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Graph universal cycles of combinatorial objects

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ABSTRACT

A connected digraph in which the in-degree of any vertex equals its out-degree is Eulerian; this baseline result is used as the basis of existence proofs for universal cycles (also known as ucycles or generalized deBruijn cycles or U-cycles) of several combinatorial objects. The existence of ucycles is often dependent on the specific representation that we use for the combinatorial objects. For example, should we represent the subset $\{2, 5\}$ of $\{1, 2, 3, 4, 5\}$ as “25” in a linear string? Is the representation “52” acceptable? Or is it tactically advantageous (and acceptable) to go with $\{0, 1, 0, 0, 1\}$? In this paper, we represent combinatorial objects as graphs, as in [3], and exhibit the flexibility and power of this representation to produce *graph universal cycles*, or *Gucycles*, for k-subsets of an n-set; permutations (and classes of permutations) of $[n] = \{1, 2, \ldots, n\}$, and partitions of an n-set, thus revisiting the classes first studied in [5]. Under this graphical scheme, we will represent $\{2, 5\}$ as the subgraph A of C_5 with edge set consisting of $\{2, 3\}$ and $\{5, 1\}$, namely the “second” and “fifth” edges in C_5. Permutations are represented via

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their permutation graphs, and set partitions through disjoint unions of complete graphs.

1. Introduction

A somewhat loose definition of universal cycles was used in a recent VCU seminar talk by Glenn Hurlbert in which he stated that "Broadly, universal cycles are special listings of combinatorial objects in which *codes for the objects* are written in an overlapping, cyclic manner." By "special" Hurlbert means "without repetitions", i.e., so that each linear window of a specific length represents a different object. We stress that the "codes for the objects" are not pre-dictated but are often chosen in a conventional way. See [1] and [7] for exhaustive treatments.

Example 1. The cyclic string 112233 encodes each of the six multisets of size 2 from the set $\{1, 2, 3\}$, with, e.g., 31 and 22 representing $\{1, 3\}$ and $\{2, 2\}$ respectively. The window size is 2.

Example 2. The string 11101000 encodes each of the binary three-letter words in the obvious way, but is also a ucycle of the eight subsets of $\{1, 2, 3\}$, with the *binary string coding* – in which membership in the set is indicated by a 1, e.g., 101 represents the subset $\{1, 3\}$. The string can also be a representation of all subgraphs of the complete graph K_3. The window length is 3.

Other examples will be given throughout the paper.

A ucycle is usually shown to exist by showing that an arc digraph D is Eulerian, which in turn holds if it is (a) balanced (i.e., the indegree $i(v)$ of every vertex $v \in D$ equals its outdegree $o(v)$) and (b) weakly connected. Weak connectedness is often showed by exhibiting a path from any starting vertex to a strategically chosen sink vertex. The edge set of the arc digraph consists of the objects that we are trying to ucycle, and the vertices are most often taken to be the "overlaps" between consecutive edges. Alternately, edges are labeled as the concatenation of adjacent vertex labels. We illustrate this strategy by showing that the set of n-letter words on a k-letter alphabet admits a ucycle (this is the classical deBruijn theorem). Vertices of D are $(n-1)$-letter words with an edge from v_1 to v_2 if the last $(n-2)$ letters of v_1 coincide with the first $n-2$ letters of v_2. The edge label, obtained by concatenation, are the desired objects we seek to ucycle. Connectedness is easy to establish, and in- and out-degrees may both be seen to be k, so D is Eulerian and the Eulerian cycle spells out the ucycle. The arc digraph that leads to the ucycle in Example 2 is given in Fig. 1.

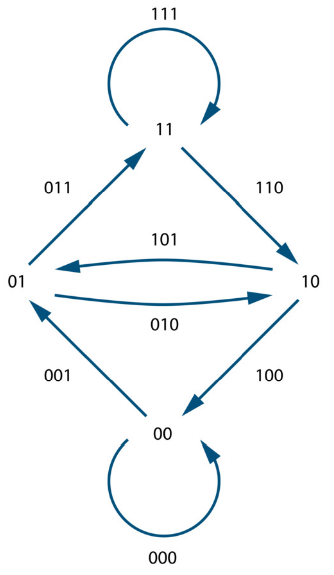

Fig. 1. The arc digraph for Example 2.

In the deBruijn theorem example above, we can alternately think of the vertices as multisets of $[n-1]$, with each entry appearing between 0 and $(k-1)$-times, and with, e.g., a word such as 30221 ($n=6, k=4$) representing the multiset $\{1,1,1,3,3,4,4,5\}$. Moreover, under this interpretation, there is an edge from v_1 to v_2 if the multiset frequencies of $\{2,\ldots,n-1\}$ in v_1 coincide with the multiset frequencies of $\{1,\ldots,n-2\}$ in v_2, and the concatenated edge is a multiset of $[n]$, with each element occurring $\leq k-1$ times. This notion of a window shift necessitating a relabeling of the vertices is crucial in the context of graph universal cycles, Gucycles, which we turn to next. But first some motivation. Note that Gucycles are simply ucycles with graph encodings. Why study Gucycles? First we shall see that we are able to use graph encodings to exhibit Gucycles in situations where regular ucycles do not exist. More importantly, however, graph encodings might provide another device that may be used to prove existence of ucycles in many other situations, yet unstudied. We start with some key definitions from [3] and an example:

Definition 1.1. Given a labeled graph G with vertex set $V(G)=\{v_1,v_2,...,v_N\}$ with vertices labeled by the rule $v_j \to j$ and an integer $n \in \{1,\ldots,N-1\}$, an **n-window of** G is the subgraph of G **induced** by the vertex set $V=\{v_i,v_{i+1},\ldots,v_{i+n-1}\}$ for some i, where vertex subscripts are reduced modulo N as appropriate, and vertices are **relabeled** such that $v_i \to 1; v_{i+1} \to 2, \ldots v_{i+n-1} \to n$. For each $i \in \{1,\ldots,N\}$, we denote the corresponding ith n-window of G as $W_{G,n}(i)$, or as $W_n(i)$ if G is clear from the context.

Fig. 2. Gucycle of all labeled graphs on 3 vertices.

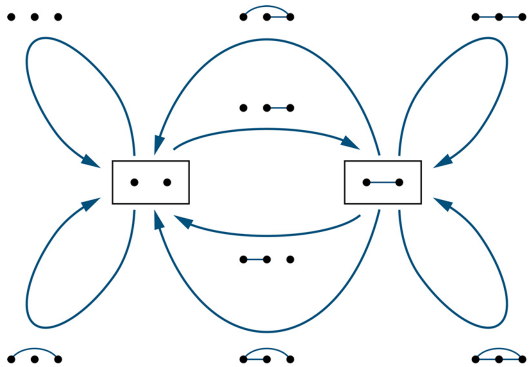

Fig. 3. The arc digraph that produces the Gucycle in Fig. 2.

Definition 1.2. Given $\mathcal{F}$, a family of labeled graphs on n vertices, a **graph universal cycle** (Gucycle) of $\mathcal{F}$, is a labeled graph G on N vertices such that the sequence of n-windows of G contains each graph in $\mathcal{F}$ precisely once. That is, $\{W_n(i)|1 \leq i \leq N\} = \mathcal{F}$, and $W_n(i) = W_n(j) \Rightarrow i = j$.

Brockman et al. [3] prove the existence of Gucycles of classes of labeled graphs on n vertices, including all simple graphs, trees, graphs with k edges, graphs with loops, graphs with multiple edges (with up to d duplications of each edge), directed graphs, hypergraphs, and r-uniform hypergraphs. Fig. 2 shows the Gucycle of the 8 labeled graphs on $n = 3$ labeled vertices. For simplicity we have not labeled the $N = 8$ vertices from 1 to 8. For example, the fifth graph is the complete graph K_3 on the vertices $\{5, 6, 7\}$, which have been relabeled as $\{1, 2, 3\}$. The Gucycle is, in turn, is constructed from the arc digraph in Fig. 3 in the following fashion: Each arc in the arc digraph induces an edge label. For example, the two arcs going from K_2 to itself are labeled, as shown, by K_3 and P_3, depending on whether the edge between v_3 to v_1 is present or not (the edge between v_2 and v_3 is always present). Since the arc digraph is Eulerian, we use the Eulerian circuit in Fig. 3 to spell out the Gucycle in Fig. 2. In other words, we concatenate the vertex labels, and, for the edge labels, we consider all possibilities of how edges may or may not exist between the new vertex and previous ones. Thus we use exactly the same process to go from the arc digraph to the Gucycle as we do with the ucycles in deBruijn's theorem.

In this paper, we extend the family of classes $\mathcal{F}$ that admit Gucycles to

- Subgraphs of size k of the cycle C_n. This leads to Theorem 2.5, namely the existence of Gucycles for k-subsets of an n-set, *for all values of* k, n.

- A class of multiple edge subgraphs of C_n, which yields Theorem 2.6 (Gucycles for k-multisets of an n-set $\forall k, n$).
- Permutation graphs on n vertices. This leads to Theorem 3.1, which exhibits the existence of Gucycles for all permutations of $[n]$.
- Transposition graphs on n vertices, leading to the existence of Gucycles for permutation involutions (Theorem 3.2).
- Disjoint unions of cliques on n vertices. This leads to Theorem 4.1 on Gucycles for set partitions.

2. Subsets and multisets

Example 3. (Ucycle for 3-subsets of [8]; [12]):

$$1356725\ 6823472\ 3578147\ 8245614\ 5712361\ 2467836\ 7134582\ 4681258,$$

where each block is obtained from the previous one by addition of 5 modulo 8, is an encoding of the 56 3-subsets of the set [8]. The window is of length 3, and, e.g., the block 836 represents the subset $\{3, 6, 8\}$. If we are to represent k-subsets of $[n]$ via their elements, then, as in the above example, we clearly must have each element of $[n]$ appear the same number of times in the ucycle, and thus

$$n \Big| \binom{n}{k}$$

is a necessary condition for the existence of a ucycle. In [5], Chung et al. conjectured that for each k, there exists an $n_0(k)$ such that ucycles exist for k-subsets of $[n]$ provided that $n \geq n_0(k)$ is such that the divisibility condition above holds. This conjecture was proved by [9]. For work prior to [9] see also [13] and [15], in addition to [5] and [12].

We are more interested, however, in binary string-type codings that dispense with divisibility conditions, and next guide the discussion in this direction.

Ucycles for k-subsets of $[n]$ cannot exist in the binary string coding unless $k = n - 1$ when we have the ucycle $111\ldots10$; for other values of k, we are forced into an incomplete cycle as seen by the example

$$a = 110000 \to 100001 \to 000011 \to 000110 \to 001100 \to 011000 \to a$$

for $k = 2; n = 6$.

Several authors have proved results that produce ucycles for objects related to $\binom{\mathbf{n}}{\mathbf{k}}$, the set of all k-subsets of n. For example, in [10], the authors prove that under certain conditions, there exists an s-ocycle of all the permutations of a fixed multiset, where an ocycle, or overlap cycle, is one in which the overlap between consecutive k windows is not $k - 1$ – but rather equals s. If we take the multiset to be one with k ones and $n - k$ zeros we see that the following is true

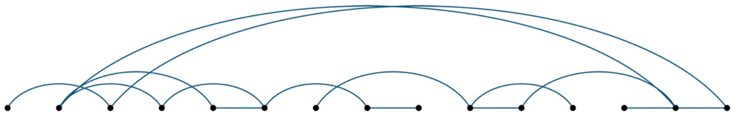

Fig. 4. Gucycle of the 2-subsets of [6].

Proposition 2.1. *([10]) There exist, for* $1 \leq s \leq n-2$*;* $\gcd(n,s) = 1$*, s-ocycles of* $\binom{\mathbf{n}}{\mathbf{k}}$ *in the binary string coding.*

In another paper [6], we see that

Proposition 2.2. *([6]) Universal packings of length* $\binom{n}{k}(1-o(1))$ *exist for each* k*.*

Note that there is no divisibility condition that hampers the universality of Propositions 2.1 and 2.2. The same is true of the next result from [2]:

Proposition 2.3. *([2]) For each* $1 \leq s < t \leq n$ *there exists a ucycle of binary words of length* n *with between* s *and* t *ones, i.e., of subsets of size in the range* $[s,t]$*, in the binary string coding.*

For example, the binary string 1110011010 is a ucycle of all subsets of size 2 and 3 of a 4-element set, using a window of length 4, and the binary string coding. Proposition 2.3 will prove to be critical in the proof of Theorem 2.5.

Now we have seen above that ucycles of 2-subsets of [6] do not exist in the binary string coding nor in the traditional sense, since 6 does not divide $\binom{6}{2}$. Consider, however, the 15-vertex graph in Fig. 3, which is a Gucycle of all 2-edge subsets of K_4. It is not surprising that this Gucycle exists – by virtue of Theorem 3.5 in [3], which states that Gucycles exist for all graphs on n vertices with precisely k edges. If we next label the six edges of K_4 lexicographically according to the code

$$\{1,2\} = 1; \{1,3\} = 2; \{1,4\} = 3; \{2,3\} = 4, \{2,4\} = 5; \{3,4\} = 6,$$

we see that the Gucycle in Fig. 4 can be viewed as a Gucycle of the fifteen 2-subsets of [6]. Once again, to reduce clutter in the diagram, we have not labeled the fifteen vertices in Fig. 4.

Since K_n has $\binom{n}{2}$ edges, we quickly see that the following is a corollary of the above theorem, and a special case of Theorem 2.5 below:

Proposition 2.4. *There exists a Gucycle of all* k*-subsets of an* N*-element set, where* $N = \binom{n}{2}$ *for some* n*.*

Notice that in the above Gucycle each of the six edges of K_4 appears five times, even though a specific edge, e.g., the one joining v_5 and v_6 plays the role of a $\{1,2\}$ edge, a $\{2,3\}$ edge, and a $\{3,4\}$ edge, and thus represents the numbers 1, 4, and 6. On the

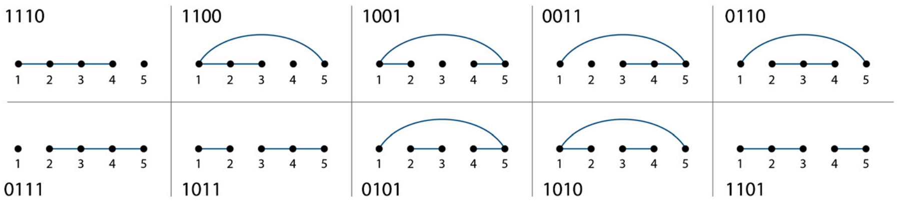

Fig. 5. Gucycle windows of the 3-subsets of [5].

other hand the edge joining v_2 and v_5 only plays the role of the number 3 (i.e., the edge $\{1,4\}$). By contrast, in the traditional approach to ucycles of k-element subsets of $[n]$, an element such as "4" always represents itself, i.e., the number 4.

Theorem 2.5. *For each $1 \leq k \leq n$ there exists a Gucycle for k-element subsets of $[n]$.*

Proof. By Proposition 2.3, there exists a binary string coding ucycle of all $k-1$ and k-subsets of $[n-1]$. Label the edges of C_n as $e_1 = \{1,2\}, e_2 = \{2,3\}, \dots, e_n = \{n,1\}$. Identify the binary string of the k-subset A of $[n-1]$, consisting of ones in the r_ith positions; $1 \leq i \leq k$ with the edges $\{e_i : 1 \leq i \leq k\}$. Identify the binary string of the $(k-1)$-subset A of $[n-1]$, consisting of ones in the r_ith positions; $1 \leq i \leq k-1$ with the edges $\{e_i : 1 \leq i \leq k-1; e_n\}$.

For example, let $n = 5, n-1 = 4$, and $k = 3$. We identify the binary string $(0,1,1,1)$ of the 3-subset $\{2,3,4\}$ of $\{1,2,3,4\}$ by the second, third, and fourth edges of C_5, namely $\{2,3\}$, $\{3,4\}$, and $\{4,5\}$. We identify the binary string $(1,0,0,1)$ of the 2-subset $\{1,4\}$ of $\{1,2,3,4\}$ by the first, fourth, and *fifth* edges of C_5, namely $\{1,2\}$, $\{4,5\}$, and $\{5,1\}$.

The key idea of the proof is that we are identifying both $(k-1)$- and k- subsets of $[n-1]$ with a k-subgraph of C_n. Note that the edge e_n, if present, is not part of the graph induced by the next (or any other) window. Thus, there is a bijection between binary-string coded subsets of size $k-1$ or k, of $[n-1]$, and the k-subgraphs of C_n. The latter, in turn, can be identified with the collection of k-subsets of $[n]$, via the bijection $j \to e_j$. □

Fig. 5 provides an example which shows how the ucycle 1110011010 of 2- and 3-element subsets of a 4-element set, obtained via Proposition 2.3, translates to a Gucycle of the 3-subsets of [5].

Theorem 2.6. *For each k and $n \geq 2$, there exists a Gucycle for k-multisets of $[n]$.*

Proof. We start with the following result in [4], which improves on Theorem 4.1 in [2]:

Proposition 2.7. *Let $k, n \in \mathbb{Z}^+$. Consider n letter words $w = (w_1, w_2, \dots, w_n)$ on the $k+1$-letter alphabet $\Lambda = \{0, 1, \dots, k\}$, and define the weight $h(w)$ of w by $h(w) = \sum_{i=1}^{n} w_i$. Let $s, t \in \mathbb{Z}^+$ satisfy $0 \leq s < s+k \leq t \leq nk$. Let $\mathcal{W}$ be the collection of*

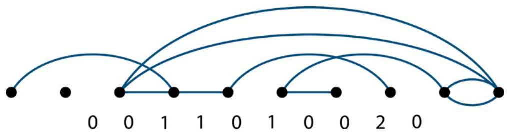

Fig. 6. Gucycle of the 2-multisets of [4].

all such words with weights between s and t. Then there exists a ucycle of the elements of $\mathcal{W}$.

To prove Theorem 2.6, we will let $s = 0$ and $t = k$ in Proposition 2.7. By Proposition 2.7, for $n \geq 2$ there exists an alphabet coding ucycle of all multisets of $[n-1]$ of size between 0 and k. Label the potential multi-edges of C_n as $e_1 = \{1, 2\}, e_2 = \{2, 3\}, \ldots, e_n = \{n, 1\}$. Identify the alphabet string of the k-multiset A of $[n - 1]$, consisting of the number a_i in the r_ith positions; $1 \leq i \leq n - 1$ with the multi-edge sets $\{e_i : 1 \leq i \leq n - 1\}$, where e_i contains a_i edges. Identify the alphabet string of the m-multiset A of $[n - 1]$ $(0 \leq m \leq k - 1)$, consisting of a_i in the r_ith positions; with the multiedges edges $\{e_i : 1 \leq i \leq n - 1; e_n\}$ (where e_i contains a_i edges for $1 \leq i \leq n - 1$ and e_n contains $k - m$ edges). Since the edge e_n, if present, is not part of the graph induced by any other window, we see that there is a bijection between alphabet-string coded multisets of size between 0 and k, of $[n - 1]$, and a Gucycle of k-multigraphs of C_n. The latter, in turn, can be identified with the collection of k-multisets of $[n]$, via the bijection $j \to e_j$. Fig. 6 illustrates how Proposition 2.7 may be used for $n = 4; k = 2$. In this figure, the ucycle of ten 0-, 1-, and 2- multisets of $\{1, 2, 3\}$ is given by 0011010020, and the Gucycle of ten 2-multisets of $\{1, 2, 3, 4\}$, encoded by the multigraphs of C_4, is pictured. Once again, we do not label the vertices in Fig. 6 from 1 to 10. □

Remark. As noted, the traditional way of exhibiting the existence of ucycles is to identify a digraph (the "arc digraph") and show that it is both balanced ($i(v) = o(v)$ for each v) and weakly connected. We do not do that in the proofs of Theorems 2.5 and 2.6, noting that this work has already been done in [2] and [4].

3. Permutations

If one attempts to create a ucycle of the permutations of $\{1, 2, 3\}$ using the letters 1, 2, and 3, then a full ucycle cannot result, since the cycle terminates before all six permutations are included, as one sees with the example

$$123 \to 231 \to 312 \to 123.$$

One needs to enhance the alphabet in order to provide a ucycle of S_n via an order-isomorphic representation. For example, the string 124324 encodes each of the six permutations of $\{1, 2, 3\}$ in an order isomorphic fashion (e.g. 243 represents the permutation 132). In [5] it was shown that between 1 and $5n$ additional symbols suffice, and

Fig. 7. Gucycles of all permutations of [3].

it was shown in [14] that only one extra symbol was actually needed. In [10] s-ocycles of S_n were shown to exist for suitable values of s. Also in [5] a ucycle was exhibited with n symbols provided that the $n-1$ overlaps were order-isomorphic but not identical. For $n = 3$, such a ucycle is given by

$$132 \to 312 \to 123 \to 231 \to 321 \to 213.$$

In this example the last two elements of the first vertex, namely 32, are not identical to the first two vertices of the second vertex, i.e., 31 – but they are both order isomorphic to 21, which is what ultimately allows the ucycle to exist under these more relaxed circumstances.

We recall that the *permutation graph* of $\pi \in S_n$ has vertex set $[n]$ with an edge present between i and j if $i < j$ but $\pi(i) > \pi(j)$, i.e., if $\{i, j\}$ is an inversion. For example K_4, is the permutation graph of $\pi = 4321$. In this section we represent an permutation via its permutation graph and exhibit the fact that the class of permutation graphs can be placed in a Gucycle for each n. Fig. 7 illustrates this for $n = 3$, where the Gucycle lists the six permutations in the order

$$321 \to 231 \to 312 \to 213 \to 123 \to 132,$$

on six numbered vertices.

Theorem 3.1. *For each n, the set S_n of all permutations on n elements can be placed in a Gucycle via their permutation graphs.*

Proof. We define an arc digraph as follows: Let V be the set of all permutation graphs on $\{1, 2, \ldots, n-1\}$. The n edges emanating from vertex $\pi = \pi_1 \ldots \pi_{n-1}$ are labeled according as how the index n is inserted into π, and the vertices that they point towards are labeled as the reduction of the edge label, minus the index 1, to the set $[n-1]$. For example, there are two edges from $\pi = 123$ to $\eta = 312$, and these are labeled as 4123 and 1423; the edge between π and $\nu = 132$ is labeled as 1243; and, finally, the edge between π and $\mu = 123$ is labeled as 1234. In general, the insertion of the new element must be done in a way that respects the inversion structure of the starting vertex if one is to exploit the permutation graph representation that we are employing. To give another example, the degree structure of the vertex 132 is shown in Fig. 8.

It is clear that the outdegree of π is n for each vertex. We note that the "overlap" between adjacent vertices is not of the customary form; for example π above ends in 23, which is neither equal, nor order isomorphic, to the starting segment 31 of η. In general, we have that π_2 follows π_1 if the *inversion structure* of $\{1, 2, \ldots, n-2\}$ in π_2 is the same

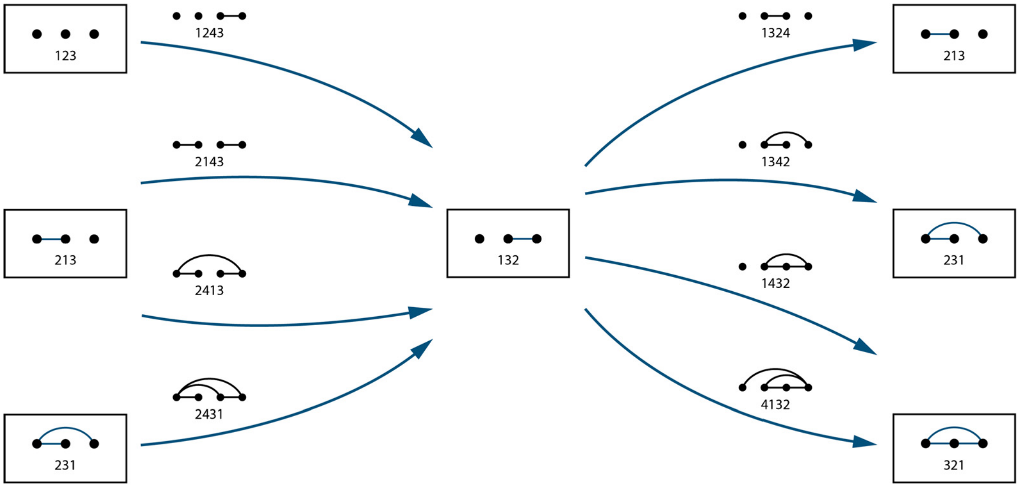

Fig. 8. The degree structure of the vertex 132 for $n = 4$.

as the inversion structure of $\{2, \ldots, n-1\}$ in π_1. This is in contrast to (but echoes) the existence of an edge in the deBruijn cycle scheme, in which word w_2 follows word w_1 if the *letters* in the positions $\{1, 2, \ldots, n-2\}$ of w_2 are the same as the letters $\{2, \ldots, n-1\}$ of w_1.

Turning to the indegree of π, the n edges pointing towards vertex $\pi = \pi_1 \ldots \pi_{n-1}$ are labeled according as how the index 1 is inserted into $\pi' = (\pi_1 + 1) \ldots (\pi_{n-1} + 1)$, with corresponding vertices equaling the edge label minus the index n. The indegree of π is thus n as well. For example, the vertices 123, 213, and 231 point towards 123, with edge labels 1234; 2134; and 2314&2341 respectively.

We claim that the digraph D defined above is weakly connected and exhibit this easily by creating a path between any starting vertex and the identity permutation (that has an empty permutation graph). To do this, we merely insert the symbol n at the end of a vertex to create a vertex π in which for which $\pi_{n-1} = n - 1$. This process is repeated until all the symbols are in their natural order. For example, we have

$$43152 \to 32415 \to 21345 \to 12345.$$

Since D is balanced ($i(v) = o(v)\ \forall v$) and weakly connected, it is Eulerian, and the Eulerian circuit spells out the Gucycle. □

The reader will recall that, in the context of permutations, an involution is a permutation that is its own inverse, and which consists therefore of transpositions and fixed points, i.e. 1- and 2-cycles. To represent an involution graphically, perhaps the simplest device is to do so using unions of K_1 s (the fixed points) and K_2 s (the transposition). The *transposition graph* of an involution is the labeled graph thus obtained. The next main result of this section is that all involutions on n elements can be placed in a Gucycle. For example, for $n = 4$, a Gucycle of the ten involutions of [4] can be seen in Fig. 9,

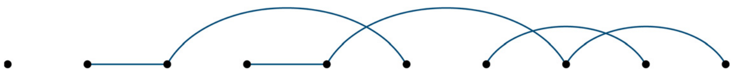

Fig. 9. Gucycle of the 10 involutions on $\{1, 2, 3, 4\}$.

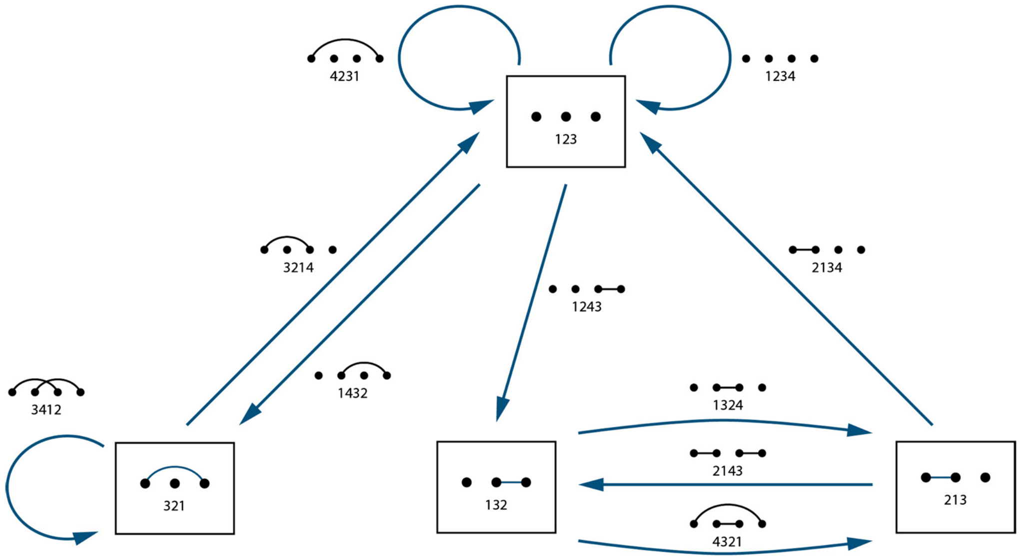

Fig. 10. Arc digraph for involutions on 1,2,3,4.

in the order 1324, 2143, 4321, 2134, 4231, 1432, 3412, 3214, 1234, 1243, and drawn as a labeled graph on ten vertices.

Theorem 3.2. *The "transposition graphs" of all involutions of S_n can be placed in a Gucycle.*

Proof. Let the vertices of a digraph be represented by the transposition graphs of involutions on $n-1$ elements. Now the element n in the next vertex will yield the edge label to be an involution on $[n]$ if n pairs up with a previous fixed point, or else forms a fixed point of its own. For example, the involution 1324 can yield edge label 53241, 13254, or 13245, which are incident to the vertices 2134, 2143, and 2134 respectively. Accordingly, the outdegree of any vertex v equals the number of fixed points of v plus one. (In general, we have that π_2 follows π_1 if the *transposition/fixed point structure* of $\{1, 2, \ldots, n-2\}$ in π_2 is the same as the transposition/fixed point structure of $\{2, \ldots, n-1\}$ in π_1.) Likewise, given an involution π on $\{2, 3, \ldots, n\}$ the element 1 might have formed a pair with any of the fixed points of π or else been in a fixed point by itself. This shows its indegree is the same as its outdegree. Finally, we can easily show weak connectedness by successively adding the fixed point n to any vertex until we end with the identity permutation. Fig. 10 illustrates the arc digraph for $n = 4$. □

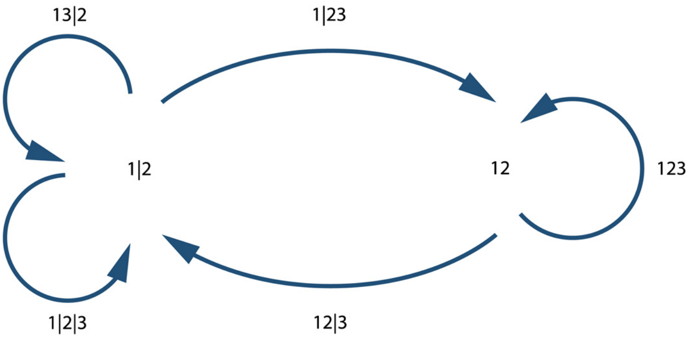

Fig. 11. Arc digraph for $\mathcal{P}(3)$.

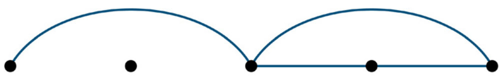

Fig. 12. Gucycle for $\mathcal{P}(3)$.

4. Partitions

Let $B(n)$ denote the ordered Bell numbers. In [5] the authors showed that for $n \geq 4$ a ucycle exists for the $B(n)$ partitions of $[n]$ into an arbitrary number of parts using an enhanced alphabet. See [8] and [11] for other results for partition ucycles. For example, we have the ucycle *abcbccccddcdeec* of $\mathcal{P}(4)$, the set of all partitions of $\{1, 2, 3, 4\}$ into an arbitrary number of parts, where, for example, the string *dcde* encodes the partition 13|2|4. Note that the alphabet used was in this case of size 5, though an alphabet of (minimum) size 5 is shown to suffice to encode $\mathcal{P}(5)$ as

$$DDDDDCHHHCCDDCCCHCHCSHHSDSSDSSHSDDCH$$
$$SSCHSHDHSCHSJCDC.$$

In this section, we show that for Gucycles, the result from [5] holds for $n \geq 1$ and that no alphabet augmentation is necessary. We will represent a set partition by a union of complete subgraphs of K_n. The subgraphs will, of course, be on the elements of the parts. For example, the partition 134|256|7 will be represented by two K_3 s, on the vertices $\{1, 3, 4\}$ and $\{2, 5, 6\}$; and the isolated vertex 7.

For example, the arc digraph and Gucycle for $\mathcal{P}(3)$ (on five labeled vertices), and the Gucycle for $\mathcal{P}(4)$ (on fifteen labeled vertices) may be seen in Figs. 11, 12, and 13 respectively.

Theorem 4.1. *For each $n \geq 1$, there exists a Gucycle of the partitions $\mathcal{P}(n)$ of $[n]$ into an arbitrary number of parts.*

Proof. The result is obvious for $n \leq 2$. For $n \geq 3$, we define a digraph as having vertex set equal to the graph representations of set partitions of $[n-1]$. If the partition

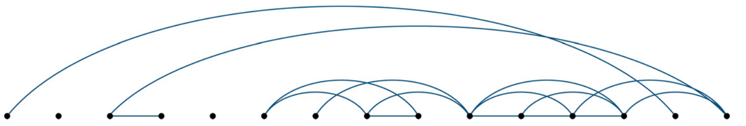

Fig. 13. Gucycle for $\mathcal{P}(4)$.

corresponding to a vertex has r parts, then the element n can either augment the r complete graphs, or else form an isolated vertex. These are the out-edge labels, which point towards the vertices (unions of complete graphs) induced by the elements $\{2, 3, \ldots, n\}$. This yields $o(v) = r + 1$ for any vertex with r parts. On the other hand, to see that $i(v) = r+1$ as well, we note that the element '1' could have been in a part by itself, or in one of the r parts of $\{2, 3, \ldots, n\}$. Weak connectedness is easy to establish via the path of any vertex to $1|2|\ldots|n-1$, with empty graph. To accomplish this, we consecutively add the vertex $n-1$ until we reach the desired destination. This may be seen by the example

$$12345 \to 1234|5 \to 123|4|5 \to 12|3|4|5 \to 1|2|3|4|5. \quad \square$$

5. Open problems

The overarching open problem that arises from this paper is the following: "Find a diversity of examples of other combinatorial structures that admit Gucycles." In a similar spirit, one may ask "what combinatorial structures may be fruitfully expressed as labeled graphs?" Likewise, if we have a subclass of structures that we seek to Gucycle, should we or should we not use the same graph representation used to successfully Gucycle the parent class? (Notice that we did not use permutation graphs to Gucycle involutions).

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