown in Proposition 4.1, a free set $\mathcal{K}$ is y^2-convex if and only if $(X_1, Y), (X_2, Y) \in \mathcal{K}(n)$ implies $(\frac{X_1+X_2}{2}, Y) \in \mathcal{K}(n)$. Using either of these criteria, it is readily verified that, for d a positive integer, the TV screen $\mathrm{TV}^d = (\mathrm{TV}^d(n))_n$ defined by

$$\mathrm{TV}^d(n) = \{(X, Y) : I - X^2 - Y^{2d} \succeq 0\} \subset \mathbb{S}_n(\mathbb{C})^2 \tag{1.3}$$

is a free set that is y^2-convex. Section 4 treats y^2-convexity. In these directions, see also [2,10,16]. □

Example 1.2 Let $\Gamma = \{x, y, xy + yx, i(xy - yx))\}$. It is straightforward to verify that $((X, Y), V)$ is a Γ-pair if and only if $V^*XYV = V^*XV\,V^*YV$. Thus it is sensible, for notational purposes, to write xy in place of the more cumbersome $(xy+yx, i(xy-yx))$ and xy-convex in place of $\{x, y, xy + yx, i(xy - yx)\}$-convex. The convexity in this example is intimately connected with Bilinear Matrix Inequalities (BMIs)[3] as explained in Sect. 2.2, which were previously studied in [22]. □

Of course a theory of convexity should contain Hahn–Banach separation results. In the case of matrix convex sets, this role is played by monic linear pencils and the Effros–Winkler Matricial Hahn–Banach separation theorem [11], or just the **Effros–Winkler theorem** for short. A **monic linear pencil** $M \in M_\mu(\mathbb{C}\langle x\rangle)$ is a symmetric matrix-valued polynomial of the form

$$M(x) = I_\mu + \sum_1^{\mathtt{g}} A_j x_j,$$

where $A \in \mathbb{S}_\mu(\mathbb{C})^{\mathtt{g}}$. We refer to μ as the **size of the pencil** M. The following version of the Effros–Winkler theorem can be found in [17,19].

Theorem 1.3 *If $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^{\mathtt{g}}$ is a closed matrix convex set containing the origin and if $Y \notin \mathcal{K}$, then there is a monic linear pencil M such that $M(\mathcal{K}) \succeq 0$, but $M(Y) \not\succeq 0$. Furthermore, if Y has size ℓ, then M can be chosen to have size ℓ.*

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[2] Explicitly, if $X \in \mathcal{K}(n)$ and $Y \in \mathcal{K}(m)$, then $X \oplus Y \in \mathcal{K}(n + m)$; if U is a $n \times n$ unitary, then $U^*XU = (U^*X_1U, \ldots, U^*X_{\mathtt{g}}U) \in \mathcal{K}(n)$; and if $\mathcal{K} \subset \mathbb{C}^n$ is k dimensional reducing subspace for X, then $X|_{\mathcal{K}} \in \mathcal{K}(k)$.

[3] See for instance the MATLAB toolbox, https://set.kuleuven.be/optec/Software/bmisolver-a-matlab-package-for-solving-optimization-problems-with-bmi-constraints.

For Γ-convex sets, the analog of a monic linear pencil is a **monic Γ-pencil**—a symmetric $L \in M_\mu(\mathbb{C}\langle x\rangle)$ of the form

$$L(x) = I_\mu + \sum_{j=1}^{\mathfrak{r}} A_j \gamma_j(x),$$

where $A \in \mathbb{S}_\mu(\mathbb{C})^{\mathfrak{r}}$. In Sect. 2, analogs of the Effros–Winkler theorem for Γ-convex sets are established. For instance, a basic separation result that follows from Theorem 2.4 is the following.

Theorem 1.4 *Suppose $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^g$ is Γ-convex, contains 0, and $\Gamma(0) = 0$. If the matrix convex hull of $\Gamma(\mathcal{K}) \subset \mathbb{S}(\mathbb{C})^{\mathfrak{r}}$ is closed and if $Y \in \mathbb{S}_\ell(\mathbb{C})^g \setminus \mathcal{K}(\ell)$, then there is a monic Γ-pencil L of size ℓ such that L is positive semidefinite on $\mathcal{K}$, but $L(Y)$ is not positive semidefinite.*

Operator convexity is defined at the outset of Sect. 3. In the operator setting, the closedness hypothesis on the (operator) convex hull of $\Gamma(\mathcal{K})$ is not needed. It is closed automatically by Theorem 3.3.

Theorem 1.5 (Theorem 3.8) *Suppose $\mathfrak{K}$ is a bounded, strong operator topology closed, Γ-convex set that contains 0 and $\Gamma(0) = 0$. If $Y \notin \mathfrak{K}$, then there exists a positive integer N and a monic Γ-pencil L of size N such that L takes positive semidefinite values on $\mathfrak{K}$, but $L(Y)$ is not positive semidefinite.*

By direct summing all Γ-pencils with rational coefficients that are positive semidefinite on $\mathfrak{K}$, one obtains a single operator Γ-pencil L with bounded coefficients whenever 0 is in the interior of the convex hull of $\Gamma(\mathfrak{K})$. That is, by this routine argument we obtain the following corollary.

Corollary 1.6 *Suppose $\mathfrak{K}$ is strong operator topology closed, Γ-convex, bounded, contains 0, and $\Gamma(0) = 0$. Suppose also that 0 is in the interior of the convex hull of $\Gamma(\mathfrak{K})$. Then there exists a monic operator Γ-pencil L such that $\mathfrak{K} = \{X : L(X) \succeq 0\}$.*

The hypotheses of Corollary 1.6 are met whenever the span of Γ does not contain a positive polynomial, which is the content of Theorem 2.6. For example, this holds whenever the coordinates of Γ are multilinear.

1.3 Reader's Guide

Section 2 develops the framework of Γ-convexity. Section 2.1 discusses Γ-convex hulls and proves an analog of the Effros–Winkler theorem for Γ-convex sets in Theorem 2.4. Free semialgebraic sets and Γ-convex polynomials are introduced and treated in Sect. 2.3. Section 3 deals with a more robust class of Γ-convex sets. Here, by adding extra points, further structure is obtained. The article concludes with an investigation of y^2-convexity in Sect. 4.

2 General Theory of Γ-Convex Sets

We now introduce the basic notions related to Γ-convexity. The main result of this section is Theorem 2.4, giving an Effros–Winkler type separation result for Γ-convex sets. In Sect. 2.3, we touch upon Γ-convex polynomials, a topic we explore further in the accompanying paper [20].

2.1 Convex Hulls

The **Γ-convex hull** of a free set $\mathcal{K}$, the intersection of all Γ-convex sets containing $\mathcal{K}$, is denoted Γ-co($\mathcal{K}$). When $r = g$ (and thus $\Gamma(x) = x$) Γ-co($\mathcal{K}$) is the ordinary matrix convex hull of $\mathcal{K}$, denoted matco($\mathcal{K}$).

Proposition 2.1 *If $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^g$ is a free set, then*

$$\Gamma\text{-co}(\mathcal{K}) = \{V^*XV : X \in \mathcal{K}, \, (X, V) \in \mathcal{C}_\Gamma\}.$$

Proof Let $\mathscr{H} = \{V^*XV : X \in \mathcal{K}, \, (X, V) \in \mathcal{C}_\Gamma\}$. It is readily verified that $\mathscr{H}$ is a Γ-convex set that contains $\mathcal{K}$. On the other hand, by definition, Γ-co($\mathcal{K}$) must contain $\mathscr{H}$. ☐

Proposition 2.2 *Suppose $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^g$ is a free set and $X \in \mathbb{S}(\mathbb{C})^g$. The point X is in Γ-co($\mathcal{K}$) if and only if $\Gamma(X)$ is in matco($\Gamma(\mathcal{K})$). Equivalently,*

$$\Gamma^{-1}(\text{matco}(\Gamma(\mathcal{K}))) = \Gamma\text{-co}(\mathcal{K}).$$

Proof First suppose $X \in \Gamma\text{-co}(\mathcal{K})$. By Proposition 2.1, there exists a Y in $\mathcal{K}$ and an isometry V such that $V^*\Gamma(Y)V = \Gamma(V^*YV)$ and $X = V^*YV$. Thus, $\Gamma(X) = V^*\Gamma(Y)V$, and therefore $\Gamma(X) \in \text{matco}(\Gamma(\mathcal{K}))$.

Conversely, suppose $\Gamma(X) \in \text{matco}(\Gamma(\mathcal{K}))$. There is a $Y \in \mathcal{K}$ and an isometry V such that $\Gamma(X) = V^*\Gamma(Y)V$. Comparing the first g coordinates gives $X = V^*YV$ and hence (Y, V) is Γ-pair. Since $Y \in \mathcal{K}$ and (Y, V) is a Γ-pair, Proposition 2.1 implies $X \in \Gamma\text{-co}(\mathcal{K})$. ☐

Proposition 2.3 *The projection of* matco($\Gamma(\mathcal{K})$) *onto the first g coordinates is* matco($\mathcal{K}$).

Proof The set matco($\Gamma(\mathcal{K})$) is matrix convex. Hence its projection onto the first g coordinates is matrix convex and contains $\mathcal{K}$. Therefore this projection contains matco($\mathcal{K}$). On the other hand, by definition, this projection must be contained in matco($\mathcal{K}$). ☐

2.2 Hahn–Banach Separation and Pencils

As a special case of a matrix-valued free polynomial, a **Γ-pencil (of size d)** is a (affine) linear pencil L in $(\gamma_1, \ldots, \gamma_r)$ whose coefficients lie in $\mathbb{S}_d(\mathbb{C})$. Thus, there is a positive integer d and $A_0, A_1, \ldots, A_r \in \mathbb{S}_d(\mathbb{C})$ such that

$$L(x) = A_0 + \sum A_j \gamma_j(x). \tag{2.1}$$

Since $A_j \in \mathbb{S}_d(\mathbb{C})$ and γ_j are symmetric polynomials, L is symmetric, and hence at times we refer to L as a symmetric Γ-pencil. The pencil L is **monic** if $A_0 = I_d$. For example, in the case of two variables (x, y), a symmetric monic xy-pencil can be expressed as

$$L(x, y) = I + A_x x + A_y y + B x y + B^* y x,$$

where A_x, A_y are self-adjoint. A symmetric monic y^2-pencil is of the form

$$L(x, y) = I + A_x x + A_y y + B y^2,$$

where A_x, A_y, and B are self-adjoint. In the special case that $r = g$ (equivalently $\Gamma(x) = x$), $L(x) = I + \sum A_j x_j$ is known as a **monic linear pencil**.

A pencil L with coefficients in $\mathbb{S}_d(\mathbb{C})$ as in Eq. (2.1) is **evaluated** at a tuple $X \in \mathbb{S}_n(\mathbb{C})^g$ using the tensor product as

$$L(X) = A_0 \otimes I_n + \sum A_j \otimes \gamma_j(X) \in \mathbb{S}_d(\mathbb{C}) \otimes \mathbb{S}_n(\mathbb{C}).$$

The **free semialgebraic sets** associated to a symmetric Γ-pencil L,

$$\widehat{\mathcal{D}}_L := \{X \in \mathbb{S}(\mathbb{C})^g : L(X) \succeq 0\} \ (\text{resp. } \mathcal{P}_L := \{X \in \mathbb{S}(\mathbb{C})^g : L(X) \succ 0\}),$$

are Γ-convex. A difficult question is to determine when a closed (resp. open) Γ-convex set is the positivity (resp. strict positivity) set of a Γ-linear pencil.

Given a subset $\mathcal{S} \subset \mathbb{S}(\mathbb{C})^g$, its (levelwise) closed matrix convex hull is denoted by $\overline{\text{matco}}(\mathcal{S})$. Thus $\overline{\text{matco}}(\mathcal{S})(n)$ is the closure of $\text{matco}(\mathcal{S})(n)$ in $\mathbb{S}_n(\mathbb{C})^g$. A routine argument shows $\overline{\text{matco}}(\mathcal{S})$ is also matrix convex.

Theorem 2.4 *Suppose $\mathcal{K}$ is Γ-convex, contains 0, and $\Gamma(0) = 0$. If $Y \in \mathbb{S}_\ell(\mathbb{C})^g$ and $\Gamma(Y) \notin \overline{\text{matco}}(\Gamma(\mathcal{K}))(\ell)$, then there exists a monic Γ-pencil L of size ℓ such that $L(\mathcal{K}) \succ 0$, but $L(Y) \not\succeq 0$. In particular, if $\text{matco}(\Gamma(\mathcal{K}))$ is closed, then for each $Y \notin \mathcal{K}(\ell)$ there exists a monic Γ-pencil L of size ℓ such that $L(\mathcal{K}) \succ 0$, but $L(Y) \not\succeq 0$.*

Proof Since $\Gamma(Y) \notin \overline{\text{matco}}(\Gamma(\mathcal{K}))(\ell)$ and $\overline{\text{matco}}(\Gamma(\mathcal{K}))$ is a closed matrix convex subset of $\mathbb{S}(\mathbb{C})^r$ containing 0, Theorem 1.3 implies there is a monic linear pencil

$$M(z) = I_\ell + \sum_{j=1}^{r} A_j z_j$$

of size ℓ such that $M(Z) \succeq 0$ for all $Z \in \overline{\text{matco}}(\Gamma(\mathcal{K}))$, but $M(\Gamma(Y)) \not\succeq 0$. Thus $L' = M \circ \Gamma$ is a monic Γ-pencil of size ℓ that is indefinite at Y and positive semidefinite on $\mathcal{K}$. Replacing L' by

$$L(z) = tI + (1-t)L'(z) = I_\ell + \sum t A_j z_j$$

for small enough $t \in (0, 1)$ produces a monic Γ-pencil of size ℓ indefinite at Y such that $L(\mathcal{K}) \succ 0$.

To complete the proof, suppose matco($\Gamma(\mathcal{K})$) is closed and $Y \notin \mathcal{K} = \Gamma$-co($\mathcal{K}$). Since, by Proposition 2.2, $\Gamma(Y) \notin$ matco($\Gamma(\mathcal{K})$) $= \overline{\text{matco}}(\Gamma(\mathcal{K}))$ the existence of L follows from what has already been proved. $\square$

Remark 2.5 In Theorem 2.4, matco($\Gamma(\mathcal{K})$) can be replaced by any matrix convex set $\mathcal{R}$ containing matco($\Gamma(\mathcal{K})$) such that

$$\Gamma\text{-co}(\mathcal{K}) = \Gamma^{-1}(\mathcal{R} \cap \text{range}(\Gamma)).$$

This ambiguity complicates the problem of determining when a Γ-convex set is the positivity set of a monic Γ-pencil. $\square$

Theorem 2.6 *Suppose $\mathcal{K}$ is Γ-convex, contains 0, and $\Gamma(0) = 0$. The real span of $\{\gamma_j : 1 \leq j \leq r\}$ contains a polynomial $q \in \mathbb{C}\langle x \rangle$ such that $q(X) \succeq 0$ for $X \in \mathcal{K}$ if and only if 0 is not in the interior of* matco($\Gamma(\mathcal{K})$)$(1) \subset \mathbb{R}^r$.

Proof Suppose 0 is not in the interior of matco($\Gamma(\mathcal{K})$)(1). In this case 0 is in the boundary of matco($\Gamma(\mathcal{K})$)(1), since $\Gamma(0) = 0$. Hence, as matco($\Gamma(\mathcal{K})$)(1) is convex, there exists a linear functional $\lambda : \mathbb{R}^r \to \mathbb{R}$ such that λ is nonnegative on matco($\Gamma(\mathcal{K})$)(1). Thus $\lambda(z) = \sum_{j=1}^r \lambda_j z_j$ for some $\lambda_j \in \mathbb{R}$. Set $q = \sum_{j=1}^r \lambda_j \gamma_j$.

Suppose n is a positive integer, $Y \in \mathbb{S}_n(\mathbb{C})^g$, and $h \in \mathbb{C}^n$ is a unit vector. Identify h as an isometry $h : \mathbb{C} \to \mathbb{C}^n$, let $y = h^* \Gamma(Y) h$ and observe,

$$
\begin{aligned}
\lambda(y) &= \sum_{j=1}^r \lambda_j y_j = \sum_{j=1}^r \lambda_j h^* \gamma_j(Y) h \\
&= h^* \left[\sum_{j=1}^r \lambda_j \gamma_j(Y) \right] h = h^* q(Y) h.
\end{aligned}
\tag{2.2}
$$

If $Y \in \mathcal{K}(n)$ and $h \in \mathbb{C}^n$ is any unit vector then, by Proposition 2.1, $y \in$ matco($\Gamma(\mathcal{K})$)(1). Thus,

$$0 \leq \lambda(y) = h^* q(Y) h$$

and it follows that $q(Y) \succeq 0$. Hence $q(Y) \succeq 0$ on $\mathcal{K}$.

To prove the converse, suppose 0 is in the interior of matco($\Gamma(\mathcal{K})$)(1) and q is in the real span of $\{\gamma_1, \ldots, \gamma_r\}$. Thus there is a $\lambda \in \mathbb{R}^r$ such that $q = \sum_{j=1}^r \lambda_j \gamma_j$. View $\lambda : \mathbb{R}^r \to \mathbb{R}$ as the linear map $\lambda(z) = \sum_{j=1}^r \lambda_j z_j$. Since $\lambda(0) = 0$ and 0 is in the interior of matco($\Gamma(\mathcal{K})$), there exists $y \in$ matco($\Gamma(\mathcal{K})$)(1) such that $\lambda(y) < 0$. Since $y \in$ matco($\Gamma(\mathcal{K})$)(1), by Proposition 2.1 there exists an n, a vector $h \in \mathbb{C}^n$, and $Y \in \mathcal{K}(n)$ such that $y = h^* \Gamma(Y) h$. Thus, by Eq. (2.2),

$$0 > \lambda(y) = h^* q(Y) h.$$

Therefore $q(Y) \not\succeq 0$ and the proof is complete. $\square$

2.3 Γ-Convex Polynomials

A symmetric $p \in M_\mu(\mathbb{C}\langle x \rangle)$ is a Γ-**convex polynomial** if, for each $(X, V) \in \mathcal{C}_\Gamma$,

$$(I_\mu \otimes V)^* p(X)(I_\mu \otimes V) - p(V^* X V) \succeq 0.$$

It is a Γ-**concave polynomial** if $-p$ is Γ-convex.

2.3.1 Free Semialgebraic Sets

Given a symmetric polynomial $p \in M_\mu(\mathbb{C}\langle x \rangle)$ with $p(0) \succ 0$ and a positive integer n, let

$$\widehat{\mathcal{D}}_p(n) = \{X \in \mathbb{S}_n(\mathbb{C})^g : p(X) \succeq 0\}$$

and let $\mathcal{D}_p(n)$ denote the closure of

$$\mathcal{P}_p(n) = \{X \in \mathbb{S}_n(\mathbb{C})^g : p(X) \succ 0\}.$$

Let $\widehat{\mathcal{D}}_p$ denote the sequence $(\widehat{\mathcal{D}}_p(n))_n$. Likewise let $\mathcal{P}_p = (\mathcal{P}_p(n))_n$ and $\mathcal{D}_p = (\mathcal{D}_p(n))_n$. The sets $\widehat{\mathcal{D}}_p$, $\mathcal{D}_p$, and $\mathcal{P}_p$ are free analogs of basic semialgebraic sets. We refer to all of these (possibly distinct) sets as **free semialgebraic sets**. As an example, the sets TV^d of (1.3) are free semialgebraic. The inequalities arising from signal flow diagrams give rise to free semialgebraic sets, or more generally sets defined by rational inequalities.

Free semialgebraic sets that have additional geometric properties, such as being star-like, satisfy cleaner versions of our main results, e.g., Corollary 3.15. One of our main goals along the lines of [17] is to develop constrained simple representations of semialgebraic sets with certain geometric properties, that is, represent them as a positivity set of a Γ-pencil.

Proposition 2.7 *If $p \in M_\mu(\mathbb{C}\langle x \rangle)$ is a Γ-concave polynomial, then $\widehat{\mathcal{D}}_p$ and $\mathcal{P}_p$ are Γ-convex.*

Proof If $X \in \widehat{\mathcal{D}}_p$ and (X, V) is a Γ-pair, then $(I_\mu \otimes V)^* p(X)(I_\mu \otimes V) \succeq 0$ and, since p is Γ-concave,

$$p(V^* X V) \succeq (I_\mu \otimes V)^* p(X)(I_\mu \otimes V) \succeq 0.$$

Therefore $V^* X V \in \widehat{\mathcal{D}}_p$ and hence $\widehat{\mathcal{D}}_p$ is Γ-convex. The same argument shows $\mathcal{P}_p$ is Γ-convex. □

An $f \in M_{\nu \times \mu}(\mathbb{C}\langle x \rangle)$ is a Γ-**concomitant** if

$$(I_\mu \otimes V)^* f(X)(I_\nu \otimes V) = f(V^* X V)$$

for every $(X, V) \in \mathcal{C}_\Gamma$.

Corollary 2.8 *If $f \in M_\mu(\mathbb{C}\langle x \rangle)$ is a Γ-concomitant, then $\widehat{\mathcal{D}}_f$ and $\mathcal{P}_f$ are Γ-convex.*

Proof If f is a Γ-concomitant, then f is Γ-concave and hence Proposition 2.7 applies.
□

Remark 2.9 If L is a monic Γ-pencil and $f_j \in M_{\mu_j \times \mu}(\mathbb{C}\langle x \rangle)$ for $j = 1, \ldots, N$ are Γ-concomitant, then

$$M(x) = \begin{pmatrix} & & f_1 \\ & I_N & \vdots \\ & & f_N \\ f_1^* & \cdots & f_N^* & L \end{pmatrix}$$

is a Γ-concomitant. Hence $\mathcal{D}_M$ is Γ-convex. By taking a Schur complement, $\mathcal{D}_M = \mathcal{D}_p$, for

$$p = L - \sum f_j^* f_j,$$

which has the form of a monic Γ-pencil minus a sum of hermitian squares of a Γ-concomitant polynomials.
□

Rudimentary classification results for partially convex free polynomials exist in [16]; several classes of Γ-convex functions for specific cases will be given in the sequel [20].

3 The Operator Setting

In this section, the notion of Γ-convexity is extended to tuples of operators. While the matrix case is our primary interest, apparent defects in the geometry, such as level sets not "varying continuously," necessitates an appeal to the penumbral operator case. We will see in Sect. 3.4 that tools from the operator setting lead to results for matrix Γ-convexity. The remainder of this section is organized as follows. Section 3.1 contains preliminary results, including an analog of Proposition 2.2 (see Proposition 3.1) and the key fact that, for a strong operator topology (SOT)-closed and bounded free set $\mathfrak{K}$, the operator convex hull of $\Gamma(\mathfrak{K})$ and the Γ-operator convex hull of $\mathfrak{K}$ are again SOT-closed (see Theorem 3.3). Versions of the Effros–Winkler theorem for operator convex sets and operator Γ-convex sets are established in Sects. 3.2 and 3.3, respectively. The section concludes with the desired applications of operator Γ-convexity to matrix Γ-convexity for free semialgebraic sets in Sect. 3.4.

3.1 Operator Γ-Convexity and the Strong Operator Topology

Fix an infinite dimensional separable complex Hilbert space $\mathcal{H}$ and let $B(\mathcal{H})_{sa}^g$ denotes the g-tuples of self-adjoint bounded operators on $\mathcal{H}$. We equip $B(\mathcal{H})_{sa}^g$ with the maximum norm $\|X\| = \max\{\|X_1\|, \ldots, \|X_g\|\}$. A subset $\mathfrak{K} \subset B(\mathcal{H})^g$ is a **free set** if it is

closed under unitary similarity and closed under direct sums, where $\mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ is identified with $\mathcal{H}$.

By analogy with the matricial theory from Sect. 2, let $\Gamma = (\gamma_1, \ldots, \gamma_r)$ denote a tuple of *symmetric* free polynomials with $\gamma_j = x_j$ for $1 \leq j \leq g \leq r$, and let $\Gamma : B(\mathcal{H})_{sa}^g \to B(\mathcal{H})_{sa}^r$ denote the resulting mapping on self-adjoint operator tuples. As before, (X, V) is called a Γ-**pair** provided $X \in B(\mathcal{H})_{sa}^g$, $V : \mathcal{H} \to \mathcal{H}$ is an isometry, and

$$V^* \Gamma(X) V = \Gamma(V^* X V).$$

Let $\mathfrak{C}_\Gamma$ denote the collection of (operator) Γ-pairs.

A free set $\mathfrak{K} \subset B(\mathcal{H})_{sa}^g$ is called **operator convex** if whenever $X \in \mathfrak{K}$ and $V : \mathcal{H} \to \mathcal{H}$ is an isometry, then $V^* X V \in \mathfrak{K}$. It is called **operator Γ-convex** if

$$X \in \mathfrak{K} \text{ and } (X, V) \in \mathfrak{C}_\Gamma \implies V^* X V \in \mathfrak{K}.$$

In the special case that $r = g$ (equivalently $\Gamma(x) = x$) Γ-convexity reduces to ordinary operator convexity. The **operator Γ-convex hull** of a free set $\mathfrak{K} \subset B(\mathcal{H})_{sa}^g$ is the intersection of all operator Γ-convex sets containing $\mathfrak{K}$ and is denoted Γ-opco$(\mathfrak{K})$.

It is immediate that Propositions 2.1 and 2.2 have operator analogs.

Proposition 3.1 *If $\mathfrak{K} \subset B(\mathcal{H})_{sa}^g$ is a free set, then*

$$\Gamma\text{-opco}(\mathfrak{K}) = \{V^* X V : X \in \mathfrak{K}, (X, V) \in \mathfrak{C}_\Gamma\} \quad and$$
$$\Gamma^{-1}(\text{opco}(\Gamma(\mathfrak{K}))) = \Gamma\text{-opco}(\mathfrak{K}).$$

We now show that the operator convex hull of a bounded SOT-closed free set is again SOT-closed, eliminating certain technical difficulties in absence of a Caratheodory-type theorem for Γ-convexity. The proof uses a Heine–Borel type compactness principle from operatorial noncommutative function theory, which was previously applied by [24].

Lemma 3.2 ([24, Lemma 4.5 and Remark 4.6]) *Let X_n be a bounded sequence of operator tuples in $B(\mathcal{H})^g$. Then there exists a sequence U_n of unitary operators on $\mathcal{H}$ and a subsequence along which $U_n^* X_n U_n$ and U_n^* both converge in SOT.*

Theorem 3.3 *Suppose $\mathfrak{K} \subset B(\mathcal{H})_{sa}^g$ is a free set. If $\mathfrak{K}$ is bounded and SOT-closed, then* opco$(\Gamma(\mathfrak{K}))$ *and Γ-opco$(\mathfrak{K})$ are SOT-closed.*

Proof By the set equality in Proposition 3.1, it suffices to show opco$(\Gamma(\mathfrak{K}))$ is SOT-closed, since Γ is SOT-continuous on bounded sets as it is a free polynomial mapping.

Suppose Y is in the SOT-closure of opco$(\Gamma(\mathfrak{K}))$. There exist isometries V_n and $X_n \in \mathfrak{K}$ such that $V_n^* \Gamma(X_n) V_n \overset{SOT}{\to} Y$. By Lemma 3.2 applied to $(V_n, X_n) \in B(\mathcal{H})^{g+1}$, there exist unitaries U_n such that, after passing to a subsequence, $U_n^* V_n U_n \overset{SOT}{\to} V$, $U_n^* X_n U_n \overset{SOT}{\to} X$, and $U_n^* \overset{SOT}{\to} W$, where V, W are isometries, and $X \in \mathfrak{K}$ since $\mathfrak{K}$ is free and SOT-closed.

Since $U_n^* V_n = (U_n^* V_n U_n) U_n^*$, we have $U_n^* V_n \overset{SOT}{\to} VW$ and *a fortiori* that $V_n^* U_n \overset{WOT}{\to} W^* V^*$. Moreover, $\Gamma(U_n^* X_n U_n) U_n^* V_n \overset{SOT}{\to} \Gamma(X) VW$ since multiplication is SOT-continuous on bounded sets. Note that if $S_n \overset{WOT}{\to} S$ and $T_n \overset{SOT}{\to} T$, then $S_n T_n \overset{WOT}{\to} ST$. Hence

$$
\begin{aligned}
Y &= \text{SOT-}\lim_n V_n^* \Gamma(X_n) V_n \\
&= \text{SOT-}\lim_n V_n^* U_n \Gamma(U_n^* X_n U_n) U_n^* V_n \\
&= \text{WOT-}\lim_n V_n^* U_n [\Gamma(U_n^* X_n U_n) U_n^* V_n] \\
&= W^* V^* [\Gamma(X) VW] = (VW)^* \Gamma(X) VW.
\end{aligned}
$$

Hence $Y = (VW)^* \Gamma(X) VW \in \text{opco}(\Gamma(\mathfrak{K}))$. $\qquad \square$

Remark 3.4 Note that, in the context of Theorem 3.3, as $\text{opco}(\Gamma(\mathfrak{K}))$ is convex and SOT-closed, it is also WOT-closed. The proof that $\text{opco}(\Gamma(\mathfrak{K}))$ is SOT-closed shows in fact that if $P : B(\mathcal{H})_{sa}^{\mathfrak{g}} \to B(\mathcal{H})_{sa}^{\mathfrak{r}}$ is any free polynomial mapping, then $\text{opco}(P(\mathfrak{K}))$ is SOT-closed. $\qquad \square$

Given a symmetric noncommutative polynomial $p \in M_\mu(\mathbb{C}\langle x \rangle)$, in addition to $\widehat{\mathcal{D}}_p$, $\mathcal{P}_p$, and $\mathcal{D}_p$ defined in Sect. 2.3.1, we consider the following sets describing its positivity on the operator level:

$$
\begin{aligned}
\widehat{\mathfrak{D}}_p &= \{X \in B(\mathcal{H})_{sa}^{\mathfrak{g}} : p(X) \succeq 0\}; \\
\mathfrak{P}_p &= \{X \in B(\mathcal{H})_{sa}^{\mathfrak{g}} : p(X) \succ 0\}; \\
\mathfrak{D}_p &= \overline{\mathfrak{P}_p}^{SOT}.
\end{aligned}
$$

We also refer to all of these (possibly distinct) sets as **free semialgebraic sets**. Observe that, whenever they are bounded, $\mathfrak{D}_p$ and $\widehat{\mathfrak{D}}_p$ are SOT-closed and hence Theorem 3.3 applies.

3.2 The Effros–Winkler Theorem for Operator Convex Sets

We now show that a version of the Effros–Winkler theorem holds for bounded, SOT-closed free sets in the operator convex case. In Sect. 3.3, we prove a version for operator Γ-convex sets. Given a monic Γ-pencil $L = I_N + \sum A_j \gamma_j$ of size N, or an operator Γ-pencil $\widetilde{L} = I_{\widetilde{\mathcal{H}}} + \sum B_j \gamma_j$ for a separable infinite dimensional Hilbert space $\widetilde{\mathcal{H}}$ and $B_j \in B(\widetilde{\mathcal{H}})_{sa}$, its **evaluation on operator tuples** is defined as

$$
\begin{aligned}
L(X) &= I_N \otimes I_{\mathcal{H}} + \sum A_j \otimes \gamma_j(X), \\
\widetilde{L}(X) &= I_{\widetilde{\mathcal{H}}} \otimes I_{\mathcal{H}} + \sum B_j \otimes \gamma_j(X),
\end{aligned}
$$

respectively, for $X \in B(\mathcal{H})_{sa}^{\mathfrak{g}}$.

The **positivity set** (resp. **strict positivity set**) of a Γ-pencil L in the operator setting is

$$\widehat{\mathfrak{D}}_L := \left\{ X \in B(\mathcal{H})_{sa}^g : L(X) \succeq 0 \right\} \ \left(\text{resp.} \ \mathfrak{P}_L := \left\{ X \in B(\mathcal{H})_{sa}^g : L(X) \succ 0 \right\} \right).$$

Theorem 3.5 *Suppose* $\mathfrak{K} \subset B(\mathcal{H})_{sa}^g$ *is operator convex, SOT-closed, and contains* 0. *If* $Y \notin \mathfrak{K}$, *then there is a positive integer* N *and a monic linear pencil*

$$L = I_N + \sum_{j=1}^{g} A_j x_j,$$

where $A_j \in \mathbb{S}_N(\mathbb{C})$, *such that* $L(\mathfrak{K}) \succeq 0$, *but* $L(Y) \not\succeq 0$.

In particular, $\mathfrak{K}$ *is the intersection* $\bigcap \widehat{\mathfrak{D}}_L$, *where the intersection is over monic linear pencils* L *such that* $\widehat{\mathfrak{D}}_L \supset \mathfrak{K}$.

Finally, if ℓ *is a positive integer,* $\mathscr{F}$ *is an* ℓ *dimensional subspace of* $\mathcal{H}$ *and* $Y = Y' \oplus 0$ *with respect to the orthogonal decomposition* $\mathscr{F} \oplus \mathscr{F}^\perp$ *of* $\mathcal{H}$, *then* L *can be chosen to have size* ℓ.

The proof of Theorem 3.5 uses Proposition 3.7 below, which in turn uses the following Lemma.

Lemma 3.6 *If* $\mathfrak{K} \subset B(\mathcal{H})_{sa}^g$ *is operator convex and contains* 0, *then* $\mathfrak{K}$ *is closed under conjugation by contractions: if* $X \in \mathfrak{K}$, $C \in B(\mathcal{H})$, *and* $\|C\| \leq 1$, *then* $C^* X C \in \mathfrak{K}$.

In particular, if $P \in B(\mathcal{H})$ *is a projection and* $Y \in \mathfrak{K}$, *then* $PYP = Y' \oplus 0 \in \mathfrak{K}$.

Proof If $X \in \mathfrak{K}$, then $Z := \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{K}$. For a contraction C, define the isometry

$$W := \begin{pmatrix} C \\ (1 - C^*C)^{1/2} \end{pmatrix}.$$

Now, $C^* X C = W^* Z W \in \mathfrak{K}$. $\square$

To a free subset $\mathfrak{K} \subset B(\mathcal{H})_{sa}^g$ we associate a matrix convex set $\mathfrak{K}_{\text{mat}}$. Given a positive integer n, let $\mathfrak{K}_{\text{mat}}(n)$ denote the set of tuples $X \in \mathbb{S}_n(\mathbb{C})^g$ of the form $V^* Y V$, where $V : \mathbb{C}^n \to \mathcal{H}$ is an isometry and $Y \in \mathfrak{K}$. Note that, by Lemma 3.6, $VV^* Y VV^* \in \mathfrak{K}$ and hence if $Y' \in \mathfrak{K}_{\text{mat}}$, then $Y' \oplus 0 \in \mathfrak{K}$.

Proposition 3.7 *If* $\mathfrak{K} \subset B(\mathcal{H})_{sa}^g$ *is operator convex, SOT-closed, and contains* 0, *then*

(a) $\mathfrak{K}_{\text{mat}}$ *is a closed matrix convex set containing* 0;

(b) $\mathfrak{K} = \overline{\{Y' \oplus 0 \mid Y' \in \mathfrak{K}_{\text{mat}}\}}^{SOT}$;

(c) *if* $Y \notin \mathfrak{K}$, *then there is an* N *and an isometry* $V : \mathbb{C}^N \to \mathcal{H}$ *so that* $V^* Y V \notin \mathfrak{K}_{\text{mat}}(N)$;

(d) *if* $Y' \in \mathbb{S}_N(\mathbb{C})^g$ *and* $Y' \oplus 0 \notin \mathfrak{K}$, *then there is a monic linear pencil* M *of size* N *such that* $M(\mathfrak{K}) \succeq 0$, *but* $M(Y') \not\succeq 0$.

Proof A routine argument shows $\mathfrak{K}_{mat}$ is (a free set and) matrix convex. If (Y_m) is a sequence from $\mathfrak{K}_{mat}(n)$ that converges to Y, then $(Y_m \oplus 0)$ is a sequence from $\mathfrak{K}$ that converges in norm, and hence in SOT, to $Y \oplus 0$. It follows that $Y \in \mathfrak{K}_{mat}(n)$ and hence $\mathfrak{K}_{mat}$ is closed.

If $Y' \in \mathfrak{K}_{mat}$, then $Y' \oplus 0 \in \mathfrak{K}$ as already noted. Conversely, given $X \in \mathfrak{K}$ and a sequence P_n of finite rank projections onto the first n basis vectors of $\mathcal{H}$, we have $P_n X P_n \overset{SOT}{\to} X$, $P_n X P_n \in \mathfrak{K}$ by Lemma 3.6 and $P_n X P_n$ has the form $Y_n \oplus 0$ for $Y_n \in \mathfrak{K}_{mat}(n)$. Item (b) follows.

Item (c) follows from item (b).

To prove item (d), since $Y' \oplus 0 \notin \mathfrak{K}$, it follows that $Y' \notin \mathfrak{K}_{mat}(N)$. Since $\mathfrak{K}_{mat}$ is closed and matrix convex, Theorem 1.3 implies there is a monic linear pencil M of size N such that $M(\mathfrak{K}_{mat}) \succeq 0$, but $M(Y') \nsucceq 0$. By item (b) and SOT-continuity of M, it follows that $M(\mathfrak{K}) \succeq 0$. □

Proof of Theorem 3.5 By Proposition 3.7, there is an N and an isometry $V : \mathbb{C}^N \to \mathcal{H}$ such that $Y' = V^* Y V \notin \mathfrak{K}_{mat}(N)$. Hence $Y' \oplus 0 \notin \mathfrak{K}$ and therefore, by Proposition 3.7(d), there is a monic linear pencil M of size N such that $M(\mathfrak{K}) \succeq 0$, but $M(Y) = M(Y') \oplus I \nsucceq 0$.

To prove the last statement, note that, given the form of Y and the hypotheses of the theorem, $Y \notin \mathfrak{K}$ if and only if $Y' \notin \mathfrak{K}_{mat}(\ell)$. □

3.3 Hahn–Banach Separation for Operator Γ-Convex Sets

Combining Theorems 3.3 and 3.5 yields Theorem 3.8 below. It may be seen as an improvement of Theorem 2.4 for bounded, SOT-closed operator Γ-convex sets since it does not require that the (operator) convex hull of $\Gamma(\mathfrak{K})$ be closed a priori.

Theorem 3.8 *Suppose $\mathfrak{K} \subset B(\mathcal{H})_{sa}^g$ is SOT-closed, operator Γ-convex, bounded, and that $0 \in \mathrm{opco}(\Gamma(\mathfrak{K}))$. For each $Y \notin \mathfrak{K}$, there is a positive integer N an a monic linear pencil*

$$M = I_N + \sum_{j=1}^{r} A_j x_j,$$

of size N such that $M(\mathrm{opco}(\Gamma(\mathfrak{K}))) \succeq 0$, but $M(\Gamma(Y)) \nsucceq 0$. Thus the Γ-pencil $L = M \circ \Gamma = I_N + \sum A_j \gamma_j$ is positive semidefinite on $\mathfrak{K}$, but $L(Y) \nsucceq 0$.

In particular, $\mathfrak{K} = \bigcap \widehat{\mathfrak{D}}_L$, where the intersection is over all monic Γ-pencils L such that $\widehat{\mathfrak{D}}_L \supset \mathfrak{K}$.

Finally, suppose $0 \in \mathfrak{K}$ and that $\Gamma(0) = 0$. If ℓ is a positive integer, $\mathcal{F}$ is an ℓ dimensional subspace of $\mathcal{H}$ and $Y = Y' \oplus 0$ with respect to the orthogonal decomposition $\mathcal{F} \oplus \mathcal{F}^\perp$ of $\mathcal{H}$, then L and M can be chosen to have size ℓ.

Proof The assumptions imply $\mathrm{opco}(\Gamma(\mathfrak{K}))$ is a bounded, operator convex set containing 0. It is SOT-closed by Theorem 3.3. If $Y \notin \mathfrak{K} = \Gamma\text{-opco}(\mathfrak{K})$, then by the set equality in Proposition 3.1, $\Gamma(Y) \notin \mathrm{opco}(\Gamma(\mathfrak{K}))$. Therefore, by Theorem 3.5, there exists a positive integer N and a monic linear pencil

$$M = I_N + \sum_{j=1}^{r} A_j x_j$$

of size N such that $M \succeq 0$ on opco($\Gamma(\mathfrak{K})$), but $M(\Gamma(Y)) \not\succeq 0$.

In the case of the final assertion, $\Gamma(Y) = \Gamma(Y') \oplus \Gamma(0) = \Gamma(Y') \oplus 0$. Since $\Gamma(Y')$ has size ℓ and $\Gamma(Y) \notin$ opco($\Gamma(\mathfrak{K})$), the monic Γ-pencil L can be chosen to have size ℓ by Theorem 3.5. □

Remark 3.9 In Theorems 3.5 and 3.8, while the convex sets consist of operator tuples, outliers are separated from these sets by a monic pencil of *finite* size; that is, a matrix pencil. In the particular cases described in their final statements, the theorems assert further control over this finite size. □

Lemma 3.10 *For an SOT-closed operator convex set $\mathfrak{K} \subset B(\mathcal{H})_{sa}^g$, the following are equivalent.*

(i) 0 *is in the (norm) interior of $\mathfrak{K}$;*
(ii) *there is a constant C such that if $L = I + \sum A_j x_j$ is monic linear pencil that is positive semidefinite on $\mathfrak{K}$, then $\|A_j\| < C$ for all j;*
(iii) *there is an $\epsilon > 0$ such that $\pm \epsilon e_j \in \mathfrak{K}$, where $e_j \in B(\mathcal{H})_{sa}^g$ has k-th entry 0 if $k \neq j$ and I if $k = j$.*

Further, if M is a monic linear pencil such that $M(\mathfrak{K}) \succeq 0$, then $M \succ 0$ on int($\mathfrak{K}$).

Proof The implication (i) implies (iii) is evident. To pass from item (iii) to item (ii), choose $C = \frac{1}{\epsilon}$.

Now suppose item (i) does not hold; that is 0 is not in the norm interior of $\mathfrak{K}$. Given $\delta > 0$, there is a tuple $Y \notin \mathcal{K}$ with $\|Y\| < \delta$. By Theorem 3.5, there is a monic linear pencil $L = I + \sum A_j x_j$ of finite size such that $L \succeq 0$ on $\mathcal{K}$, but $L(Y) \not\succeq 0$. Since $I \not\preceq -\sum A_j \otimes Y_j$ and

$$-\sum A_j \otimes Y_j \preceq \|\sum A_j \otimes Y_j\| I \preceq \sum \|A_j\| \|Y_j\| I \preceq \delta \sum \|A_j\| I,$$

it follows that there is a j such that $\|A_j\| \geq \frac{1}{g\delta}$. Hence item (ii) does not hold and the proof that items (i), (ii), and (iii) are equivalent is complete.

To complete the proof, suppose $X \in \mathfrak{K}$ and $M(X) \succeq 0$, but $M(X) \not\succ 0$. Thus, there is a nonzero vector h such that $\langle M(X)h, h \rangle = 0$. Since M is monic linear, $\langle M(tX)h, h \rangle = \|h\|^2 - t\|h\|^2$. Hence, $\langle M(tX)h, h \rangle < 0$ for $t > 1$. Thus X is not in the interior of $\mathfrak{K}$. □

In Corollary 3.11 below, the topological notions of boundary and interior are with respect to the relative norm topology on $B(\mathcal{H})_{sa}^g$. Note that, in the case $r = g$ (equivalently $\Gamma(x) = x$) the condition of Eq. (3.1) in Corollary 3.11 is automatically satisfied.

Corollary 3.11 *Suppose $\mathfrak{K} \subset B(\mathcal{H})_{sa}^g$ is SOT-closed, operator Γ-convex, bounded, contains 0, and $\Gamma(0) = 0$. Suppose also that 0 is in the (norm) interior of* opco($\Gamma(\mathfrak{K})$) *and that*

$$\text{int}(\mathfrak{K}) \subset \Gamma^{-1}(\text{int}(\text{opco}(\Gamma(\mathfrak{K})))). \tag{3.1}$$

If Y is in the (norm) boundary of $\mathfrak{K}$, then there exists a monic operator Γ-pencil $L = I + \sum A_j \gamma_j$ such that $L(X) \succ 0$ for all X in the interior of $\mathfrak{K}$ and such that $L(Y)$ is not bounded below by any positive multiple of the identity.

Proof There is a sequence (Y_n) from $B(\mathcal{H})_{sa}^{\mathfrak{g}}$ such that $Y_n \notin \mathfrak{K}$ and $(Y_n) \to Y$ in norm. By the first statement in Theorem 3.8, there are monic linear pencils M_n of finite size such that $M_n(\text{opco}(\Gamma(\mathfrak{K}))) \succeq 0$, but $M_n(\Gamma(Y_n)) \not\succeq 0$. Since 0 is in the (norm) interior of $\text{opco}(\Gamma(\mathfrak{K}))$, Lemma 3.10 implies $M := \bigoplus_n M_n$ defines a monic pencil whose coefficients are bounded operators on a separable Hilbert space. Put $L := M \circ \Gamma$.

By construction, $M \succeq 0$ on $\text{opco}(\Gamma(\mathfrak{K}))$. By Lemma 3.10 applied to $\text{opco}(\Gamma(\mathfrak{K}))$, it follows that $M(Z) \succ 0$ for $Z \in \text{int}(\text{opco}(\Gamma(\mathfrak{K})))$. By the inclusion in (3.1), if $X \in \text{int}(\mathfrak{K})$, then $\Gamma(X) \in \text{int}(\text{opco}(\Gamma(\mathfrak{K})))$, and $M(\Gamma(X)) = L(X) \succ 0$.

Finally, we show $L(Y)$ is not bounded below by a positive multiple of the identity. Since the sequence Y_n tends to Y in norm, $\Gamma(Y_n) \to \Gamma(Y)$ in norm. As $M_n(\Gamma(Y_n))$ is a summand of $M(\Gamma(Y_n))$, it follows that $M(\Gamma(Y_n)) \not\succeq 0$. Since $M(\Gamma(Y_n))$ converges to $M(\Gamma(Y))$ in norm, there does not exist an $\epsilon > 0$ such that $L(Y) = M(\Gamma(Y)) \succeq \epsilon$. □

Remark 3.12 In the case of y^2-convexity and without hypotheses that guarantee a uniform bound on the coefficients, it is not clear how to avoid, in the proof of Corollary 3.11 and after scaling, having the pencils $L_n = M_n \circ \Gamma$ converge to a Γ-pencil L that is not equivalent to a monic Γ-pencil, such as

$$L(x, y) = \begin{pmatrix} 1 & y \\ y & y^2 \end{pmatrix},$$

which is always positive semidefinite, but never positive definite. We view this as a symptom of the fact that the image of Γ in this case lies in the boundary of its operator convex hull. The same issue arises in describing operator Γ-convex sets defined by a single (operator-valued) monic Γ-pencil. □

3.4 Applications of Theorem 3.8 to Matricial Γ-Convexity

We say $p \in M_\mu(\mathbb{C}\langle x \rangle)$ is **regular** if $\mathcal{D}_p = \widehat{\mathcal{D}}_p$, the set $\widehat{\mathfrak{D}}_p$ is bounded and $\mathfrak{D}_p = \widehat{\mathfrak{D}}_p$.

It is clear that $\mathcal{D}_p \subset \widehat{\mathcal{D}}_p$ and $\mathfrak{D}_p \subset \widehat{\mathfrak{D}}_p$. However equality may not hold: consider $p(x) = -(1 - x^2)^2$. An example of a regular polynomial is $p(x, y) = 1 - x^2 - y^{2d}$.

A simple and natural geometric sufficient condition for regularity may be described as follows. We say $p \in M_\mu(\mathbb{C}\langle x \rangle)$ is **star-like** if the set $\widehat{\mathfrak{D}}_p$ is bounded and if $X \in B(\mathcal{H})_{sa}^{\mathfrak{g}}$ and $p(X) \succeq 0$, then $p(tX) \succ 0$ for all $0 \leq t < 1$. If p is star-like and $\widehat{\mathfrak{D}}_p \neq \varnothing$, then $p(0) \succ 0$.

Example 3.13 For d a positive integer, $p(x, y) = 1 - x^2 - y^{2d}$ is star-like since

$$p(t(x, y)) = (1 - t^2) + t^2 p(x, y) + t^2(1 - t^{2d-2})y^{2d}.$$

Thus, if $p(X, Y) \succeq 0$, then $p(t(X, Y)) \succeq 1 - t^2$ for $0 \leq t < 1$.

Proposition 3.14 *If $p \in M_\mu(\mathbb{C}\langle x \rangle)$ is star-like, then p is a regular polynomial.*

Proof Clearly, $t\widehat{\mathfrak{D}}_p \subset \text{int } \mathfrak{D}_p$ for $0 < t < 1$, and $X = \lim_{t \nearrow 1} tX$. Similarly, the matrix case holds. □

For a regular polynomial p, Corollary 3.15—a separation result for $\mathcal{D}_p$ under the assumption that $\mathfrak{D}_p$ is operator Γ-convex—is an immediate consequence of Theorem 3.8.

Corollary 3.15 *Let $p \in M_\mu(\mathbb{C}\langle x \rangle)$ be a regular polynomial such that $\mathfrak{D}_p$ is operator Γ-convex with $0 \in \mathcal{D}_p$ and suppose $\Gamma(0) = 0$. If $Y \in \mathbb{S}_\ell(\mathbb{C})^g$ and $Y \notin \mathcal{D}_p(\ell)$, then there is a monic Γ-pencil L of size ℓ such that $L \succeq 0$ on $\mathcal{D}_p$ but $L(Y) \not\succeq 0$. In particular, $\mathcal{D}_p = \cap \widehat{\mathcal{D}}_L$, where the intersection is over all monic Γ-pencils L such that $L(\mathcal{D}_p) \succeq 0$.*

Example 3.16 In light of Example 3.13 and Corollary 3.15, if $p(x, y) = 1 - x^2 - y^{2d}$ and $(X, Y) \notin \mathcal{D}_p$, then there is a monic y^2-pencil L such that $L \succeq 0$ on $\mathcal{D}_p$ and such that $L(X, Y) \not\succeq 0$. Hence $\mathcal{D}_p = \cap \widehat{\mathcal{D}}_L$, where the intersection is over all monic y^2-pencils L that are positive semidefinite on $\mathcal{D}_p$. This example is explored further in Proposition 4.2.

If p is regular with $p(0) = 1$ and $\mathfrak{D}_p$ is operator xy-convex (see Example 1.2), then Corollary 3.15 says $\mathcal{D}_p$ arises from Bilinear Matrix Inequalities (BMIs). In this case more can be said.

Proposition 3.17 *Suppose $p \in M_\mu(\mathbb{C}\langle x \rangle)$ is regular, $p(0) > 0$, and $\mathfrak{D}_p$ is operator xy-convex. If ℓ is a positive integer and Y is in the boundary of $\mathcal{D}_p(\ell)$, then there is a monic xy-pencil L of size ℓ such that $L(X) \succ 0$ for X in the interior of $\mathcal{D}_p$, but $L(Y) \not\succ 0$.*

Lemma 3.18 *If $p \in M_\mu(\mathbb{C}\langle x \rangle)$ and $p(0) \succ 0$, then there is a constant β such that if*

$$L(x, y) = I + Ax + By + Cxy + C^* yx,$$

is a monic xy-pencil that is positive semidefinite on $\mathcal{D}_p$, then

$$\|A\|, \|B\|, \|C + C^*\|, \|C - C^*\| \leq \beta.$$

Proof Since $p(0)$ is positive, there is an $\epsilon > 0$ such that $(\pm\epsilon, 0), (0, \pm\epsilon), \pm(\epsilon, -\epsilon) \in \mathcal{D}_p(1)$ and $\pm(X, Y) \in \mathcal{D}_p(2)$, where

$$(X, Y) = \left(\begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}, \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \right).$$

It follows that $\|A\|, \|B\| \leq \frac{1}{\epsilon}$. Likewise,

$$0 \preceq L(\epsilon, \epsilon) + L(-\epsilon, -\epsilon) = 2I + \epsilon^2(C + C^*)$$

and

$$0 \preceq L(\epsilon, -\epsilon) + L(-\epsilon, \epsilon) = 2I - \epsilon^2(C + C^*).$$

Thus $\|C + C^*\| \leq \frac{2}{\epsilon^2}$. Further,

$$0 \preceq L(X, Y) + L(-(X, Y)) = 2I + \epsilon^2(C - C^*)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence $\|C - C^*\| \leq \frac{2}{\epsilon^2}$. Thus, there is a constant β such that if L is a monic xy-pencil and L is positive semidefinite on $\mathcal{D}_p$, then the coefficients of L are all bounded (in norm) by β. □

Lemma 3.19 *Suppose $S \subset \mathbb{S}_N(\mathbb{C})^2$. If L is a monic xy-pencil that is positive semidefinite on S, then L is positive definite on the interior of S.*

Proof Suppose L is a monic xy-pencil,

$$L(x, y) = I + Ax + By + Cxy + C^*yx,$$

that is positive semidefinite on S. Arguing by contradiction, suppose (X, Y) is in the interior of S, but $L(X, Y) \not\succ 0$. Hence there is a vector h such that $\|h\| = 1$ and $L(X, Y)h = 0$. Set

$$\begin{aligned} q_1(t) &= \langle L(tX, tY)h, h \rangle \\ &= 1 + t\langle[A \otimes X + B \otimes Y]h, h\rangle + t^2\langle[C \otimes XY + C^* \otimes YX]h, h\rangle. \end{aligned}$$

Thus q_1 is quadratic, $q_1(0) = 1$, $q_1(1) = 0$, and $q_1(t) \geq 0$ for t near 1. Hence the coefficient of t^2 is positive; that is

$$\alpha := \langle[C \otimes XY + C^* \otimes YX]h, h\rangle > 0.$$

Let

$$\begin{aligned} q_2(t) &= \langle L(tX, Y)h, h\rangle \\ &= \langle L(X, Y)h, h\rangle + (t - 1)[\langle A \otimes Xh, h\rangle + \alpha] \\ &= (t - 1)[\langle A \otimes Xh, h\rangle + \alpha]. \end{aligned}$$

Since $q_2(t) \geq 0$ for t real and near 1,

$$\langle A \otimes Xh, h\rangle + \alpha = 0 \tag{3.2}$$

A similar argument shows,

$$\langle B \otimes Yh, h\rangle + \alpha = 0. \tag{3.3}$$

Combining Eqs. (3.2) and (3.3) gives,

$$-\langle A \otimes Xh, h \rangle = -\langle B \otimes Yh, h \rangle = \alpha > 0.$$

Since $\langle L(X, Y)h, h \rangle = 0$, it follows that $\alpha = 1$. Hence,

$$q_3(t) := t \left\langle L\left(tX, \frac{1}{t}Y\right)h, h \right\rangle = 2t - t^2 - 1 = -(t-1)^2.$$

On the other hand, $(tX, \frac{1}{t}Y) \in S$ for t real and near 1 and hence $q_3(t) \geq 0$ for such t and we have reached a contradiction. Thus, $L(X, Y) \succ 0$. □

Proof of Proposition 3.17 Suppose Y is in the boundary of $\mathcal{D}_p(\ell)$. Thus, there is a sequence (Y_n) converging to Y with each $Y_n \notin \mathcal{D}_p(\ell)$. By Corollary 3.15, there exists monic xy-pencils L_n of size ℓ such that L_n is positive semidefinite on $\mathcal{D}_p$ and $L_n(Y_n) \not\succeq 0$. Since the coefficients of L_n all have size ℓ and are, by Lemma 3.18, uniformly bounded, by passing to a subsequence if necessary, we may assume L_n converges (coefficient-wise) to a monic xy-pencil L of size ℓ. Thus L is positive semidefinite on $\mathcal{D}_p$ and $L_n(Y_n)$ converges to $L(Y)$. Hence $L(Y) \not\succ 0$ and, since L is monic and $L(\mathcal{D}_p) \succeq 0$, Lemma 3.19 implies $L \succ 0$ on the interior of $\mathcal{D}_p$ and the proof is complete. □

4 y^2-Convex Sets

This section treats y^2-convex sets, where it is shown that a free set $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^g \times \mathbb{S}(\mathbb{C})^h$ is y^2-convex if and only if, for each n and $Y \in \mathbb{S}_n(\mathbb{C})^h$, the **slice** $\{X : (X, Y) \in \mathcal{K}(n)\}$ is convex in the ordinary sense. It is also shown that the y^2-convex sets TV^d of equation (1.3) are the positivity set of a single (finite) y^2-pencil. By comparison, as noted in Example 3.16, the general theory only guarantees that $\mathrm{TV}^d = \mathcal{D}_p$ is the intersection of, possibly infinitely many, positivity sets of (finite) monic y^2-pencils.

Suppose $S \subset \mathbb{S}(\mathbb{C})^g \times \mathbb{S}(\mathbb{C})^h$ and write elements Z of $S(n)$ as $Z = (X, Y)$ with $X \in \mathbb{S}_n(\mathbb{C})^g$ and $Y \in \mathbb{S}_n(\mathbb{C})^h$. In the case S is free, it is called **convex in** x if, for each n and $Y \in \mathbb{S}_n(\mathbb{C})^h$, the slice

$$S[Y] := \left\{ X \in \mathbb{S}_n(\mathbb{C})^g : (X, Y) \in S(n) \right\} \subset \mathbb{S}_n(\mathbb{C})^g,$$

is convex (in the usual sense as a subset of $\mathbb{S}_n(\mathbb{C})^g$). In this setting y^2-convex means Γ-convex for

$$\Gamma = \left\{ x_1, \ldots, x_g, y_1, \ldots, y_h, y_1^2, \ldots, y_h^2 \right\}.$$

Proposition 4.1 *A free set $S \subset \mathbb{S}(\mathbb{C})^g \times \mathbb{S}(\mathbb{C})^h$ is y^2-convex if and only if it is convex in x.*

Proof Suppose $\mathcal{S}$ is a y^2-convex free set. Fix n and $E \in \mathbb{S}_n(\mathbb{C})^{\mathsf{h}}$. Given $A, B \in \mathbb{S}_n(\mathbb{C})^{\mathsf{g}}$ such that $(A, E), (B, E) \in \mathcal{S}(n)$, observe that, since $\mathcal{S}$ is a free set

$$(X, Y) := (A, E) \oplus (B, E) = \left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \right) \in \mathcal{S}(2n).$$

Since $\mathcal{S}$ is y^2-convex and, for $0 \le t \le 1$, $V = \left(\sqrt{t} I_n \ \sqrt{1-t} I_n \right)^*$ is an isometry satisfying $V^* Y^2 V = (V^* Y V)^2$ (so that $((X, Y), V)$ is a y^2-pair),

$$V^*(X, Y)V = (tA + (1 - t)B, E) \in \mathcal{S}(n).$$

Thus $\mathcal{S}$ is convex in x.

Now suppose $\mathcal{S}$ is a free set that is convex in x. To prove $\mathcal{S}$ is y^2-convex, suppose $(X, Y) \in \mathcal{S}(n + m)$ and $V : \mathbb{C}^n \to \mathbb{C}^{n+m}$ is an isometry such that $((X, Y), V)$ is a y^2-pair. Explicitly $V^* Y^2 V = (V^* Y V)^2$ and thus the range of V reduces Y. Hence, with respect to the direct sum $\mathbb{C}^n \oplus \mathbb{C}^m$,

$$(X, Y) = \left(\begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix}, \begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{pmatrix} \right).$$

Letting U denote the unitary matrix,

$$U = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix},$$

$U^*(X, Y)U \in \mathcal{S}(n + m)$ since free sets are closed under unitary similarity. Since $\mathcal{S}$ is convex in x, the slice $\mathcal{S}[Y]$ is convex and thus

$$(X', Y') := \frac{1}{2}[U^*(X, Y)U + (X, Y)] = \left(\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}, \begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{pmatrix} \right) \in \mathcal{S}(n + m).$$

Finally, since $\mathbb{C}^n \oplus \{0\}$ reduces (X', Y') and free sets are closed with respect to restrictions to reducing subspaces, $(X_{11}, Y_{11}) \in \mathcal{S}(n)$ and hence $\mathcal{S}$ is y^2-convex. $\qquad \square$

A fundamental question is: if $\mathcal{K}$ is free semialgebraic and Γ-convex, then is $\mathcal{K}$ the positivity set of (a) a Γ-concomitant or, more restrictively, (b) a Γ-pencil? Since, for positive integers d, the symmetric polynomial $x^2 + y^{2d}$ is y^2-convex, $p = 1 - x^2 - y^{2d}$ is y^2-concave and hence, by Proposition 2.7, $\widehat{\mathcal{D}}_p = \mathrm{TV}^d$ is y^2-convex.

Proposition 4.2 *For $d \in \mathbb{N}$ and $p_d = 1 - x^2 - y^{2d}$, there is a monic y^2-pencil L_d of size $d + 1$ such that $\widehat{\mathcal{D}}_{L_d} = \widehat{\mathcal{D}}_{p_d}$.*

Proof The pencils L_1 and L_2 are trivial to construct. For $d = 3, 4$ one can take

$$L_3(x, y) = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & y & y^2 \\ 0 & y & 1 + y^2 & \frac{y}{2} \\ x & y^2 & \frac{y}{2} & 1 + \frac{y^2}{4} \end{pmatrix},$$

$$L_4(x, y) = \begin{pmatrix} 1 & 0 & 0 & 0 & x \\ 0 & 1 & y & 0 & y^2 \\ 0 & y & y^2 + 1 & y & 0 \\ 0 & 0 & y & y^2 + 1 & \frac{1}{8}(-4y^2 - 5) \\ x & y^2 & 0 & \frac{1}{8}(-4y^2 - 5) & \frac{5y^2}{8} + \frac{89}{64} \end{pmatrix}.$$

That L_4 is not monic is easily remedied since its constant term is positive definite.

A recipe for constructing such pencils is the following. Fix d. For $0 \le k \le d - 2$, set

$$\alpha_k = \sqrt{\frac{d - 1 - k}{d - 1}}, \quad c_0 = 1, \text{ and } c_k = \frac{\alpha_k}{\alpha_{k-1}} \text{ if } k > 0.$$

Then $\prod_{j=1}^{k} c_j = \alpha_k$. Let $q = \sqrt{d - 1}(y^2 - 1)$ and let

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -c_1 y & 1 & 0 & 0 & \cdots & 0 \\ 0 & -c_2 y & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -c_{d-2}y & 1 \\ \alpha_0 q & \alpha_1 yq & \alpha_2 y^2 q & \cdots & \cdots & \alpha_{d-2}y^{d-2}q \end{pmatrix} \in M_{d \times (d-1)}(\mathbb{C}).$$

Then

$$WW^* = \begin{pmatrix} 1 & -c_1 y & 0 & 0 & 0 & \cdots & q \\ -c_1 y & 1 + c_1^2 y^2 & -c_2 y & 0 & 0 & \cdots & 0 \\ 0 & -c_2 y & 1 + c_2^2 y^2 & -c_3 y & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 + c_{d-3}^2 y^2 & -c_{d-2}y & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 1 + c_{d-2}^2 y^2 & 0 \\ q & 0 & 0 & 0 & \cdots & 0 & q^2 (\sum_{j=0}^{d-2} \alpha_j^2 y^{2j}) \end{pmatrix}$$

$\in M_d(\mathbb{C})$.

Next observe, for $0 \leq k \leq d - 4$,

$$0 = \alpha_k^2 - 2\alpha_{k+1}^2 + \alpha_{k+2}^2$$
$$0 = \alpha_{d-3}^2 - 2\alpha_{d-2}^2.$$

Hence

$$q^2 \left(\sum_{j=0}^{d-2} \alpha_j^2 y^{2j} \right) = (d - 1) - dy^2 + y^{2d}.$$

Let

$$M = WW^* + \begin{pmatrix} 0_{d-1,d-1} & 0_{1,d-1} \\ 0_{d-1,1} & 1 - y^{2d} \end{pmatrix}.$$

Finally, set

$$L_d = \begin{pmatrix} 1 & \begin{pmatrix} 0 & 0 & \dots & x \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x \end{pmatrix} & M \end{pmatrix}.$$

Now L_d is not monic but its constant term is positive definite, so a simple scaling produces an equivalent monic linear pencil. $\square$

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