A SHORT PROOF OF THE CANONICAL POLYNOMIAL VAN DER WAERDEN THEOREM

JACOB FOX, YUVAL WIGDERSON, AND YUFEI ZHAO

Abstract. We present a short new proof of the canonical polynomial van der Waerden theorem, recently established by Girão.

Girão [3] recently proved the following canonical version of the polynomial van der Waerden theorem. Here a set is *rainbow* if all elements have distinct colors. We write $[N] := \{1, \dots, N\}$.

Theorem 1 ([3]). *Let $p_1, \dots, p_k$ be distinct polynomials with integer coefficients and $p_i(0) = 0$ for each i. For all sufficiently large N, every coloring of $[N]$ contains a sequence $x+p_1(y), \dots, x+p_k(y)$ (for some $x, y \in \mathbb{N}$) that is monochromatic or rainbow.*

Girão's proof uses a color-focusing argument. Here we give a new short proof of Theorem 1, deducing it from the polynomial Szemerédi's theorem of Bergelson and Leibman [1].

Theorem 2 ([1]). *Let $p_1, \dots, p_k$ be distinct polynomials with integer coefficients and $p_i(0) = 0$ for each i. Let $\varepsilon > 0$. For all N sufficiently large, every $A \subset [N]$ with $|A| \geq \varepsilon N$ contains $x + p_1(y), \dots, x + p_k(y)$ for some $x, y \in \mathbb{N}$.*

Our proof of Theorem 1 follows the strategy of Erdős and Graham [2], who deduced a canonical van der Waerden theorem (i.e., for arithmetic progressions) using Szemerédi's theorem [6].

We quote the following result, proved by Linnik [5] in his elementary solution of Waring's problem (see [4, Theorem 19.7.2]). Note the left-hand side below counts the number of solutions $f(x_1) + \cdots + f(x_{s/2}) = f(x_{s/2+1}) + \cdots + f(x_s)$ with $x_1, \dots, x_s \in [n]$.

Theorem 3 ([5]). *Fix a polynomial f of degree $d \geq 2$ with integer coefficients. Let $s = 8^{d-1}$. Then*

$$\int_0^1 \left| \sum_{x=1}^{n} e^{2\pi i \theta f(x)} \right|^s \mathrm{d}\theta = O(n^{s-d}).$$

Lemma 4. *Fix a polynomial f of degree $d \geq 2$ with integer coefficients. For every $A \subset \mathbb{Z}$ and $n \in \mathbb{N}$, the number of pairs $(x, y) \in A \times [n]$ with $x + f(y) \in A$ is $O(|A|^{1+\frac{1}{s}} n^{1-\frac{d}{s}})$, where $s = 8^{d-1}$.*

Proof. We write

$$\widehat{1}_A(\theta) = \sum_{x \in A} e^{2\pi i \theta x} \qquad \text{and} \qquad F(\theta) = \sum_{y=1}^{n} e^{2\pi i \theta f(y)}.$$

Fox is supported by a Packard Fellowship and by NSF award DMS-1855635. Wigderson is supported by NSF GRFP grant DGE-1656518. Zhao is supported by NSF award DMS-1764176, the MIT Solomon Buchsbaum Fund, and a Sloan Research Fellowship.

Then the number of solutions to $z = x + f(y)$ with $x, z \in A$ and $y \in [n]$ is

$$\begin{aligned}
\int_0^1 |\widehat{1}_A(\theta)|^2 F(\theta)\,\mathrm{d}\theta &\le \left(\int_0^1 |\widehat{1}_A(\theta)|^{\frac{2s}{s-1}}\,\mathrm{d}\theta\right)^{1-\frac{1}{s}} \left(\int_0^1 |F(\theta)|^s\,\mathrm{d}\theta\right)^{\frac{1}{s}} && \text{[Hölder]}\\
&\le \left(|A|^{\frac{2}{s-1}} \int_0^1 |\widehat{1}_A(\theta)|^2\,\mathrm{d}\theta\right)^{1-\frac{1}{s}} \cdot O(n^{1-\frac{d}{s}}) && [|\widehat{1}_A(\theta)| \le |A| \text{ and Theorem 3}]\\
&= \left(|A|^{\frac{2}{s-1}}\,|A|\right)^{1-\frac{1}{s}} \cdot O(n^{1-\frac{d}{s}}) && \text{[Parseval]}\\
&= O(|A|^{1+\frac{1}{s}}\, n^{1-\frac{d}{s}}). && \square
\end{aligned}$$

Lemma 5. *Fix a polynomial f of degree $d \ge 1$ with integer coefficients. Let $A \subset \mathbb{Z}$. Suppose that $|A \cap [x, x+L)| \le \varepsilon L$ for every $L \ge n^d$ and x. Then the number of pairs $(x, y) \in A \times [n]$ with $x + f(y) \in A$ is $O(\varepsilon^{1/s}\,|A|\,n)$, where $s = 8^{d-1}$.*

Proof. If $d = 1$, then for every $x \in A$, the number of $y \in [n]$ so that $x + f(y) \in A$ is $O(\varepsilon n)$ by the local density condition on A. Summing over all $x \in A$ yields the desired bound $O(\varepsilon|A|n)$ on the number of pairs. From now on assume $d \ge 2$.

Let $m = O(n^d)$ so that $|f(y)| \le m$ for all $y \in [n]$. Let $A_i = A \cap [im, (i+2)m)$. Then $|A_i| = O(\varepsilon m)$. Every pair $x, x + f(y) \in A$ with $y \in [n]$ is contained in some A_i, and, by Lemma 4, the number of pairs contained in each A_i is $O(|A_i|^{1+\frac{1}{s}} n^{1-\frac{d}{s}}) = O((\varepsilon m)^{\frac{1}{s}} |A_i| n^{1-\frac{d}{s}}) = O(\varepsilon^{1/s}|A_i|n)$. Summing over all integers i yields the lemma (each element of A lies in precisely two different A_i's). $\square$

Proof of Theorem 1. Choose a sufficiently small $\varepsilon > 0$ (depending on $p_1, \dots, p_k$). Consider a coloring of $[N]$ without monochromatic progressions $x + p_1(y), \dots, x + p_k(y)$. By Theorem 2, every color class has density at most ε on every sufficiently long interval.

Let $D = \max_{i \ne j} \deg(p_i - p_j)$. Let n be an integer on the order of $N^{1/D}$ so that $x + p_1(y), \dots, x + p_k(y) \in [N]$ only if $y \in [n]$. For each color class A, applying Lemma 5 to $f = p_i - p_j$ and summing over all $i \ne j$, we see that the number of pairs $(x, y) \in \mathbb{Z} \times [n]$ where at least two of $x + p_1(y), \dots, x + p_k(y)$ lie in A is $O(\varepsilon^{1/8^{D-1}}|A|n)$. Summing over all color classes A, we see that the number of non-rainbow progressions $x + p_1(y), \dots, x + p_k(y) \in [N]$ is $O(\varepsilon^{1/8^{D-1}} Nn)$. Since the total number of sequences $x + p_1(y), \dots, x + p_k(y) \in [N]$ is on the order of Nn, some such sequence must be rainbow, as long as $\varepsilon > 0$ is small enough and N is large enough. $\square$

References

[1] V. Bergelson and A. Leibman, *Polynomial extensions of van der Waerden's and Szemerédi's theorems*, J. Amer. Math. Soc. **9** (1996), 725–753.

[2] P. Erdős and R. L. Graham, *Old and new problems and results in combinatorial number theory*, vol. 28, Université de Genève, L'Enseignement Mathématique, Geneva, 1980.

[3] A. Girão, *A canonical polynomial van der Waerden's theorem*, arXiv:2004.07766.

[4] L. K. Hua, *Introduction to number theory*, Springer-Verlag, Berlin-New York, 1982.

[5] U. V. Linnik, *An elementary solution of the problem of Waring by Schnirelman's method*, Rec. Math. [Mat. Sbornik] N.S. **12(54)** (1943), 225–230.

[6] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245.

Fox, Wigderson: Department of Mathematics, Stanford University, Stanford, CA, USA
E-mail address: `{jacobfox,yuvalwig}@stanford.edu`

Zhao: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA
E-mail address: `yufeiz@mit.edu`