

Polarization modes of gravitational waves in quadratic gravity

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The observation of the inspiral and merger of compact binaries by the LIGO-Virgo Collaboration has allowed for new tests of Einstein's theory in the extreme gravity regime, where gravitational interactions are simultaneously strong, nonlinear, and dynamical. Theories beyond Einstein's can also be constrained by detecting the polarization modes of gravitational waves. In this paper, we show that dynamical Chern-Simons and Einstein-dilaton-Gauss-Bonnet gravity cannot be differentiated from general relativity based on the detection of polarization modes alone. To prove this result, we use the Newman-Penrose method and an irreducible decomposition method to find that only the tensorial modes can be detected in both these theories.

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I. INTRODUCTION

General relativity (GR) has passed a plethora of tests in the Solar System [1] and in binary pulsars [2], thus making Einstein's theory one of the most well-verified models in nature. However, these tests have only probed systems in which either the gravitational field is weak, as in the Solar System, or the field is strong but the system is weakly dynamical, as in binary pulsars. Meanwhile, there are some observational and theoretical anomalies that standard GR does not provide a full answer to, such as the late-time acceleration of the Universe, the anomalous galaxy rotation curves [3], the matter-antimatter asymmetry of the Universe, and the existence of singularities.

A resolution to these anomalies may reside in a modification to Einstein's theory that passes all current tests, yet yields deviations in other extreme regimes, such as where the gravitational interaction is simultaneously strong, nonlinear, and highly dynamical. On the theoretical side, the intrinsic incompatibility of GR with quantum mechanics has prompted efforts at a variety of unified theories, from string theory [4,5] to loop quantum gravity [6–8]. On the phenomenological side, the observational anomalies described above have led to a variety of extensions to GR, such as tensor-vector-scalar theories or TeVeS [9–11], modified gravity or MoG [12–14], and massive gravity [15] and bigravity [16]. Whether any of these attempts at modifying GR has anything to do with reality can only be determined through further experiment and observation.

A class of theories that have been proposed to resolve some of these anomalies, yet pass current constraints, are those that correct the Einstein-Hilbert action through a scalar field that is nonminimally coupled to squared

curvature. One subset of these theories, dynamical Chern-Simons (dCS) gravity, was proposed as a possible way to explain the matter-antimatter asymmetry of the Universe by introducing additional parity-violating, gravitational interactions [6,7]. Another subset, Einstein-dilaton-Gauss-Bonnet (EdGB) gravity, was proposed to explain late-time acceleration [17,18]. Both of these theories can in principle escape current constraints because large deviations from GR are activated only near spacetime singularities [19–21].

With the observation of gravitational waves (GWs) by the LIGO and Virgo Collaboration, it is now possible to probe the highly dynamical and strongly curved, extreme gravity regime [1,22–24]. The observations of GWs from the mergers of binary compact objects, such as black holes and neutron stars, have allowed us to place constraints on a number of modified theories [25,26]. GWs are unique and versatile probes to test extreme gravity, as they are weakly interacting, and thus, travel unhindered from their sources to our detectors. The addition of more ground-based and space-based detectors in the near future will lead to numerous observations that will yield the most stringent tests for GR, as well as the most stringent constraints on modified gravity.

In principle, GW observations can also be used to carry out precision studies of their polarizations, particularly allowing for constraints on nontensorial modes [27–29]. GR admits only two modes of polarization, i.e., the + (plus) and $\times$ (cross) polarization modes. A general theory of gravity allows up to six polarization modes; in addition to the two tensorial ones, the other four correspond to two scalar modes (a longitudinal and a “breathing”

mode) and two vector (transverse) modes. The response of an interferometer depends strongly on the polarization content of the impinging GW. Therefore, if enough detectors receive a signal, one can in principle separate all polarization modes from the data [27]. The presence of only tensorial modes in GW signals could then both verify GR and stringently constrain beyond-Einstein theories that predict additional polarizations.

One can analytically obtain the polarization modes of a GW in a particular theory by a number of methods. One such method is the study of polarization modes of weak, plane, and nearly null GWs using the Newman-Penrose (NP) formalism [30]. This method can be employed along with the E(2) classification to calculate NP scalars corresponding to different polarization modes [31–33]. Another method to discover the polarization content of GWs in a given theory is through an irreducible decomposition [34,35]. In this formalism, the metric is reduced into irreducible components, namely a scalar component, two vector component (a longitudinal and a transverse part), and four tensor components (a trace, a longitudinal, a trace-free, and a longitudinal-transverse and transverse-tracefree part). Correspondingly, the field equations are reduced into independent scalar, vector, and tensorial components, which can be identified with radiative and nonradiative degrees of freedom (d.o.f.).

In this paper, we study EdGB and dCS gravity and calculate its polarization content explicitly. We employ both the NP formalism and the irreducible decomposition method and find that in both theories, GWs possess the two tensorial modes, namely the $+$ and the $\times$ modes just as one would observe for GWs in GR. Therefore, EdGB and dCS gravity are examples of theories for which a polarization test would be completely unconstraining.

The remainder of this paper deals with the details of the results summarized above. Sections II A and II B provide a basic introduction to dCS and EdGB gravity, respectively. Sections III and III A introduce the NP formalism and the E(2) classification, respectively, and we apply it to GR, dCS, and EdGB gravity provided in Secs. IV A, IV B, and IV C, respectively. Section V provides a very brief introduction to the irreducible decomposition method, followed by application and analysis for GR, dCS, and EdGB in Secs. V A, V B, and V D, respectively. Section VI concludes and points to future research.

Henceforth, we adopt the following conventions throughout the paper unless otherwise mentioned: we work in four dimensions with metric signature $(-, +, +, +)$ as used in [36], Latin indices $(a, b, c, \dots, j, k, \dots)$ in the index list represent spatial indices, whereas Greek indices $(\alpha, \beta, \dots)$ represent spacetime indices, round brackets around indices represent symmetrization, ∂_μ represents a partial derivative, $\square = \partial_\mu \partial^\mu$, whereas $\nabla^2 = \partial_j \partial^j$, the Einstein summation convention is employed, and we work in geometric units in which $G = 1 = c$.

II. QUADRATIC GRAVITY THEORIES

A. Dynamical Chern-Simons gravity

This subsection provides a brief review of dCS gravity and establishes some notation. We will be presenting a minimal review here and direct the interested reader to the recent review paper [7] for a more complete discussion. The action is given by

$$S = S_{\text{EH}} + S_{\text{CS}} + S_\vartheta + S_{\text{mat}}, \quad (1)$$

where the Einstein Hilbert term is

$$S_{\text{EH}} = \kappa \int_\nu d^4x \sqrt{-g} R, \quad (2)$$

with $\kappa = (16\pi)^{-1}$, R the Ricci scalar, and g the determinant of the metric tensor g_{ab} . The CS term is

$$S_{\text{CS}} = \frac{\alpha}{4} \int_\nu d^4x \sqrt{-g} \vartheta^* RR, \quad (3)$$

where α is a coupling constant, *RR is the Pontryagin density, defined via

$$^*RR \equiv ^*R^\mu{}_\nu{}^{\kappa\delta} R^\nu{}_{\mu\kappa\delta}, \quad (4)$$

with $^*R^a{}_b{}^{cd}$ the dual Riemann tensor defined as

$$^*R^\mu{}_\nu{}^{\kappa\delta} \equiv \frac{1}{2} \epsilon^\mu{}_{\nu\alpha\beta} R^{\alpha\beta\kappa\delta}, \quad (5)$$

ϑ is a pseudoscalar field, and ϵ^{abcd} is the Levi-Civita tensor. The Pontryagin density can also be expressed as a total divergence of a topological current that contains a combination of the product of Christoffel symbols and its derivatives [21]. The action for the scalar field is

$$S_\vartheta = -\frac{\beta}{2} \int_\nu d^4x \sqrt{-g} [g^{\mu\nu} (\nabla_\mu \vartheta) (\nabla_\nu \vartheta) + 2V(\vartheta)], \quad (6)$$

where ∇_μ is the covariant derivative operator compatible with the metric, β is a constant that determines the gravitational strength of the CS scalar field stress energy distribution, while $V(\vartheta)$ is a potential for the scalar that we set to zero. In addition to these terms, one must also include a matter action that couples directly to the metric tensor only.

The field equations for dCS gravity can be obtained by varying the action with respect to the metric tensor and the scalar field. These equations are

$$G_{\mu\nu} + \frac{\alpha}{\kappa} C_{\mu\nu} = \frac{1}{2\kappa} (T_{\mu\nu}^{\text{mat}} + T_{\mu\nu}^\vartheta), \quad (7)$$

$$\beta \square \vartheta + \frac{\alpha}{4} ^*RR = 0, \quad (8)$$

where $\square \equiv \nabla_\alpha \nabla^\alpha$ is the d'Alembertian operator, $T_{\mu\nu}^{\text{mat}}$ is the matter stress energy tensor, $T_{\mu\nu}^\vartheta$ is the scalar field stress-energy tensor, $G_{\mu\nu}$ is the Einstein tensor, and C_{ab} is the C tensor, which contains derivatives of the scalar field and the metric and is also trace-free in nature. The stress energy tensor of the scalar field is given by

$$T_{\mu\nu}^\vartheta = \beta \left[(\nabla_\mu \vartheta)(\nabla_\nu \vartheta) - \frac{1}{2} g_{\mu\nu} (\nabla^\sigma \vartheta)(\nabla_\sigma \vartheta) \right]. \quad (9)$$

The C tensor in Eq. (7) can be split into two separate parts, $C^{\mu\nu} = C_1^{\mu\nu} + C_2^{\mu\nu}$, where

$$\begin{aligned} C_1^{\mu\nu} &= (\nabla_\sigma \vartheta) \epsilon^{\sigma\delta\alpha(\mu} \nabla_\alpha R^{\nu)}_{\delta}, \\ C_2^{\mu\nu} &= (\nabla_\sigma \nabla_\delta \vartheta)^* R^{\delta(\mu\nu)\sigma}. \end{aligned} \quad (10)$$

B. Einstein dilaton Gauss-Bonnet gravity

In this subsection, we provide a brief overview of EdGB gravity. The action in this theory is given by

$$S = S_{\text{EH}} + S_{\text{EdGB}} + S_\vartheta + S_{\text{mat}}, \quad (11)$$

where S_{EH} and S_ϑ are given by Eq. (2) and Eq. (6), respectively. The matter action couples only to the metric. The EdGB term is given by

$$S_{\text{EdGB}} = \int d^4x \sqrt{-g} \lambda \vartheta \mathcal{G}, \quad (12)$$

where λ is a coupling constant. The Gauss-Bonnet scalar $\mathcal{G}$ can be written in terms of the Riemann tensor as

$$\mathcal{G} = \frac{1}{4} \delta^{\mu\nu\alpha\beta}_{\rho\sigma\gamma\delta} R^{\rho\sigma}_{\mu\nu} R^{\gamma\delta}_{\alpha\beta}, \quad (13)$$

with $\delta^{\mu\nu\alpha\beta}_{\rho\sigma\gamma\delta}$ the generalized Kronecker delta. The field equations in EdGB gravity take the form

$$\begin{aligned} G_{\mu\nu} + 2\lambda \delta^{\gamma\delta\kappa\epsilon}_{\alpha\beta\rho\sigma} R^{\rho\sigma}_{\kappa\epsilon} (\nabla^\alpha \nabla_\gamma \vartheta) \delta^\beta_{(\mu} g_{\nu)\delta} \\ = \nabla_\mu \vartheta \nabla_\nu \vartheta - \frac{1}{2} g_{\mu\nu} (\nabla_\rho \vartheta \nabla^\rho \vartheta), \end{aligned} \quad (14)$$

$$\square \vartheta + \lambda \mathcal{G} = 0. \quad (15)$$

These equations are obtained by varying the action in Eq. (11) with respect to the metric $g_{\mu\nu}$ and the scalar field ϑ , respectively.

A note of caution regarding notation is now due. It is customary to represent the field that couples to squared curvature with the symbol ϑ in both dCS gravity and EdGB gravity. However, these fields are not the same. In dCS gravity, ϑ is a pseudoscalar field, while in EdGB gravity, ϑ is a scalar field. We will never consider a theory where both

the dCS and EdGB corrections to the action are included simultaneously, so it should be straightforward to see what ϑ represents in any subsequent section of the paper by context.

III. NEWMAN PENROSE FORMALISM

The study of GWs using tetrad and spinor calculus gained prominence in the 1960s. Ezra Newman and Roger Penrose came up with a formalism that combines these calculus techniques to derive a very compact and useful set of equations that are equivalent to the Einstein equations. This set of equations consists of a linear combination of equations for the Riemann tensor in terms of Ricci rotation coefficients or spinor affine connections [30]. The different possible components of the Riemann tensor or the Weyl tensor in a null tetrad or a null basis were then associated with certain quantities, called NP coefficients or NP scalars. These coefficients provided physicists with a new tool to understand GWs, especially since they relate directly to GW polarization. Later, in [37,38], several authors investigated a formalism to transform from Cartesian coordinates to null tetrads. In this section, we present a brief introduction to the NP formalism and the E(2) classification. We refer the interested reader to [39] and [37,38] for a more in-depth discussion.

A. E(2) classification

The most general GW that a theory may predict can be composed of six polarization modes in total, which are characterized by the six “electric” components of the Riemann tensor R_{0i0j} , which govern the driving forces in a detector [37]. Indeed, the geodesic deviation equation states that the acceleration of a test particle with spatial coordinates x^j with respect to the origin is

$$a_i = -R_{0i0j} x^j, \quad (16)$$

where R_{0i0j} are the electric components of the Riemann tensor, due to, e.g., impinging GWs or other external gravitational influences. One can therefore characterize a GW just in terms of the Riemann tensor it produces.

A weak, plane, nearly null GW in any metric theory can be defined to be a weak, propagating vacuum gravitational field characterized by a linearized Riemann tensor that depends only on the retarded time $\tilde{u}$, i.e.,

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}(\tilde{u}), \quad (17)$$

with the wave vector normal to the surfaces of constant u ,

$$\tilde{l}_\mu = -\tilde{u}_{,\mu}. \quad (18)$$

This wave vector is *almost* null with respect to a certain local Lorentz metric,

$$\eta^{\mu\nu}\tilde{l}_\mu\tilde{l}_\nu = \epsilon, \quad |\epsilon| \ll 1, \quad (19)$$

where ϵ is related to the difference in speed as measured in a local Lorentz frame at rest in the universe rest frame, between light and the propagating GW.

Let us now be more formal and begin by introducing a null tetrad as a basis instead of a locally Lorentz orthonormal basis (t, x^j) . For a null plane wave propagating in the $+z$ direction, we define retarded time as $u = t - z$, while if the wave is traveling in the $-z$ direction, then the advanced time is $v = t + z$. We then define a (completely) null basis $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ with

$$l_\mu = -u_{,\mu}, \quad n_\mu = -\frac{1}{2}v_{,\mu}, \quad (20)$$

and in the (t, x^j) basis, our null tetrad vectors can be expressed as

$$\begin{aligned} l^\mu &= (1, 0, 0, 1), \\ n^\mu &= \frac{1}{2}(1, 0, 0, -1), \\ m^\mu &= \frac{1}{\sqrt{2}}(0, 1, i, 0), \end{aligned} \quad (21)$$

with $\bar{m}^\mu$ the complex conjugate of m^μ . These form a null tetrad as each of the individual vectors are orthogonal with respect to themselves, i.e.,

$$\begin{aligned} l^\mu l_\mu &= 0 = n^\mu n_\mu, \\ m^\mu m_\mu &= 0 = \bar{m}^\mu \bar{m}_\mu. \end{aligned} \quad (22)$$

Also, these null vectors obey the orthonormality conditions

$$-l^\mu n_\mu = m^\mu \bar{m}_\mu = 1. \quad (23)$$

The Minkowski metric in such a null tetrad can be expressed as

$$\eta^{\mu\nu} = -2l^{(\mu}n^{\nu)} + 2m^{(\mu}\bar{m}^{\nu)}, \quad (24)$$

which in matrix form is simply

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \quad (25)$$

Using such a null tetrad, Newman and Penrose found a set of coefficients (NP coefficients) that describe the radiative modes of the gravitational field [30]. These coefficients depend on the Weyl tensor, the traceless Ricci tensor, and the Ricci scalar. Correspondingly, these coefficients can also be expressed in terms of the Riemann

tensor. Using the geodesic deviation equation and the fact that the Riemann tensor for a GW as defined above just depends on retarded time, it can be shown that the only nonvanishing components of the Riemann tensor are of the form R_{npnq} with $p, q \in (l, m, \bar{m})$ in the chosen null tetrad. We have here introduced the contracted tetrad notation, where, for example,

$$R_{nlnl} = R_{\mu\nu\alpha\beta}n^\mu l^\nu n^\alpha l^\beta. \quad (26)$$

In general, a tensor in the null basis can be expressed in the Cartesian basis as

$$A_{apb\dots} = A_{\alpha\beta\gamma\dots}a^\alpha p^\beta b^\gamma \dots, \quad (27)$$

where $(a, b, c, \dots, o)$ can be any of $(l, n, m, \bar{m})$, while $(p, q, r, \dots, w)$ can only be one of $(l, m, \bar{m})$, while the Greek indices run over (t, x, y, z) .

We can also define null vector fields such that $\mathbf{l} = l^\mu \mathbf{e}_\mu$, $\mathbf{n} = n^\mu \mathbf{e}_\mu$, $\mathbf{m} = m^\mu \mathbf{e}_\mu$, and similar for $\bar{\mathbf{m}}$. Since these form a complete set of basis vectors, we can expand the GW vector $\tilde{l}^\mu$ in terms of them. However, since we are working with a nearly null GW, the expansion of $\tilde{l}^\mu$ depends on the velocity of the observer's local frame relative to the global rest frame. Choosing a preferred observer whose frame is at rest with respect to the global rest frame, we have

$$\tilde{l}^\mu = l^\mu(1 + \epsilon_l) + \epsilon_n n^\mu + \epsilon_m m^\mu + \epsilon_{\bar{m}} \bar{m}^\mu. \quad (28)$$

This chosen observer is free to orient her spatial basis such that GWs and her null wave are parallel, and she can choose a frequency such that it is equal to that of GW. These conditions reduce Eq. (28) to the form,

$$\tilde{l}^\mu = l^\mu - \epsilon_n \left(\frac{1}{2} l^\mu - n^\mu \right). \quad (29)$$

There are clearly no components of $\tilde{l}^\mu$ along $\mathbf{m}$ or $\bar{\mathbf{m}}$.

Putting all of this information together, the independent nonvanishing coefficients for a nearly null, plane GW in the preferred tetrad have the form

$$\begin{aligned} \Psi_2 &= -\frac{1}{6}R_{nlnl}, & \Psi_3 &= -\frac{1}{2}R_{nlm\bar{m}}, \\ \Psi_4 &= -R_{n\bar{m}m\bar{m}}, & \Phi_{22} &= -R_{nmn\bar{m}}, \end{aligned} \quad (30)$$

where Ψ_3 and Ψ_4 are complex scalars. One can further show that each of these scalars represents a different polarization mode, each of which affects the way matter responds to an impinging GW differently, as shown in Fig. 1.

Let us now consider the functional form NP scalars take for GWs. A GW can be represented in metric form via the decomposition

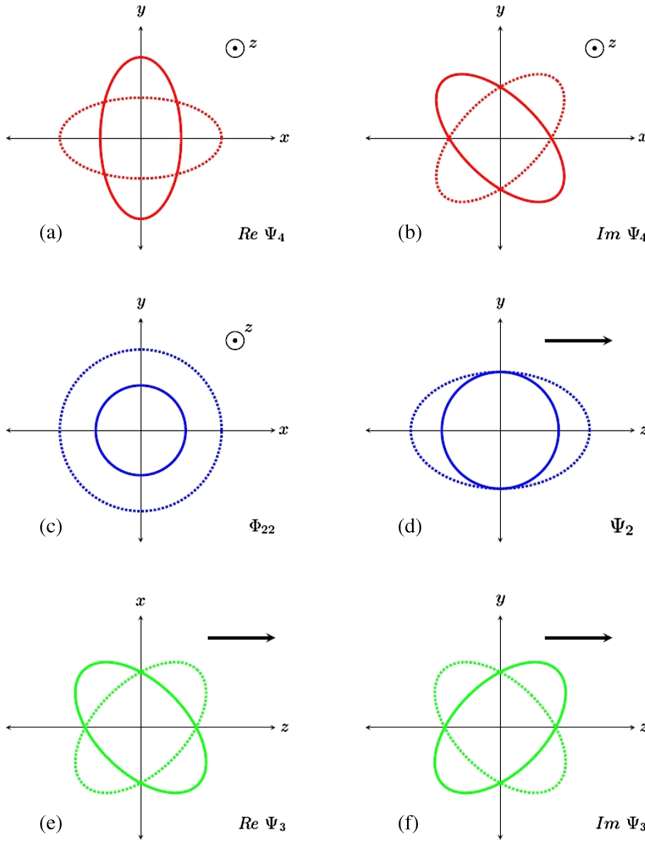


FIG. 1. The impact of the six polarization modes [(a) plus mode, (b) cross mode, (c) breathing mode, (d) longitudinal mode, (e) vector- x mode, and (f) vector- y mode] of weak, plane, nearly null GW, permitted in a general, four-dimensional theory of gravity, on a ring of test particles. The red, blue, and green colors correspond to the tensor, scalar, and vector modes, respectively. The circled dots in (a), (b), and (c) indicate the wave propagating out of the page. All modes (a)–(f) are propagating in the $+z$ direction. The solid line shows the displacement that each mode induces on a ring of test particles in the x - y plane, while the dashed line indicates the displacement after half a period.

$$g_{\mu\nu} = \eta_{\mu\nu} + p_{\mu\nu}, \quad (31)$$

where $p_{\mu\nu}$ is the GW metric perturbation. The Riemann tensor for such a linearized metric takes the form

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}(p_{\mu\beta,\alpha\nu} - p_{\mu\alpha,\beta\nu} + p_{\nu\alpha,\beta\mu} - p_{\nu\beta,\alpha\mu}), \quad (32)$$

but the Riemann tensor, and thus the GW metric perturbation, must be a function of the retarded time u . Therefore, in the null basis, Eq. (32) can be expressed as

$$R_{abcd} = \frac{1}{2}(\tilde{l}_c \tilde{l}_b \ddot{p}_{ad} - \tilde{l}_d \tilde{l}_b \ddot{p}_{ac} + \tilde{l}_d \tilde{l}_a \ddot{p}_{bc} - \tilde{l}_c \tilde{l}_a \ddot{p}_{bd}), \quad (33)$$

where (a, b, c, d) can again be any of $(l, n, m, \bar{m})$. With this at hand, Eq. (30) can be rewritten in terms of the

corresponding Ricci tensor components or even in terms of the metric perturbation as

$$\begin{aligned} \Psi_2 &= -\frac{1}{6}R_{nl} = \frac{1}{12}\ddot{p}_{ll}, & \Psi_3 &= -\frac{1}{2}R_{n\bar{m}} = \frac{1}{4}\ddot{p}_{l\bar{m}}, \\ \Psi_4 &= \frac{1}{2}\ddot{p}_{\bar{m}\bar{m}}, & \Phi_{22} &= -\frac{1}{2}R_{nn} = \frac{1}{2}\ddot{p}_{m\bar{m}}. \end{aligned} \quad (34)$$

Based on this, we can now organize gravity theories into different classes. For an exactly null wave, these classes are as follows:

- (i) Class II_6 : $\Psi_2 \neq 0$. All other NP scalars are observer dependent.
- (ii) Class III_5 : $\Psi_2 = 0, \Psi_3 \neq 0$. All other NP scalars are observer dependent.
- (iii) Class N_3 : $\Psi_2 \equiv \Psi_3 = 0, \Psi_4 \neq 0, \Phi_{22} \neq 0$.
- (iv) Class N_2 : $\Psi_2 \equiv \Psi_3 \equiv \Phi_{22} = 0, \Psi_4 \neq 0$.
- (v) Class O_1 : $\Psi_2 \equiv \Psi_3 \equiv \Psi_4 = 0, \Phi_{22} \neq 0$.
- (vi) Class O_0 : $\Psi_2 \equiv \Psi_3 \equiv \Phi_{22} \equiv \Psi_4 = 0$.

The GWs of GR are therefore of class N_2 , while those of scalar-tensor theories, which contain a breathing mode in addition to the two tensorial modes, are of class N_3 . More details about E(2) classification can be found in [39].

IV. GW POLARIZATION THROUGH THE NP FORMALISM

In this section, we present a calculation using the techniques presented in the previous subsection. We begin by presenting a brief calculation to obtain the polarization modes in GR followed by application of the NP formalism to obtain the polarization modes in dCS and EdGB.

A. Polarization modes in GR

Before we start with quadratic theories of gravity such as dCS and EdGB, let us first try to calculate the polarization modes using NP scalars in GR as a pedagogical warm-up. We begin by considering the field equation for GR in trace-reversed form,

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \quad (35)$$

The right-hand side of this equation is zero since we assume GWs are propagating in vacuum, and thus $R_{\mu\nu} = 0$. Therefore, in the chosen null tetrad, we have

$$R_{nn} = 0, \quad R_{nl} = 0, \quad R_{n\bar{m}} = 0, \quad (36)$$

and thus, $\Psi_2 = 0 = \Psi_3 = \Phi_{22}$. The only unconstrained NP scalar is Ψ_4 . From this, we conclude that GWs in GR are purely tensorial; i.e., only the $+$ and $\times$ modes exist, and the theory is of class N_2 as already anticipated.

B. Polarization modes in dCS gravity

Let us now focus on the polarization modes of GWs in dCS gravity using NP formalism and the E(2) classification formalism discussed in Sec. III A. The essence of this calculation lies in expressing Eq. (7) such that we isolate the Ricci tensor on the left-hand side, which we can achieve by reversing the trace. Doing so, Eq. (7) becomes

$$R_{\mu\nu} = \frac{1}{2\kappa} \left[T_{\mu\nu}^M - \frac{1}{2} g_{\mu\nu} T^M \right] + \frac{1}{2\kappa} [\beta (\nabla_\mu \vartheta) (\nabla_\nu \vartheta)] - \frac{\alpha}{\kappa} [(\nabla_\sigma \vartheta) \epsilon^{\sigma\delta\alpha} (\nabla_\alpha R_{\nu})_\delta + (\nabla_\sigma \nabla_\delta \vartheta)^* R^\delta_{(\mu\nu)}{}^\sigma], \quad (37)$$

where the first term is a combination of the matter stress energy tensor and its trace, both of which we set to zero henceforth, since again we focus only on GWs propagating in vacuum. The second term is the trace-reversed form of the stress energy tensor of the scalar field in Eq. (9), and the last term is simply the C tensor in Eq. (10) because this quantity is naturally trace-free.

Now that we have the first field equation in the form we require, let us analyze the scalar field evolution in Eq. (8). The right-hand side of Eq. (8) is the Pontryagin density, but when considering GWs, we must work in the far field limit, where this density vanishes. Thus, we have a second equation of the form,

$$\square \vartheta = 0. \quad (38)$$

The above equation simply tells us that the scalar field ϑ is a free wave, which we can represent as

$$\vartheta = A e^{iq^\mu x_\mu}, \quad (39)$$

where A is its amplitude and q^μ is its 4-wave number (or familiarly, (ω, k_x, k_y, k_z) in Cartesian coordinates). Moreover, since the wave operator is that of Minkowski in the far zone, we must have that the scalar wave is null:

$$q^\mu q_\mu = 0. \quad (40)$$

With this at hand, and using Eq. (39) in Eq. (37), the field equations become

$$R_{\mu\nu} = \frac{\beta}{2\kappa} [-A^2 q_\mu q_\nu e^{2iq \cdot x}] - \frac{\alpha}{\kappa} [(Aiq_\sigma e^{iq \cdot x}) \epsilon^{\sigma\delta\alpha} (\nabla_\alpha R_{\nu})_\delta - (Aq_\sigma q_\delta e^{iq \cdot x})^* R^\delta_{(\mu\nu)}{}^\sigma], \quad (41)$$

where $q \cdot x := q^\mu x_\mu$. The different polarization modes contained in Eq. (34) can be obtained by considering the individual, independent components of Eq. (41), which we analyze individually below.

I. Analysis of Ψ_2

From Eq. (34), we know that

$$\begin{aligned} \Psi_2 &= -\frac{1}{6} R_{nl} \\ &= -\frac{\beta}{12\kappa} [-A^2 q_n q_l e^{2iq \cdot x}] \\ &\quad + \frac{\alpha}{6\kappa} [(Aiq_\sigma e^{iq \cdot x}) \epsilon^{\sigma\delta\alpha} (\nabla_\alpha R_l)_\delta - (Aq_\sigma q_\delta e^{iq \cdot x})^* R^\delta_{(nl)}{}^\sigma]. \end{aligned} \quad (42)$$

Recall that we are considering a weak, plane, nearly null GW, and so the Riemann tensor is only a function of the retarded time [as stated in Eq. (17)]. Combining this with Eqs. (8), (38), and (39), we can conclude that the wave vector q^μ will only have a nonvanishing component along the retarded time, or equivalently along l^μ in terms of the null tetrad under consideration. Thus, the only nonvanishing component is q^l or q_n by means of Eq. (25). This then implies that the first term in Eq. (42) does not contribute at all.

Let us now consider the second term of Eq. (42). The Levi-Civita tensor in the second term is nonvanishing only when the superscript indices α and δ are equal to m or $\bar{m}$. This is because one of the superscript indices of the Levi-Civita tensor is either n or l (due to the symmetrizer), while the σ superscript index must contract onto q_σ , which is nonvanishing only in the n^μ direction. Since the metric perturbation is a function of retarded time only, the Ricci tensor must also be a function of retarded time, which means we can write

$$\partial_\alpha R_{\nu\delta} = \tilde{l}_\alpha \dot{R}_{\nu\delta}, \quad (43)$$

and this is the only nonvanishing derivative of the Ricci tensor. By definition of $\tilde{l}^\mu$, we have that $\tilde{l}^\mu = l^\mu$ for a perfectly null GW, whereas for a nearly null GW, we have Eq. (29). However, the second term of Eq. (42) needs the α index to be either m or $\bar{m}$, which means that upon contraction with the $\tilde{l}_\alpha$ generated from the partial derivative one finds either $\tilde{l}_m$ or $\tilde{l}_{\bar{m}}$, both of which are zero. Therefore, the second term of Eq. (29) also vanishes. Applying a similar treatment to the third term in Eq. (42), one can easily show that it also vanishes.

With all of this at hand, we then have that $R_{nl} = 0$ and thus

$$\Psi_2 = 0 \quad (44)$$

in dCS gravity. The physical implication of this mathematical result is that GWs in dCS gravity have no longitudinal modes.

2. Analysis of Ψ_3

Let us now follow a similar approach to study Ψ_3 . Equation (41) says that

$$\Psi_3 = -\frac{\beta}{4\kappa}[-A^2 q_n q_{\bar{m}} e^{2iq \cdot x}] + \frac{\alpha}{2\kappa}[(Aiq_\sigma e^{iq \cdot x})\epsilon^{\sigma\delta\alpha}({}_n\partial_\alpha R_{\bar{m}})_\delta - (Aq_\sigma q_\delta e^{iq \cdot x})^* R^\delta_{(n\bar{m})}{}^\sigma]. \quad (45)$$

The first term in the above equation vanishes since $q_m = 0$. Following the same arguments as those used for Ψ_2 one can also show that the second and the third terms of Eq. (45) vanish, using Eqs. (27) and (33) and the orthogonality conditions in Eq. (22).

Combining these results with Eq. (34), we then find

$$\Psi_3 = 0. \quad (46)$$

The physical interpretation of this mathematical result is that GWs in dCS gravity have no vector modes.

3. Analysis of Φ_{22}

Let us now study the breathing mode. Equation (41) says that

$$\Phi_{22} = -\frac{\beta}{4\kappa}[-A^2 q_n q_n e^{2iq \cdot x}] + \frac{\alpha}{2\kappa}[(Aiq_\sigma e^{iq \cdot x})\epsilon^{\sigma\delta\alpha}({}_n\partial_\alpha R_n)_\delta - (Aq_\sigma q_\delta e^{iq \cdot x})^* R^\delta_{(nn)}{}^\sigma]. \quad (47)$$

As before, the second term in the above equation vanishes by arguments similar to those presented in Sec. IV B 1, whereas the third term vanishes by the definition of the wave 4-vector, the dual Riemann tensor, and the Levi-Civita tensor. However, the first term does not vanish by the characteristics of the GW established previously.

The above arguments imply that, in general, Φ_{22} , and thus the breathing mode of GWs in dCS gravity is not vanishing. However, GWs are always defined in terms of the $1/r$ part of the radiative field. Since q_n falls off as $\mathcal{O}(r^{-1})$, it is then clear that Φ_{22} falls off as $\mathcal{O}(r^{-2})$. In the far field, then, we have that

$$\Phi_{22} \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty \quad (48)$$

and the breathing mode of GWs in dCS gravity vanishes.

4. Analysis of Ψ_4

Combining Eqs. (30), (34), and (41), it can be seen that there are no constraints possible on the Ψ_4 mode. Thus, in dCS gravity, Ψ_4 , or the $+$ and $\times$ polarization modes, cannot be constrained with the use of the field equations.

From the above analysis, we can see that Ψ_4 and Φ_{22} are the nonvanishing NP scalars in general with Φ_{22} having a $\mathcal{O}(r^{-2})$ dependence. Therefore, one might be able to conclude that dCS is a Class N_3 theory behaving as a Class N_2 theory in the far field limit. However, this is not entirely true. The reason lies in the definitions of Eq. (34). GWs in general are defined as the $1/r$ part of the radiative field far away from the source. This means that we only need to take into consideration the terms of Eq. (41) that are nonvanishing and scale as $\mathcal{O}(r^{-1})$. We can therefore conclude that the nonvanishing part of R_{nn} or Φ_{22} is not the term that actively contributes to the GW. With all these arguments, we can conclude that for a weak, plane, nearly null GW, an observer can only detect the $+$ and $\times$ polarization modes, just as one would observe in GR. Under the E(2) classification, dCS therefore always behaves as a Class N_2 theory.

At this junction, it is convenient to compare the above result to that of scalar-tensor theories. In such theories, the field equations are quite similar to those in Eq. (37), except for three observations. First, there is no nonminimal coupling between the scalar field and quadratic curvature invariants, so the third term on the right-hand side of Eq. (37) is absent. Second, the second term on the right-hand side of this equation is multiplied by ϑ^{-2} , but since the field is typically assumed to have some cosmological boundary value, this term is still quadratic in the amplitude of the scalar field perturbation. Third, the field equations of scalar-tensor theories have an extra term on the right-hand side of Eq. (37) that is proportional to $\vartheta^{-1}\partial_{\mu\nu}\vartheta$, which is linear in the amplitude of the scalar field perturbation. This term arises because the scalar-tensor action has a scalar field multiplying the Ricci scalar, which then leads to nonvanishing contributions when varying the action with respect to the metric tensor and integrating by parts. It is this term in the action of scalar-tensor theories that generates a nonvanishing Φ_{22} NP scalar, and thus a nonvanishing breathing mode. In dCS, however, this extra term is not present because the Ricci scalar is not multiplied by ϑ , with the field only coupling to the metric through squared curvature invariants, thus explaining why dCS gravity does not possess a breathing mode.

C. Polarization modes in EdGB gravity

Let us finally consider the polarization modes of GWs in EdGB gravity. We will not provide here as many details as in the dCS gravity case, as the procedure is fairly similar in essence. Instead, we refer the interested reader to the Appendix.

The evolution equation for the scalar field is identical to that in dCS gravity. This is because the geometry of the source of our GWs does not affect the observer, so once more, one finds a wave equation in flat spacetime for the scalar field, whose solution is of the form of Eq. (39). The evolution of the GW metric perturbation is controlled by

the trace-reversed form of field equations in Eq. (14). Such a trace-reversed form is

$$R_{\mu\nu} = \left[T_{\mu\nu}^M - \frac{1}{2} g_{\mu\nu} T^M \right] + [(\partial_\mu \vartheta)(\partial_\nu \vartheta)] - 2\lambda \delta_{\alpha\beta\rho\sigma}^{\gamma\delta\kappa\epsilon} R^{\rho\sigma}_{\kappa\epsilon} (\partial^\alpha \partial_\gamma \vartheta) \delta^\beta_{(\mu} g_{\nu)\delta}, \quad (49)$$

which clearly takes a form quite similar to Eq. (8). Again, since we are working in vacuum, the first term of the above equation is zero. The second term is the same as that in dCS gravity, and thus, it is formally nonvanishing only for the breathing mode, but then again it vanishes in the far field limit. The third term also vanishes for a plane, null GW propagating along the null direction associated with retarded time. This can be shown through Eqs. (25), (39), and (33), along with the orthogonality conditions in Eq. (22). Just as in the case for dCS, the Ψ_4 mode remains unconstrained.

These arguments lead us to the conclusion that GWs emitted in EdGB gravity can only possess GW polarization modes associated with Ψ_4 and Φ_{22} . However, as discussed earlier, Φ_{22} does not contribute to the GW perturbation, because GWs are defined as the $1/r$ part of the radiative perturbation. Therefore, the only true nonvanishing NP scalar is Ψ_4 . This means that, just as for GR, only the $+$ and $\times$ polarization modes are nonvanishing, thereby making EdGB a class N_2 theory under the E(2) classification.

V. IRREDUCIBLE DECOMPOSITION

In this section, we present a brief overview of an alternate way of identifying the polarization modes by decomposing the metric into irreducible components. Such a decomposition allows us to clearly identify the d.o.f. present in any theory. As far as we know, this treatment for a linearized theory was first suggested in [34].

The metric perturbation transforms as a tensor field under Lorentz transformations in Minkowski spacetime. Such a transformation includes boosts and rotations. Ignoring the boosts and just focusing on pure rotations, p_{00} transforms as a scalar, p_{0j} transforms as a 3-vector, and p_{jk} transforms as a 3-tensor, where $p_{\mu\nu}$ is the metric perturbation as defined in Eq. (31). In the Cartesian basis, one can think of the (0,0) component of the metric perturbation as the (t, t) component, while the subscripts $(j, k) \in (x, y, z)$. These quantities can be decomposed further into their irreducible pieces. Thus, p_{0j} decomposes into a longitudinal and a transverse piece, whereas p_{jk} decomposes into a trace, a longitudinal and trace-free piece, a longitudinal and transverse piece, and a transverse and trace-free piece. Thus, such a decomposition portrays all possible d.o.f. contained in the metric perturbation $p_{\mu\nu}$.

Following [35], one can express these components of the metric perturbation as

$$\begin{aligned} p_{00} &= 2U, \\ p_{0j} &= -4U_j - \partial_j A, \\ p_{jk} &= 2\delta_{jk} V + \left(\partial_{jk} - \frac{1}{3} \delta_{jk} \nabla^2 \right) B + (\partial_j B_k + \partial_k B_j) + h_{jk}^{TT}, \end{aligned} \quad (50)$$

where U, V, A, B are scalars, U_j and B_j are 3-vectors, and h_{jk}^{TT} is a transverse-traceless 3-tensor. These quantities satisfy the conditions

$$\begin{aligned} \partial_j U^j &= 0, \\ \partial_j B^j &= 0, \\ \partial_k h_{TT}^{jk} &= 0 = \delta_{jk} h_{TT}^{jk}, \end{aligned} \quad (51)$$

which imply that U^j and B^j are transverse 3-vectors.

All these individual pieces of the metric perturbation are also gauge invariant. Thus, without loss of generality, we can choose the Coulomb gauge, in which

$$A = B = B_j = 0, \quad (52)$$

such that the components of metric perturbation become

$$\begin{aligned} p_{00} &= 2U, \\ p_{0j} &= -4U_j, \\ p_{jk} &= 2\delta_{jk} V + h_{jk}^{TT}. \end{aligned} \quad (53)$$

From these components, we can also construct gauge-invariant potentials, which are given by

$$\Phi = U, \quad \Phi_j = U_j, \quad \Psi = V, \quad (54)$$

and which are clearly essentially equivalent to the gravitational potentials in the Coulomb gauge, making this gauge meaningful and convenient. The gauge invariant potentials represent the d.o.f. of the gravitational field. The two scalar potentials, Φ and Ψ , the 3-vector potential Φ_j with its two independent components, and the 3-tensor potential h_{jk}^{TT} with its two independent components, represent the six independent d.o.f. required to describe all possible GW polarizations in a generic theory of gravity (see also Fig. 1).

A. Irreducible decomposition in GR

We can now apply the treatment mentioned above to reduce the field equation of GR into independent irreducible pieces. For a GW, the linearized Einstein tensor depends on the metric perturbation via

$$G_{\alpha\beta} = -\frac{1}{2}(\Box p_{\alpha\beta} + \partial_{\alpha\beta} p - \partial_{\alpha\mu} p^\mu{}_\beta - \partial_{\beta\mu} p^\mu{}_\alpha) + \frac{1}{2}\eta_{\alpha\beta}(\Box p - \partial_{\mu\nu} p^{\mu\nu}), \quad (55)$$

where $p = \eta_{\alpha\beta} p^{\alpha\beta}$ is the flat spacetime trace. Using Eqs. (53)–(55), we can express the individual components of the Einstein tensor as

$$\begin{aligned} G_{00} &= -2\nabla^2\Psi, \\ G_{0j} &= -2\partial_{ij}\Psi + 2\nabla^2\Phi_j, \\ G_{jk} &= -\delta_{jk}\nabla^2(\Phi - \Psi) - 2\delta_{jk}\partial_{tt}\Psi + \partial_{jk}(\Phi - \Psi) \\ &\quad + 2(\partial_{ij}\Phi_k + \partial_{ik}\Phi_j) - \frac{1}{2}\Box h_{jk}^{TT}. \end{aligned} \quad (56)$$

The Einstein tensor is now fully decomposed into its irreducible pieces. Now, the right-hand side of the field equation, Eq. (35) involves a stress energy tensor.

Let us now focus on the right-hand side of the Einstein equations. The stress energy tensor can be decomposed into its own irreducible pieces as

$$\begin{aligned} T^{00} &= \rho, \\ T^{0j} &= (s^j + \partial^j s), \\ T^{jk} &= \tau\delta^{jk} + \partial^{jk}\sigma - \frac{1}{3}\delta^{jk}\nabla^2\sigma + 2\partial^{(j}\sigma^{k)} + \sigma^{jk}, \end{aligned} \quad (57)$$

where ρ is the mass density of the matter distribution measured by an observer at rest, $(s^j + \partial^j s)$ or T^{0j} is the momentum density, and T^{jk} is the stress tensor. These quantities satisfy the conditions

$$\partial_j s^j = 0, \quad \partial_j \sigma^j = 0, \quad \partial_k \sigma^{jk} = 0 = \delta_{jk} \sigma^{jk}, \quad (58)$$

which imply that s^j and σ^{jk} are transverse.

Energy-momentum conservation in linearized theory reveals that not all of the ten fields in Eq. (57) are independent. Using that $\partial_\beta T^{\alpha\beta} = 0$, one finds that

$$\begin{aligned} \nabla^2 s &= -\partial_t \rho, \\ \nabla^2 \sigma^j &= -\partial_t s^j, \\ \nabla^2 \sigma &= -\frac{3}{2}(\partial_t s + \tau). \end{aligned} \quad (59)$$

This implies that only ρ , s^j , τ , and σ_{jk} are independent, while the other four fields can be determined in terms of them through the equations above. Combining these expressions with the linearized Einstein equations implies that

$$\begin{aligned} \nabla^2 \Psi &= -4\pi\rho, \\ \nabla^2(\Phi - \Psi) &= -12\pi(\partial_t s + \tau), \\ \nabla^2 \Phi_j &= -4\pi s_j, \\ \Box h_{jk}^{TT} &= -16\pi\sigma_{jk}. \end{aligned} \quad (60)$$

The first three equations above are (elliptic) Poisson equations, and therefore, the solutions at a particular time depend only on the matter configuration at that particular time. In this sense, the 4 d.o.f. contained in (Ψ, Φ, Φ_j) are constrained by the field equations and do not represent radiative modes. On the other hand, the last equation is a (hyperbolic) wave equation in flat spacetime, which means that h_{jk}^{TT} clearly represents a radiative mode, corresponding to the two polarization modes (h_+ , $h_\times$) of GR. Such a gauge invariant formulation of GWs thus separates the radiative modes from the nonradiative ones.

B. Irreducible decomposition in dCS gravity

Let us now focus on dCS gravity and compare the results of an irreducible decomposition to those obtained from the NP method in Sec. IV B.

Consider the field equations in dCS gravity in Eq. (7). The left-hand side of this equation is completely geometric in nature, whereas the right-hand side depends on the matter-energy and scalar field content of the system under consideration. We can again decompose our metric perturbation into scalar, vector, and tensor parts, as in Eq. (53), which allows us to decompose the geometric part of Eq. (7). The right-hand side of the field equations consists of two independent parts—matter stress-energy tensor and a scalar field stress-energy tensor—both of which we can again decompose into irreducible pieces using Eqs. (57), (9), and (53). Since Eq. (8) behaves as a wave equation in the far zone, the scalar field can be expressed by means of Eq. (39) as before.

By following the steps mentioned above, we obtain the decomposed field equations for dCS gravity

$$\begin{aligned} \nabla^2 \Psi + \frac{\alpha}{\kappa} \epsilon_{\alpha\gamma\delta\eta} n^\alpha n^\beta q^\gamma \vartheta(\partial^{\eta\delta} \Psi) \\ = \frac{\beta}{8\kappa} q_\mu q^\mu \vartheta^2 + \frac{\beta}{4\kappa} n^\alpha n^\beta q_\alpha q_\beta \vartheta^2 - \frac{\rho}{4\kappa}, \end{aligned} \quad (61)$$

$$\begin{aligned} 2\nabla^2 \Phi_j - \frac{4\alpha}{\kappa} \epsilon_{j\beta\gamma}{}^\delta q^\alpha q^\beta \vartheta \partial_{\alpha\delta} \Phi^\gamma + \frac{2\alpha}{\kappa} \epsilon_{j\beta\delta\eta} n^\beta n^\alpha q^\gamma q^\delta \vartheta \partial_\alpha{}^\eta \Phi_\gamma \\ - \frac{2\alpha}{\kappa} \epsilon_{\beta\gamma\delta\eta} n^\beta n^\alpha q^\gamma \vartheta \partial_\alpha{}^{(\delta} \Phi_{|j)} - \frac{\alpha}{\kappa} \epsilon_{j\gamma\delta\eta} n^\alpha n^\beta q^\gamma \vartheta \partial_{\alpha\beta}{}^\eta \Phi^\delta \\ - \frac{2\alpha}{\kappa} \epsilon_{\beta\gamma\delta\eta} n^\alpha n^\beta q_\alpha q^\gamma \vartheta \partial_j{}^\eta \Phi^\delta = \frac{\beta}{2\kappa} n^\alpha q_\alpha q_j \vartheta^2 - \frac{s_j}{2\kappa}, \end{aligned} \quad (62)$$

$$\begin{aligned} \delta_{jk} \nabla^2(\Phi - \Psi) - \frac{\alpha}{2\kappa} \epsilon_{(k|\alpha\beta\gamma} q^\alpha \vartheta \partial^{\gamma\beta}{}_{|j)}(\Phi - \Psi) \\ = -\frac{3}{4\kappa} \delta_{jk}(\partial_t s + \tau) + \frac{\beta}{2\kappa} q_j q_k \vartheta^2, \end{aligned} \quad (63)$$

$$\square h_{jk}^{TT} + \frac{4\alpha}{\kappa} \left[\epsilon_{(k|\beta}^{\gamma\delta} q^\alpha q^\beta \partial_{\alpha\delta} h_{|j)\gamma}^{TT} + \frac{1}{2} \epsilon_{(k|\alpha}^{\beta\delta} q^\alpha \partial_{\beta\delta} \square h_{|j)\beta}^{TT} - \epsilon_{(k|\beta}^{\gamma\delta} q^\alpha q^\beta \partial_{|j)\delta} h_{\alpha\gamma}^{TT} \right] = -\frac{\sigma_{jk}}{\kappa}, \quad (64)$$

where we have used the notation $\partial_{\alpha\beta} = \partial_\alpha \partial_\beta$ and $\partial_{\alpha\beta\gamma} = \partial_\alpha \partial_\beta \partial_\gamma$, and where n^α is the normal 4-vector pointing along the direction of propagation. These equations are analogous to Eq. (60), with certain modifications that depend on the scalar field, so let us analyze them term by term.

Before doing so, however, it is useful to remember a few facts we discovered in the NP method section. For the GWs under consideration, we have already established that the Riemann tensor, and therefore the metric perturbation, are functions of retarded time u only. Therefore, $\partial_u p_{\alpha\beta} \neq 0$, where ∂_u is the partial derivative with respect to retarded time. When written in terms of null coordinates, by virtue of the chain rule, only the partial derivative of the metric perturbation with respect to the l^μ tetrad is nonvanishing. Another important fact is that $q^\alpha = (q^l, 0, 0, 0)$.

Using these facts, one can use tensor manipulations to show that a number of terms in Eqs. (61)–(64) vanish identically. The reader familiar with these kinds of manipulations should skip to below Eq. (71).

C. Detailed manipulation of Eqs. (61)–(64)

We will analyze each term in Eqs. (61)–(64) separately. Let us begin the analysis with Eq. (61). Without loss of generality, we can use a coordinate system of the form (u, v, x, y) where u is retarded time and v is advanced time, as defined in previous sections. The normal 4-vector is then of the form $n^\alpha = (1, 0, 0, 0)$, as it points along the direction of propagation. Therefore, Eq. (61) takes the form

$$\nabla^2 \Psi + \frac{\alpha}{\kappa} \epsilon_{uu\delta\eta} n^\mu n^\mu q^\mu \partial(\partial^\delta \Psi) = \frac{\beta}{8\kappa} q_\mu q^\mu \partial^2 + \frac{\beta}{4\kappa} n^\mu n^\mu q_u q_u \partial^2 - \frac{\rho}{4\kappa}. \quad (65)$$

The second term on the left-hand side vanishes by definition of the Levi-Civita tensor, while the first term on the right-hand side vanishes because wave 4-vector is null, and the second term on the right-hand side vanishes because $q_u = 0$ [recall that $q^\mu \neq 0$, but $q_u = \eta_{uu} q^\mu$ and $\eta_{uu} = 0$, similar to what we have in Eq. (25)]. Therefore, Eq. (61) is of the form

$$\nabla^2 \Psi = -\frac{\rho}{4\kappa}. \quad (66)$$

We will now continue to use a similar approach for Eq. (62). It is important to realize that Φ_j is transverse, which means it only has nonvanishing x and y components in the Cartesian basis or in our (u, v, x, y) coordinate

system. Let us begin by setting $j = x$ in Eq. (62) (similar arguments would hold under the transformation $x \leftrightarrow y$), so that using the definitions of n^α and q^α , Eq. (62) reduces to

$$2\nabla^2 \Phi_j - \frac{4\alpha}{\kappa} \epsilon_{xuy}^v q^\mu q^\mu \partial_{uv} \Phi^y + \frac{2\alpha}{\kappa} \epsilon_{juu\eta} n^\mu n^\mu q^\mu \partial_u \Phi_u - \frac{2\alpha}{\kappa} \epsilon_{uu\delta\eta} n^\mu n^\mu q^\mu \partial_u \partial^\delta \Phi_{|j)} - \frac{\alpha}{\kappa} \epsilon_{xuy}^v n^\mu n^\mu q^\mu \partial_{vuu} \Phi^y - \frac{2\alpha}{\kappa} \epsilon_{uu\delta\eta} (n^\mu q_u) n^\mu q^\mu \partial_j \Phi^\delta = \frac{\beta}{2\kappa} n^\mu q_u q_j \partial^2 - \frac{s_j}{2\kappa}. \quad (67)$$

Since Φ_j is just a function of retarded time, it is clear that the second term and the fifth term on the left-hand side of Eq. (67) vanish. The third, fourth, and sixth terms on the left-hand side are zero by the properties of the Levi-Civita tensor, while the first term on the right-hand side is zero because n^μ points along q^μ and the latter is null by the equation of motion of the scalar field. We can thus see that Eq. (62) is of the form

$$\nabla^2 \Phi_j = -\frac{s_j}{4\kappa}. \quad (68)$$

Equation (63) can be rewritten as

$$\delta_{jk} \nabla^2 (\Phi - \Psi) - \frac{\alpha}{2\kappa} \epsilon_{(k|u\beta\gamma} q^\mu \partial \partial^\gamma \Phi_{|j)} (\Phi - \Psi) = -\frac{3}{4\kappa} \delta_{jk} (\partial_t s + \tau) + \frac{\beta}{2\kappa} q_j q_k \partial^2, \quad (69)$$

where here $(j, k) \in (x, y)$. Since Ψ and Φ are functions of retarded time u only, the second term on the left-hand side vanishes, since $\Psi_{,x} = 0 = \Psi_{,y}$ and $\Phi_{,x} = 0 = \Phi_{,y}$. The second term on the right-hand side also vanishes because the wave 4-vector q^μ points in the direction of propagation (along u). Therefore, with the above analysis, Eq. (63) can be expressed as

$$\nabla^2 (\Phi - \Psi) = -\frac{3}{4\kappa} (\partial_t s + \tau). \quad (70)$$

Let us finally look at Eq. (64). The field $h_{\alpha\beta}^{TT}$ is the transverse-traceless part of the metric perturbation in our irreducible decomposition. The transverse nature of this term means that α and β must be either x or y for a GW propagating along the u direction. This implies that only the (x, x) , (x, y) , and (y, y) components of $h_{\alpha\beta}^{TT}$ can be nonvanishing. This, in turn, implies that the fourth term on the left-hand side of Eq. (64) vanishes because $h_{\alpha\beta}^{TT}$ is contracted onto the wave vector q^α which points in the u direction. Similarly, the second and the third terms of Eq. (64) can be shown to vanish using the arguments above and the dependence of h^{TT} on retarded time. Therefore, we are left with

$$\square h_{jk}^{TT} = -\frac{\sigma_{jk}}{\kappa}. \quad (71)$$

We have now shown in excruciating detail that the complicated Eqs. (61)–(64) reduce to Eqs. (66), (68), (70), and (71). The latter are exactly the same as Eq. (60) of GR. Equations (66), (68), and (70) are (elliptic) Poisson equations, and therefore, at a particular time, the solutions depend on the matter configuration only, and the 4 d.o.f. described by Ψ , Φ , and Φ_j do not represent radiative modes. Equation (71), on the other hand, is a hyperbolic equation, and it must thus

represent a radiative mode corresponding to tensorial polarization modes $+$ and $\times$. These are the same GW modes that survive in GR, which confirms the results of Sec. IV B.

D. Irreducible decomposition in EdGB

In this subsection, we present a brief calculation to obtain the polarization modes for EdGB gravity by decomposition of the metric into irreducible pieces. This subsection is very similar in essence to Sec. V B. The decomposed equations of motion in EdGB gravity take the form

$$2\nabla^2\Psi + 8\lambda q^\beta q^\alpha \vartheta(\partial_\beta\partial_\alpha\Psi) + 32\lambda n^\alpha n^\beta q_\alpha q^\gamma \vartheta(\partial_\gamma\partial_\beta\Psi) - 16\lambda n^\alpha n^\beta q_\alpha q_\beta \vartheta(\nabla^2\Psi) = -\rho + n^\alpha n^\beta q_\alpha q_\beta \vartheta^2, \quad (72)$$

$$\begin{aligned} 2\nabla^2\Phi_j - 16\lambda n^\alpha n^\beta q^\gamma q_j \vartheta\partial_{\alpha\beta}\Phi_\gamma - 16\lambda q^\alpha q^\beta \vartheta\partial_{\alpha\beta}\Phi_j - 16\lambda n^\alpha n^\beta q_\alpha q^\gamma \vartheta\partial_{\gamma\beta}\Phi_j \\ + 16\lambda q^\alpha q^\beta \vartheta\partial_{j\beta}\Phi_\alpha + 16\lambda n^\alpha n^\beta q_\alpha q^\gamma \vartheta\partial_{j\beta}\Phi_\gamma - 16\lambda n^\alpha n^\beta q_\alpha q^\gamma \vartheta\partial_{\gamma\beta}\Phi_j + 16\lambda n^\alpha n^\beta q_\alpha q_j \vartheta\nabla^2\Phi_\beta \\ + 16\lambda n^\alpha n^\beta q_\alpha q_\beta \vartheta\nabla^2\Phi_j = -s_j + n^\alpha q_\alpha q_j, \end{aligned} \quad (73)$$

$$\delta_{jk}\nabla^2(\Phi - \Psi) = -\frac{3}{4}\delta_{jk}(\partial_t s + \tau) + q_j q_k \vartheta^2, \quad (74)$$

$$\frac{1}{2}\square h_{jk}^{TT} + 8\lambda q^\alpha q_{(k} \vartheta\square h_{j)\alpha}^{TT} - 2\lambda q^\alpha q^\beta \vartheta\partial_{(j|\beta} h_{k)\alpha}^{TT} - 4\lambda\delta_{jk} q^\alpha q^\beta \vartheta\square h_{\alpha\beta}^{TT} = -\sigma_{jk}. \quad (75)$$

Using the definitions of q^α and n^α , along with the fact that the metric perturbation is only a function of retarded time, and noting that Φ_α ($h_{\alpha\beta}^{TT}$) are transverse and thus they only possess nonvanishing x and y [(x, x), (x, y), and (y, y)] components, the above equations can be shown to reduce to the form

$$2\nabla^2\Psi + 8\lambda q^\mu q^\mu \vartheta(\partial_\mu\partial_\mu\Psi) = -\rho, \quad (76)$$

$$2\nabla^2\Phi_j - 16\lambda q^\mu q^\mu \vartheta\partial_{\mu\mu}\Phi_j = -s_j, \quad (77)$$

$$\delta_{jk}\nabla^2(\Phi - \Psi) = -\frac{3}{4}\delta_{jk}(\partial_t s + \tau), \quad (78)$$

$$\frac{1}{2}\square h_{jk}^{TT} = -\sigma_{jk}, \quad (79)$$

after some tensor manipulations. However, recall that when we solved the wave equation for the scalar field [Eq. (38)], we worked in the far field limit and kept only the leading $1/r$ term in the solution. Therefore, in the far field limit, the second terms on the left-hand sides of Eqs. (76) and (77) are subdominant because they fall off a factor of $1/r$ faster than the first terms on the left-hand sides of Eqs. (76) and (77).

As in the dCS gravity case, Eqs. (76)–(79) of EdGB gravity are the same as those in GR [see Eq. (60)], after discarding subdominant terms. Equations (76)–(78) are elliptic, and thus, their solutions only depend on the matter

configuration at a particular time instant, implying that Ψ , Φ , and Φ_j are not radiative d.o.f. On the other hand, Eq. (79) is hyperbolic, and thus, it describes a radiative d.o.f. We can therefore conclude that, as GR, GWs in EdGB only possess two modes of polarization, namely the $+$ and $\times$ modes. This result is also in agreement with those obtained in Sec. IV C.

VI. DISCUSSIONS

We have here used two distinct methods to calculate and verify the polarization content of weak, plane, nearly null GWs in two different quadratic theories of gravity, namely dCS and EdGB gravity. The methods consisted of the Newman-Penrose formalism coupled to the E(2) classification, as well as an irreducible decomposition. We have found out that in both theories, the nonvanishing polarizations are the two tensorial modes of GR (the $+$ and $\times$ GW modes).

This work, of course, is not the first to calculate the nonvanishing polarization modes in dCS and EdGB gravity. In the early 2000s, Jackiw and Pi [40] calculated the polarization modes for nondynamical Chern-Simons theory. This theory, however, is quite distinct from dCS gravity, because the scalar in the former was prescribed *a priori* and not allowed to vary dynamically, rendering the theory overconstrained in certain scenarios [41,42]. The polarization modes in dCS gravity were studied later in [41], considering a pp -wave spacetime and finding the

same results obtained in our paper. The work of [41], however, was limited to a pp -wave spacetime and only considered the evolution of the breathing mode, without studying the possibility of vector modes. Finally and more recently, the authors of [43] showed that as one approaches future null infinity $\mathcal{I}^+$, the trace of the metric perturbation obeys a wave equation, and thus, one can use the TT gauge to model GW polarizations in the far zone, implying that only the $+$ and $\times$ GW modes survive at $\mathcal{I}^+$. The work of [43], however, did not study why or how the other potential polarization modes are suppressed in dCS gravity.

The results presented here confirm and extend the results of earlier papers on dCS gravity, arriving at the same conclusions by exploring the why and the how in more detail through two techniques that had not been explored before. Moreover, we applied the same techniques to EdGB gravity, arriving again at the same results as in dCS gravity, though it seems this is the first time these results appear in the literature. Our work therefore shows, in a pedagogical way, how to calculate the evolution equation for different polarization modes in modified theories of gravity.

Our results also have important implications for gravitational wave tests of GR. In the near future, the detection of GWs through multiple interferometers, or through space-based instruments, hold the key to measure the polarization content of GWs. If future observations can show that only the two $+$ and $\times$ modes are present in nature, this could be a death blow to many modified theories. We here show clearly that this is not the case in general. In dCS and EdGB gravity, as well as probably in other theories of gravity, the polarization content of GWs remains the same as in GR, and thus, polarization tests of GR with GWs are uninformative. The best avenue to constrain these theories, therefore, continues to be the dynamical late inspiral and merger phase of coalescing binaries [44].

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APPENDIX: DETAILED CALCULATION OF THE POLARIZATION MODES IN EDGB GRAVITY USING THE NP METHOD

Let us begin by considering the scalar field evolution controlled by Eq. (15). Since the Gauss-Bonnet invariant $\mathcal{G}$, defined by Eq. (13), decays fast in the far zone, the interaction term vanishes. One is then left with a free wave equation in flat spacetime, whose solution is simply

$$\vartheta = B e^{iq^\mu x_\mu}, \quad (\text{A1})$$

with B an amplitude and q^μ the 4-wave vector, which is also null, i.e., $q^\mu q_\mu = 0$.

Let us now consider the trace-reversed form of the field equations given in Eq. (49). Using Eq. (A1) in Eq. (49), we have

$$R_{\mu\nu} = -B^2 q_\mu q_\nu e^{2iq \cdot x} + 2\lambda B \delta_{\alpha\beta\rho\sigma}^{\gamma\delta\kappa\epsilon} R^{\rho\sigma}_{\kappa\epsilon} q^\alpha q_\gamma e^{iq \cdot x} \delta^\beta_{(\mu} \eta_{\nu)\delta}. \quad (\text{A2})$$

Since we are working with vacuum, the first term in the equation above vanishes. With this at hand, we can now compute the different NP scalars.

1. Analysis for Ψ_2

Equation (A2) implies that

$$\begin{aligned} \Psi_2 &= -\frac{1}{6} R_{nl} \\ &= \frac{1}{6} B^2 q_n q_l e^{2iq \cdot x} - \frac{1}{3} \lambda B \delta_{\alpha\beta\rho\sigma}^{\gamma\delta\kappa\epsilon} R^{\rho\sigma}_{\kappa\epsilon} q^\alpha q_\gamma e^{iq \cdot x} \delta^\beta_{(n} \eta_{l)\delta}. \end{aligned} \quad (\text{A3})$$

For a weak, plane, nearly null GW, the Riemann tensor is only dependent on retarded time. Thus, Eq. (15) along with Eq. (A1) implies that the only nonvanishing component is along retarded time, i.e., either q^l [or q_n when using the metric in Eq. (25)] are the only nonvanishing components. Using this, the first term of Eq. (A3) vanishes, and one is left with

$$\Psi_2 = -\frac{1}{3} \lambda B \delta_{\alpha\beta\rho\sigma}^{\gamma\delta\kappa\epsilon} R^{\rho\sigma}_{\kappa\epsilon} q^\alpha q_\gamma e^{iq \cdot x} \delta^\beta_{(n} \eta_{l)\delta}. \quad (\text{A4})$$

Using Eq. (25) and the nonvanishing components of the wave 4-vector, we see that Eq. (A4) vanishes due to the symmetries of the generalized Kronecker delta. The conclusion then is that $R_{nl} = 0$, and thus, $\Psi_2 = 0$.

2. Analysis for Ψ_3

Equation (34) implies that

$$\begin{aligned} \Psi_3 &= -\frac{1}{2} R_{n\bar{m}} \\ &= \frac{1}{2} B^2 q_n q_{\bar{m}} e^{2iq \cdot x} - \lambda B \delta_{\alpha\beta\rho\sigma}^{\gamma\delta\kappa\epsilon} R^{\rho\sigma}_{\kappa\epsilon} q^\alpha q_\gamma e^{iq \cdot x} \delta^\beta_{(n} \eta_{\bar{m})\delta}. \end{aligned} \quad (\text{A5})$$

The first term in the above equation vanishes again, since q_n is the only nonzero component of the 4-wave vector for the GW under consideration. Using Eq. (25), Eq. (A5) can be written as

$$\begin{aligned} \Psi_3 &= \frac{1}{2} \lambda B \delta_{l\bar{m}\rho\sigma}^{n\bar{m}\kappa\epsilon} R^{\rho\sigma}_{\kappa\epsilon} q^l q_n e^{iq \cdot x} \\ &\quad - \frac{1}{2} \lambda B \delta_{l\bar{m}\rho\sigma}^{n\bar{m}\kappa\epsilon} R^{\rho\sigma}_{\kappa\epsilon} q^l q_n e^{iq \cdot x}. \end{aligned} \quad (\text{A6})$$

The properties of the generalized Kronecker delta restrict the values that the indices ρ, κ, ϵ , and σ can take. We can therefore use Eq. (33) to rewrite the Riemann tensor in Eq. (A6). Since the metric or the metric perturbation is only a function of retarded time, we find that the right-hand side of Eq. (A6) is zero identically. Therefore, with Eq. (34), we have that

$$\Psi_3 = 0. \quad (\text{A7})$$

3. Analysis for Φ_{22}

Equation (A2) implies that

$$\begin{aligned} \Phi_{22} &= -\frac{1}{2} R_{nn} \\ &= \frac{1}{2} B^2 q_n q_n e^{2iq \cdot x} \\ &\quad - \lambda B \delta_{\alpha\beta\rho\sigma}^{\gamma\delta\kappa\epsilon} R^{\rho\sigma}_{\kappa\epsilon} q^\alpha q_\gamma e^{iq \cdot x} \delta^\beta_{(n} \eta_{n)\delta}. \end{aligned} \quad (\text{A8})$$

The second term in the above equation vanishes by the same argument presented in the paragraph below Eq. (A6). The first term does not vanish, but it scales as $\mathcal{O}(r^{-2})$ in the far zone. Therefore, we have that far from the source

$$\Phi_{22} = 0. \quad (\text{A9})$$

4. Analysis of Ψ_4

Using the same line of reasoning, there are no constraints one can place on Ψ_4 using the field equations for EdGB gravity. This in turn means that, just as in dCS gravity, Ψ_4 remains unconstrained or nonvanishing in EdGB gravity. Therefore, the transverse-traceless tensorial modes, i.e., the $+$ and $\times$ GW polarizations, are nonvanishing in EdGB gravity for a weak, plane, nearly null GW.

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