Distributed Tracking in Heterogeneous Networks With Asynchronous Sampled-Data Control

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***Abstract*—This article investigates distributed coordinated tracking problems of networked heterogeneous systems. Based on asynchronous sampling information, distributed sampled-data protocols are employed to realize leader-following synchronization and containment tracking in networked heterogeneous systems. In asynchronous sampled-data protocols, each node has different sampling instants with other nodes and only samples itself information at its own sampling instants. By utilizing the input-delay approach and Lyapunove–Krasovskii functional approach, some sufficient conditions for guaranteeing the coordinated tracking are presented. First, quasi-synchronization criteria are obtained for networked heterogeneous oscillator systems with a dynamic leader over the directed graph. Second, in the presence of multiple heterogeneous leaders for networked heterogeneous systems, sufficient conditions of quasi-containment tracking are derived. In a word, all followers can converge into a bounded level of convex hull spanned by the leader(s). The upper bounds of tracking errors are estimated for both quasi-synchronization and quasi-containment tracking. Finally, two numerical examples are given to verify the theoretical results.**

***Index Terms*—Asynchronous sampling, containment control, harmonic oscillators, heterogeneous networks, leader-following control.**

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I. Introduction

RECENTLY, the problem of distributed cooperative control for networked systems has become a hot topic due to its broad potential applications in spacecraft formation control, flocking, and sensor networks. Synchronization and consensus are the most ubiquitous phenomena in distributed cooperative control and have gained a great attention among researchers from various disciplines. For example, synchronization in complex networks and coupled systems [1]–[3], consensus in integrator multiagent systems [4]–[6], linear multiagent systems [7], [8], and nonlinear multiagent systems [9] have been widely investigated in recent years.

In reality, the presence of leader(s) can generalize the applications as a distributed system can be guided by one leader or multiple leaders. Under the situation that there exists a single leader for distributed systems, a multiagent consensus problem with an active leader is studied in [5] by a neighbor-based local controller. Leader-following consensus of second-order multiagent systems with both fixed and switching topologies was addressed in [6]. By applying a variable structure approach in [10], a distributed coordinated tracking problem was studied. In the presence of multiple leaders, the corresponding coordination problem is called the containment control problem where the followers will asymptotically converge into the convex hull spanned by the leaders. In the pioneering work [11], a hybrid stop-go control policy was presented to drive all the followers to the convex polytope spanned by the leaders. In [12], the distributed containment control problem was studied for a group of mobile autonomous agents with multiple stationary or dynamic leaders under both fixed and switching topologies. In [13], the containment control problem of first-order multiagent systems was investigated over both fixed and switching topologies. To solve the containment control problem for second-order multiagent systems, two distributed containment control algorithms only based on the position measurements were proposed in [14]. For general linear multiagent systems, the distributed containment control problems were studied in [15] based on the relative outputs of neighboring agents and in [16] based on distributed observer-type protocols.

Because of individual difference, external disturbance, and parameter uncertainty, the nodes in the practical networks are often governed by different dynamics. These kinds of networks are called heterogeneous networks. The heterogeneity of such networks can lead to the failure of synchronization [17]. Consequently, it is worth exploring distributed coordinated performance for heterogeneous networks. To synchronize heterogeneous networks, weighted average state of all nodes is adopted as synchronization target [17], [18]. By introducing the known reference signals directly, impulsive consensus in heterogeneous complex networks was studied in [19]. By hypothesizing the presence of a leader in [20], the robust cooperative tracking problem was studied for heterogeneous second-order nonlinear systems based on a distributed discontinuous control algorithm. As noted from above, heterogeneous networks need one or more targets to lead all the nodes. Thus, leader-following networks are the natural ways to study the coordinated control problems for the heterogenous networks [21], [22], [23]. Recently, the containment control problem of heterogeneous linear multiagent systems has been studied based on output regulation framework in [24]. In [25], the containment control has been addressed for heterogeneous second-order multiagent systems, where the position topology and velocity topology are different. Recent work [26] has investigated the output containment problem of heterogeneous linear multiagent systems by designing a protocol based on internal model principle.

It should be noted that the majority of the aforementioned results concerning leader-following control are focused on the continuous-time control protocols. Due to the digital technology of controller implementation and low communication cost, the sampled-data protocols usually are used for coordination for networked dynamical systems [27], [28]. Since the networks only sample information at some discrete sampling instants rather than continuous intervals by adopting sampled-data control, sampling rate and sampling period are critical for achieving coordination performance. Therefore, various sampling schemes are designed to coordinate the networked systems, such as deterministic sampling controls [29], [30] and stochastic sampling controls [31]. It is worth pointing out that most sampled-data control protocols for networked systems are synchronous, that is, all the nodes are sampled at the same sampling instants. Synchronous sampling protocol is not reasonable for heterogeneous networks since each node has different dynamics with each other. It is reasonable to expect that each node should only be sampled at its own sampling instants but need not be sampled at sampling instants of other nodes in heterogeneous networks. Results in [23] showed that quasi (bounded) leader-following consensus can be achieved for heterogeneous multiagent systems by an asynchronous sampled-data protocol. Compared with the extensive study of the synchronous sampled-data coordinated control, the asynchronous sampled-data coordinated control for networked systems especially for heterogeneous dynamical networks so far received very little attention.

In this article, the distributed tracking problems for heterogeneous dynamical networks are addressed by asynchronous sampled-data protocols. The main contributions can be summarized as follows. First, both leader-following synchronization with a single leader and containment tracking with multiple leaders are studied for heterogeneous dynamical networks, which generalize the homogeneous dynamical networks. Second, asynchronous sampled-data protocols are designed to solve the leader-following synchronization and containment tracking in heterogeneous networks. Since networked systems are heterogeneous, each node should update information at its own rate, that is, the sampling instants of all the nodes in heterogeneous networks should be asynchronous. Based on asynchronous sampled-data protocols, sampling instants for different oscillators are independent of each other. In addition, in the partial literatures on asynchronous sampling, different nodes have independent sampling sequences, and each node samples itself information at its own sampling instants but also simultaneously samples information of each of its neighbors synchronously. Different from this case, this article considers complete asynchronous sampling protocols for the heterogeneous dynamical networks. In other words, each oscillator only samples itself information at its own sampling instants and utilizes each of its neighbors the latest completed sampling information, which is sampled at its neighbor's own sampling instants. Third, based on asynchronous sampled-data protocols, theoretical analysis and sufficient criteria of quasi-synchronization and quasi-containment tracking for heterogeneous dynamical networks are derived. It should be pointed out that the asynchronous protocols lead to the new heterogeneities. The heterogeneities arising from both nonidentical nodes' dynamics and asynchronous sampling protocols are analyzed synthetically. The upper bounds of quasi-synchronization errors and quasi-containment tracking errors are estimated analytically. Furthermore, the theoretical results can be applied to the electrical networks.

Notations: Let $\mathbb{R}$ be the set of the real numbers, 1_N and 0_N be the $N \times 1$ column vectors of all ones and all zeros, respectively, O and I be the zero matrix and identity matrix with appropriate dimension, respectively. $\text{diag}\{A_1, A_2, \ldots, A_n\}$ stands for a block-diagonal matrix with square matrix A_i being its ith diagonal block matrix. A^T means transpose for a real matrix A. The symbol $*$ in a symmetric matrix represents the symmetric elements. A symmetric matrix $P > 0$ means that P is positive definite. Let $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ be the minimal and maximal eigenvalue of a symmetric square matrix P, respectively. For a point $x \in \mathbb{R}^n$ and a set $\mathcal{C} \subseteq \mathbb{R}^n$, denote the distance between x and $\mathcal{C}$ by $\text{d}(x, \mathcal{C}) = \inf_{y \in \mathcal{C}} \|x - y\|$, where $\| \cdot \|$ is the Euclidian norm.

II. Preliminaries and Problem Formulations

A. Communication Graphs

Denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ the directed graph, where $\mathcal{V} = \{\nu_1, \nu_2, \ldots, \nu_N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the sets of nodes and edges, respectively. A directed edge from ν_j to ν_i of $\mathcal{G}$ is represented by $e_{ji} = (\nu_j, \nu_i)$. The set of neighbors of node ν_i is $\mathcal{N}_i = \{\nu_j | e_{ji} \in \mathcal{E}\}$. $\mathcal{A} = [a_{ij}]_{N \times N} \in \mathbb{R}^{N \times N}$ is a weighted adjacency matrix with (i, j)th entry $a_{ij} \geq 0$, and it is assumed that $a_{ii} = 0$ and $a_{ij} > 0$ if $e_{ji} \in \mathcal{E}$. A directed path from ν_j to ν_i is a sequence of distinct nodes $\{\nu_{\ell_1}, \nu_{\ell_2}, \ldots, \nu_{\ell_r}\}$ with $\nu_{\ell_1} = \nu_j$ and $\nu_{\ell_r} = \nu_i$ such that $(\nu_{\ell_i}, \nu_{\ell_{i+1}}) \in \mathcal{E}, i = 1, 2, \ldots, \ell$. A digraph $\mathcal{G}$ has a spanning tree if there exists at least one node that has a directed path to all the other nodes in $\mathcal{G}$. The Laplacian matrix $L = [l_{ij}]_{N\times N}$ of $\mathcal{G}$ is defined as $l_{ii} = \sum_{j\neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$.

B. Model Formulations

Consider a heterogeneous networked system coupled by N heterogeneous harmonic oscillators. The dynamics of the ith oscillator is given by

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = -\omega_i^2 x_i(t) - u_i, \quad i = 1, 2, \dots, N \end{cases} \tag{1}$$

where constant ω_i is the frequency of the ith node, $x_i(t), v_i(t) \in \mathbb{R}$ are the position and velocity states of the ith node, respectively, and $u_i \in \mathbb{R}$ is the control protocol to be designed for the ith node, $i = 1, 2, \dots, N$. It is assumed that $\omega_1, \omega_2, \dots, \omega_N$ are N nonidentical constants. Therefore, network (1) is heterogeneous.

Remark 1: Different from homogeneous networks studied in [3] and [32]–[34], the heterogeneous networks (1) cannot achieve complete synchronization if the network only relies on a static control. Therefore, by applying complicated dynamic controllers, complete synchronization was studied for heterogeneous networks in [19]. Otherwise, quasi-synchronization may be reached for heterogeneous networks if only by the internal coupling without any external controllers [17].

Consider the following leader as a synchronization signal:

$$\begin{cases} \dot{x}_0(t) = v_0(t) \\ \dot{v}_0(t) = -\omega_0^2 x_0(t) \end{cases} \tag{2}$$

where ω_0 is the frequency of the leader, and $x_0(t), v_0(t) \in \mathbb{R}$ are the position and velocity states of the leader, respectively. Throughout this article, the leader is assumed to start from a bounded region, that is, there exists a compact $\mathcal{C}_0$ such that $[x_0(0), v_0(0)]^T \in \mathcal{C}_0 \subset \mathbb{R}^2$.

In order to drive the heterogeneous networked system (1) to synchronize with leader (2), controllers $u_i(t)$ should be designed. Because of the robustness and low communication cost, sampled-data control is widely used for the communication in complex networks or multiagent systems [28], [29], [31], [32]. Most studies on sampled-data communication are based on an assumption that all nodes are sampled synchronously. Since the network is heterogeneous, all nonidentical nodes may have different sampling schemes. Therefore, it is reasonable to adopt an asynchronous sampled-data control input. Specifically, suppose that agent i is sampled at the sampling instant t_k^i, $k = 0, 1, 2, \dots$, where $0 = t_0^i < t_1^i < t_2^i < \cdots < t_k^i < \cdots$ with $\lim_{k\to\infty} t_k^i = \infty$ and $0 < t_{k+1}^i - t_k^i \leq h_i$, $i = 0, 1, 2, \dots, N$. Here, $h_0, h_1, \dots, h_N$ are $N+1$ positive constants and represent the upper bounds of sampling intervals. Let $h = \max\{h_0, h_1, h_2, \dots, h_N\}$. The control input $u_i(t)$ of node i will update the information at sampling instants of itself and its neighbors, and will keep a constant by the zero-order hold before any new sampling information arrives during the sampling interval $[t_k^i, t_{k+1}^i), k = 0, 1, 2, \dots$. In what follows, all the oscillators are assumed to have different sampling rates, namely, the following distributed asynchronous sampled-data protocol is adopted:

$$\begin{aligned} u_i(t) = \alpha \sum_{j\in\mathcal{N}_i} a_{ij} \left[v_i\left(t_k^i\right) - v_j\left(t_{k_j(t)}^j\right)\right] \\ + \alpha b_i \left[v_i\left(t_k^i\right) - v_0\left(t_{k_0(t)}^0\right)\right] \end{aligned} \tag{3}$$

for $t \in [t_k^i, t_{k+1}^i), k \in \mathbb{N}, i = 1, 2, \dots, N$, where α is coupling strength, b_i are pinning gains with the property that $b_i > 0$ if the ith node can receive information from the leader (otherwise $b_i = 0$), $k_j(t) = \max\{k | t_k^j \leq t\}$ represents that node j has already completed the $k_j(t)$th sampling at any time t.

Remark 2: For $t \in [t_k^i, t_{k+1}^i)$, node i has completed kth sampling $v_i(t_k^i)$, whereas t_k^i may be not a sampling instant for its neighbor node $j \in \mathcal{N}_i$. In other words, sampling sequence $\{t_k^i\}_{k=1}^\infty$ of node i is independent of sampling sequence $\{t_k^j\}_{k=1}^\infty$ of node j. Therefore, the control input $u_i(t)$ of node i only can use the latest completed sampling information from itself and its neighbors at time t. In addition, the latest completed sampling information of itself and its neighbors may be sampled at different moments.

Remark 3: If $t_k^i = t_k^j = t_k$ for all $i, j = 1, 2, \dots, N$, then all the oscillators sample the information at the same rate, that is, the sampled-data protocol is synchronous [28], [29], [31]. In [13], [28], and [35], each node i has its own sampling instants $\{t_k^i\}_{k=1}^\infty$, however, the sampling information of its neighbor node $j \in \mathcal{N}_i$ need be available at each t_k^i. Therefore, they are essentially the synchronous sampling cases. Besides, the literature [35] considers asynchronous sampling delays. In view of the heterogeneity of oscillators, this article employs complete asynchronous sampling protocols, where each agent i only needs to be sampled at its own sampling instants and does not need to be sampled at its neighbors' sampling instants. In particular, agent i is sampled at t_k^i and its neighbor agent j is sampled at $t_{k_j(t)}^j$ in protocol (3). The design of the distributed asynchronous protocol (3) can be introduced to the stochastic sampling control, event-triggered sampling control, and impulsive control. In this way, one can extend the synchronous sampling sequences to the asynchronous cases, which are more consistent with the reality.

Definition 1 (see [22]): The heterogeneous networked system (1) and leader (2) are said to reach quasi-synchronization with a level $\varepsilon > 0$ if $\|x_i(t) - x_0(t)\|$ and $\|v_i(t) - v_0(t)\|$ converge into a compact set $\mathcal{C}$ as $t \to \infty$ for any initial values of followers (1) and bounded initial values of leader (2), i.e., $\lim_{t\to\infty} d(x_i(t) - x_0(t), \mathcal{C}) = 0$ and $\lim_{t\to\infty} d(v_i(t) - v_0(t), \mathcal{C}) = 0$, $i = 1, 2, \dots, N$, where $\mathcal{C} = \{e \in \mathbb{R} : |e| \leq \varepsilon\}$.

Lemma 1 (Jensen's Ineuality [36]): For any positive definite matrix $M \in \mathbb{R}^{n\times n}$ and a scalar ρ, and vector function $z : [0, \rho] \to \mathbb{R}^n$ such that the integrations in the following are well defined, then one has

$$\rho \int_0^\rho z^T(t) M z(t) dt \geq \left(\int_0^\rho z(t) dt\right)^T M \left(\int_0^\rho z(t) dt\right).$$

III. Leader-Following Control With a Leader

In this section, synchronization tracking of heterogenous networked systems with a dynamic leader is addressed.

For heterogeneous networked systems (1)–(2), define synchronization errors $\widehat{x}_i(t) = x_i(t) - x_0(t)$, $\widehat{v}_i(t) = v_i(t) - v_0(t)$. Combining (1) and (2) with protocol (3) derives error systems

$$\begin{cases}\dot{\widehat{x}}_i(t) = \widehat{v}_i(t) \\ \dot{\widehat{v}}_i(t) = -\omega_i^2\widehat{x}_i(t) - \alpha\sum_{j\in\mathcal{N}_i} a_{ij}\left[\widehat{v}_i\left(t_k^i\right) - \widehat{v}_j\left(t_{k_j(t)}^j\right)\right] \\ - \alpha b_i\widehat{v}_i\left(t_k^i\right) + r_i(t), t\in[t_k^i, t_{k+1}^i)\end{cases} \tag{4}$$

where

$$r_i(t) = \left(\omega_0^2 - \omega_i^2\right)x_0(t) - \alpha\sum_{j\in\mathcal{N}_i} a_{ij}\left[v_0\left(t_k^i\right) - v_0\left(t_{k_j(t)}^j\right)\right] - \alpha b_i\left[v_0\left(t_k^i\right) - v_0\left(t_{k_0(t)}^0\right)\right], i = 1, 2, \ldots, N.$$

Remark 4: To the authors' knowledge, the heterogeneous harmonic oscillator systems have been rarely studied. Considering the existence of the asynchronous sampled-data protocols, the heterogeneities are induced from both the nonidentities of nodes' dynamics and the asynchronization of nodes' sampling. Though the models of harmonic oscillators seem relatively simple, heterogeneous harmonic oscillators cause additional terms that cannot be counteracted each other in the corresponding error systems. The similar consequences are caused by the asynchronous sampling. Heterogeneous harmonic oscillator systems have similar complex structures with other linear or nonlinear heterogeneous systems. For the other heterogeneous multiagent systems, the leader is usually assumed to be bounded at any time [22], [23]. Due to the dynamic properties of harmonic oscillator, the assumption on the leader is weakened to be only started from the bounded set in this article.

Let $\tau_i(t) = t - t_k^i$ for $t\in[t_k^i, t_{k+1}^i)$, $i = 1, 2, \ldots, N$. Hence, $0\le\tau_i < h_i$. By introducing the term $\tau_i(t)$, it follows from (4) that for $t\in[t_k^i, t_{k+1}^i)$

$$\begin{cases}\dot{\widehat{x}}_i(t) = \widehat{v}_i(t) \\ \dot{\widehat{v}}_i(t) = -\omega_i^2\widehat{x}_i(t) - \alpha\sum_{j=1}^N l_{ij}\widehat{v}_j\left(t - \tau_j(t)\right) \\ \qquad - \alpha b_i\widehat{v}_i\left(t - \tau_i(t)\right) + r_i(t), i = 1, 2, \ldots, N\end{cases}. \tag{5}$$

Define $B = \mathrm{diag}\{b_1, b_2\ldots, b_N\}$ and $H = L + B = [H_1^c, H_2^c, \ldots, H_N^c]$, where $H_1^c, H_2^c, \ldots, H_N^c$ are N column vectors of H. Denote $H_n = [\underbrace{0_N, \ldots, 0_N}_{n-1}, H_n^c, \underbrace{0_N, \ldots, 0_N}_{N-n}]$. Let $\widehat{x}(t) = [\widehat{x}_1(t), \widehat{x}_2(t), \ldots, \widehat{x}_N(t)]^T$, $\widehat{v}(t) = [\widehat{v}_1(t), \widehat{v}_2(t), \ldots, \widehat{v}_N(t)]^T$. Then, error systems (5) can be further rewritten as follows:

$$\begin{cases}\dot{\widehat{x}}(t) = \widehat{v}(t) \\ \dot{\widehat{v}}(t) = W\widehat{x}(t) - \alpha\sum_{n=1}^N H_n\widehat{v}\left(t - \tau_n(t)\right) + r(t)\end{cases} \tag{6}$$

where $W = \mathrm{diag}\{-\omega_1^2, -\omega_2^2, \ldots, -\omega_N^2\}$ and $r(t) = [r_1(t), r_2(t), \ldots, r_N(t)]^T$. Let $y(t) = [\widehat{x}^T(t), \widehat{v}^T(t)]^T$. Then, one can obtain the following compact systems for $t\ge 0$:

$$\dot{y}(t) = \widetilde{W}y(t) - \alpha\sum_{n=1}^N\widetilde{H}_n y\left(t - \tau_n(t)\right) + \widetilde{\mathcal{R}}(t) \tag{7}$$

where

$$\widetilde{W} = \begin{bmatrix}O_N & I_N\\ W & O_N\end{bmatrix}, \widetilde{H}_n = \begin{bmatrix}O_N & O_N\\ O_N & H_n\end{bmatrix}, \widetilde{\mathcal{R}}(t) = \begin{bmatrix}0_N\\ r(t)\end{bmatrix}.$$

Assumption 1: The leader has a directed path to each follower oscillator.

Proposition 1: $r(t)$ is bounded for any initial values $[x_0(0), v_0(0)]^T\in\mathcal{C}_0$ of leader (2).

Proof: It is easy to obtain the solution of (2) for any given initial values $[x_0(0), v_0(0)]^T\in\mathcal{C}_0$

$$x_0(t) = x_0(0)\cos\left(\omega_0 t\right) + \frac{v_0(0)}{\omega_0}\sin\left(\omega_0 t\right)$$

$$v_0(t) = -\omega_0 x_0(0)\sin\left(\omega_0 t\right) + v_0(0)\cos\left(\omega_0 t\right).$$

Therefore, $x_0(t)$ and $v_0(t)$ are bounded. Thus, there exists a constant $\delta > 0$ such that $\|\widetilde{\mathcal{R}}(t)\| = \|r(t)\|\le\delta$. ■

Assume that the initial values of (7) are of form

$$y(s) = \phi(s) = \left[\widehat{x}^T(0), \widehat{v}^T(0)\right]^T, s\in[-h, 0]. \tag{8}$$

Construct the Lyapunov–Krasovskii functional

$$V\left(t, y(t)\right) = V_1\left(t, y(t)\right) + V_2\left(t, y(t)\right) + V_3\left(t, y(t)\right) \tag{9}$$

where $V_1(t, y(t)) = y^T(t)Py(t)$

$$V_2\left(t, y(t)\right) = \sum_{n=1}^N\int_{t-h_n}^t e^{a(s-t)}y^T(s)Q_n y(s)ds$$

$$V_3\left(t, y(t)\right) = \sum_{n=1}^N 2h_n\int_{-h_n}^0\int_{t+\theta}^t e^{a(s-t)}\dot{y}^T(s)R_n\dot{y}(s)dsd\theta$$

with positive definite matrices $P > 0, Q_n > 0$, and $R_n > 0$, $n = 1, 2, \ldots, N$.

For delay system (7), the following result holds.

Proposition 2: If there exist positive constants a and b such that the Lyapunov–Krasovskii functional (9) along the trajectories of (7) satisfies the following condition:

$$U(t) = \dot{V}(t) + aV(t) - b\widetilde{\mathcal{R}}^T(t)\widetilde{\mathcal{R}}(t) < 0 \tag{10}$$

then the solutions of (7) with initial values (8) satisfy the following inequality:

$$y^T(t)Py(t)\le e^{-at}V\left(0, y(0)\right) + \frac{b\delta^2}{a}\left(1 - e^{-at}\right).$$

Proof: The proof is similar to [37] and thus is omitted. ■

Based on Proposition 2, the following result holds.

Theorem 1: Suppose that Assumption 1 holds. For given constants $a > 0$ and $h_n > 0$, quasi-synchronization is reached in heterogeneous coupled harmonic oscillators (1)–(2) with the asynchronous sampled-data protocol (3), if there exist matrices $P > 0, Q_n > 0, R_n > 0, S_1, S_2$, and a constant $b > 0$ such that

$$\Psi = \begin{bmatrix}\Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & S_2\\ * & \Psi_{22} & O & \Psi_{24} & S_1\\ * & * & -\psi_q - 2\psi_r & \psi_r & O\\ * & * & * & -2\psi_r & O\\ * & * & * & * & -bI\end{bmatrix} < 0 \tag{11}$$

where $n = 1, 2, \ldots, N$

$$\Psi_{11} = aP + \sum_{n=1}^{N} Q_n - 2\sum_{n=1}^{N} e^{-ah_n} R_n + S_2\widetilde{W} + \widetilde{W}^T S_2^T$$
$$\Psi_{12} = P + \widetilde{W}^T S_1^T - S_2$$
$$\Psi_{13} = \left[e^{-ah_1}R_1, e^{-ah_2}R_2, \ldots, e^{-ah_N}R_N\right]$$
$$\Psi_{14} = \left[e^{-ah_1}R_1 - \alpha S_2\widetilde{H}_1, \ldots, e^{-ah_N}R_N - \alpha S_2\widetilde{H}_N\right]$$
$$\Psi_{22} = 2\sum_{n=1}^{N} h_n^2 R_n - \left(S_1 + S_1^T\right)$$
$$\Psi_{24} = \left[-\alpha S_1\widetilde{H}_1, -\alpha S_1\widetilde{H}_2, \ldots, -\alpha S_1\widetilde{H}_N\right]$$
$$\psi_q = \text{diag}\left\{e^{-ah_1}Q_1, e^{-ah_2}Q_2, \ldots, e^{-ah_N}Q_N\right\}$$
$$\psi_r = \text{diag}\left\{e^{-ah_1}R_1, e^{-ah_2}R_2, \ldots, e^{-ah_N}R_N\right\}.$$

Proof: Taking the derivative of $V(t)$ along the trajectory of (7) and substituting the derivative of $V(t)$ into $U(t)$ derive

$$\begin{aligned} U(t) \le\ & 2y^T(t)P\dot{y}(t) + y^T(t)\left(aP + \sum_{n=1}^{N} Q_n\right)y(t) \\ & - \sum_{n=1}^{N}\left[e^{-ah_n}y^T(t-h_n)Q_n y(t-h_n) + 2h_n^2\dot{y}^T(t)R_n\dot{y}(t)\right] \\ & - \sum_{n=1}^{N} 2h_n e^{-ah_n}\int_{t-h_n}^{t}\dot{y}^T(s)R_n\dot{y}(s)ds - b\widetilde{\mathcal{R}}^T(t)\widetilde{\mathcal{R}}(t). \end{aligned} \tag{12}$$

Simple calculation yields the following inequality:

$$\begin{aligned} 2h_n\int_{t-h_n}^{t}\dot{y}^T(s)R_n\dot{y}(s)ds \ge\ & h_n\int_{t-h_n}^{t}\dot{y}^T(s)R_n\dot{y}(s)ds \\ & + (h_n - \tau_n(t))\int_{t-h_n}^{t-\tau_n(t)}\dot{y}^T(s)R_n\dot{y}(s)ds \\ & + \tau_n(t)\int_{t-\tau_n(t)}^{t}\dot{y}^T(s)R_n\dot{y}(s)ds. \end{aligned} \tag{13}$$

Appling Jensen's inequality of Lemma 1, one has

$$\begin{aligned} & 2h_n\int_{t-h_n}^{t}\dot{y}^T(s)R_n\dot{y}(s)ds \\ & \ge [y(t) - y(t-h_n)]^T R_n [y(t) - y(t-h_n)] \\ & \quad + [y(t-\tau_n) - y(t-h_n)]^T R_n [y(t-\tau_n) - y(t-h_n)] \\ & \quad + [y(t) - y(t-\tau_n)]^T R_n [y(t) - y(t-\tau_n)]. \end{aligned} \tag{14}$$

Defining the vectorial variable

$$\eta(t) = \left[y^T(t), \dot{y}^T(t), \gamma_1^T(t), \gamma_2^T(t), \widetilde{\mathcal{R}}^T(t)\right]^T$$

where $\gamma_1(t) = [y^T(t-h_1), y^T(t-h_2), \ldots, y^T(t-h_N)]^T$ and $\gamma_2(t) = [y^T(t-\tau_1(t)), y^T(t-\tau_2(t)), \ldots, y^T(t-\tau_N(t))]^T$. Combining (11)–(14) yields $U(t) \le \eta^T(t)\Psi\eta(t) < 0$.

It follows from Proposition 2 that $y^T(t)Py(t) < b\delta^2/a$ with a exponential decay rate a as $t \to \infty$, that is, $\|y(t)\|^2 < b\delta^2/[a\lambda_{\min}(P)]$ as $t \to \infty$. Therefore, quasi-synchronization in (1) and (2) is achieved with the upper bound $b\delta^2/[a\lambda_{\min}(P)]$. ■

Corollary 1: Suppose that Assumption 1 holds. For given constants $a > 0$ and $h > 0$, quasi-synchronization is reached in heterogeneous coupled harmonic oscillators (1)–(2) with the asynchronous sampled-data protocol (3), if there exist matrices $P > 0$, $Q > 0$, $R > 0 S_1, S_2$, and a constant $b > 0$ such that

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & e^{-ah}R & \Phi_{14} & S_2 \\ * & \Phi_{22} & O & \Phi_{24} & S_1 \\ * & * & -e^{-ah}(Q+2R) & \Phi_{34} & O \\ * & * & * & \Phi_{44} & O \\ * & * & * & * & -bI \end{bmatrix} < 0 \tag{15}$$

where $\Phi_{12} = \Psi_{12}, \Phi_{24} = \Psi_{24}$

$$\Phi_{11} = aP + Q - 2e^{-ah}R + S_2\widetilde{W} + \widetilde{W}^T S_2^T$$
$$\Phi_{14} = \left[\frac{1}{N}e^{-ah}R - \alpha S_2\widetilde{H}_1, \ldots, \frac{1}{N}e^{-ah}R - \alpha S_2\widetilde{H}_N\right]$$
$$\Phi_{22} = 2h^2R - \left(S_1 + S_1^T\right)$$
$$\Phi_{34} = \left[\frac{1}{N}e^{-ah}R, \frac{1}{N}e^{-ah}R, \ldots, \frac{1}{N}e^{-ah}R\right]$$
$$\Phi_{44} = \text{diag}\left\{-\frac{2}{N}e^{-ah}R, -\frac{2}{N}e^{-ah}R, \ldots, -\frac{2}{N}e^{-ah}R\right\}.$$

Proof: Choose the Lyapunov–Krasovskii functional

$$V(t, y(t)) = \widetilde{V}_1(t, y(t)) + \widetilde{V}_2(t, y(t)) + \widetilde{V}_3(t, y(t)) \tag{16}$$

where $\widetilde{V}_1(t, y(t)) = y^T(t)Py(t)$

$$\widetilde{V}_2(t, y(t)) = \int_{t-h}^{t} e^{a(s-t)}y^T(s)Qy(s)ds$$
$$\widetilde{V}_3(t, y(t)) = 2h\int_{-h}^{0}\int_{t+\theta}^{t} e^{a(s-t)}\dot{y}^T(s)R\dot{y}(s)dsd\theta.$$

Taking the derivative of $V(t)$ along the trajectory of (7) and substituting the derivative of $V(t)$ into $U(t)$ obtains

$$\begin{aligned} U(t) \le\ & 2y^T(t)P\dot{y}(t) + y^T(t)(aP+Q)y(t) - b\widetilde{\mathcal{R}}^T(t)\widetilde{\mathcal{R}}(t) \\ & - e^{-ah}y^T(t-h)Qy(t-h) + 2h^2\dot{y}^T(t)R\dot{y}(t) \\ & - 2he^{-ah}\int_{t-h}^{t}\dot{y}^T(s)R\dot{y}(s)ds. \end{aligned} \tag{17}$$

Defining the vectorial variable

$$\widetilde{\eta}(t) = \left[y^T(t), \dot{y}^T(t), y^T(t-h), \gamma_2^T(t), \widetilde{\mathcal{R}}^T(t)\right]^T.$$

The remainder proof is similar to Theorem 1 and hence is omitted here. ■

Remark 5: Corollary 1 only depends on the maximal upper bound of sampling intervals and only needs to find solutions

P, Q, R, S_1, S_2 by solving linear matrix inequality (LMI) (15), which is more simple than LMI (11) in Theorem 1.

Remark 6: From Corollary 1, for given constants $a > 0$ and $h > 0$, one can find the minimum and maximum of α denoted by $\underline{\alpha}$ and $\overline{\alpha}$, respectively, by solving (15). Then, quasi-synchronization can be reached in heterogeneous coupled harmonic oscillators (1)–(2) with the asynchronous sampled-data protocol (3) for any $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ by [38, Lemma 4].

Remark 7: Since leader (2) is dynamic and sampled-data protocol (3) is asynchronous, $v_0(t_k^i)$ usually is not equal to $v_0(t^j_{k_j(t)})$, that is, the norm $\|r(t)\|$ will not approach zero for error systems (5). Therefore, only quasi-synchronization can be reached by adopting asynchronous sampled-data protocol even if network (1)–(2) is homogeneous. If network (1)–(2) is homogeneous and protocol (3) is synchronous, then complete synchronization is achieved in (1)–(2).

IV. Containment Control With Multiple Leaders

In this section, containment control for heterogeneous networked oscillator systems is investigated. Assume that the directed graph $\overline{\mathcal{G}}$ is composed of N heterogeneous follower oscillators and M heterogeneous leader oscillators. Without loss of generality, we assume that oscillators indexed by $1, 2, \ldots, N$ are followers and by $N+1, \ldots, N+M$ are leaders. Denote the sets of followers and leaders by $\mathcal{F} = \{1, 2, \ldots, N\}$ and $\mathcal{L} = \{N+1, N+2, \ldots, N+M\}$, respectively. The leaders are assumed to not be affected by other oscillators, that is, the leaders have no neighbors. Each follower has at least one neighbor.

Consider a networked heterogeneous system coupled by $N+M$ nonidentical nodes

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = -\omega_i^2 x_i(t) - u_i, i = 1, 2, \ldots, N+M \end{cases}. \tag{18}$$

According to the definition, the control input of each leader is set as zero, that is, $u_i(t) = 0,\ \ i = N+1, N+2, \ldots, N+M$.

This section aims to address the containment control problem of heterogeneous coupled oscillators with multiple dynamic leaders (18) by designing a distributed asynchronous sampled-data protocol for the followers.

Definition 2 (see [39]): A set $\mathcal{C} \subseteq \mathbb{R}^n$ is convex if $(1-\lambda)x + \lambda y \in \mathcal{C}$ for any $x, y \in \mathcal{C}$ and $\lambda \in [0, 1]$. The convex hull $\mathrm{Co}(X)$ spanned by a finite set $X = \{x_1, x_2, \ldots, x_q\}$ is the minimal convex set containing all points of X, that is, $\mathrm{Co}(X) = \{\sum_{i=1}^q \sigma_i x_i | x_i \in X, \sigma_i \geq 0, \sum_{i=1}^q \sigma_i = 1\}$.

Definition 3: Quasi-containment tracking in heterogenous networked oscillator systems (18) is said to be reached if there exists a positive constant ε such that for all $i \in \mathcal{F}$, $\mathrm{d}(x_i(t), \mathrm{Co}(x_{N+1}, \ldots, x_{N+M})) \leq \varepsilon$ and $\mathrm{d}(v_i(t), \mathrm{Co}(v_{N+1}, \ldots, v_{N+M})) \leq \varepsilon$ hold. If $\varepsilon = 0$, then accurate containment tracking is reached.

To achieve containment tracking in heterogeneous network (18), the following distributed asynchronous sampled-data protocol is proposed for the follower oscillators:

$$u_i(t) = \alpha \sum_{j=1}^{N} a_{ij} \left[v_i\left(t_k^i\right) - v_j\left(t^j_{k_j(t)}\right)\right] + \alpha \sum_{r=N+1}^{N+M} a_{ir} \left[v_i\left(t_k^i\right) - v_r\left(t^r_{k_r(t)}\right)\right] \tag{19}$$

for $t \in [t_k^i, t_{k+1}^i), k \in \mathbb{N}, i = 1, 2, \ldots, N$, where α is coupling strength, a_{ir} are pinning gains with the property that $a_{ir} > 0$ if the ith node can receive information from the rth leader (otherwise $a_{ir} = 0$), $r \in \{N+1, N+2, \ldots, N+M\}$.

Note that the leaders have no neighbors, the Laplacian matrix $\overline{L}$ of the directed graph $\overline{\mathcal{G}}$ can be rewritten as a partitioned matrix

$$\overline{L} = \begin{bmatrix} L_1 & L_2 \\ O_{M\times N} & O_{M\times M} \end{bmatrix} \in \mathbb{R}^{(N+M)\times(N+M)} \tag{20}$$

where $L_1 \in \mathbb{R}^{N\times N}$ and $L_2 \in \mathbb{R}^{N\times M}$.

Assumption 2: For each follower oscillator, there exists at least one leader that has a directed path to the follower.

Lemma 2 (see [12]): Under Assumption 2, all the eigenvalues of L_1 have positive real parts. Moreover, each entry of $-L_1^{-1}L_2$ is nonnegative and the sum of each row of $-L_1^{-1}L_2$ is 1, that is $-L_1^{-1}L_2 1_M = 1_M$.

Set

$$x_f = [x_1, x_2, \ldots, x_N]^T, x_l = [x_{N+1}, x_{N+2}, \ldots, x_{N+M}]^T$$
$$v_f = [v_1, v_2, \ldots, v_N]^T, v_l = [v_{N+1}, v_{N+2}, \ldots, v_{N+M}]^T$$
$$W_f = -\mathrm{diag}\left\{\omega_1^2, \omega_2^2, \ldots, \omega_N^2\right\}$$
$$W_l = -\mathrm{diag}\left\{\omega_{N+1}^2, \omega_{N+2}^2, \ldots, \omega_{N+M}^2\right\}.$$

Note that $u_i(t) = 0$ for $i \in \mathcal{L}$, one has

$$\begin{cases} \dot{x}_l(t) = v_l(t) \\ \dot{v}_l(t) = W_l x_l \end{cases}. \tag{21}$$

Combining (18) and (19) yields

$$\begin{cases} \dot{x}_f(t) = v_f(t) \\ \dot{v}_f(t) = W_f x_f - \alpha \sum_{n=1}^N H_n^1 v_f\left(t - \tau_n(t)\right) \\ \qquad - \alpha \sum_{m=1}^M H_m^2 v_l\left(t - \tau_{N+m}(t)\right) \end{cases} \tag{22}$$

where the columns of H_n^1 are zero vectors except the nth column, which is the same as the nth column of L_1; the columns of H_m^1 are zero vectors except the mth column, which is the same as the mth column of L_2. Denote

$$\widetilde{W}_f = \begin{bmatrix} O_N & I_N \\ W_f & O_N \end{bmatrix} \text{ and } \widetilde{H}_n^1 = \begin{bmatrix} O_N & O_N \\ O_N & H_n^1 \end{bmatrix}.$$

Let $\overline{x}(t) = x_f(t) + L_1^{-1}L_2 x_l(t)$ and $\overline{v}(t) = v_f(t) + L_1^{-1}L_2 v_l(t)$. According to Definition 4, containment tracking in heterogenous coupled oscillator network (18) will be reached if

$\overline{x}(t)$ and $\overline{v}(t)$ asymptotically converge to zero. By calculating, one has

$$\begin{cases}\dot{\overline{x}}(t) = \overline{v}(t) \\ \dot{\overline{v}}(t) = W_f\overline{x}(t) - \alpha\sum_{n=1}^{N} H_n^1\overline{v}\left(t-\tau_n(t)\right) + s(t)\end{cases} \tag{23}$$

where

$$\begin{aligned} s(t) = &\left(L_1^{-1}L_2W_l - W_fL_1^{-1}L_2\right)x_l(t) \\ &+ \alpha\sum_{n=1}^{N} H_n^1L_1^{-1}L_2v_l\left(t-\tau_n(t)\right) \\ &- \alpha\sum_{m=1}^{M} H_m^2v_l\left(t-\tau_{N+m}(t)\right). \end{aligned}$$

Similar to Proposition 1, $s(t)$ has the following property.

Proposition 3: $s(t)$ is bounded for given initial values of the leaders $\mathcal{L}$, that is, there exists $\ell > 0$ such that $\|s(t)\| \le \ell$.

Based on the previous analysis in Section III, one can establish the following results where the detailed proof is omitted here.

Theorem 2: Suppose that Assumption 2 holds. For given constants $a > 0$ and $h_n > 0$, quasi-containment tracking is reached in heterogeneous coupled harmonic oscillators (21)–(22) with the asynchronous sampled-data protocol (19), if there exist matrices $P > 0$, $Q_n > 0$, $R_n > 0$, S_1, S_2, and a constant $b > 0$ such that

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & S_2 \\ * & \Omega_{22} & O & \Omega_{24} & S_1 \\ * & * & -\psi_q - 2\psi_r & \psi_r & O \\ * & * & * & -2\psi_r & O \\ * & * & * & * & -bI \end{bmatrix} < 0$$

where

$$\begin{aligned} \Omega_{11} &= aP + \sum_{n=1}^{N} Q_n - 2\sum_{n=1}^{N} e^{-ah_n}R_n + S_2\widetilde{W}_f + \widetilde{W}_f^TS_2^T \\ \Omega_{12} &= P + \widetilde{W}_f^TS_1^T - S_2 \\ \Omega_{13} &= \left[e^{-ah_1}R_1, e^{-ah_2}R_2, \ldots, e^{-ah_N}R_N\right] \\ \Omega_{14} &= \left[e^{-ah_1}R_1 - \alpha S_2\widetilde{H}_1^1, \ldots, e^{-ah_N}R_N - \alpha S_2\widetilde{H}_N^1\right] \\ \Omega_{22} &= 2\sum_{n=1}^{N} h_n^2R_n - \left(S_1 + S_1^T\right) \\ \Omega_{24} &= \left[-\alpha S_1\widetilde{H}_1^1, -\alpha S_1\widetilde{H}_2^1, \ldots, -\alpha S_1\widetilde{H}_N^1\right]. \end{aligned}$$

Corollary 2: Suppose that Assumption 2 holds. For given constants $a > 0$ and $h > 0$, quasi-containment tracking is reached in heterogeneous coupled harmonic oscillators (21)–(22) with the asynchronous sampled-data protocol (19), if there exist matrices $P > 0$, $Q > 0$, $R > 0$, S_1, S_2, and a constant $b > 0$ such that

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & e^{-ah}R & \Lambda_{14} & S_2 \\ * & \Lambda_{22} & O & \Lambda_{24} & S_1 \\ * & * & -e^{-ah}(Q+2R) & \Lambda_{34} & O \\ * & * & * & \Lambda_{44} & O \\ * & * & * & * & -bI \end{bmatrix} < 0 \tag{24}$$

where $\Lambda_{12} = \Omega_{12}$, $\Lambda_{24} = \Omega_{24}$

$$\begin{aligned} \Lambda_{11} &= aP + Q - 2e^{-ah}R + S_2\widetilde{W}_f + \widetilde{W}_f^TS_2^T \\ \Lambda_{14} &= \left[\frac{1}{N}e^{-ah}R - \alpha S_2\widetilde{H}_1^1, \ldots, \frac{1}{N}e^{-ah}R - \alpha S_2\widetilde{H}_N^1\right] \\ \Lambda_{22} &= \left[2h^2R - \left(S_1 + S_1^T\right)\right] \\ \Lambda_{34} &= \left[\frac{1}{N}e^{-ah}R, \frac{1}{N}e^{-ah}R, \ldots, \frac{1}{N}e^{-ah}R\right] \\ \Lambda_{44} &= \text{diag}\left\{-\frac{2}{N}e^{-ah}R, -\frac{2}{N}e^{-ah}R, \ldots, -\frac{2}{N}e^{-ah}R\right\}. \end{aligned}$$

Remark 8: Similar to Remark 6, one can derive the minimum and maximum of α denoted by $\underline{\alpha}$ and $\overline{\alpha}$, respectively, by solving (24). Then, quasi-synchronization can be reached in heterogeneous coupled harmonic oscillators (21)–(22) with the asynchronous sampled-data protocol (19) for any $\alpha \in [\underline{\alpha}, \overline{\alpha}]$.

Remark 9: Based on Remark 7, $s(t)$ cannot vanish even if network (18) is homogeneous. Therefore, if the asynchronous sampled-data protocol (19) is adopted, then only quasi-containment tracking can be achieved for network (18) whatever it is heterogeneous or homogeneous.

Remark 10: There are two basic forms of tracking control. One is the leader-following control with one leader, and the other is the containment control with multiple leaders. They look similar in form, however, the leader-following control refers to node-to-node and the containment control refers to group-to-group. It is worth mentioning that the leaders' dynamics in system (18) are also heterogeneous for the containment control.

V. Numerical Examples

A. Quasi-Synchronization With a Dynamic Leader

Example 1: Consider the heterogeneous network (1) with the coefficient matrix $W = \text{diag}\{-1.5, -1.2, -5.5, -0.85\}$. Set $\omega_0 = 1$ for leader (2).

Suppose that the graph topology is a directed network as shown in Fig. 1(a), where the follower oscillators are labeled by 1, 2, 3, 4 and the leader is labeled by 0. Therefore the pinning matrix $B = \text{diag}\{1, 0, 0, 0\}$. Solving LMI in Corollary 1 with $a = 0.1$ and $h = 0.15$ derives an allowable interval of coupling strength [0.42, 1.31]. Set $\alpha = 0.9$. For the sake of simplicity, this example considers the asynchronous periodic sampling case. Design the sampling periods of oscillators are $h_0 = 0.05$, $h_1 = 0.1$, $h_2 = 0.15$, $h_3 = 0.12$, and $h_4 = 0.09$. Take initial values $[x_0(0), v_0(0)]^T = [0.25, 0.5]^T$ for the leader, and $x(0) = [29, -18, 12, -8]^T$ and $v(0) = [2, 10, 5, 8]^T$ for the follower

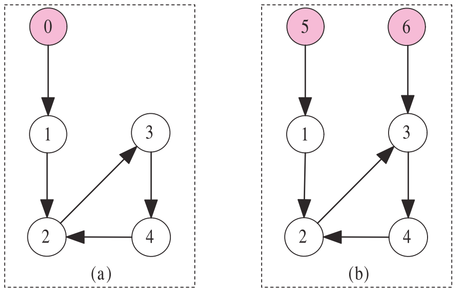

Fig. 1. Network topologies.

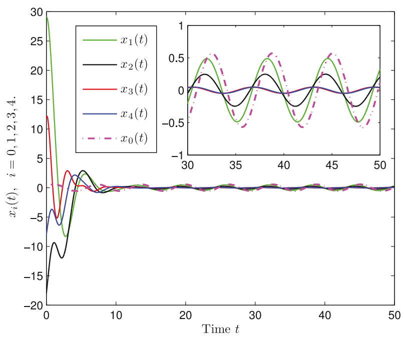

Fig. 2. Trajectories $x_i(t)$ of all oscillators, $i = 0, 1, \ldots, 4$.

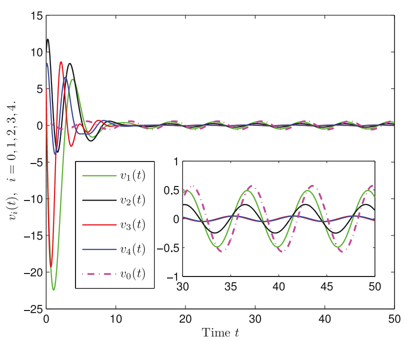

Fig. 3. Trajectories $v_i(t)$ of all oscillators, $i = 0, 1, \ldots, 4$.

oscillators. The state trajectories of position and velocity of both the leader and the followers are depicted in Figs. 2 and 3, respectively. By Corollary 1 and Figs. 2 and 3, quasi-synchronization tracking is successfully reached in heterogeneous network (1) and leader (2), that is, the follower oscillators can track the leader with a bounded error as the network is heterogeneous. If $\omega_0 = \omega_1 = \cdots = \omega_4 = 1$, then the networked system (1)–(2) is homogeneous. By employing asynchronous sampled-data protocol (3), the curves of the norm $\|r(t)\|$ for heterogeneous and homogeneous networks are drawn in Fig. 4(a) and (b), respectively. According to Remark 7 and Fig. 4(b), $\|r(t)\|$ cannot approach zero even if the network is homogeneous.

B. Containment Tracking With Two Dynamic Leaders

Example 2: Consider the heterogeneous network (18) composed of six nodes with a directed topology, as shown in

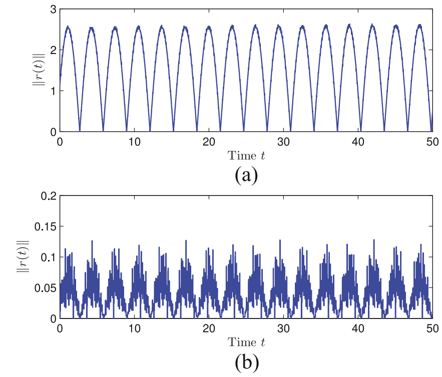

Fig. 4. Curves of $\|r(t)\|$ of (a) heterogeneous networks and (b) homogeneous networks.

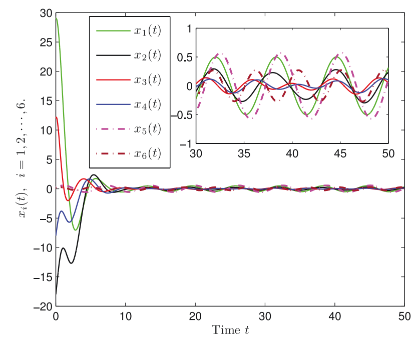

Fig. 5. Trajectories $x_i(t)$ of all oscillators, $i = 1, 2, \ldots, 6$.

Fig. 1(b). Four follower oscillators are labeled by 1, 2, 3, 4 and two leader oscillators are labeled by 5,6. The coefficient matrices of the follower and the leader oscillators are set as $W_f = \text{diag}\{-1.5, -1.2, -5.5, -0.85\}$ and $W_l = \text{diag}\{-1, -2\}$, respectively.

Similarly, solving LMI in Corollary 2 with $a = 0.1$ and $h = 0.15$ derives an allowable interval of coupling strength [0.21, 1.23]. Set coupling strength $\alpha = 1$. For the sake of simplicity, this example also considers the asynchronous periodic sampling case. Take the same asynchronous sampling periods and initial values as in Example 1 for four follower oscillators. Choose the sampling periods $h_5 = 0.05, h_6 = 0.02$ and initial values $x_l(0) = [0.25, 0.15]^T, v_l(0) = [0.5, 0.3]^T$ for two leader oscillators. The state trajectories of position and velocity of heterogeneous network (18) with the asynchronous sampling protocol (19) are shown in Figs. 5 and 6, respectively. If one takes $W_f = -I_4$ and $W_l = -I_2$, then network (18) becomes homogeneous. However, $\|s(t)\|$ will not tend to zero by Remark 9. The curves of the norm $\|s(t)\|$ for heterogeneous and homogeneous networks are sketched in Fig. 7(a) and (b), respectively. Therefore, only quasi-containment tracking can be achieved if asynchronous sampled-data protocol is applied.

Remark 11: $r(t)$ and $s(t)$ indicate the network heterogeneities arising from the nonidentity of nodes' dynamics and the asynchronism of sampling. Fig. 7(a) shows more oscillation than Fig. 4(a) due to the influence of multiple heterogeneous leaders. When the networks are homogeneous, the nonidentity

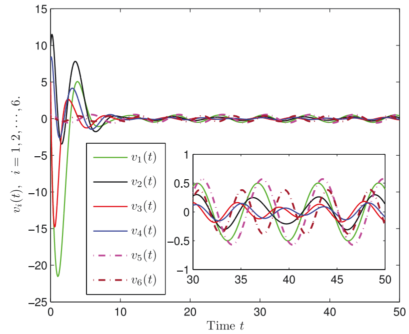

Fig. 6. Trajectories $v_i(t)$ of all oscillators, $i = 1, 2, \dots, 6$.

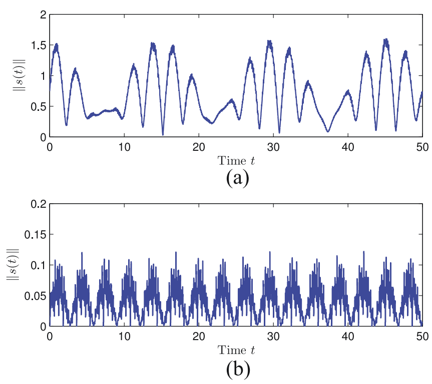

Fig. 7. Curves of $\|s(t)\|$ of (a) heterogeneous networks and (b) homogeneous networks.

of nodes' dynamics vanishes and only the asynchronism of sampling exists. Therefore, Figs. 4(b) and 7(b) show the similar oscillation.

C. Application to Electrical Networks

Consider a group of *LC* oscillators

$$c\ddot{z}_i + \frac{1}{l_i} z_i = 0 \tag{25}$$

where z_i denotes the voltage of the ith oscillator, $c > 0$ is the common capacitance, l_i is the inductance of the ith oscillator, $i = 1, 2, \dots, N+M$, $\mathcal{F} = \{1, 2, \dots, N\}$, and $\mathcal{L} = \{N+1, N+2, \dots, N+M\}$.

Certain two *LC* oscillators are coupled by LTI resistors with conductances a_{ij}. By Kirchhoff's current law, an electrical network with asynchronous coupling can be described as follows [40]:

$$\begin{aligned}\dot{z}_i(t) &= w_i(t)\\ \dot{w}_i(t) &= -\frac{1}{cl_i} z_i(t) - \frac{1}{c}\sum_{j=1}^{N} a_{ij}\left[w_i\left(t_k^i\right) - w_j\left(t_{k_j(t)}^j\right)\right]\\ &\quad - \frac{1}{c}\sum_{r=N+1}^{N+M} a_{ij}\left[w_i\left(t_k^i\right) - w_r\left(t_{k_r(t)}^r\right)\right]\end{aligned} \tag{26}$$

$i = 1, 2, \dots, N$. Therefore, network (26) is similar to network (18) with protocol (19). Therefore, the theoretical results of this article can be applied to the electrical network (26).

VI. Conclusion

In this article, the problems of both leader-following synchronization and containment tracking of networked heterogeneous systems were investigated. Based on asynchronous sampled-data, two distributed protocols were adopted to realize coordinated tracking. All the nodes have different sampling instants with each other, and each node only samples the information itself at its own sampling instants. In the case of asynchronous sampling protocol and one single dynamic leader, all the nonidentical followers can track the leader with an upper bound, which can be estimated. In the presence of multiple heterogeneous leaders for networked heterogeneous systems, the asynchronous sampling protocol is still effective to ensure that all the nonidentical followers can converge into a bounded level of convex hull spanned by heterogeneous leaders. Numerical simulations were given to illustrate the effectiveness of the theoretical results. The theoretical results have the applicability to the electrical networks. Possible extensions of the present work for event-triggered sampling, stochastic sampling, and time-varying topology will be explored in the future. Especially, future work will mainly focus on the asynchronous control for the event-triggered sampling control, stochastic sampling control, and impulsive control.

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