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EXPONENTIAL IMPROVEMENTS FOR SUPERBALL PACKING UPPER BOUNDS

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Abstract. We prove that for all fixed $p > 2$, the translative packing density of unit ℓ_p-balls in $\mathbb{R}^n$ is at most $2^{(\gamma_p+o(1))n}$ with $\gamma_p < -1/p$. This is the first exponential improvement in high dimensions since van der Corput and Schaake (1936).

1. Introduction

The sphere packing problem asks for the densest packing of non-overlapping unit balls in $\mathbb{R}^n$. This is an old and difficult problem whose exact solution is only known in dimensions 1, 2, 3, 8, and 24. The problem is already non-trivial in two dimensions (see [8] for a short proof). The three-dimensional sphere packing problem is known as Kepler's conjecture, and it was solved by Hales [9] via a monumental computer-assisted proof. The problem in eight dimensions was recently resolved by Viazovska [23] in a stunning breakthrough, and the method was then quickly extended to solve the problem in twenty-four dimensions [3]. Dimensions 8 and 24 are special due to the existence of highly dense and symmetric lattices known as the E_8 lattice (dimension 8) and the Leech lattice (dimension 24). See the survey [2] and its references for background and recent developments.

In this paper, we study translative packings of ℓ_p-balls in high dimensions. Denote the ℓ_p-balls with radius R in $\mathbb{R}^n$ by $\boldsymbol{B}_p^n(R) := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_p \le R\}$ and the unit ℓ_p-ball by $\boldsymbol{B}_p^n := \boldsymbol{B}_p^n(1)$. Here $\|(x_1, \dots, x_n)\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ is the ℓ_p-norm. The name *superball* refers to ℓ_p-balls with $p > 2$ [19]. Superballs are more cube-like compared to the familiar ℓ_2-balls. See [10, 11, 5] for studies of ℓ_p-ball packings in $\mathbb{R}^3$. Although ℓ_p-balls do not possess rotational symmetry, in this paper we only consider translations of identical ℓ_p-balls, not allowing rotations. The best known lower bounds on high dimensional superball packing densities do not use rotations [6] (see Section 3.2).

Let $\Delta_p(n)$ denote the maximum translative packing density of copies of $\boldsymbol{B}_p^n$ in $\mathbb{R}^n$. Here *density* is the fraction of space occupied by these balls. For fixed $p \in [1, \infty)$, let

$$\gamma_p := \limsup_{n\to\infty} \frac{1}{n} \log_2 \Delta_p(n)$$

be the exponential rate of optimal packing densities in high dimensions. The precise value of γ_p is unknown for any $p \in [1, \infty)$, and the current best upper and lower bounds are quite far apart. For Euclidean balls, $p = 2$, the best high dimensional upper bound (apart from constant factors) is due to Kabatiansky and Levenshtein [12]:

$$\Delta_2(n) \le 2^{(\kappa_{\mathrm{KL}}+o(1))n}, \quad \text{where } \kappa_{\mathrm{KL}} := -0.5990\dots.$$

See Cohn and Zhao [4] and Sardari and Zargar [20] for constant factor improvements over Kabatiansky and Levenshtein [12]. For lower bounds, we have $\Delta_p(n) \ge 2^{-n}$ for all n and $p \ge 1$ since every maximal packing has density at least 2^{-n}. For $p = 2$, there have only been subexponential improvements, with the current best lower bound due to Venkatesh [22]. In summary, the best bounds on γ_2 are $-1 \le \gamma_2 \le \kappa_{\mathrm{KL}} = -0.5990\dots$.

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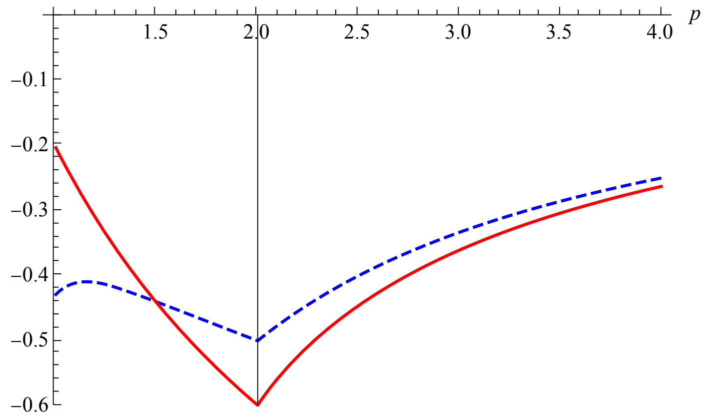

FIGURE 1. Upper bounds on the exponential rate γ_p of translative packing densities of identical ℓ_p-balls in high dimensions. For $p > 2$, the dashed blue curve is the previous upper bound $-1/p$ and the solid red curve is our new upper bound. For $1 \le p < 2$, discussed in Section 3, the dashed blue curve is (3.1) due to Rankin [15] and the solid red curve is (3.3) derived from the Kabatiansky–Levenshtein [12] sphere packing bound.

For $p > 2$, the current best upper bound on the exponential rate of superball packing densities was first proved by van der Corput and Schaake [21] via Blichfeldt's method [1] (e.g., see [24, Section 6.3]), giving

$$\gamma_p \le -1/p \quad \text{for } p > 2.$$

There have been subsequent subexponential upper bound improvements on $\Delta_p(n)$ for $p > 2$, e.g., Rankin [16, 17]. We defer to Section 3 for a discussion of known bounds on γ_p in other regimes.

In this paper, we prove a new upper bound on γ_p for all $p > 2$, giving the first exponential improvement since 1936 on the upper bound of superball packing densities in high dimensions.

Theorem 1.1. *For all $p \ge 2$,*

$$\gamma_p \le \inf_{0<\theta<\pi/2} \left(\frac{1+\sin\theta}{2\sin\theta} \log_2 \frac{1+\sin\theta}{2\sin\theta} - \frac{1-\sin\theta}{2\sin\theta} \log_2 \frac{1-\sin\theta}{2\sin\theta} + \frac{2}{p} \log_2 \sin\frac{\theta}{2} \right).$$

In particular, $\gamma_p < -1/p$ for all $p \ge 2$.

See Figure 1 for a plot of the bounds.

Remark. Theorem 1.1 with $p = 2$ recovers $\gamma_p \le \kappa_{\mathrm{KL}}$. Our upper bound on γ_p is continuous with p, whereas the previous best bounds were not continuous[1] at $p = 2$. The fact that our bound at $p = 2$ recovers the Kabatiansky–Levenshtein bound is not a coincidence, as our proof relies on the Kabatiansky–Levenshtein bound for spherical codes.

2. PROOF OF MAIN THEOREM

2.1. **Kabatiansky–Levenshtein spherical code bound.** Denote the ℓ_p-sphere in $\mathbb{R}^n$ of radius R by $\boldsymbol{S}_p^{n-1}(R) := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_p = R\}$ and the unit ℓ_p-sphere by $\boldsymbol{S}_p^{n-1} := \boldsymbol{S}_p^{n-1}(1)$. Let $A_p(n,d)$ to be the maximum number of points on $\boldsymbol{S}_p^{n-1}$ with pairwise ℓ_p-distance at least $2d$, i.e, an ℓ_p-spherical code. Note that $A_p(n,d) = 1$ unless $d \in [0,1]$. Note that $A_2(n, \sin(\theta/2))$ is the maximum size of a

[1] It is unknown whether $p \mapsto \gamma_p$ is continuous. Lemma 3.1 implies that γ_p is continuous at all but at most countably many points.

spherical code in $\mathbb{R}^n$ with pairwise angle at least θ. Kabatiansky and Levenshtein [12] proved that for all[2] $0 < \theta < \pi/2$,

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 A_2(n, \sin(\theta/2)) \le a(\theta) \tag{2.1}$$

where

$$a(\theta) := \frac{1+\sin\theta}{2\sin\theta}\log_2\frac{1+\sin\theta}{2\sin\theta} - \frac{1-\sin\theta}{2\sin\theta}\log_2\frac{1-\sin\theta}{2\sin\theta}.$$

A projection argument (see [4, Section 2]) shows that

$$\Delta_2(n) \le \sin^n(\theta/2) A_2(n+1, \sin(\theta/2)),$$

so (2.1) gives

$$\lim_{n\to\infty} \frac{1}{n}\log_2 \Delta_2(n) \le a(\theta) + \log_2 \sin\frac{\theta}{2}.$$

The bound $\gamma_2 \le \kappa_{\mathrm{KL}} = -0.5990\dots$ is obtained by choosing $\theta = \theta_{\mathrm{KL}} = 1.0995\dots$ to minimize the upper bound above.

2.2. **ℓ_p-twist.** Fix $p \ge 2$. Define

$$x^* := \operatorname{sgn}(x)|x|^{p/2}, \qquad x \in \mathbb{R}.$$

For $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, write $\boldsymbol{x}^* := (x_1^*, \dots, x_n^*)$, and for $X \subseteq \mathbb{R}^n$, write $X^* := \{\boldsymbol{x}^* : \boldsymbol{x} \in X\}$.

Observe that for all $x, y \in \mathbb{R}$,

$$|x^* - y^*| \ge 2^{1-p/2}|x-y|^{p/2}. \tag{2.2}$$

Indeed, without loss of generality it suffices to consider two cases: $x \ge 0 \ge y$ and $x \ge y \ge 0$. The former case is an immediate consequence of Hölder's inequality (or the convexity of $x \mapsto x^{p/2}$). In the latter case, we have

$$x^{p/2} - y^{p/2} \ge (x-y)^{p/2} \ge 2^{1-p/2}(x-y)^{p/2}.$$

Here we use $(w+z)^{p/2} \ge w^{p/2} + z^{p/2}$ for $w, z > 0$, which can be proved by first normalizing to $w + z = 1$ and noting that $w^{p/2} + z^{p/2} \le w + z = 1$.

Lemma 2.1. *For all $p \ge 2$ and $d \in (0, 1]$, we have $A_p(n, d) \le A_2(n, d^{p/2})$.*

Proof. Let $X \subseteq \boldsymbol{S}_p^{n-1}$ with $|X| = A_p(n,d)$ and $\|\boldsymbol{x} - \boldsymbol{y}\|_p \ge 2d$ for all distinct $\boldsymbol{x}, \boldsymbol{y} \in X$. We have $\|\boldsymbol{x}^*\|_2 = \|\boldsymbol{x}\|_p = 1$ for all $\boldsymbol{x} \in X$, so $X^* \subseteq \boldsymbol{S}_2^{n-1}$. For distinct $\boldsymbol{x}, \boldsymbol{y} \in X$, we have

$$\|\boldsymbol{x}^* - \boldsymbol{y}^*\|_2^2 = \sum_{i=1}^n |x_i^* - y_i^*|^2 \ge 2^{2-p}\sum_{i=1}^n |x_i - y_i|^p \ge 2^2 d^p,$$

by (2.2). Thus X^* is a subset of $\boldsymbol{S}_2^{n-1}$ whose points have pairwise ℓ_2-distance at least $2d^{p/2}$. Hence $|X| = |X^*| \le A_2(n, d^{p/2})$. □

Remark. The same argument shows that $A_p(n,d) \le A_q(n, d^{p/q})$ for all $1 \le q \le p$ and $d \in (0,1]$.

Lemma 2.2. *For every $p \ge 1$, $d \in (0,1]$, and $n \in \mathbb{N}$, we have $\Delta_p(n) \le d^n A_p(n+1, d)$.*

Proof. Let $\rho < \Delta_p(n)$ be arbitrary. Consider a translative packing $\{\boldsymbol{x} + \boldsymbol{B}_p^n(d) : \boldsymbol{x} \in X\}$ in $\mathbb{R}^n$ with density greater than ρ, where $X \subseteq \mathbb{R}^n$ is the set of centers of the ℓ_p-balls. By an averaging argument[3], there exists some translate of a unit ℓ_p-ball that contains at least $d^{-n}\rho$ points of X. Translating X if necessary, we may assume that $|X \cap \boldsymbol{B}_p^n| \ge d^{-n}\rho$. Add an $(n+1)$-st coordinate to each point in $X \cap \boldsymbol{B}_p^n$ to obtain a set X' of points on the unit ℓ_p-sphere in $\mathbb{R}^{n+1}$. In other words, X'

[2] A simple geometric argument (see [13, (17)]) shows that the upper bound (2.1) can be improved for $\theta < \theta_{\mathrm{KL}} := 1.0995\dots$ to $a(\theta_{\mathrm{KL}}) + \log_2\sin(\theta_{\mathrm{KL}}/2) - \log_2\sin(\theta/2)$, but this improvement does not benefit our bounds.

[3] A uniform random translation of a unit ℓ_p-ball inside $[-R, R]^n$ contains more than $d^{-n}\rho + o_{R\to\infty}(1)$ points of X.

is obtained by projecting the points of X contained in the unit ball "upward" to the hemisphere one dimension higher. Since the points in X are pairwise at least $2d$ apart in ℓ_p-distance, the same holds for X'. So X' is an ℓ_p-spherical code whose points are pairwise separated by ℓ_p-distance at least $2d$, and hence $d^{-n}\rho \le |X \cap \boldsymbol{B}_p^n| = |X'| \le A_p(n+1,d)$. Since ρ can be arbitrarily close to $\Delta_p(n)$, we obtain the claimed inequality. □

Remark. As in [4], the above argument can be modified so that we do not need to add a new dimension when $d \in [1/2, 1]$, resulting in a slightly better bound $\Delta_p(n) \le d^{-n} A_p(n,d)$. We omit the details of this modification since this improvement does not affect the exponential asymptotics.

Proof of Theorem 1.1. Applying Lemmas 2.1 and 2.2, we have, for every $0 < \theta < \pi/2$,

$$\Delta_p(n) \le \sin(\theta/2)^{2n/p} A_p(n+1, \sin(\theta/2)^{2/p}) \le \sin(\theta/2)^{2n/p} A_2(n+1, \sin(\theta/2)).$$

Applying (2.1), we obtain

$$\gamma_p = \limsup_{n\to\infty} \frac{1}{n} \log_2 \Delta_p(n) \le a(\theta) + \frac{2}{p}\log_2 \sin(\theta/2).$$

The main result follows by taking the infimum of the bound over $\theta \in (0, \pi/2)$.

Setting $\theta = \pi/2 - \eta$, we have, with $p \ge 2$ fixed and $\eta \to 0^+$,

$$\gamma_p \le -\frac{1}{p} - \frac{\eta}{p\ln 2} + o(\eta).$$

So choosing $\eta > 0$ sufficiently small gives $\gamma_p < -1/p$ for all $p \ge 2$. □

3. Remarks

3.1. **Asymptotics.** Setting $\theta = \pi/2 - (p\ln p)^{-1}$, we obtain

$$\gamma_p \le -\frac{1}{p} - \frac{1}{\ln 4}\cdot\frac{1}{p^2 \ln p} + O\left(\frac{1}{p^2\ln^2 p}\right), \quad \text{as } p\to\infty.$$

Taking $\theta = \theta_{\mathrm{KL}}$ gives

$$\gamma_p \le \kappa_{\mathrm{KL}} + \frac{2-p}{p}\log_2 \sin\frac{\theta_{KL}}{2}, \quad \text{for all } p \ge 2.$$

Thus, as $\epsilon \to 0^+$,

$$\gamma_{2+\epsilon} \le \gamma_{\mathrm{KL}} - \left(\frac{1}{2}\log_2 \sin\frac{\theta_{\mathrm{KL}}}{2}\right)\epsilon + O(\epsilon^2) = (-0.5990\ldots) + (0.4650\ldots)\epsilon + O(\epsilon^2).$$

3.2. **Review of other bounds on γ_p.** Here we survey other existing bounds on γ_p.

For $p = 2$, the best known bounds are $-1 \le \gamma_2 \le \kappa_{\mathrm{KL}} = -0.5990\ldots$ as discussed earlier.

For $p > 2$, the best known upper bounds are the ones given in this paper. For lower bounds, extending on methods developed by Rush [18] and Rush–Sloane [19], Elkies, Odlyzko, and Rush [6] proved $\gamma_p > -1$ for all $p > 2$, thereby exponentially beating the Minkowski–Hlawka lower bound. See [6] for the precise bound. Their bounds have the following asymptotics:

$$\gamma_p \ge -(1+o(1))\frac{\ln\ln p}{p\ln 2}, \quad \text{as } p \to \infty,$$

and

$$\gamma_{2+\epsilon} \ge -1 + \left(\frac{\sqrt{\pi}\zeta(3)}{2\ln 2} + o(1)\right)\frac{\epsilon}{\ln^{3/2}(1/\epsilon)}, \quad \text{as } \epsilon \to 0^+.$$

Here ζ denotes the Riemann zeta function. See [14] for some later improvements using algebraic-geometric codes for some specific integers p.

For $1 \le p < 2$, no improvement over the Minkowski–Hlawka lower bound $\gamma_p \ge -1$ is known. The best upper bound on γ_p is due to Rankin [15], based on Blichfeldt's method [1]:

$$\gamma_p \le \inf_{\frac{1}{2} \le \frac{1}{q} \le \frac{1}{3}\left(1+\frac{1}{p}\right)} \left(b(p) - b(q) - 1 + 1/p + (1/q - 1/p) \log_2 \left(\frac{2 - 1/q}{1 - 1/q} \right) \right) \tag{3.1}$$

where

$$b(p) := \lim_{n\to\infty} \frac{1}{n} \log_2 \operatorname{vol} \boldsymbol{B}_p^n(n^{1/p}) = 1 + \log_2 \Gamma\left(1 + \frac{1}{p}\right) + \frac{1}{p}\log_2(pe). \tag{3.2}$$

Recall that $\operatorname{vol} \boldsymbol{B}_p^n = 2^n \Gamma(1+1/p)^n / \Gamma(1+n/p)$.

For packings of congruent cross-polytopes (i.e., unit ℓ_1-balls) allowing rotations, Fejes Tóth, Fodor, and Vígh [7] proved an exponentially decaying upper bound in high dimensions. For translative packing of unit ℓ_1-balls, the upper bound (3.1) remains best known in high dimensions.

We note that the above bound (3.1) can be improved on the region $p \in [1.494\ldots, 2)$ using the Kabatiansky–Levenshtein bound via the following folklore observation.

Lemma 3.1. *For $1 \le p \le q \le \infty$, $\gamma_p - b(p) \le \gamma_q - b(q)$.*

Proof. By monotonicity of norms, we have $n^{-1/p}\|\boldsymbol{x}\|_p \le n^{-1/q}\|\boldsymbol{x}\|_q$, so $\boldsymbol{B}_p^n(n^{1/p}) \supseteq \boldsymbol{B}_q^n(n^{1/q})$. Any packing of $\boldsymbol{B}_p^n(n^{1/p})$ can be shrunk into a packing of $\boldsymbol{B}_q^n(n^{1/q})$. Hence

$$\frac{\Delta_p(n)}{\operatorname{vol} \boldsymbol{B}_p^n(n^{1/p})} \le \frac{\Delta_q(n)}{\operatorname{vol} \boldsymbol{B}_p^n(n^{1/q})}.$$

Taking log, dividing by n, and letting $n \to \infty$ yields the lemma. ☐

Using $\gamma_2 \le \kappa_{\mathrm{KL}}$, we find that

$$\gamma_p \le \kappa_{\mathrm{KL}} - b(2) + b(p) = (-0.5990\ldots) - b(2) + b(p) \quad \text{for } 1 \le p < 2. \tag{3.3}$$

Thus

$$\gamma_p \le \min\{\text{RHS of (3.1)}, \text{RHS of (3.3)}\}$$

See Figure 1 for an illustration of the above bounds.

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