

ON STOKES–RITZ PROJECTION AND MULTISTEP BACKWARD DIFFERENTIATION SCHEMES IN DECOUPLING THE STOKES–DARCY MODEL*

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Abstract. We analyze a parallel, noniterative, multiphysics domain decomposition method for decoupling the Stokes–Darcy model with multistep backward differentiation schemes for the time discretization and finite elements for the spatial discretization. Based on a rigorous analysis of the Ritz projection error shown in this article, we prove almost optimal L^2 convergence of the numerical solution. In order to estimate the Ritz projection error on the interface, which plays a key role in the error analysis of the Stokes–Darcy problem, we derive L^∞ error estimate of the Stokes–Ritz projection under the stress boundary condition for the first time in the literature. The k -step backward differentiation schemes, which are important to improve the accuracy in time discretization with unconditional stability, are analyzed in a general framework for any $k \leq 5$. The unconditional stability and high accuracy of these schemes can allow relatively larger time step sizes for given accuracy requirements and hence save a significant amount of computational cost.

Key words. Stokes–Darcy flow, finite elements, Stokes–Ritz projection, backward differentiation, domain decomposition

AMS subject classifications. 65M12, 65M15, 65M55, 65M60, 35M10, 35Q35, 76D07, 76S05

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1. Introduction. We consider a coupled Stokes–Darcy system in a polygonal domain $\Omega = \Omega_D \cup \Omega_S \subset \mathbb{R}^2$, which is divided into subdomains Ω_D and Ω_S by a smooth interface Γ . In the porous medium region Ω_D , the Darcy flow is described by

$$(1.1) \quad \begin{aligned} \vec{u}_D &= -\mathbb{K} \nabla \phi_D, \\ \frac{\partial \phi_D}{\partial t} + \nabla \cdot \vec{u}_D &= f_D, \end{aligned}$$

where $\vec{u}_D$ and ϕ_D denote the unknown fluid discharge rate and hydraulic head, respectively; $\mathbb{K}$ is the hydraulic conductivity tensor; and f_D is the sink/source term. Eliminating $\vec{u}_D$, we obtain a second-order equation for the Darcy flow:

$$(1.2) \quad \frac{\partial \phi_D}{\partial t} - \nabla \cdot (\mathbb{K} \nabla \phi_D) = f_D.$$

In the free-flow region Ω_S , fluid velocity $\vec{u}_S$ is governed by

$$(1.3) \quad \begin{aligned} \frac{\partial \vec{u}_S}{\partial t} - \nabla \cdot \mathbb{T}(\vec{u}_S, p_S) &= \vec{f}_S \\ \nabla \cdot \vec{u}_S &= 0, \end{aligned}$$

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where p_S is the kinematic pressure, $\vec{f}_S$ denotes the density of external body force,

$$\mathbb{T}(\vec{u}_S, p_S) = 2\nu\mathbb{D}(\vec{u}_S) - p_S\mathbb{I}$$

denotes the stress tensor, ν is the kinematic viscosity, $\mathbb{D}(\vec{u}_S) = \frac{1}{2}(\nabla\vec{u}_S + \nabla^T\vec{u}_S)$ is the deformation tensor, and $\mathbb{I}$ is the identity tensor.

On the interface Γ , the mass conservation and balance of normal forces require the following interface conditions:

$$(1.4) \quad \vec{u}_S \cdot \vec{n}_S = -\vec{u}_D \cdot \vec{n}_D \quad \text{and} \quad -\vec{n}_S \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) = g(\phi_D - z),$$

where $\vec{n}_S$ and $\vec{n}_D$ denote the unit outer normal to the free flow and porous medium regions at the interface Γ , respectively, and z denotes the vertical Cartesian coordinate. In the tangential direction on the interface, the Beavers–Joseph–Saffman–Jones interface condition [46, 48, 74]

$$(1.5) \quad -\vec{\tau} \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) = \beta \vec{\tau} \cdot \vec{u}_S$$

is often used, where $\beta = \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}}$, $\vec{\tau}$ denotes the unit tangential vector on the interface Γ and Π denotes the permeability of the porous media.

On the boundary of the domain, we impose no-flux boundary condition on $\partial\Omega_D \setminus \Gamma$ and the free stress condition on $\partial\Omega_S \setminus \Gamma$, respectively, i.e.,

$$(1.6) \quad \mathbb{K}\nabla\phi_D \cdot \vec{n}_D = 0 \quad \text{on } \partial\Omega_D \setminus \Gamma$$

$$(1.7) \quad \mathbb{T}(\vec{u}_S, p_S)\vec{n}_S = 0 \quad \text{on } \partial\Omega_S \setminus \Gamma.$$

To summarize, the Stokes–Darcy system we consider is given by

$$(1.8) \quad \begin{cases} \frac{\partial\phi_D}{\partial t} - \nabla \cdot (\mathbb{K}\nabla\phi_D) = f_D & \text{in } \Omega_D, \\ \mathbb{K}\nabla\phi_D \cdot \vec{n}_D = 0 & \text{on } \partial\Omega_D \setminus \bar{\Gamma}, \\ \mathbb{K}\nabla\phi_D \cdot \vec{n}_D = \vec{u}_S \cdot \vec{n}_S & \text{on } \Gamma, \end{cases}$$

and

$$(1.9) \quad \begin{cases} \frac{\partial\vec{u}_S}{\partial t} - \nabla \cdot \mathbb{T}(\vec{u}_S, p_S) = \vec{f}_S & \text{in } \Omega_S, \\ \nabla \cdot \vec{u}_S = 0 & \text{in } \Omega_S, \\ -\mathbb{T}(\vec{u}_S, p_S)\vec{n}_S = 0 & \text{on } \partial\Omega_S \setminus \bar{\Gamma}, \\ -\mathbb{T}(\vec{u}_S, p_S)\vec{n}_S \cdot \vec{n}_S = g(\phi_D - z) & \text{on } \Gamma, \\ -\mathbb{T}(\vec{u}_S, p_S)\vec{n}_S \cdot \vec{\tau} = \beta\vec{u}_S \cdot \vec{\tau} & \text{on } \Gamma, \end{cases}$$

with certain initial conditions

$$(1.10) \quad \phi_D(0, x, y) = \phi_0(x, y) \quad \text{and} \quad \vec{u}_S(0, x, y) = \vec{u}_0(x, y).$$

Due to its wide range of applications in groundwater systems [24, 26, 44, 54], industrial filtrations [29, 42, 68], petroleum extraction [7, 6, 19, 45], etc., many different numerical methods have been proposed and analyzed for the Stokes–Darcy model, see [8, 9, 11, 12, 13, 14, 20, 23, 25, 27, 28, 31, 32, 34, 36, 43, 64, 65, 67, 73, 80] and references therein. It is well known that Ritz projection plays a key role in the

optimal-order L^2 -norm error estimates of finite element solutions for parabolic problems [79, 82, 83]. Therefore, Ritz projections have been often utilized in the existing articles of the Stokes–Darcy model [15, 16, 17, 18, 21, 49, 55, 66, 72, 78, 81].

On the other hand, a three-step backward differentiation (BDF) scheme was directly used, without rigorous analysis, for a domain decomposition method of the Stokes–Darcy model [15, 30]. Compared with the backward Euler scheme analyzed in [15], this three-step BDF scheme can improve the accuracy in the temporal discretization to be third order. It can also fully make use of the unconditional stability to significantly reduce the computational cost with many fewer time iteration steps. Another third order IMEX-type scheme in time discretization was proposed for the coupled weak formulation of the Stokes–Darcy model and analyzed for long time stability and high accuracy in time [22]. And there is no spatial discretization involved in this analysis. To our best knowledge, for this sophisticated coupled model, there exists no rigorous analysis on k -step ($3 \leq k \leq 5$) BDF schemes, which rely on a different multiplier technique compared with the one-step and two-step BDFs. Furthermore, the analysis of the domain decomposition method with finite element spatial discretization requires rigorous estimates of the Darcy and Stokes–Ritz projection errors on the interface, which are also not available in the literature.

In this paper, we analyze a parallel, noniterative, and multiphysics domain decomposition method which was originally introduced and numerically demonstrated in [15]. This domain decomposition method decouples the Stokes–Darcy system (1.8)–(1.9) and makes it convenient to use separate Ritz projections for the Darcy and Stokes equations, respectively. Instead of the three-step BDF scheme used in [15], we consider a more general k -step ($k \leq 5$) BDF scheme for time discretization in the analysis, with finite elements for the spatial discretization. As usual, in order to carry out an analysis of high-order accuracy, the solution of the Stokes–Darcy system is assumed to be sufficiently smooth. In particular, we present rigorous analysis of the $L^\infty(\Omega_S)$ -norm and $L^2(\Gamma)$ -norm error estimates of the Stokes–Ritz projection and prove almost optimal-order L^2 convergence of the finite element solutions of the domain decomposition method.

There are two major difficulties in the analysis: the analysis of the Ritz-projection errors on the interface and error estimates of the multistep BDF schemes. Instead of using a joint Ritz projection [15, 16, 21, 78], we consider separate Ritz projections for the Stokes and Darcy equations, respectively, which are particularly suitable for the analysis of our decoupled scheme. In particular, we define Ritz projections inside Ω_D and Ω_S , respectively (see (3.1)–(3.3) on the definition of Ritz projections), without involving any interface terms. Then one major difficulty arises from estimating the $L^2(\Gamma)$ -norm errors of Ritz projections on the interface, which will play a key role in the analysis of the interface terms. To overcome this difficulty, we carry out an L^∞ -norm estimate of the Stokes–Ritz projection error in the domain under the stress boundary condition. To our best knowledge, this is the first result for the L^∞ -norm estimate of the Stokes–Ritz projection error with the stress boundary condition. Once we obtain this estimate, we apply it to the $L^2(\Gamma)$ -norm error estimation of the Stokes–Ritz projection. Finally, after establishing error estimates for Ritz projections, we analyze the error of the multistep scheme by utilizing the multiplier technique of Nevanlinna and Odeh [69], which has been used to analyze parabolic problems recently in [1, 3, 4]. For the recent advances in L^∞ and L^p estimates for linear parabolic equations, see [33, 50, 51, 53, 56, 59, 57, 60, 62, 63] and references therein.

The rest of the paper is organized as follows. In section 2, we present the multistep BDF scheme for the domain decomposition method, which will be recalled from [15],

and then present the main result of the error estimate. In section 3, we introduce and analyze Ritz projection as a critical preparation for the error estimate. In section 4, we prove the almost optimal L^2 -norm convergence rate of the numerical solution. In section 5, we conclude our major contributions in this article.

2. Notations and main result. In this section, we briefly recall the parallel, noniterative, and multiphysics domain decomposition method with finite elements for the spatial discretization from [15]. But we will consider the more general k -step BDF instead of the three-step BDF used in [15] for the time discretization. Then we present the main theoretical result on the error estimate, which will be analyzed in the next two sections.

Let $0 = t_0 < t_1 < \dots < t_N = T$ be a uniform partition of the time interval $[0, T]$, with step size $\tau = T/N$, and for any function φ defined on the time interval $[0, T]$, we denote $\varphi^n := \varphi(t_n)$ for simplicity. For a given integer $k \geq 1$, the k -step backward difference operator and extrapolation operator are defined by

$$(2.1) \quad D_{\tau,k}\varphi^{n+1} := \sum_{j=0}^k \alpha_{k,j}\varphi^{n+1-k+j} \quad \text{and} \quad I_{\tau,k}\varphi^{n+1} := \sum_{j=0}^{k-1} \gamma_{k,j}\varphi^{n+1-k+j},$$

respectively, where the coefficients $\alpha_{k,j}$ and $\gamma_{k,j}$ are defined by

$$(2.2) \quad \sum_{j=0}^k \alpha_{k,j}z^j := \sum_{j=1}^k \frac{1}{j} z^{k-j}(z-1)^j \quad \text{and} \quad \sum_{j=0}^{k-1} \gamma_{k,j}z^j := z^k - (z-1)^k.$$

Then $D_{\tau,k}\varphi^{n+1}$ and $I_{\tau,k}\varphi^{n+1}$ are k th-order approximations of $\partial_t\varphi(t_{n+1})$ and $\varphi(t_{n+1})$, respectively [2, 41].

With the following three Robin–Robin boundary conditions proposed on the interface for the Stokes and Darcy equations in [15],

$$(2.3) \quad \mathbb{K}\nabla\phi_D \cdot \vec{n}_D + g\phi_D = \xi_D \quad \text{on } \Gamma$$

$$(2.4) \quad \vec{n}_S \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) + \vec{u}_S \cdot \vec{n}_S = \zeta_S \quad \text{on } \Gamma$$

$$(2.5) \quad -\vec{\tau} \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) = \beta \vec{\tau} \cdot \vec{u}_S \quad \text{on } \Gamma$$

the solution of (1.8)–(1.9) satisfies

$$(2.6) \quad \begin{aligned} & (\partial_t\phi_D, \varphi)_{\Omega_D} + (\mathbb{K}\nabla\phi_D, \nabla\varphi)_{\Omega_D} + (g\phi_D, \varphi)_{\Gamma} \\ & = (f_D, \varphi)_{\Omega_D} + (\xi_D, \varphi)_{\Gamma} \end{aligned} \quad \forall \varphi \in H^1(\Omega_D)$$

$$(2.7) \quad \begin{aligned} & (\partial_t\vec{u}_S, \vec{v})_{\Omega_S} + (2\nu\mathbb{D}(\vec{u}_S), \mathbb{D}(\vec{v}))_{\Omega_S} - (p_S, \nabla \cdot \vec{v})_{\Omega_S} \\ & + (\vec{u}_S \cdot \vec{n}_S, \vec{v}_S \cdot \vec{n}_S)_{\Gamma} + (\beta\vec{u}_S \cdot \vec{\tau}_S, \vec{v}_S \cdot \vec{\tau}_S)_{\Gamma} \\ & = (\vec{f}_S, \vec{v})_{\Omega_S} + (\zeta_S, \vec{v} \cdot \vec{n}_S)_{\Gamma} \end{aligned} \quad \forall \vec{v} \in \vec{H}^1(\Omega_S)$$

$$(2.8) \quad (\nabla \cdot \vec{u}_S, q)_{\Omega_S} = 0 \quad \forall q \in L^2(\Omega_S)$$

where

$$(2.9) \quad \xi_D := \vec{u}_S \cdot \vec{n}_S + g\phi_D$$

$$(2.10) \quad \zeta_S := \vec{u}_S \cdot \vec{n}_S - g(\phi_D - z).$$

Let the domain Ω be divided into quasi-uniform triangles which fit the interface. Correspondingly, Ω_D and Ω_S are divided into quasi-uniform triangles, respectively. For any given integer $r \geq 2$, we define the Lagrange finite element spaces $V_h^r(\Omega_D) \subset H^1(\Omega_D)$ and $\vec{V}_h^r(\Omega_S) \subset \vec{H}^1(\Omega_S)$, which consist of continuous piecewise polynomials of degree r , and let $V_h^{r-1}(\Omega_S) \subset L^2(\Omega_S)$ denote the Lagrange finite element space consisting of continuous piecewise polynomials of degree $r-1$. It is well known that the Taylor–Hood finite element space $\vec{V}_h^r(\Omega_S) \times V_h^{r-1}(\Omega_S)$, with $r \geq 2$, satisfies the inf-sup condition

$$(2.11) \quad \|q_h\|_{L^2(\Omega_S)} \leq \sup_{\vec{v}_h \in \vec{V}_h^r(\Omega_S)} \frac{C|(q_h, \nabla \cdot \vec{v}_h)|}{\|\nabla \vec{v}_h\|_{L^2(\Omega_S)}} \quad \forall q_h \in V_h^{r-1}(\Omega_S).$$

For given initial data $\{(\phi_h^n, \vec{u}_h^n) : n = 0, \dots, k-1\}$, the k -step BDF scheme seeks $\phi_h^{n+1} \in V_h^r(\Omega_D)$, $\vec{u}_h^{n+1} \in \vec{V}_h^r(\Omega_S)$ and $p_h^{n+1} \in V_h^{r-1}(\Omega_S)$, $n = k, \dots, N-1$, such that

$$(2.12) \quad \begin{aligned} & (D_{\tau,k}\phi_h^{n+1}, \varphi_h)_{\Omega_D} + (\mathbb{K}\nabla\phi_h^{n+1}, \nabla\varphi_h)_{\Omega_D} + (g\phi_h^{n+1}, \varphi_h)_\Gamma \\ & = (f_D^{n+1}, \varphi_h)_{\Omega_D} + (I_{\tau,k}\xi_h^{n+1}, \varphi_h)_\Gamma \quad \forall \varphi_h \in V_h^r(\Omega_D), \end{aligned}$$

$$(2.13) \quad \begin{aligned} & (D_{\tau,k}\vec{u}_h^{n+1}, \vec{v}_h)_{\Omega_S} + (2\nu\mathbb{D}(\vec{u}_h^{n+1}), \mathbb{D}(\vec{v}_h))_{\Omega_S} - (p_h^{n+1}, \nabla \cdot \vec{v}_h)_{\Omega_S} \\ & \quad + (\vec{u}_h^{n+1} \cdot \vec{n}, \vec{v}_h \cdot \vec{n})_\Gamma + (\beta\vec{u}_h^{n+1} \cdot \vec{\tau}, \vec{v}_h \cdot \vec{\tau})_\Gamma \\ & = (\vec{f}_S^{n+1}, \vec{v}_h)_{\Omega_S} + (I_{\tau,k}\zeta_h^{n+1}, \vec{v}_h \cdot \vec{n}_S)_\Gamma, \quad \forall \vec{v}_h \in \vec{V}_h^r(\Omega_S) \end{aligned}$$

$$(2.14) \quad (\nabla \cdot \vec{u}_h^{n+1}, q_h)_{\Omega_S} = 0 \quad \forall q_h \in V_h^{r-1}(\Omega_S)$$

where

$$(2.15) \quad \xi_h^j := \vec{u}_h^j \cdot \vec{n}_S + g\phi_h^j$$

$$(2.16) \quad \zeta_h^j := \vec{u}_h^j \cdot \vec{n}_S - g(\phi_h^j - z).$$

In the following, we present our final error estimate for the above scheme, which will be proved in section 4 by using the Ritz-projection error estimates in section 3 and the multiplier technique of multistep BDFs.

THEOREM 2.1. *For $1 \leq k \leq 5$, if the solution of (1.8)–(1.9) is sufficiently smooth, i.e.,*

$$(2.17) \quad \partial_t^{k+1}\phi_D \in L^\infty(0, T; L^2(\Omega_D)), \quad \phi_D, \partial_t\phi_D \in L^\infty(0, T; W^{r+1,\infty}(\Omega_D))$$

$$(2.18) \quad \partial_t^{k+1}\vec{u}_S \in L^\infty(0, T; L^2(\Omega_S)^2), \quad \vec{u}_S, \partial_t\vec{u}_S \in L^\infty(0, T; W^{r+1,\infty}(\Omega_S)^2)$$

$$(2.19) \quad p_S \in L^\infty(0, T; W^{r,\infty}(\Omega_S))$$

then the finite element solution given by (2.12)–(2.14) satisfies

$$(2.20) \quad \begin{aligned} & \max_{k \leq n \leq N} (\|\phi_h^n - \phi_D^n\|_{L^2(\Omega_D)} + \|\vec{u}_h^n - \vec{u}_S^n\|_{L^2(\Omega_S)}) \\ & \quad + \left(\sum_{n=k}^N \tau \|\nabla(\phi_h^n - R_{h,D}\phi_D^n)\|_{L^2(\Omega_D)}^2 + \sum_{n=k}^N \tau \|\nabla(\vec{u}_h^n - R_{h,S}(\vec{u}_S^n, p_S^n))\|_{L^2(\Omega_S)}^2 \right)^{\frac{1}{2}} \\ & \leq C(\tau^k + h^{r+1}|\ln h|) + C\mathcal{I}_h^k, \end{aligned}$$

where

$$(2.21) \quad \begin{aligned} \mathcal{I}_h^k &:= \max_{0 \leq n \leq k-1} (\|\phi_h^n - \phi_D^n\|_{L^2(\Omega_D)} + \|\vec{u}_h^n - \vec{u}_S^n\|_{L^2(\Omega_S)}) \\ &+ \left(\sum_{n=0}^{k-1} \tau \|\phi_h^n - R_{h,D} \phi_D^n\|_{H^1(\Omega_D)}^2 + \sum_{n=0}^{k-1} \tau \|\vec{u}_h^n - R_{h,S}(\vec{u}_S^n, p_S^n)\|_{H^1(\Omega_S)}^2 \right)^{\frac{1}{2}} \end{aligned}$$

denotes the errors of the initial data, in which $R_{h,D}$ and $R_{h,S}$ are Ritz projections defined in the next section.

Theorem 2.1 states that the global error of the multistep method is bounded by an optimal-order error bound plus a constant multiple of the errors of initial data. Our analysis is based on the following regularity result [5, 38].

LEMMA 2.2. *Assume the boundaries of the domains Ω_D and Ω_S are either smooth or piecewise smooth. For the piecewise smooth case, assume the angles of the corners of Ω_D and Ω_S are all between 0 and π . Then any weak solution of the equation*

$$(2.22) \quad \begin{cases} -\nabla \cdot (\mathbb{K} \nabla \phi) = f_1 & \text{in } \Omega_D, \\ \mathbb{K} \nabla \phi \cdot \vec{n}_D = f_2 & \text{on } \partial\Omega_D \end{cases}$$

with the normalization condition $\int_{\Omega_D} \phi(x) dx = 0$ satisfies

$$(2.23) \quad \|\phi\|_{H^2(\Omega_D)} \leq C \|f_1\|_{L^2(\Omega_D)} + C \sum_{j=1}^{J_D} \|f_2\|_{H^{1/2}(\Gamma_{D,j})},$$

where $\Gamma_{D,j}$, $j = 1, \dots, J_D$, denote the smooth pieces of the boundary $\partial\Omega_D$, and any weak solution of the stationary Stokes equation

$$(2.24) \quad \begin{cases} -\nabla \cdot (2\nu \mathbb{D}(\vec{w}) - q\mathbb{I}) = \vec{f}_1 & \text{in } \Omega_S, \\ \nabla \cdot \vec{w} = f_2 & \text{in } \Omega_S, \\ 2\nu \mathbb{D}(\vec{w})\vec{n} - q\vec{n} = \vec{f}_3 & \text{on } \partial\Omega_S, \end{cases}$$

with the normalization $\int_{\Omega_S} \vec{w} dx = 0$ satisfies

$$(2.25) \quad \begin{aligned} \|\vec{w}\|_{H^2(\Omega_S)} + \|q\|_{H^1(\Omega_S)} &\leq C \left(\|\vec{f}_1\|_{L^2(\Omega_S)} + \|f_2\|_{H^1(\Omega_S)} + \|f_2/\rho\|_{L^2(\Omega_S)} \right) \\ &+ C \sum_{j=1}^{J_S} \left(\|\vec{f}_3\|_{H^{1/2}(\Gamma_{S,j})} + \|\vec{f}_3/\sqrt{\rho}\|_{L^2(\Gamma_{S,j})} \right), \end{aligned}$$

where $\rho(x)$ denotes the minimal distance between x and the corners of the domain Ω_S and $\Gamma_{S,j}$, $j = 1, \dots, J_S$, denote the smooth pieces of the boundary $\partial\Omega_S$. If the boundary of Ω_S is smooth (without corners), then $\rho(x) \equiv 1$.

Remark 2.3. The regularity (2.23) is a special case of [5, Lemma 2.4 with $\beta_j = 0$], and (2.25) is a special case of [38, Theorem 4.15 with $\beta = 0$ and $\ell = 1$]. In the rest of this paper, to simplify our notations, we denote by C a generic positive constant which may have different values at different locations and denote by ϵ an arbitrary small enough constant such that $0 < \epsilon < 1$.

3. Ritz projection. In this section, we define and analyze Ritz projections for the above decoupling method of the Stokes–Darcy model, which is a critical preparation for the proof of Theorem 2.1 in section 4.

First, we let $R_{h,D} : H^1(\Omega_D) \rightarrow V_h^r(\Omega_D)$ and $(R_{h,S}, P_{h,S}) : \vec{H}^1(\Omega_S) \times L^2(\Omega_S) \rightarrow \vec{V}_h^r(\Omega_S) \times V_h^{r-1}(\Omega_S)$ be projection operators defined by

$$(3.1) \quad (\mathbb{K} \nabla(\phi - R_{h,D}\phi), \nabla \varphi_h)_{\Omega_D} = 0 \quad \forall \varphi_h \in V_h^r(\Omega_D)$$

and

$$(3.2) \quad (2\nu \mathbb{D}(\vec{u} - R_{h,S}(\vec{u}, p)), \mathbb{D}(\vec{v}_h))_{\Omega_S} - (p - P_{h,S}(\vec{u}, p), \nabla \cdot \vec{v}_h)_{\Omega_S} = 0 \quad \forall \vec{v}_h \in \vec{V}_h^r(\Omega_S)$$

$$(3.3) \quad (\nabla \cdot (\vec{u} - R_{h,S}(\vec{u}, p)), q_h)_{\Omega_S} = 0 \quad \forall q_h \in V_h^{r-1}(\Omega_S)$$

respectively, with the normalization conditions

$$(3.4) \quad \int_{\Omega_D} (\phi - R_{h,D}\phi) dx = 0 \quad \text{and} \quad \int_{\Omega_S} (\vec{u} - R_{h,S}(\vec{u}, p)) dx = 0$$

for the well-posedness of the projection operators. Unlike the existing literature on the Stokes–Darcy model, Ritz projections defined above do not contain interface terms. Therefore, it is well known that the following L^2 and L^∞ error estimates hold for the Poisson–Ritz projection:

$$(3.5) \quad \|\phi - R_{h,D}\phi\|_{L^2(\Omega_D)} + h \|\nabla(\phi - R_{h,D}\phi)\|_{L^2(\Omega_D)} \leq C \|\phi\|_{H^{m+1}(\Omega_D)} h^{m+1}, \quad 1 \leq m \leq r$$

$$(3.6) \quad \begin{aligned} & \|\phi - R_{h,D}\phi\|_{L^\infty(\Omega_D)} + h \|\nabla(\phi - R_{h,D}\phi)\|_{L^\infty(\Omega_D)} \\ & \leq C \|\phi\|_{W^{m+1,\infty}(\Omega_D)} h^{m+1} |\ln h|, \end{aligned} \quad 1 \leq m \leq r$$

where (3.5) is a standard consequence of the H^2 estimate (2.23) and (3.6) is a consequence of [33, Theorem A.3] and the standard duality argument used in [71, p. 2] (as $p \rightarrow \infty$ in the duality argument a logarithmic fact “ $|\ln h|$ ” appears). The corresponding maximum-norm error estimate of Ritz projection with Dirichlet boundary condition can be found in [75], where we can see that the logarithmic factor can be removed when $r \geq 2$.

Since we do not include the interface terms in Ritz projections, we will need to control the Ritz-projection errors on the interface Γ in the later analysis. For this purpose, we need the following error estimates of the Stokes–Ritz projection.

THEOREM 3.1. *The Stokes–Ritz projection satisfies*

$$(3.7) \quad \begin{aligned} & \|\vec{u} - R_{h,S}(\vec{u}, p)\|_{L^2(\Omega_S)} + h \|\vec{u} - R_{h,S}(\vec{u}, p)\|_{H^1(\Omega_S)} + h \|p - P_{h,S}(\vec{u}, p)\|_{L^2(\Omega_S)} \\ & \leq C (\|\vec{u}\|_{H^{l+1}(\Omega_S)} + \|p\|_{H^l(\Omega_S)}) h^{l+1} \end{aligned}$$

$$(3.8) \quad \|\vec{u} - R_{h,S}(\vec{u}, p)\|_{L^\infty(\Omega_S)} \leq C (\|\vec{u}\|_{W^{l+1,\infty}(\Omega_S)} + \|p\|_{W^l(\Omega_S)}) h^{l+1} |\ln h|$$

for any $1 \leq l \leq r$.

Since the maximum-norm error estimate of the Stokes–Ritz projection with the stress boundary condition has not been proved in the literature, we shall prove Theorem 3.1 in the next subsection.

For ϕ_D and $\vec{u}_S$ satisfying the requirements of Theorem 2.1, the inequality (3.6), Theorem 3.1, and trace theory imply

$$\begin{aligned}
(3.9) \quad & \|\phi_D - R_{h,D}\phi_D\|_{L^2(\Gamma)} \leq C\|\phi_D - R_{h,D}\phi_D\|_{L^\infty(\Omega_D)} \\
& \leq Ch^{r+1}|\ln h|\|\phi_D\|_{W^{r+1,\infty}(\Omega_D)} \\
(3.10) \quad & \|\vec{u}_S - R_{h,S}(\vec{u}_S, p_S)\|_{L^2(\Gamma)} \leq C\|\vec{u}_S - R_{h,S}(\vec{u}_S, p_S)\|_{L^\infty(\Omega_S)} \\
& \leq Ch^{r+1}|\ln h|(\|\vec{u}_S\|_{W^{r+1,\infty}(\Omega_S)} + \|p_S\|_{W^{r,\infty}(\Omega_S)})
\end{aligned}$$

which are crucial error estimates on the interface for our proof of Theorem 2.1 in section 4.

In the rest of this section, we will prove the above theorem. We will first prepare the related concepts and lemmas in subsections 3.1 and 3.2 and then show the proof of Theorem 3.1 in subsection 3.3.

For the simplicity of notations, we denote $\vec{u}_h = R_{h,S}(\vec{u}, p)$ and $p_h = P_{h,S}(\vec{u}, p)$ so that $(\vec{u}_h, p_h)$ is a finite element approximation of $(\vec{u}, p)$ in the sense that

$$\begin{aligned}
(3.11) \quad & (2\nu\mathbb{D}(\vec{u} - \vec{u}_h), \mathbb{D}(\vec{v}_h)) - (p - p_h, \nabla \cdot \vec{v}_h) = 0 \quad \forall \vec{v}_h \in \vec{V}_h^r(\Omega_S) \\
(3.12) \quad & (\nabla \cdot (\vec{u} - \vec{u}_h), q_h) = 0 \quad \forall q_h \in V_h^{r-1}(\Omega_S).
\end{aligned}$$

By the definition of the projection operators $R_{h,S}$ and $P_{h,S}$, we also have $\int_{\Omega_S} (\vec{u} - \vec{u}_h) dx = 0$.

3.1. Regularized and discrete Green functions. To estimate the error in the maximum norm, we consider an arbitrary point x_0 contained in the mesh element $\mathcal{K}_{x_0}$ and present estimates for $|\vec{u}_h(x_0) - \Pi_h \vec{u}(x_0)|$, where $\Pi_h \vec{u}$ denotes the Fortin projection of $\vec{u}$; see Lemma 3.4. Our estimates will not depend on the point x_0 or the element $\mathcal{K}_{x_0}$. We choose a regularized delta function $\tilde{\delta}(x, x_0) \in C_0^3(\mathcal{K}_{x_0})$ such that

$$(3.13) \quad (\tilde{\delta}, \chi_h) = \chi_h(x_0) \quad \forall \chi_h \in V_h^r(\Omega_S)$$

$$(3.14) \quad \int_{\Omega_S} \tilde{\delta} dx = 1$$

$$(3.15) \quad \|\tilde{\delta}\|_{W^{l,p}(\Omega_S)} \leq Ch^{-l-2(1-1/p)} \quad \text{for } 1 \leq p \leq \infty, \quad l = 0, 1, 2, 3.$$

Existence of such a regularized delta function has been proved in [77], which was also used in many other works [39, 61, 76]. With this regularized delta function, we can define a regularized Green function $(\vec{G}_m, p_m) = (\vec{G}_m(\cdot, x_0), p_m(\cdot, x_0))$, $m = 1, 2$, as a solution of

$$(3.16) \quad \begin{cases} -\nabla \cdot (2\nu\mathbb{D}(\vec{G}_m) - p_m\mathbb{I}) = (\delta_{1,m}, \delta_{2,m})\tilde{\delta} - \frac{1}{|\Omega_S|}(\delta_{1,m}, \delta_{2,m}) & \text{in } \Omega_S, \\ \nabla \cdot \vec{G}_m = 0 & \text{in } \Omega_S, \\ 2\nu\mathbb{D}(\vec{G}_m)\vec{n}_S - p_m\vec{n}_S = 0 & \text{on } \partial\Omega_S, \end{cases}$$

satisfying the normalization condition $\int_{\Omega_S} \vec{G}_m dx = 0$, where $\delta_{i,m}$ is the Kronecker symbol.

Let δ_h be the discrete delta function, i.e., the L^2 projection of $\tilde{\delta}$ onto the finite element space, and let $(\vec{G}_{h,m}, p_{h,m})$ be the discrete Green function, i.e., the finite element solution of (3.16). Then

$$\begin{aligned}
(3.17) \quad & (2\nu\mathbb{D}(\vec{G}_{h,m}), \mathbb{D}(\vec{v}_h)) - (p_{h,m}, \nabla \cdot \vec{v}_h) \\
&= (\delta_{1,m}, \delta_{2,m}) \cdot \vec{v}_h(x_0) - (\delta_{1,m}, \delta_{2,m}) \cdot \frac{1}{|\Omega_S|} \int_{\Omega_S} \vec{v}_h \, dx \\
&= \left(\left(\delta_h - \frac{1}{|\Omega_S|} \right) (\delta_{1,m}, \delta_{2,m}), \vec{v}_h \right) \quad \forall \vec{v}_h \in \vec{V}_h^r(\Omega_S)
\end{aligned}$$

$$(3.18) \quad (\nabla \cdot \vec{G}_{h,m}, q_h) = 0 \quad \forall q_h \in \vec{V}_h^{r-1}(\Omega_S).$$

Here $(\vec{G}_{h,m}, p_{h,m})$ is equivalent to the Stokes–Ritz projection of $(\vec{G}_m, p_m)$. Let the corners of Ω_S be denoted by x_i ($i = 1, 2, \dots, c_I$), define $\rho_j = \text{diam}(\Omega_S)2^{-j}$ ($j = 0, 1, 2, \dots$), and choose $J = \lceil \log_2(\text{diam}(\Omega_S)/(\kappa h)) \rceil$ so that $\rho_J \sim \kappa h$, where the symbol $\sim$ denotes equivalence. The constant κ needs to be large enough, such as $\kappa = 32$, so that the Ω_j , Ω'_j , and Ω''_j ($j = 0, 1, \dots, J$), which are defined below, have no intersection with the element $\mathcal{K}_{x_0}$. Let the domain Ω_S be divided into

$$(3.19) \quad \Omega_S = \cup_{j=0}^J \Omega_j(x_0) \cup \Omega_*(x_0)$$

with

$$(3.20) \quad \Omega_j(x_0) := \{x \in \Omega_S : \rho_j/2 \leq \min_{0 \leq i \leq c_I} \text{dist}(x, x_i) < \rho_j\}$$

$$(3.21) \quad \Omega_*(x_0) := \{x \in \Omega_S : \min_{0 \leq i \leq c_I} \text{dist}(x, x_i) < \rho_J/2\}.$$

For simplicity of notations, we shall denote $\Omega_j = \Omega_j(x_0)$ and $\Omega_* = \Omega_*(x_0)$ in the rest of this paper. Then $|\Omega_j| \sim \rho_j^2$ and $|\Omega_*| \sim \kappa^2 h^2$. Let

$$\begin{aligned}
(3.22) \quad & \Omega'_j = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}, \quad \Omega''_j = \Omega'_{j-1} \cup \Omega'_j \cup \Omega'_{j+1}, \\
& \text{and } \Omega'''_j = \Omega''_{j-1} \cup \Omega''_j \cup \Omega''_{j+1}, \quad j = 0, 1, 2, \dots
\end{aligned}$$

We refer the reader to [39, 58, 77] for these notations. In the following (using the notations of [76]), we denote by $\sum_j$ the summation over all Ω_j ($j = 1, \dots, J$), excluding Ω_* , and denote by $\sum_{j,*}$ the summation, including Ω_* . The generic positive constant C in the rest part of this section will be independent of x_0 and κ (until κ is determined).

With the notations introduce above, we have the following local energy estimates for the regularized Green function.

LEMMA 3.2. *The regularized Green function satisfies*

$$(3.23) \quad \|\vec{G}_m\|_{H^2(\Omega_j)} + \|p_m\|_{H^1(\Omega_j)} \leq C\rho_j^{-1}, \quad j = 0, 1, 2, \dots$$

$$(3.24) \quad \|\vec{G}_m\|_{H^2(\Omega_S)} + \|p_m\|_{H^1(\Omega_S)} \leq Ch^{-1}$$

where the constant C does not depend on x_0 ; (3.23) is also true if Ω_j is replaced by Ω'_j or Ω''_j .

Proof. Note that (3.24) is a consequence of the estimate (2.25) together with (3.15) and (3.16):

$$(3.25) \quad \|\vec{G}_m\|_{H^2(\Omega_S)} + \|p_m\|_{H^1(\Omega_S)} \leq C\|\tilde{\delta} - 1/|\Omega_S|\|_{L^2(\Omega_S)} \leq Ch^{-1} + C.$$

It remains to prove (3.23). To this end, we shall prove that the local H^2 norm in a subdomain away from the singular point x_0 can be controlled by a local L^∞ norm in a

Note that the subdomain Ω_j can be covered by a finite number of balls of radius $\rho_j/24$, say, $B_{j,k}$, $k = 1, 2, \dots, M$, where M is a constant (depending only on the domain Ω_S) and $B_{j,k} \cap \Omega_j$ is not empty for each $k = 1, 2, \dots, M$. If we let $B'_{j,k}$ denote the ball of radius $\rho_j/12$ (double of the radius of $B_{j,k}$) with the same center as $B_{j,k}$, then by the triangle inequality,

We shall derive the estimates for $\vec{G}_m$ and p_m on each of these balls and then combine them together to obtain (3.23).

For $l = 1, 2$, let ω_j^l be a smooth cutoff function such that $\omega_j^l = 1$ in $B_{j,k}^l$ but $\omega_j^l = 0$ outside $B_{j,k}^{l+1}$ with

for any multi-index α . Such smooth cutoff functions have been widely used in the literature; see [39, 61, 76, 77, 79]. For any fixed point $y_0 \in B_{j,k}^1$, define $\tilde{\mathcal{G}}_m := \tilde{G}_m(\cdot, x_0) - \tilde{G}_m(y_0, x_0)$. One can easily verify that $\tilde{\mathcal{G}}_m$ is also a solution of (3.16). In other words, we have

Moreover, due to the product rule $\mathbb{D}(\omega_j^1 \vec{\mathcal{G}}_m) = \omega_j^1 \mathbb{D}(\vec{\mathcal{G}}_m) + \frac{1}{2} \vec{\mathcal{G}}_m \otimes_s \nabla \omega_j^1$ and (3.28), $(\omega_j^1 \vec{\mathcal{G}}_m, \omega_j^1 p_m)$ is the solution of

where $\vec{\mathcal{G}}_m \otimes_s \nabla \omega_j^1 = \vec{\mathcal{G}}_m \otimes \nabla \omega_j^1 + \nabla \omega_j^1 \otimes \vec{\mathcal{G}}_m$, and we have denoted $\vec{u} \otimes \vec{v}$ as the matrix with component $(\vec{u} \otimes \vec{v})_{ij} = u_i v_j$ for any two vectors $\vec{u}$ and $\vec{v}$.

By the definition of $B_{j,k}^2$, one can easily see that $B_{j,k}^2 \subset \Omega'_j$. Since κ is large enough such that the Ω_j and Ω'_j ($j = 0, 1, \dots, J$) have no intersection with the element $\mathcal{K}_{x_0}$, then $B_{j,k}^2$ has no intersection with $\mathcal{K}_{x_0}$. Since $\omega_j^1 = 0$ outside of $B_{j,k}^2$, then $\omega_j^1 = 0$ on the element $\mathcal{K}_{x_0}$, which is the support of $\tilde{\delta}$. Hence, the term $\omega_j^1(\delta_{1,m}, \delta_{2,m})\tilde{\delta}$ can

be removed in (3.29). Applying (2.25) to the equation above (with $\vec{w} = \omega_j^1 \vec{\mathcal{G}}_m - \frac{1}{|\Omega_S|} \int_{\Omega_S} \omega_j^1 \vec{\mathcal{G}}_m \, dx$ and $q = \omega_j^1 p_m$) and using (3.27), product rule, and trace inequality, we have

(3.30)

$$\begin{aligned}
& \|\omega_j^1 \vec{\mathcal{G}}_m\|_{H^2(\Omega_S)} + \|\nabla(\omega_j^1 p_m)\|_{L^2(\Omega_S)} \\
& \leq C \left| \frac{1}{|\Omega_S|} \int_{\Omega_S} \omega_j^1 \vec{\mathcal{G}}_m \, dx \right| \\
& \quad + C \left(\left\| -\mathbb{D}(\vec{\mathcal{G}}_m) \nabla \omega_j^1 + p_m \nabla \omega_j^1 - \nabla \cdot [\vec{\mathcal{G}}_m \otimes_s \nabla \omega_j^1] - \omega_j^1 \frac{1}{|\Omega_S|} (\delta_{1,m}, \delta_{2,m}) \right\|_{L^2(\Omega_S)} \right. \\
& \quad \left. + \|\vec{\mathcal{G}}_m \cdot \nabla \omega_j^1\|_{H^1(\Omega_S)} + \|\vec{\mathcal{G}}_m \cdot \nabla \omega_j^1 / \rho\|_{L^2(\Omega_S)} \right. \\
& \quad \left. + \|\vec{\mathcal{G}}_m \otimes_s \nabla \omega_j^1\|_{H^{1/2}(\partial\Omega_S)} + \|\vec{\mathcal{G}}_m \otimes_s \nabla \omega_j^1 / \sqrt{\rho}\|_{L^2(\partial\Omega_S)} \right) \\
& \leq C \|\vec{\mathcal{G}}_m\|_{L^2(B_{j,k}^2)} + C + C \rho_j^{-1} \left(\|\nabla \vec{\mathcal{G}}_m\|_{L^2(B_{j,k}^2)} + \|p_m\|_{L^2(B_{j,k}^2)} \right) \\
& \quad + C \rho_j^{-2} \|\vec{\mathcal{G}}_m\|_{L^2(B_{j,k}^2)} + C \rho_j^{-3/2} \|\vec{\mathcal{G}}_m\|_{L^2(\partial\Omega_S \cap B_{j,k}^2)} \\
& \leq C \rho_j^{-1} \left(\|\nabla(\omega_j^2 \vec{\mathcal{G}}_m)\|_{L^2(\Omega_S)} + \|\omega_j^2 p_m\|_{L^2(\Omega_S)} \right) \\
& \quad + C \rho_j^{-1} \|\vec{\mathcal{G}}_m\|_{L^\infty(B_{j,k}^2)} + C \rho_j^{-1} \|\vec{\mathcal{G}}_m\|_{L^\infty(\partial\Omega_S \cap B_{j,k}^2)},
\end{aligned}$$

where the last inequality is due to $\omega_j^2 = 1$ on $B_{j,k}^2$ and the Hölder inequality.

Now we need to estimate $\|\nabla(\omega_j^2 \vec{\mathcal{G}}_m)\|_{L^2(\Omega_S)} + \|\omega_j^2 p_m\|_{L^2(\Omega_S)}$ on the right-hand side of (3.30). In order to do this, integrating the first equation of (3.28) with the test function $(\omega_j^2)^2 \vec{\mathcal{G}}_m$, applying integration by parts and product rule, and using the second equation in (3.28) and $\omega_j^2 = 0$ on the support of $\tilde{\delta}$, we obtain

$$\begin{aligned}
& 2\nu(\omega_j^2 \mathbb{D}(\vec{\mathcal{G}}_m), \mathbb{D}(\omega_j^2 \vec{\mathcal{G}}_m)) + \nu \left(\mathbb{D}(\vec{\mathcal{G}}_m) \omega_j^2, \vec{\mathcal{G}}_m \otimes_s \nabla \omega_j^2 \right) - (p_m, 2\omega_j^2 \nabla \omega_j^2 \cdot \vec{\mathcal{G}}_m) \\
& = \int_{\Omega_S} (\omega_j^2)^2 \vec{\mathcal{G}}_m \cdot (\delta_{1,m}, \delta_{2,m}) \tilde{\delta} \, dx - \int_{\Omega_S} \frac{1}{|\Omega_S|} (\omega_j^2)^2 \vec{\mathcal{G}}_m \cdot (\delta_{1,m}, \delta_{2,m}) \, dx \\
(3.31) \quad & = - \int_{\Omega_S} \frac{1}{|\Omega_S|} (\omega_j^2)^2 \vec{\mathcal{G}}_m \cdot (\delta_{1,m}, \delta_{2,m}) \, dx.
\end{aligned}$$

Plugging the product rule $\omega_j^2 \mathbb{D}(\vec{\mathcal{G}}_m) = \mathbb{D}(\omega_j^2 \vec{\mathcal{G}}_m) - \frac{1}{2} \vec{\mathcal{G}}_m \otimes_s \nabla \omega_j^2$ into the first two terms of (3.31), we obtain

$$\begin{aligned}
(3.32) \quad 2\nu \|\mathbb{D}(\omega_j^2 \vec{\mathcal{G}}_m)\|_{L^2(\Omega_S)}^2 &= \frac{\nu}{2} \|\vec{\mathcal{G}}_m \otimes_s \nabla \omega_j^2\|_{L^2(\Omega_S)}^2 + (p_m, 2\omega_j^2 \nabla \omega_j^2 \cdot \vec{\mathcal{G}}_m) \\
&\quad - \int_{\Omega_S} \frac{1}{|\Omega_S|} (\omega_j^2)^2 \vec{\mathcal{G}}_m \cdot (\delta_{1,m}, \delta_{2,m}) \, dx.
\end{aligned}$$

By using (3.27), $\omega_j^2 = 0$ outside $B_{j,k}^3$, the Cauchy–Schwarz inequality, and Young’s inequality, the last equation leads to

(3.33)

$$\begin{aligned}
2\nu \|\mathbb{D}(\omega_j^2 \vec{\mathcal{G}}_m)\|_{L^2(\Omega_S)}^2 &\leq \frac{\nu}{2} \|\vec{\mathcal{G}}_m \otimes_s \nabla \omega_j^2\|_{L^2(\Omega_S)}^2 \\
&\quad + \epsilon \|\omega_j^2 p_m\|_{L^2(\Omega_S)}^2 + C \epsilon^{-1} \|\nabla \omega_j^2 \cdot \vec{\mathcal{G}}_m\|_{L^2(\Omega_S)}^2 + C \|\omega_j^2 \vec{\mathcal{G}}_m\|_{L^2(\Omega_S)}^2 \\
&\leq \epsilon \|\omega_j^2 p_m\|_{L^2(\Omega_S)}^2 + C \epsilon^{-1} \rho_j^{-2} \|\vec{\mathcal{G}}_m\|_{L^2(B_{j,k}^3)}^2 + C \|\omega_j^2 \vec{\mathcal{G}}_m\|_{L^2(\Omega_S)}^2 + C \\
&\leq \epsilon \|\omega_j^2 p_m\|_{L^2(\Omega_S)}^2 + C \epsilon^{-1} \|\vec{\mathcal{G}}_m\|_{L^\infty(B_{j,k}^3)}^2 + C.
\end{aligned}$$

In order to estimate $\|\omega_j^2 p_m\|_{L^2(\Omega_S)}^2$ on the right-hand side above, we replace ω_j^1 by ω_j^2 in (3.29) and multiply the first equation of (3.29) by $\vec{v} - \vec{c}$ with a constant vector $\vec{c}$ to obtain

$$\begin{aligned}
|(\omega_j^2 p_m, \nabla \cdot \vec{v})| &= \left| (2\nu \mathbb{D}(\omega_j^2 \vec{\mathcal{G}}_m), \mathbb{D}(\vec{v})) + (\delta_{1,m}, \delta_{2,m}) \cdot \frac{1}{|\Omega_S|} \int_{\Omega_S} \omega_j^2 (\vec{v}(x) - c) dx \right. \\
&\quad \left. + (2\nu \mathbb{D}(\vec{\mathcal{G}}_m) \nabla \omega_j^2 - p_m \nabla \omega_j^2, \vec{v} - \vec{c}) + (\nu \vec{\mathcal{G}}_m \otimes_s \nabla \omega_j^2, \nabla \vec{v}) \right| \\
&\leq C \|\mathbb{D}(\omega_j^2 \vec{\mathcal{G}}_m)\|_{L^2(\Omega_S)} \|\mathbb{D}(\vec{v})\|_{L^2(\Omega_S)} + \|\vec{v} - \vec{c}\|_{L^2(\Omega_S)} \\
&\quad + C \rho_j^{-1} (\|\nabla \vec{\mathcal{G}}_m\|_{L^{2,\infty}(\Omega_S)} + \|p_m\|_{L^{2,\infty}(\Omega_S)}) \|\vec{v} - \vec{c}\|_{L^{2,1}(B_{j,k}^3)} \\
&\quad + C \rho_j^{-1} \|\vec{\mathcal{G}}_m\|_{L^2(B_{j,k}^3)} \|\nabla \vec{v}\|_{L^2(\Omega_S)} \\
&\leq C(\epsilon \|\omega_j^2 p_m\|_{L^2(\Omega_S)} + C\epsilon^{-1} \|\vec{\mathcal{G}}_m\|_{L^\infty(B_{j,k}^3)} + C) \\
&\quad \|\nabla \vec{v}\|_{L^2(\Omega_S)} + C \|\vec{v} - \vec{c}\|_{L^{2,1}(B_{j,k}')} \\
&\quad + C \rho_j^{-1} (\|\nabla \vec{\mathcal{G}}_m\|_{L^{2,\infty}(\Omega_S)} + \|p_m\|_{L^{2,\infty}(\Omega_S)}) \|\vec{v} - \vec{c}\|_{L^{2,1}(B_{j,k}')} \\
(3.34) \quad &\quad + C \|\vec{\mathcal{G}}_m\|_{L^\infty(B_{j,k}^3)} \|\nabla \vec{v}\|_{L^2(\Omega_S)},
\end{aligned}$$

where we extend $\vec{v}$ to be zero outside Ω_S for any $\vec{v} \in H_0^1(\Omega_S)$, use the duality between the two Lorentz spaces $L^{2,\infty}(B_{j,k}^3)$ and $L^{2,1}(B_{j,k}^3)$, and substitute (3.33) into the inequality above in estimating $\|\mathbb{D}(\omega_j^2 \vec{\mathcal{G}}_m)\|_{L^2(\Omega_S)}$. By choosing $\vec{c} = \frac{1}{|B_{j,k}'|} \int_{B_{j,k}'} \vec{v} dx$, we have

$$\begin{aligned}
\|\vec{v} - \vec{c}\|_{L^{2,1}(B_{j,k}')} &\leq C \|1\|_{L^{6,\frac{3}{2}}(B_{j,k}')} \|\vec{v} - \vec{c}\|_{L^3(B_{j,k}')} \leq C |B_{j,k}'|^{\frac{1}{6}} \|\vec{v} - \vec{c}\|_{L^3(B_{j,k}')} \\
(3.35) \quad &\leq C \rho_j^{\frac{1}{3}} \|\vec{v} - \vec{c}\|_{L^3(B_{j,k}')} \leq C \rho_j \|\nabla \vec{v}\|_{L^2(B_{j,k}')} \leq C \rho_j \|\nabla \vec{v}\|_{L^2(\Omega_S)},
\end{aligned}$$

where the first inequality uses Hölder's inequality for the Lorentz spaces (cf. [52, Proposition 5.1.1]); the second inequality uses such basic property of the Lorentz space as $\|1\|_{L^{6,\frac{3}{2}}(B_{j,k}')} \leq C |B_{j,k}'|^{\frac{1}{6}}$, with $|B_{j,k}'|$ denoting the area of the ball $B_{j,k}'$ (cf. [37, Example 1.4.8]); and the fourth inequality is due to the (3,2)-type Poincaré inequality (cf. [10, Theorem 5.1]).

Since $\|\nabla \vec{\mathcal{G}}_m\|_{L^{2,\infty}(\Omega_S)} + \|p_m\|_{L^{2,\infty}(\Omega_S)} \leq C$ (cf. [70, Theorem 1.1]), then (3.34) and (3.35) lead to

$$\begin{aligned}
(3.36) \quad |(\omega_j^2 p_m, \nabla \cdot \vec{v})| &\leq (C\epsilon \|\omega_j^2 p_m\|_{L^2(\Omega_S)} + C\epsilon^{-1} \|\vec{\mathcal{G}}_m\|_{L^\infty(B_{j,k}^3)} + C) \\
&\quad + C \|\vec{\mathcal{G}}_m\|_{L^\infty(B_{j,k}^3)} \|\nabla \vec{v}\|_{L^2(\Omega_S)}.
\end{aligned}$$

Applying the inf-sup condition (2.11) to the last inequality yields

$$(3.37) \quad \|\omega_j^2 p_m\|_{L^2(\Omega_S)} \leq \left(C\epsilon \|\omega_j^2 p_m\|_{L^2(\Omega_S)} + C\epsilon^{-1} \|\vec{\mathcal{G}}_m\|_{L^\infty(B_{j,k}^3)} + C + C \|\vec{\mathcal{G}}_m\|_{L^\infty(B_{j,k}^3)} \right).$$

By choosing a sufficiently small ϵ , the last inequality implies

$$(3.38) \quad \|\omega_j^2 p_m\|_{L^2(\Omega_S)} \leq C \|\vec{\mathcal{G}}_m\|_{L^\infty(B_{j,k}^3)} + C,$$

which together with (3.33) implies

$$(3.39) \quad \|\nabla(\omega_j^2 \vec{\mathcal{G}}_m)\|_{L^2(\Omega_S)} + \|\omega_j^2 p_m\|_{L^2(\Omega_S)} \leq C \|\vec{\mathcal{G}}_m\|_{L^\infty(B_{j,k}^3)} + C.$$

Plugging (3.39) into (3.30) and using the definitions of $\vec{\mathcal{G}}_m$, ω_j^l , and $B_{j,k}^l$ with the property $B_{j,k}^1 \subset B_{j,k}^2 \subset B_{j,k}^3$, we obtain

$$(3.40) \quad \begin{aligned} \|\vec{\mathcal{G}}_m\|_{H^2(B_{j,k} \cap \Omega_S)} + \|\nabla p_m\|_{L^2(B_{j,k} \cap \Omega_S)} &\leq C \rho_j^{-1} \|\vec{\mathcal{G}}_m(\cdot, x_0) \\ &\quad - \vec{\mathcal{G}}_m(y_0, x_0)\|_{L^\infty(B_{j,k} \cap \Omega_S)} + C \rho_j^{-1}. \end{aligned}$$

A fundamental pointwise estimate for the Green function is (see [70, Theorem 1.1], which also holds for the regularized Green function)

$$(3.41) \quad |\vec{\mathcal{G}}_m(x, x_0) - \vec{\mathcal{G}}_m(y_0, x_0)| \leq \frac{C|x - y_0|^\alpha}{|x_0 - y_0|^\alpha} \quad \text{when } |x_0 - y_0| > 2|x - y_0|.$$

Since (3.26) leads to $\text{dist}(B'_{j,k}, x_0) \geq \rho_j/3 = 4 \text{ radius}(B'_{j,k})$, it follows that $|x_0 - y_0| > 2|x - y_0|$ when $x \in B'_{j,k}$. Hence, (3.41) implies that $\|\vec{\mathcal{G}}_m(\cdot, x_0) - \vec{\mathcal{G}}_m(y_0, x_0)\|_{L^\infty(B'_{j,k} \cap \Omega_S)} \leq C$. Then (3.40) implies

$$(3.42) \quad \|\vec{\mathcal{G}}_m\|_{H^2(B_{j,k} \cap \Omega_S)} + \|p_m\|_{H^1(B_{j,k} \cap \Omega_S)} \leq C \rho_j^{-1}, \quad k = 1, 2, \dots, M.$$

By summing up the estimate above for $k = 1, 2, \dots, M$, we obtain (3.23).

When κ is large enough, such as $\kappa = 32$, Ω'_j , Ω''_j , and Ω'''_j ($j = 0, 1, \dots, J$) also have no intersection with the element $\mathcal{K}_{x_0}$. Then based on the definition (3.22), (3.23) is also true if Ω_j is replaced by Ω'_j or Ω''_j . $\square$

Remark 3.3. In the proof of Lemma 3.2, we have estimated $\vec{\mathcal{G}}_m := \vec{\mathcal{G}}_m(\cdot, x_0) - \vec{\mathcal{G}}_m(y_0, x_0)$ by subtracting $\vec{\mathcal{G}}_m(y_0, x_0)$ from the solution $\vec{\mathcal{G}}_m(\cdot, x_0)$. This makes it possible to control the right-hand side of (3.40) by using the local Hölder continuity estimate (3.41). Without subtracting the constant $\vec{\mathcal{G}}_m(y_0, x_0)$, we can only have

$$\|\vec{\mathcal{G}}_m\|_{H^2(B_{j,k} \cap \Omega_S)} + \|\nabla p_m\|_{L^2(B_{j,k} \cap \Omega_S)} \leq C \rho_j^{-1} \|\vec{\mathcal{G}}_m(\cdot, x_0)\|_{L^\infty(B'_{j,k} \cap \Omega_S)} + C \rho_j^{-1}$$

instead of (3.40). In this case, the best estimate for the right-hand side above is

$$\|\vec{\mathcal{G}}_m(\cdot, x_0)\|_{L^\infty(B'_{j,k} \cap \Omega_S)} \leq C \ln(2 + 1/h),$$

which is logarithmically worse than the estimate (3.42) proved in Lemma 3.2.

3.2. Two estimates. In order to prove Theorem 3.1, we also need the following two lemmas. The first one is concerned with the existence of a “Fortin projection operator” [35, 40].

LEMMA 3.4. *There exists a positive constant κ , independent of the mesh size h , such that there exists a local projection operator $\Pi_h : \vec{H}^1(\Omega_S) \rightarrow \vec{V}_h^r(\Omega_S)$, called the Fortin projection, satisfying*

$$(3.43) \quad (\nabla \cdot (\vec{v} - \Pi_h \vec{v}), q_h) = 0 \quad \forall \vec{v} \in \vec{H}^1(\Omega_S), \quad q_h \in V_h^{r-1}(\Omega_S)$$

$$(3.44) \quad \|\vec{v} - \Pi_h \vec{v}\|_{H^s(\Omega_j)} \leq Ch^{l-s} |\vec{v}|_{H^l(\Omega'_j)} \quad \forall \vec{v} \in \vec{H}^l(\Omega_S), \quad s = 0, 1, \quad 1 \leq l \leq r + 1$$

$$(3.45) \quad \|\vec{v} - \Pi_h \vec{v}\|_{H^s(\Omega_S)} \leq Ch^{l-s} |\vec{v}|_{H^l(\Omega_S)} \quad \forall \vec{v} \in \vec{H}^l(\Omega_S) \quad s = 0, 1, \quad 1 \leq l \leq r + 1$$

$$(3.46) \quad \|\Pi_h \vec{v}\|_{H^1(\Omega_S)} \leq C \|\vec{v}\|_{H^1(\Omega_S)} \quad \forall \vec{v} \in \vec{H}^1(\Omega_S),$$

$$(3.47) \quad \|\vec{v} - \Pi_h \vec{v}\|_{L^\infty(\Omega_j)} \leq Ch^l \|\vec{v}\|_{W^{l,\infty}(\Omega_j')} \quad \forall \vec{v} \in \vec{W}^{l,\infty}(\Omega_S) \quad 1 \leq l \leq r+1$$

and there exists a local projection operator $I_h : H^1(\Omega_S) \rightarrow V_h^r(\Omega_S)$ such that

$$(3.48) \quad \|q - I_h q\|_{W^{s,p}(\Omega_j)} \leq Ch^{l-s} \|\vec{v}\|_{W^{l,p}(\Omega_j')} \quad \forall \vec{v} \in \vec{W}^{l,p}(\Omega_S)$$

for all $1 \leq p \leq \infty$ and $0 \leq s \leq 1 \leq l \leq r+1$.

Remark 3.5. The Fortin projection satisfies $\Pi_h \vec{v}_h = \vec{v}_h$ for $\vec{v}_h \in \vec{V}_h^r(\Omega_S)$ (see [40, proofs of Theorems 3.7 and 4.1]). Let L_h denote the Lagrange interpolation operator. Lemma 3.4 implies the local superapproximation property: If $\tilde{\omega} = 0$ outside a subdomain D of Ω_S and $|\partial^\alpha \tilde{\omega}| \leq C_\alpha \rho^{-|\alpha|}$ for all multi-indices α and any positive number $\rho \geq h$, then by using (3.46), the Lagrange interpolation error, and the inverse inequality, we have

$$\begin{aligned} \|\tilde{\omega} \vec{v}_h - \Pi_h(\tilde{\omega} \vec{v}_h)\|_{H^1(\Omega_S)} &= \|\tilde{\omega} \vec{v}_h - L_h(\tilde{\omega} \vec{v}_h) - \Pi_h(\tilde{\omega} \vec{v}_h - L_h(\tilde{\omega} \vec{v}_h))\|_{H^1(\Omega_S)} \\ &\leq C \|\tilde{\omega} \vec{v}_h - L_h(\tilde{\omega} \vec{v}_h)\|_{H^1(\Omega_S)} \\ &\leq \sum_{\substack{|\alpha|+|\beta|=r+1 \\ \text{and } |\beta| \leq r}} Ch^r \|\partial^\alpha \tilde{\omega} \partial^\beta \vec{v}_h\|_{L^2(\Omega_S)} \\ &\leq C \sum_{\substack{|\alpha|+|\beta|=r+1 \\ 1 \leq |\beta| \leq r}} \rho^{-|\alpha|} h^{r+1-|\beta|} \|\nabla \vec{v}_h\|_{L^2(D)} \\ &\quad + C \rho^{-r-1} h^r \|\vec{v}_h\|_{L^2(D)} \\ &= C \sum_{\substack{|\alpha|+|\beta|=r+1 \\ 1 \leq |\beta| \leq r}} (h/\rho)^{r-|\beta|} h \rho^{-1} \|\nabla \vec{v}_h\|_{L^2(D)} \\ &\quad + C (h/\rho)^{r-1} h \rho^{-2} \|\vec{v}_h\|_{L^2(D)} \\ (3.49) \quad &\leq Ch \rho^{-1} \|\nabla \vec{v}_h\|_{L^2(D)} + Ch \rho^{-2} \|\vec{v}_h\|_{L^2(D)} \quad \forall \vec{v}_h \in \vec{V}_h^r(\Omega_S). \end{aligned}$$

Similarly, we have

$$(3.50) \quad \|\tilde{\omega} \varphi_h - L_h(\tilde{\omega} \varphi_h)\|_{L^2(\Omega_S)} \leq Ch \rho^{-1} \|\varphi_h\|_{L^2(D)} \quad \forall \varphi_h \in V_h^{r-1}(\Omega_S).$$

We are going to choose $D = \Omega_j'$ and $\rho = \rho_j$ in the following proof.

The second lemma is a local energy error estimate with stress boundary condition. Similar local energy error estimates have been proved for the Stokes problem with Dirichlet boundary condition in [39]. Here we will prove it for the stress boundary condition.

LEMMA 3.6 (local energy error estimates). *The following inequality holds for any $\epsilon \in (0, 1)$:*

$$\begin{aligned} (3.51) \quad &\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_j)} + \|p_m - p_{h,m}\|_{L^2(\Omega_j)} \\ &\leq \epsilon \left(\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_j')} + \|p_m - p_{h,m}\|_{L^2(\Omega_j')} \right) + C \epsilon^{-1} \|\nabla(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega_j')} \\ &\quad + C \epsilon^{-1} \|p_m - I_h p_m\|_{L^2(\Omega_j')} + C \epsilon^{-1} \rho_j^{-1} \left(\|\vec{G}_m - \Pi_h \vec{G}_m\|_{L^2(\Omega_j')} + \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega_j')} \right), \end{aligned}$$

where $j = 1, 2, \dots, J$.

Proof. Let ω be a smooth cutoff function which equals 0 outside Ω'_j but equals 1 on Ω_j with

$$(3.52) \quad |\partial^\alpha \omega| \leq C_\alpha \rho_j^{-|\alpha|}$$

for all multi-indices α . Then by using the definition of ω , Korn's inequality (see [47]), the product rule, the triangle inequality, $\omega = 0$ outside Ω'_j , and (3.52), we have

$$(3.53) \quad \begin{aligned} \|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_j)} &\leq \|\nabla[\omega(\vec{G}_m - \vec{G}_{h,m})]\|_{L^2(\Omega_S)}^2 \\ &\leq C \left(\|\mathbb{D}[\omega(\vec{G}_m - \vec{G}_{h,m})]\|_{L^2(\Omega_S)}^2 + \|\omega(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_S)}^2 \right) \\ &\leq C \left(\|\omega \mathbb{D}(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_S)}^2 \right. \\ &\quad \left. + \|(\vec{G}_m - \vec{G}_{h,m}) \nabla \omega\|_{L^2(\Omega'_j)}^2 + \|\omega(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)}^2 \right) \\ &\leq C \left(\|\omega \mathbb{D}(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_S)}^2 + \rho_j^{-2} \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}^2 \right). \end{aligned}$$

Hence, we need to estimate $\|\omega \mathbb{D}(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_S)}$ as follows. Using the Cauchy–Schwarz inequality, the product rule, and Young's inequality, we obtain

$$(3.54) \quad \begin{aligned} &2\nu \|\omega \mathbb{D}(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_S)}^2 \\ &= (2\nu \omega^2 \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), \mathbb{D}(\vec{G}_m - \vec{G}_{h,m})) \\ &= (2\nu \omega^2 \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), \mathbb{D}(\vec{G}_m - \Pi_h \vec{G}_m)) + (2\nu \omega^2 \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), \mathbb{D}(\Pi_h \vec{G}_m - \vec{G}_{h,m})) \\ &\leq \nu \|\omega \mathbb{D}(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_S)}^2 + \nu \|\omega \mathbb{D}(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega_S)}^2 \\ &\quad + (2\nu \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), \mathbb{D}[\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m})]) \\ &\quad - (2\nu \omega \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), (\Pi_h \vec{G}_m - \vec{G}_{h,m}) \otimes_s \nabla \omega). \end{aligned}$$

The first term on the right-hand side can be absorbed by the left-hand side. Thus,

$$(3.55) \quad \begin{aligned} &\nu \|\omega \mathbb{D}(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_S)}^2 \\ &\leq \nu \|\omega \mathbb{D}(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega_S)}^2 + (2\nu \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), \mathbb{D}[\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m})]) \\ &\quad - (2\nu \omega \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), (\Pi_h \vec{G}_m - \vec{G}_{h,m}) \otimes_s \nabla \omega). \end{aligned}$$

In order to estimate $(2\nu \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), \mathbb{D}[\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m})])$ by using the local superapproximation properties (3.49)–(3.50), we consider the following error equations, which can be obtained from (3.16)–(3.18) and $(\tilde{\delta} - \delta_h, \vec{v}_h) = \vec{v}_h(x_0) - \vec{v}_h(x_0) = 0$:

$$(3.56) \quad (2\nu \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), \mathbb{D}(\vec{v}_h)) - (p_m - p_{h,m}, \nabla \cdot \vec{v}_h) = 0 \quad \forall \vec{v}_h \in \vec{V}_h^r(\Omega_S)$$

$$(3.57) \quad (\nabla \cdot (\vec{G}_m - \vec{G}_{h,m}), q_h) = 0 \quad \forall q_h \in V_h^{r-1}(\Omega_S).$$

Choosing $\vec{v}_h = \Pi_h(\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m}))$ in (3.56), we obtain

$$(3.58) \quad \begin{aligned} &-(2\nu \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), \mathbb{D}[\Pi_h(\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m}))]) \\ &+ (p_m - I_h p_m, \nabla \cdot [\Pi_h(\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m}))]) \\ &+ (I_h p_m - p_{h,m}, \nabla \cdot [\Pi_h(\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m}))]) = 0. \end{aligned}$$

Adding (3.58) to (3.55), we obtain

$$\begin{aligned}
\nu \|\omega \mathbb{D}(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_S)}^2 &\leq \nu \|\omega \mathbb{D}(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega_S)}^2 \\
&\quad + (2\nu \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), \mathbb{D}[\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m}) \\
&\quad - \Pi_h(\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m}))]) \\
&\quad + (p_m - I_h p_m, \nabla \cdot [\Pi_h(\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m}))]) \\
&\quad + (I_h p_m - p_{m,h}, \nabla \cdot [\Pi_h(\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m}))]) \\
&\quad - (2\nu \omega \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), (\Pi_h \vec{G}_m - \vec{G}_{h,m}) \otimes_s \nabla \omega) \\
(3.59) \quad &=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Now we need to estimate I_i ($i = 1, \dots, 5$) and then plug (3.59) back into (3.53). First, we have

$$\begin{aligned}
(3.60) \quad &Ch \|\mathbb{D}(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)} \\
&\leq Ch \|\nabla(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega'_j)} + Ch \|\nabla(\Pi_h \vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)} \\
&\leq Ch \|\nabla(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega'_j)} + C \|\Pi_h \vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)} \quad (\text{by inverse inequality}) \\
&\leq Ch \|\nabla(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega'_j)} + C \|\Pi_h \vec{G}_m - \vec{G}_m\|_{L^2(\Omega'_j)} + C \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}.
\end{aligned}$$

Choose $\vec{v}_h = \Pi_h \vec{G}_m - \vec{G}_{h,m}$, $\tilde{\omega} = \omega^2$, $D = \Omega'_j$, and $\rho = \rho_j$ in the local superapproximation properties (3.49)–(3.50). Using $\omega = 0$ outside Ω'_j , the Cauchy–Schwarz inequality, (3.49), (3.60), Young’s inequality, (3.46), the product rule, the triangle inequality, (3.52), (3.43), (3.57), and (3.50), we have the following estimates for the right-hand side of (3.59):

$$(3.61) \quad I_1 = C \|\omega \mathbb{D}(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega_S)}^2 \leq C \|\nabla(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega'_j)}^2$$

$$\begin{aligned}
(3.62) \quad &I_2 = |(2\nu \mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), \mathbb{D}[\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m}) - \Pi_h(\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m}))])| \\
&\leq C \|\mathbb{D}(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_S)} \|\mathbb{D}[\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m}) - \Pi_h(\omega^2(\Pi_h \vec{G}_m - \vec{G}_{h,m}))]\|_{L^2(\Omega_S)} \\
&\leq Ch \|\mathbb{D}(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)} (\rho_j^{-1} \|\nabla(\Pi_h \vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)} \\
&\quad + \rho_j^{-2} \|\Pi_h \vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}) \quad (\text{by (3.49)}) \\
&\leq Ch \|\mathbb{D}(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)} (\rho_j^{-1} \|\nabla(\Pi_h \vec{G}_m - \vec{G}_m)\|_{L^2(\Omega'_j)} \\
&\quad + \rho_j^{-1} \|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)} \\
&\quad + \rho_j^{-2} \|\Pi_h \vec{G}_m - \vec{G}_m\|_{L^2(\Omega'_j)} + \rho_j^{-2} \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}) \\
&\leq \rho_j^{-1} \left(Ch \|\nabla(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega'_j)} + C \|\Pi_h \vec{G}_m - \vec{G}_m\|_{L^2(\Omega'_j)} \right. \\
&\quad \left. + C \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)} \right) \\
&\quad \cdot \left(\|\nabla(\Pi_h \vec{G}_m - \vec{G}_m)\|_{L^2(\Omega'_j)} + \|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)} + \rho_j^{-1} \|\Pi_h \vec{G}_m - \vec{G}_m\|_{L^2(\Omega'_j)} \right. \\
&\quad \left. + \rho_j^{-1} \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)} \right) \quad (\text{by (3.60)})
\end{aligned}$$

$$\begin{aligned}
&\leq \left(C\epsilon^{-1}h^2\rho_j^{-2}\|\nabla(\vec{G}_m - \Pi_h\vec{G}_m)\|_{L^2(\Omega'_j)}^2 + C\epsilon^{-1}\rho_j^{-2}\|\Pi_h\vec{G}_m - \vec{G}_m\|_{L^2(\Omega'_j)}^2 \right. \\
&\quad + C\epsilon^{-1}\rho_j^{-2}\|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}^2 \left. + \left(\epsilon\|\nabla(\Pi_h\vec{G}_m - \vec{G}_m)\|_{L^2(\Omega'_j)}^2 \right. \right. \\
&\quad + \epsilon\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)}^2 + \epsilon\rho_j^{-2}\|\Pi_h\vec{G}_m - \vec{G}_m\|_{L^2(\Omega'_j)}^2 \\
&\quad \left. \left. + \epsilon\rho_j^{-2}\|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}^2 \right) \right) \\
(3.63) \quad &I_3 = |(p_m - I_h p_m, \nabla \cdot [\Pi_h(\omega^2(\Pi_h\vec{G}_m - \vec{G}_{h,m}))])| \\
&\leq \|p_m - I_h p_m\|_{L^2(\Omega_S)} \|\nabla \cdot [\Pi_h(\omega^2(\Pi_h\vec{G}_m - \vec{G}_{h,m}))]\|_{L^2(\Omega_S)} \\
&\leq C\|p_m - I_h p_m\|_{L^2(\Omega_S)} \|\nabla[\omega^2(\Pi_h\vec{G}_m - \vec{G}_{h,m})]\|_{L^2(\Omega_S)} \quad (\text{by (3.46)}) \\
&\leq C\|p_m - I_h p_m\|_{L^2(\Omega'_j)} \|\nabla[\omega(\Pi_h\vec{G}_m - \vec{G}_{h,m})]\|_{L^2(\Omega'_j)} \\
&\quad + \rho_j^{-1}\|\omega(\Pi_h\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)} \\
&\quad (\text{by product rule, triangle inequality, (3.52), and } \omega = 0 \text{ outside } \Omega'_j) \\
&\leq C\|p_m - I_h p_m\|_{L^2(\Omega'_j)} (\|\nabla(\Pi_h\vec{G}_m - \vec{G}_m)\|_{L^2(\Omega'_j)} + \|\nabla[\omega(\vec{G}_m - \vec{G}_{h,m})]\|_{L^2(\Omega'_j)}) \\
&\quad + \rho_j^{-1}(\|\Pi_h\vec{G}_m - \vec{G}_m\|_{L^2(\Omega'_j)} + \rho_j^{-1}\|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}), \\
&\leq C\epsilon^{-1}\|p_m - I_h p_m\|_{L^2(\Omega'_j)}^2 + \left(\epsilon\|\nabla(\Pi_h\vec{G}_m - \vec{G}_m)\|_{L^2(\Omega'_j)}^2 \right. \\
&\quad \left. + \epsilon\|\nabla[\omega(\vec{G}_m - \vec{G}_{h,m})]\|_{L^2(\Omega'_j)}^2 \right. \\
&\quad \left. + \epsilon\rho_j^{-2}\|\Pi_h\vec{G}_m - \vec{G}_m\|_{L^2(\Omega'_j)}^2 + \epsilon\rho_j^{-2}\|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}^2 \right) \\
(3.64) \quad &I_4 = |(I_h p_m - p_{m,h}, \nabla \cdot [\Pi_h(\omega^2(\Pi_h\vec{G}_m - \vec{G}_{h,m}))])| \\
&= |(I_h p_m - p_{m,h}, \nabla \cdot [\omega^2(\Pi_h\vec{G}_m - \vec{G}_{h,m})])| \\
&= |(I_h p_m - p_{m,h}, \omega^2 \nabla \cdot (\Pi_h\vec{G}_m - \vec{G}_{h,m}) + 2\omega \nabla \omega \cdot (\Pi_h\vec{G}_m - \vec{G}_{h,m}))| \\
&\quad (\text{by product rule}) \\
&= |(\omega^2(I_h p_m - p_{m,h}) - L_h[\omega^2(I_h p_m - p_{m,h})], \nabla \cdot (\Pi_h\vec{G}_m - \vec{G}_{h,m}))| \\
&\quad (\text{by (3.43) and (3.57)}) \\
&\quad + (I_h p_m - p_{m,h}, 2\omega \nabla \omega \cdot (\Pi_h\vec{G}_m - \vec{G}_{h,m}))| \\
&\leq Ch\rho_j^{-1}\|I_h p_m - p_{m,h}\|_{L^2(\Omega'_j)} \|\nabla \cdot (\Pi_h\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)} \\
&\quad + C\rho_j^{-1}\|I_h p_m - p_{m,h}\|_{L^2(\Omega'_j)} \|\Pi_h\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)} \\
&\quad (\text{by (3.50), (3.52) and } \omega = 0 \text{ outside } \Omega'_j) \\
&\leq C\rho_j^{-1}\|I_h p_m - p_{m,h}\|_{L^2(\Omega'_j)} \|\Pi_h\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)} \quad (\text{by inverse inequality}) \\
&\leq C\rho_j^{-1}(\|I_h p_m - p_m\|_{L^2(\Omega'_j)} + \|p_m - p_{m,h}\|_{L^2(\Omega'_j)}) \\
&\quad \cdot (\|\Pi_h\vec{G}_m - \vec{G}_m\|_{L^2(\Omega'_j)} + \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}) \\
&\leq \epsilon\|p_m - p_{m,h}\|_{L^2(\Omega'_j)}^2 + \left(C\epsilon^{-1}\rho_j^{-2}\|\Pi_h\vec{G}_m - \vec{G}_m\|_{L^2(\Omega'_j)}^2 \right. \\
&\quad \left. + C\epsilon^{-1}\rho_j^{-2}\|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}^2 \right) + C\epsilon^{-1}\|I_h p_m - p_m\|_{L^2(\Omega'_j)}^2 \\
&\quad + (\epsilon\rho_j^{-2}\|\Pi_h\vec{G}_m - \vec{G}_m\|_{L^2(\Omega'_j)}^2 + \epsilon\rho_j^{-2}\|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}^2)
\end{aligned}$$

$$\begin{aligned}
(3.65) \quad I_5 &= |(2\nu\omega\mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), (\Pi_h \vec{G}_m - \vec{G}_{h,m}) \otimes_s \nabla\omega)| \\
&\leq C\rho_j^{-1} \|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)} \|\Pi_h \vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)} \\
&\leq \epsilon \|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)}^2 + C\epsilon^{-1}\rho_j^{-2} \|\Pi_h \vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}^2 \\
&\leq \epsilon \|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)}^2 + C\epsilon^{-1}\rho_j^{-2} \|\Pi_h \vec{G}_m - \vec{G}_m\|_{L^2(\Omega'_j)}^2 \\
&\quad + C\epsilon^{-1}\rho_j^{-2} \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}^2
\end{aligned}$$

where $\epsilon \in (0, 1)$ is arbitrary. Since $\rho_j = \text{diam}(\Omega_S)2^{-j}$, $j = 0, 1, 2, \dots, J$, and $\rho_J \sim \kappa h$, we have $h\rho_j^{-1} \leq C$. Plugging (3.61)–(3.65) into (3.59), plugging (3.59) into (3.53), and using $h\rho_j^{-1} \leq C$ and $\omega = 0$ outside Ω'_j , we obtain

$$\begin{aligned}
(3.66) \quad \|\nabla[\omega(\vec{G}_m - \vec{G}_{h,m})]\|_{L^2(\Omega_S)}^2 &\leq \epsilon \|\nabla[\omega(\vec{G}_m - \vec{G}_{h,m})]\|_{L^2(\Omega_S)}^2 + \epsilon \|p_m - p_{m,h}\|_{L^2(\Omega'_j)}^2 \\
&\quad + \epsilon \|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)}^2 \\
&\quad + C\epsilon^{-1} \|p_m - I_h p_m\|_{L^2(\Omega'_j)}^2 \\
&\quad + C\epsilon^{-1} \|\nabla(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega'_j)}^2 \\
&\quad + C\epsilon^{-1}\rho_j^{-2} \|\vec{G}_m - \Pi_h \vec{G}_m\|_{L^2(\Omega'_j)}^2 \\
&\quad + C\epsilon^{-1}\rho_j^{-2} \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}^2.
\end{aligned}$$

When ϵ is chosen to be small enough such that $0 < \frac{\epsilon}{1-\epsilon} < 1$, the term $\epsilon \|\nabla[\omega(\vec{G}_m - \vec{G}_{h,m})]\|_{L^2(\Omega_S)}^2$ can be absorbed by the left-hand side of (3.66) since $\frac{\epsilon}{1-\epsilon}$ can be defined to be a new ϵ in the following proof. Furthermore, ω was defined as a smooth cutoff function which equals 0 outside Ω'_j but equals 1 on Ω_j . Then the inequality above and (3.53) lead to

$$\begin{aligned}
(3.67) \quad \|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_j)}^2 &\leq \|\nabla[\omega(\vec{G}_m - \vec{G}_{h,m})]\|_{L^2(\Omega_S)}^2 \\
&\leq \epsilon \|p_m - p_{m,h}\|_{L^2(\Omega'_j)}^2 + \epsilon \|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)}^2 \\
&\quad + C\epsilon^{-1} \|p_m - I_h p_m\|_{L^2(\Omega'_j)}^2 + C\epsilon^{-1} \|\nabla(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega'_j)}^2 \\
&\quad + C\epsilon^{-1}\rho_j^{-2} \|\vec{G}_m - \Pi_h \vec{G}_m\|_{L^2(\Omega'_j)}^2 \\
&\quad + C\epsilon^{-1}\rho_j^{-2} \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}^2.
\end{aligned}$$

This proves the estimate of $\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_j)}$ in Lemma 3.6.

It remains to prove the estimate of $\|p_m - p_{h,m}\|_{L^2(\Omega_j)}$ in Lemma 3.6. To this end, we consider (3.56) with $\vec{v}_h \in \vec{V}_h^r(\Omega_S)$ such that $\text{supp}(\vec{v}_h) \subset \Omega_j$. For such $\vec{v}_h$, (3.56) leads to

$$\begin{aligned}
(3.68) \quad |(I_h p_m - p_{h,m}, \nabla \cdot \vec{v}_h)| &= |(2\nu\mathbb{D}(\vec{G}_m - \vec{G}_{h,m}), \mathbb{D}(\vec{v}_h)) - (p_m - I_h p_m, \nabla \cdot \vec{v}_h)| \\
&\leq C(\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_j)} \\
&\quad + \|p_m - I_h p_m\|_{L^2(\Omega_j)}) \|\nabla(\vec{v}_h)\|_{L^2(\Omega_j)}.
\end{aligned}$$

Then the local inf-sup condition [35, Lemma 3.1] implies

$$(3.69) \quad \|I_h p_m - p_{h,m}\|_{L^2(\Omega_j)} \leq C(\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_j)} + \|p_m - I_h p_m\|_{L^2(\Omega_j)}).$$

Substituting (3.67) into (3.69) and using the triangle inequality

$$\|p_m - p_{h,m}\|_{L^2(\Omega_j)} \leq \|p_m - I_h p_m\|_{L^2(\Omega_j)} + \|I_h p_m - p_{h,m}\|_{L^2(\Omega_j)},$$

we obtain the estimate of $\|p_m - p_{h,m}\|_{L^2(\Omega_j)}$ and hence complete the proof of Lemma 3.6. $\square$

3.3. Proof of Theorem 3.1. The L^2 and H^1 estimates in (3.7) are standard once we have the H^2 estimate (2.25). This subsection is devoted to the maximum-norm error estimate (3.8), for which we need to investigate the error at an arbitrary point x_0 by using the properties of the regularized/discrete delta function and equations (3.16)–(3.17) of the regularized Green function and discrete Green function.

Denote $\vec{u}$ by (u_1, u_2) and $\vec{u}_h$ by $(u_{h,1}, u_{h,2})$, and denote δ_h as the L^2 projection of $\tilde{\delta}$ onto the finite element space V_h^r . Then by the definition of δ_h and $(\delta_{1,m}, \delta_{2,m})$, we have

$$\begin{aligned} & \left| u_{h,m}(x_0) - (\Pi_h \vec{u})_m(x_0) - \frac{1}{|\Omega_S|} \int_{\Omega_S} (u_{h,m}(x) - (\Pi_h \vec{u})_m(x)) \, dx \right| \\ &= \left| \left((\delta_h - \frac{1}{|\Omega_S|}) (\delta_{1,m}, \delta_{2,m}), \vec{u}_h - \Pi_h \vec{u} \right) \right| \\ &= |(2\nu \mathbb{D}(\vec{G}_{h,m}), \mathbb{D}(\vec{u}_h - \Pi_h \vec{u})) - (p_{h,m}, \nabla \cdot (\vec{u}_h - \Pi_h \vec{u}))| \\ & \quad \text{(by (3.17) with } \vec{v}_h = \vec{u}_h - \Pi_h \vec{u}) \\ &= |(2\nu \mathbb{D}(\vec{G}_{h,m}), \mathbb{D}(\vec{u} - \Pi_h \vec{u})) - (2\nu \mathbb{D}(\vec{G}_{h,m}), \mathbb{D}(\vec{u} - \vec{u}_h)) \\ & \quad + (p_{h,m}, \nabla \cdot (\vec{u} - \vec{u}_h)) - (p_{h,m}, \nabla \cdot (\vec{u} - \Pi_h \vec{u}))|. \end{aligned}$$

Choosing $\vec{v}_h = \vec{G}_{h,m}$ in (3.11), we have

$$(2\nu \mathbb{D}(\vec{G}_{h,m}), \mathbb{D}(\vec{u} - \vec{u}_h)) = (p - p_h, \nabla \cdot \vec{G}_{h,m}).$$

Choosing $q_h = p_{h,m}$ in (3.12) and (3.43), we have

$$(p_{h,m}, \nabla \cdot (\vec{u} - \vec{u}_h)) = 0 \quad \text{and} \quad (p_{h,m}, \nabla \cdot (\vec{u} - \Pi_h \vec{u})) = 0.$$

Using the above three equations, we have

$$\begin{aligned} & \left| u_{h,m}(x_0) - (\Pi_h \vec{u})_m(x_0) - \frac{1}{|\Omega_S|} \int_{\Omega_S} (u_{h,m}(x) - (\Pi_h \vec{u})_m(x)) \, dx \right| \\ &= |(2\nu \mathbb{D}(\vec{G}_{h,m}), \mathbb{D}(\vec{u} - \Pi_h \vec{u})) - (p - p_h, \nabla \cdot \vec{G}_{h,m})| \\ &= |(2\nu \mathbb{D}(\vec{G}_{h,m}), \mathbb{D}(\vec{u} - \Pi_h \vec{u})) + (p - I_h p, \nabla \cdot (\vec{G}_m - \vec{G}_{h,m}))| \\ & \quad \text{(by (3.57) and } \nabla \cdot \vec{G}_m = 0 \text{ of (3.16))} \\ &= \left| (2\nu \mathbb{D}(\vec{G}_{h,m} - \vec{G}_m), \mathbb{D}(\vec{u} - \Pi_h \vec{u})) \right. \\ & \quad \left. + (p - I_h p, \nabla \cdot (\vec{G}_m - \vec{G}_{h,m})) + (2\nu \mathbb{D}(\vec{G}_m), \mathbb{D}(\vec{u} - \Pi_h \vec{u})) \right|. \end{aligned}$$

By using (3.16) with test function $\vec{u} - \Pi_h \vec{u}$ and (3.43), we have

$$\begin{aligned} & (2\nu \mathbb{D}(\vec{G}_m), \mathbb{D}(\vec{u} - \Pi_h \vec{u})) \\ &= (p_m, \nabla \cdot (\vec{u} - \Pi_h \vec{u})) + u_m(x_0) - (\Pi_h \vec{u})_m(x_0) - \frac{1}{|\Omega_S|} \int_{\Omega_S} (u_m(x) - (\Pi_h \vec{u})_m(x)) \, dx \end{aligned}$$

$$\begin{aligned}
& = (p_m - I_h p_m, \nabla \cdot (\vec{u} - \Pi_h \vec{u})) + u_m(x_0) - (\Pi_h \vec{u})_m(x_0) \\
& \quad - \frac{1}{|\Omega_S|} \int_{\Omega_S} (u_m(x) - (\Pi_h \vec{u})_m(x)) \, dx.
\end{aligned}$$

By using the last two equations, Hölder's inequality, and Lemma 3.4, we obtain

$$\begin{aligned}
(3.70) \quad & \left| u_{h,m}(x_0) - (\Pi_h \vec{u})_m(x_0) - \frac{1}{|\Omega_S|} \int_{\Omega_S} (u_{h,m}(x) - (\Pi_h \vec{u})_m(x)) \, dx \right| \\
& = \left| (2\nu \mathbb{D}(\vec{G}_{h,m} - \vec{G}_m), \mathbb{D}(\vec{u} - \Pi_h \vec{u})) + (p - I_h p, \nabla \cdot (\vec{G}_m - \vec{G}_{h,m})) \right. \\
& \quad + (p_m - I_h p_m, \nabla \cdot (\vec{u} - \Pi_h \vec{u})) + u_m(x_0) - (\Pi_h \vec{u})_m(x_0) \\
& \quad \left. - \frac{1}{|\Omega_S|} \int_{\Omega_S} (u_m(x) - (\Pi_h \vec{u})_m(x)) \, dx \right| \\
& \leq C \|\nabla(\vec{G}_{h,m} - \vec{G}_m)\|_{L^1(\Omega_S)} (\|\nabla(\vec{u} - \Pi_h \vec{u})\|_{L^\infty(\Omega_S)} + \|p - I_h p\|_{L^\infty(\Omega_S)}) \\
& \quad + C \|p_m - I_h p_m\|_{L^1(\Omega_S)} \|\nabla(\vec{u} - \Pi_h \vec{u})\|_{L^\infty(\Omega_S)} + C \|\vec{u} - \Pi_h \vec{u}\|_{L^\infty(\Omega_S)} \\
& \leq C \|\nabla(\vec{G}_{h,m} - \vec{G}_m)\|_{L^1(\Omega_S)} (\|\vec{u}\|_{W^{k+1,\infty}(\Omega_S)} + \|p\|_{W^{k,\infty}(\Omega_S)}) h^k \\
& \quad + C \|p_m - I_h p_m\|_{L^1(\Omega_S)} (\|\vec{u}\|_{W^{k+1,\infty}(\Omega_S)} + \|p\|_{W^{k,\infty}(\Omega_S)}) h^k + C \|\vec{u}\|_{W^{k+1,\infty}(\Omega_S)} h^{k+1}.
\end{aligned}$$

By using the decomposition $\Omega_S = \cup_{j=0}^J \Omega_j(x_0) \cup \Omega_*(x_0)$ introduced in (3.20)–(3.21), the definition of $\sum_{j,*}$, the Cauchy–Schwarz inequality, (3.48), (3.23), $J = \lceil \log_2(\text{diam}(\Omega_S)/(\kappa h)) \rceil$, $\rho_J \sim \kappa h$, (3.7), and (3.24), we have

$$\begin{aligned}
(3.71) \quad & \|I_h p_m - p_m\|_{L^1(\Omega_S)} \leq C \sum_{j,*} \|1\|_{L^2(\Omega_j)} \|I_h p_m - p_m\|_{L^2(\Omega_j)} \\
& \leq C \sum_{j,*} \rho_j \|I_h p_m - p_m\|_{L^2(\Omega_j)} \\
& \leq C \sum_{j,*} \rho_j h \|p_m\|_{H^1(\Omega'_j)} \leq C \sum_{j,*} h \leq C J h \leq C h \left| \ln \frac{C}{\kappa h} \right|
\end{aligned}$$

and

$$\begin{aligned}
(3.72) \quad & \|\nabla(\vec{G}_{h,m} - \vec{G}_m)\|_{L^1(\Omega_S)} \leq C \sum_{j,*} \rho_j \|\nabla(\vec{G}_{h,m} - \vec{G}_m)\|_{L^2(\Omega_j)} \\
& \leq C \kappa h \|\nabla(\vec{G}_{h,m} - \vec{G}_m)\|_{L^2(\Omega_*)} \\
& \quad + C \sum_j \rho_j \|\nabla(\vec{G}_{h,m} - \vec{G}_m)\|_{L^2(\Omega_j)} \\
& \leq C \kappa h^2 (\|\vec{G}_m\|_{H^2(\Omega_S)} + \|p_m\|_{H^1(\Omega_S)}) \\
& \quad + C \sum_j \rho_j \|\nabla(\vec{G}_{h,m} - \vec{G}_m)\|_{L^2(\Omega_j)} \\
& \leq C \kappa h + C \sum_j \rho_j \|\nabla(\vec{G}_{h,m} - \vec{G}_m)\|_{L^2(\Omega_j)}.
\end{aligned}$$

By using the Cauchy–Schwarz inequality, (3.7), and (3.45), we obtain

$$(3.73) \quad \left| \int_{\Omega_S} (u_{h,m}(x) - (\Pi_h \vec{u})_m(x)) \, dx \right| \leq C \|u_{h,m} - (\Pi_h \vec{u})_m\|_{L^2(\Omega_S)} \leq Ch^{k+1} \|u\|_{H^{k+1}(\Omega_S)}.$$

Substituting the last three estimates into (3.70) yields

$$(3.74) \quad \begin{aligned} |u_{h,m}(x_0) - (\Pi_h \vec{u})_m(x_0)| &\leq Ch^{k+1} \|u\|_{H^{k+1}(\Omega_S)} \\ &\quad + C_\kappa h^{k+1} \left| \ln \frac{C}{\kappa h} \right| (\|u\|_{W^{k+1,\infty}(\Omega_S)} + \|p\|_{W^{k,\infty}(\Omega_S)}) \\ &\quad + Ch^k (\|u\|_{W^{k+1,\infty}(\Omega_S)} + \|p\|_{W^{k,\infty}(\Omega_S)}) \\ &\quad \times \sum_j \rho_j \|\nabla(\vec{G}_{h,m} - \vec{G}_m)\|_{L^2(\Omega_j)}. \end{aligned}$$

It remains to prove that

$$(3.75) \quad \sum_j \rho_j \|\nabla(\vec{G}_{h,m} - \vec{G}_m)\|_{L^2(\Omega_j)} \leq Ch |\ln h|.$$

To this end, we apply Lemma 3.6. Multiplying (3.51) by ρ_j and taking the summation for $j = 1, \dots, J$ on both sides, we obtain

$$(3.76) \quad \begin{aligned} &\sum_j \rho_j \left(\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_j)} + \|p_m - p_{m,h}\|_{L^2(\Omega_j)} \right) \\ &\leq \epsilon \sum_j \rho_j \left(\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega'_j)} + \|p_m - p_{m,h}\|_{L^2(\Omega'_j)} \right) \\ &\quad + \frac{C}{\epsilon} \sum_j \rho_j \left(\|\nabla(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega'_j)} + C \|p_m - I_h p_m\|_{L^2(\Omega'_j)} \right) \\ &\quad + \frac{C}{\epsilon} \sum_j \left(\|\vec{G}_m - \Pi_h \vec{G}_m\|_{L^2(\Omega'_j)} + \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)} \right) \\ &\leq 3\epsilon \sum_j \rho_j \left(\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_j)} + \|p_m - p_{m,h}\|_{L^2(\Omega_j)} \right) \\ &\quad + \epsilon \kappa h \left(\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_*)} + \|p_m - p_{m,h}\|_{L^2(\Omega_*)} \right) \\ &\quad + \frac{C}{\epsilon} \sum_j \rho_j \left(\|\nabla(\vec{G}_m - \Pi_h \vec{G}_m)\|_{L^2(\Omega'_j)} + C \|p_m - I_h p_m\|_{L^2(\Omega'_j)} \right) \\ &\quad + \frac{C}{\epsilon} \sum_j \left(\|\vec{G}_m - \Pi_h \vec{G}_m\|_{L^2(\Omega'_j)} + \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)} \right). \end{aligned}$$

By fixing a sufficiently small ϵ , the first term on the right-hand side of (3.76) can be absorbed by its left-hand side. Now we estimate the last three terms on the right-hand side of (3.76). Recall that $J = \lceil \log_2(\text{diam}(\Omega_S)/(\kappa h)) \rceil$ and $\sum_j$ is the summation over all Ω_j ($j = 1, \dots, J$). Using (3.7), (3.19), and (3.22), we obtain

$$(3.77) \quad \begin{aligned} &\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_*)} + \|p_m - p_{m,h}\|_{L^2(\Omega_*)} \\ &\leq \|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_S)} + \|p_m - p_{m,h}\|_{L^2(\Omega_S)} \\ &\leq Ch (\|\vec{G}_m\|_{H^2(\Omega_S)} + \|p_m\|_{H^1(\Omega_S)}). \end{aligned}$$

and

$$\begin{aligned}
 \sum_j \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)} &\leq C \left(\sum_j 1^2 \right)^{1/2} \left(\sum_j \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega'_j)}^2 \right)^{1/2} \\
 &\leq C(C + |\ln h|)^{1/2} \|\vec{G}_m - \vec{G}_{h,m}\|_{L^2(\Omega_S)} \\
 (3.78) \quad &\leq C(C + |\ln h|)^{1/2} h^2 \left(\|\vec{G}_m\|_{H^2(\Omega_S)} + \|p_m\|_{H^1(\Omega_S)} \right)
 \end{aligned}$$

Using (3.44), (3.48), and (3.77)–(3.78), (3.76) becomes

$$\begin{aligned}
 &\sum_j \rho_j \left(\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_j)} + \|p_m - p_{m,h}\|_{L^2(\Omega_j)} \right) \\
 &\leq Ch^2 \left(\|\vec{G}_m\|_{H^2(\Omega_S)} + \|p_m\|_{H^1(\Omega_S)} \right) + C \sum_j \rho_j h \left(\|\vec{G}_m\|_{H^2(\Omega'_j)} + \|p_m\|_{H^1(\Omega'_j)} \right) \\
 (3.79) \quad &+ C \sum_j h^2 \|\vec{G}_m\|_{H^2(\Omega'_j)} + C(C + |\ln h|)^{1/2} h^2 \left(\|\vec{G}_m\|_{H^2(\Omega_S)} + \|p_m\|_{H^1(\Omega_S)} \right).
 \end{aligned}$$

Recall that $J = \lfloor \log_2(\text{diam}(\Omega_S)/(\kappa h)) \rfloor$ and $\sum_j$ is the summation over all Ω_j ($j = 1, \dots, J$), excluding Ω_* . With Lemma 3.2 and $h\rho_j^{-1} < C$, (3.79) can be further reduced to

$$\begin{aligned}
 &\sum_j \rho_j \left(\|\nabla(\vec{G}_m - \vec{G}_{h,m})\|_{L^2(\Omega_j)} + \|p_m - p_{m,h}\|_{L^2(\Omega_j)} \right) \\
 &\leq Ch^2 h^{-1} + C \sum_j \rho_j h \rho_j^{-1} + C \sum_j h^2 \rho_j^{-1} + C(C + |\ln h|)^{1/2} h^2 h^{-1} \\
 (3.80) \quad &\leq Ch + Ch |\ln h|.
 \end{aligned}$$

This completes the proof of (3.8).

4. Proof of Theorem 2.1. In this section, we prove Theorem 2.1 based on the Ritz-projection error estimate shown in the last section. The following lemma was proved in [69] and has been recently introduced in [1, 4] to analyze high-order BDFs for parabolic equations.

LEMMA 4.1. *For $1 \leq k \leq 5$, there exists $\eta_k \in (0, 1)$ and a symmetric positive definite matrix $G = (g_{ij}) \in \mathbb{R}^{k \times k}$ such that*

$$(4.1) \quad (D_{\tau,k} v^{n+1})(v^{n+1} - \eta_k v^n) \geq D_\tau \left(\sum_{i,j=0}^{k-1} g_{ij} v^{n+1-i} v^{n+1-j} \right)$$

for $n = k-1, \dots, N-1$, where $D_\tau f^{n+1} := (f^{n+1} - f^n)/\tau$.

Remark 4.2. Since the matrix (g_{ij}) is positive definite, there exists a positive constant κ such that

$$(4.2) \quad \sum_{i,j=0}^{k-1} g_{ij} v^{n+1-i} v^{n+1-j} \geq \kappa \sum_{i=0}^{k-1} |v^{n+1-i}|^2.$$

Second, we recall Korn's inequality (see [47]):

$$(4.3) \quad \|\nabla \vec{u}\|_{L^2(\Omega_S)} \leq C \|\mathbb{D}(\vec{u})\|_{L^2(\Omega_S)} + C \left| \int_{\Omega} \vec{u} \, dx \right|.$$

Considering (2.6)–(2.8) at t_{n+1} and using the definition of the projection operators $R_{h,D}$ and $(R_{h,S}, P_{h,S})$, we obtain

$$(4.4) \quad \begin{aligned} & (D_{\tau,k} R_{h,D} \phi_D^{n+1}, \varphi_h)_{\Omega_D} + (\mathbb{K} \nabla R_{h,D} \phi_D^{n+1}, \nabla \varphi_h)_{\Omega_D} + (g R_{h,D} \phi_D^{n+1}, \varphi_h)_{\Gamma} \\ &= - (D_{\tau,k} (\phi_D^{n+1} - R_{h,D} \phi_D^{n+1}), \varphi_h)_{\Omega_D} - (g (\phi_D^{n+1} - R_{h,D} \phi_D^{n+1}), \varphi_h)_{\Gamma} \\ & \quad + (f_D^{n+1}, \varphi_h)_{\Omega_D} + (I_{\tau,k} \xi_D^{n+1}, \varphi_h)_{\Gamma} + (E_{\phi}^{n+1}, \varphi_h)_{\Omega_D} + (E_{\xi}^{n+1}, \varphi_h)_{\Gamma} \\ & \quad \forall \varphi_h \in V_h^r(\Omega_D) \end{aligned}$$

$$(4.5) \quad \begin{aligned} & (D_{\tau,k} R_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1}), \vec{v}_h)_{\Omega_S} + 2\nu (\mathbb{D}(R_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1})), \mathbb{D}(\vec{v}_h))_{\Omega_S} \\ & \quad - (P_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1}), \nabla \cdot \vec{v}_h)_{\Omega_S} + (R_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1}) \cdot \vec{n}_S, \vec{v}_h \cdot \vec{n}_S)_{\Gamma} \\ & \quad + (\beta R_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1}) \cdot \vec{\tau}, \vec{v}_h \cdot \vec{\tau})_{\Gamma} \\ &= - (D_{\tau,k}(\vec{u}_S^{n+1} - R_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1})), \vec{v}_h)_{\Omega_S} \\ & \quad - ((\vec{u}_S^{n+1} - R_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1})) \cdot \vec{n}_S, \vec{v}_h \cdot \vec{n}_S)_{\Gamma} \\ & \quad - (\beta(\vec{u}_S^{n+1} - R_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1})) \cdot \vec{\tau}, \vec{v}_h \cdot \vec{\tau})_{\Gamma} + (\vec{f}_S^{n+1}, \vec{v}_h)_{\Omega_S} \\ & \quad + (I_{\tau,k} \zeta_S^{n+1}, \vec{v}_h \cdot \vec{n}_S)_{\Gamma} + (E_u^{n+1}, \vec{v}_h)_{\Omega_S} + (E_{\zeta}^{n+1}, \vec{v}_h \cdot \vec{n}_S)_{\Gamma} \quad \forall \vec{v}_h \in \vec{V}_h^r(\Omega_S) \end{aligned}$$

$$(4.6) \quad (\nabla \cdot R_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1}), q_h)_{\Omega_S} = 0 \quad \forall q_h \in V_h^{r-1}(\Omega_S)$$

where $E_{\phi}^{n+1} = D_{\tau,k} \phi_D^{n+1} - \partial_t \phi_D^{n+1}$, $E_{\xi}^{n+1} = \xi_D^{n+1} - I_{\tau,k} \xi_D^{n+1}$, $E_u^{n+1} = D_{\tau,k} u_S^{n+1} - \partial_t u_S^{n+1}$, and $E_{\zeta}^{n+1} = \zeta_S^{n+1} - I_{\tau,k} \zeta_S^{n+1}$ are truncation errors due to the time discretization, satisfying that

$$(4.7) \quad \|E_{\phi}^{n+1}\|_{L^2(\Omega_D)} \leq C \|\partial_t^{k+1} \phi_D\|_{L^\infty(0,T;L^2(\Omega_D))} \tau^k$$

$$(4.8) \quad \|E_{\xi}^{n+1}\|_{L^2(\Gamma)} \leq C \|\partial_t^k \xi\|_{L^\infty(0,T;L^2(\Gamma))} \tau^k$$

$$(4.9) \quad \|E_u^{n+1}\|_{L^2(\Omega_S)} \leq C \|\partial_t^{k+1} \vec{u}_S\|_{L^\infty(0,T;L^2(\Omega_D))} \tau^k$$

$$(4.10) \quad \|E_{\zeta}^{n+1}\|_{L^2(\Gamma)} \leq C \|\partial_t^k \zeta\|_{L^\infty(0,T;L^2(\Gamma))} \tau^k.$$

Let

$$(4.11) \quad e_{h,\phi}^{n+1} = \phi_h^{n+1} - R_{h,D} \phi_D^{n+1}$$

$$(4.12) \quad \vec{e}_{h,u}^{n+1} = \vec{u}_h^{n+1} - R_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1})$$

$$(4.13) \quad e_{h,p}^{n+1} = p_h^{n+1} - P_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1}).$$

Then the difference between (2.12)–(2.14) and (4.4)–(4.6) can be written as

$$(4.14) \quad \begin{aligned} & (D_{\tau,k} e_{h,\phi}^{n+1}, \varphi_h)_{\Omega_D} + (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla \varphi_h)_{\Omega_D} + (g e_{h,\phi}^{n+1}, \varphi_h)_{\Gamma} \\ &= (D_{\tau,k} \phi_D^{n+1} - R_{h,D} D_{\tau,k} \phi_D^{n+1}, \varphi_h)_{\Omega_D} + (g (\phi_D^{n+1} - R_{h,D} \phi_D^{n+1}), \varphi_h)_{\Gamma} \\ & \quad + (I_{\tau,k} (\xi_h^{n+1} - \xi_D^{n+1}), \varphi_h)_{\Gamma} - (E_{\phi}^{n+1}, \varphi_h)_{\Omega_D} - (E_{\xi}^{n+1}, \varphi_h)_{\Gamma}, \quad \varphi_h \in V_h^r(\Omega_D) \end{aligned}$$

(4.15)

$$\begin{aligned}
& (D_{\tau,k}\bar{e}_{h,u}^{n+1}, \bar{v}_h)_{\Omega_S} + 2\nu(\mathbb{D}(\bar{e}_{h,u}^{n+1}), \mathbb{D}(\bar{v}_h))_{\Omega_S} - (e_{h,p}^{n+1}, \nabla \cdot \bar{v}_h)_{\Omega_S} \\
& \quad + (\bar{e}_{h,u}^{n+1} \cdot \bar{n}_S, \bar{v}_h \cdot \bar{n}_S)_{\Gamma} + (\beta \bar{e}_{h,u}^{n+1} \cdot \bar{\tau}, \bar{v}_h \cdot \bar{\tau})_{\Gamma} \\
& = (D_{\tau,k}(\bar{u}_S^{n+1} - R_{h,S}(\bar{u}_S^{n+1}, p_S^{n+1})), \bar{v}_h)_{\Omega_S} + ((\bar{u}_S^{n+1} - R_{h,S}(\bar{u}_S^{n+1}, p_S^{n+1})) \cdot \bar{n}_S, \bar{v}_h \cdot \bar{n}_S)_{\Gamma} \\
& \quad + (\beta(\bar{u}_S^{n+1} - R_{h,S}(\bar{u}_S^{n+1}, p_S^{n+1})) \cdot \bar{\tau}, \bar{v}_h \cdot \bar{\tau})_{\Gamma} + (I_{\tau,k}(\zeta_h^{n+1} - \zeta_S^{n+1}), \bar{v}_h \cdot \bar{n}_S)_{\Gamma} \\
& \quad - (E_u^{n+1}, \bar{v}_h)_{\Omega_S} - (E_{\zeta}^{n+1}, \bar{v}_h \cdot \bar{n}_S)_{\Gamma}, \quad \bar{v}_h \in \bar{V}_h^r(\Omega_S)
\end{aligned}$$

(4.16)

$$(\nabla \cdot \bar{e}_{h,u}^{n+1}, q_h)_{\Omega_S} = 0, \quad q_h \in V_h^{r-1}(\Omega_S).$$

Choose $\varphi_h = e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n$ in (4.14). We have the inequalities

$$\begin{aligned}
& (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla(e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n))_{\Omega_D} \\
& = (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D} - \eta_k (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^n)_{\Omega_D} \\
& \geq (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D} - \eta_k \sqrt{(\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D}} \sqrt{(\mathbb{K} \nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n)_{\Omega_D}} \\
& \geq (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D} - \frac{1}{2} (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D} - \frac{\eta_k^2}{2} (\mathbb{K} \nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n)_{\Omega_D} \\
(4.17) \quad & = D_{\tau} \left(\frac{\tau \eta_k^2}{2} (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D} \right) + \frac{1 - \eta_k^2}{2} (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D}
\end{aligned}$$

and

$$\begin{aligned}
& (g e_{h,\phi}^{n+1}, e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n)_{\Gamma} = g \|e_{h,\phi}^{n+1}\|_{L^2(\Gamma)}^2 - g \eta_k (e_{h,\phi}^{n+1}, e_{h,\phi}^n)_{\Gamma} \\
& \geq g \|e_{h,\phi}^{n+1}\|_{L^2(\Gamma)}^2 - g \eta_k \|e_{h,\phi}^{n+1}\|_{L^2(\Gamma)} \|e_{h,\phi}^n\|_{L^2(\Gamma)} \\
& \geq g \|e_{h,\phi}^{n+1}\|_{L^2(\Gamma)}^2 - \frac{g}{2} \|e_{h,\phi}^{n+1}\|_{L^2(\Gamma)}^2 - \frac{\eta_k^2 g}{2} \|e_{h,\phi}^n\|_{L^2(\Gamma)}^2 \\
(4.18) \quad & = D_{\tau} \left(\frac{\tau \eta_k^2 g}{2} \|e_{h,\phi}^{n+1}\|_{L^2(\Gamma)}^2 \right) + \frac{(1 - \eta_k^2)g}{2} \|e_{h,\phi}^{n+1}\|_{L^2(\Gamma)}^2.
\end{aligned}$$

Using Lemma 4.1, (3.5), (3.9), and (4.17)–(4.18), we obtain

(4.19)

$$\begin{aligned}
& D_{\tau} \left(\sum_{i,j=0}^{k-1} g_{ij} (e_{h,\phi}^{n+1-i}, e_{h,\phi}^{n+1-j})_{\Omega_D} + \frac{\tau \eta_k^2}{2} (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D} + \frac{\tau \eta_k^2 g}{2} \|e_{h,\phi}^{n+1}\|_{L^2(\Gamma)}^2 \right) \\
& \quad + \frac{1 - \eta_3^2}{2} (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D} + \frac{(1 - \eta_3^2)g}{2} \|e_{h,\phi}^{n+1}\|_{L^2(\Gamma)}^2 \\
& \leq \|D_{\tau,k} \phi_D^{n+1} - R_{h,D} D_{\tau,k} \phi_D^{n+1}\|_{L^2(\Omega_D)} \|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Omega_D)} \\
& \quad + g \|\phi_D^{n+1} - R_{h,D} \phi_D^{n+1}\|_{L^2(\Gamma)} \|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Gamma)} \\
& \quad + \|I_{\tau,k}(\xi_h^{n+1} - \xi_D^{n+1})\|_{L^2(\Gamma)} \|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Gamma)} \\
& \quad + \|E_{\phi}^{n+1}\|_{L^2(\Omega_D)} \|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Omega_D)} + \|E_{\xi}^{n+1}\|_{L^2(\Gamma)} \|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Gamma)} \\
& \leq C h^{r+1} \|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Omega_D)} + C h^{r+1} |\ln h| \|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Gamma)} \\
& \quad + \|I_{\tau,k}(\xi_h^{n+1} - \xi_D^{n+1})\|_{L^2(\Gamma)} \|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Gamma)}
\end{aligned}$$

$$\begin{aligned}
& + C\tau^k \|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Omega_D)} + C\tau^k \|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Gamma)} \\
& \leq C\epsilon^{-1}(h^{2r+2}|\ln h|^2 + \tau^{2k}) + C\epsilon^{-1}\|I_{\tau,k}(\xi_h^{n+1} - \xi_D^{n+1})\|_{L^2(\Gamma)}^2 \\
& + \epsilon(\|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Omega_D)}^2 + \|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Gamma)}^2).
\end{aligned}$$

Choose $\vec{v}_h = \vec{e}_{h,u}^{n+1} - \eta_k \vec{e}_{h,u}^n$ in (4.15). We have the identity

$$(4.20) \quad (e_{h,p}^{n+1}, \nabla \cdot (\vec{e}_{h,u}^{n+1} - \eta_k \vec{e}_{h,u}^n))_{\Omega_S} = 0$$

and the inequalities

$$\begin{aligned}
& 2\nu(\mathbb{D}(\vec{e}_{h,u}^{n+1}), \mathbb{D}(\vec{e}_{h,u}^{n+1} - \eta_k \vec{e}_{h,u}^n))_{\Omega_S} \\
& = 2\nu(\mathbb{D}(\vec{e}_{h,u}^{n+1}), \mathbb{D}(\vec{e}_{h,u}^{n+1}))_{\Omega_S} - 2\nu\eta_k(\mathbb{D}(\vec{e}_{h,u}^{n+1}), \mathbb{D}(\vec{e}_{h,u}^n))_{\Omega_S} \\
& \geq 2\nu\|\mathbb{D}(\vec{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}^2 - 2\nu\eta_k\|\mathbb{D}(\vec{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}\|\mathbb{D}(\vec{e}_{h,u}^n)\|_{L^2(\Omega_S)} \\
& \geq 2\nu\|\mathbb{D}(\vec{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}^2 - \nu\|\mathbb{D}(\vec{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}^2 - \nu\eta_k^2\|\mathbb{D}(\vec{e}_{h,u}^n)\|_{L^2(\Omega_S)}^2 \\
(4.21) \quad & = D_\tau \left(\tau\nu\eta_k^2\|\mathbb{D}(\vec{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}^2 \right) + (1 - \eta_k^2)\nu\|\mathbb{D}(\vec{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}^2,
\end{aligned}$$

$$\begin{aligned}
& (\beta \vec{e}_{h,u}^{n+1} \cdot \tau, (\vec{e}_{h,u}^{n+1} - \eta_k \vec{e}_{h,u}^n) \cdot \tau)_\Gamma \\
& = \beta\|\vec{e}_{h,u}^{n+1} \cdot \tau\|_{L^2(\Gamma)}^2 - \beta\eta_k(\vec{e}_{h,u}^{n+1} \cdot \tau, \vec{e}_{h,u}^n \cdot \tau)_\Gamma \\
& \geq \beta\|\vec{e}_{h,u}^{n+1} \cdot \tau\|_{L^2(\Gamma)}^2 - \beta\eta_k\|\vec{e}_{h,u}^{n+1} \cdot \tau\|_{L^2(\Gamma)}\|\vec{e}_{h,u}^n \cdot \tau\|_{L^2(\Gamma)} \\
& \geq \beta\|\vec{e}_{h,u}^{n+1} \cdot \tau\|_{L^2(\Gamma)}^2 - \frac{\beta}{2}\|\vec{e}_{h,u}^{n+1} \cdot \tau\|_{L^2(\Gamma)}^2 - \frac{\eta_k^2\beta}{2}\|\vec{e}_{h,u}^n \cdot \tau\|_{L^2(\Gamma)}^2 \\
(4.22) \quad & = D_\tau \left(\frac{\tau\eta_k^2\beta}{2}\|\vec{e}_{h,u}^{n+1} \cdot \tau\|_{L^2(\Gamma)}^2 \right) + \frac{(1 - \eta_k^2)\beta}{2}\|\vec{e}_{h,u}^{n+1} \cdot \tau\|_{L^2(\Gamma)}^2,
\end{aligned}$$

$$\begin{aligned}
& (\vec{e}_{h,u}^{n+1} \cdot \vec{n}_S, (\vec{e}_{h,u}^{n+1} - \eta_k \vec{e}_{h,u}^n) \cdot \vec{n}_S)_\Gamma \\
(4.23) \quad & \geq D_\tau \left(\frac{\tau\eta_k^2}{2}\|\vec{e}_{h,u}^{n+1} \cdot \vec{n}_S\|_{L^2(\Gamma)}^2 \right) + \frac{(1 - \eta_k^2)}{2}\|\vec{e}_{h,u}^{n+1} \cdot \vec{n}_S\|_{L^2(\Gamma)}^2.
\end{aligned}$$

Using Lemma 4.1, (3.7), (3.10), and (4.20)–(4.23), we can obtain

$$\begin{aligned}
(4.24) \quad & D_\tau \left(\sum_{i,j=0}^{k-1} g_{ij}(\vec{e}_{h,u}^{n+1-i}, \vec{e}_{h,u}^{n+1-j})_{\Omega_S} + \tau\eta_k^2\nu\|\mathbb{D}(\vec{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}^2 \right) \\
& + D_\tau \left(\frac{\tau\eta_k^2}{2}\|\vec{e}_{h,u}^{n+1} \cdot \vec{n}_S\|_{L^2(\Gamma)}^2 + \frac{\tau\eta_k^2\beta}{2}\|\vec{e}_{h,u}^{n+1} \cdot \tau\|_{L^2(\Gamma)}^2 \right) \\
& + (1 - \eta_k^2)\nu\|\mathbb{D}(\vec{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}^2 + \frac{(1 - \eta_k^2)\beta}{2}\|\vec{e}_{h,u}^{n+1} \cdot \tau\|_{L^2(\Gamma)}^2 + \frac{(1 - \eta_k^2)}{2}\|\vec{e}_{h,u}^{n+1} \cdot \vec{n}_S\|_{L^2(\Gamma)}^2 \\
& \leq \|D_{\tau,k}\vec{u}_S^{n+1} - R_{h,S}(D_{\tau,k}\vec{u}_S^{n+1}, D_{\tau,k}p_S^{n+1})\|_{L^2(\Omega_S)}\|\vec{e}_{h,u}^{n+1} - \eta_k \vec{e}_{h,u}^n\|_{L^2(\Omega_S)} \\
& + \|\vec{u}_S^{n+1} - R_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1})\|_{L^2(\Gamma)}\|\vec{e}_{h,u}^{n+1} - \eta_k \vec{e}_{h,u}^n\|_{L^2(\Gamma)} \\
& + \beta\|\vec{u}_S^{n+1} - R_{h,S}(\vec{u}_S^{n+1}, p_S^{n+1})\|_{L^2(\Gamma)}\|\vec{e}_{h,u}^{n+1} - \eta_k \vec{e}_{h,u}^n\|_{L^2(\Gamma)} \\
& + \|I_{\tau,k}(\zeta_h^{n+1} - \zeta_S^{n+1})\|_{L^2(\Gamma)}\|\vec{e}_{h,u}^{n+1} - \eta_k \vec{e}_{h,u}^n\|_{L^2(\Gamma)}
\end{aligned}$$

$$\begin{aligned}
& + \|E_u^{n+1}\|_{L^2(\Omega_S)} \|\bar{e}_{h,u}^{n+1} - \eta_k \bar{e}_{h,u}^n\|_{L^2(\Omega_S)} + \|E_\zeta^{n+1}\|_{L^2(\Gamma)} \|\bar{e}_{h,u}^{n+1} - \eta_k \bar{e}_{h,u}^n\|_{L^2(\Gamma)} \\
& \leq C\epsilon^{-1}(h^{2r+2}|\ln h|^2 + \tau^{2k}) + C\epsilon^{-1}\|I_{\tau,k}(\zeta_h^{n+1} - \zeta_S^{n+1})\|_{L^2(\Gamma)}^2 \\
& \quad + \epsilon(\|\bar{e}_{h,u}^{n+1} - \eta_k \bar{e}_{h,u}^n\|_{L^2(\Omega_S)}^2 + \|\bar{e}_{h,u}^{n+1} - \eta_k \bar{e}_{h,u}^n\|_{L^2(\Gamma)}^2).
\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{E}_h^{n+1} &= \sum_{i,j=0}^{k-1} \left[g_{ij}(e_{h,\phi}^{n+1-i}, e_{h,\phi}^{n+1-j})_{\Omega_D} + g_{ij}(\bar{e}_{h,u}^{n+1-i}, \bar{e}_{h,u}^{n+1-j})_{\Omega_S} \right] \\
& \quad + \frac{\tau\eta_k^2}{2} (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D} + \tau\eta_k^2 \nu \|\mathbb{D}(\bar{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}^2 \\
(4.25) \quad & \quad + \frac{\tau\eta_k^2 g}{2} \|e_{h,\phi}^{n+1}\|_{L^2(\Gamma)}^2 + \frac{\tau\eta_k^2}{2} \|\bar{e}_{h,u}^{n+1} \cdot \bar{n}\|_{L^2(\Gamma)}^2 + \frac{\tau\eta_k^2 \beta}{2} \|\bar{e}_{h,u}^{n+1} \cdot \bar{\tau}\|_{L^2(\Gamma)}^2.
\end{aligned}$$

Since $0 < \eta_k < 1$, the sum of (4.19) and (4.24) gives that

$$\begin{aligned}
(4.26) \quad & D_\tau \mathcal{E}_h^{n+1} + \frac{1-\eta_k^2}{2} (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D} + (1-\eta_k^2) \nu \|\mathbb{D}(\bar{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}^2 \\
& \leq C\epsilon^{-1}(h^{2r+2}|\ln h|^2 + \tau^{2k}) + C\epsilon^{-1}\|I_{\tau,k}(\xi_h^{n+1} - \xi_D^{n+1})\|_{L^2(\Gamma)}^2 \\
& \quad + \|I_{\tau,k}(\zeta_h^{n+1} - \zeta_S^{n+1})\|_{L^2(\Gamma)}^2 + \epsilon(\|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Omega_S)}^2 + \|e_{h,\phi}^{n+1} - \eta_k e_{h,\phi}^n\|_{L^2(\Gamma)}^2) \\
& \quad + \epsilon(\|\bar{e}_{h,u}^{n+1} - \eta_k \bar{e}_{h,u}^n\|_{L^2(\Omega_S)}^2 + \|\bar{e}_{h,u}^{n+1} - \eta_k \bar{e}_{h,u}^n\|_{L^2(\Gamma)}^2),
\end{aligned}$$

which hold for arbitrary $0 < \epsilon < 1$.

Using (2.1), (2.9), (2.15), (3.9)–(3.10), and the trace inequality, we have

$$\begin{aligned}
& \|I_{\tau,k}(\xi_h^{n+1} - \xi_D^{n+1})\|_{L^2(\Gamma)}^2 \\
& \leq C \sum_{j=n+1-k}^n \left(\|\bar{u}_h^j - R_{h,S}(\bar{u}_S^j, p_S^j)\|_{L^2(\Gamma)}^2 + g\|\phi_h^j - R_{h,D}\phi_D^j\|_{L^2(\Gamma)}^2 \right. \\
& \quad \left. + \|R_{h,S}(\bar{u}_S^j, p_S^j) - \bar{u}_S^j\|_{L^2(\Gamma)}^2 + g\|R_{h,D}\phi_D^j - \phi_D^j\|_{L^2(\Gamma)}^2 \right) \\
& \leq C \sum_{j=n+1-k}^n \left(\|\bar{e}_{h,u}^j\|_{L^2(\Gamma)}^2 + \|\bar{e}_{h,\phi}^j\|_{L^2(\Gamma)}^2 \right) + Ch^{2r+2}|\ln h|^2 \\
& \leq C \sum_{j=n+1-k}^n \left(\epsilon_1^{-1} \|\bar{e}_{h,u}^j\|_{L^2(\Omega_S)}^2 + \epsilon_1 \|\nabla \bar{e}_{h,u}^j\|_{L^2(\Omega_S)}^2 \right) \\
(4.27) \quad & \quad + C \sum_{j=n+1-k}^n \left(\epsilon_1^{-1} \|\bar{e}_{h,\phi}^j\|_{L^2(\Omega_D)}^2 + \epsilon_1 \|\nabla \bar{e}_{h,\phi}^j\|_{L^2(\Omega_D)}^2 \right) + Ch^{2r+2}|\ln h|^2.
\end{aligned}$$

Similarly, using (2.1), (2.10), (2.16), (3.9)–(3.10), and the trace inequality, we have

$$\begin{aligned}
& \|I_{\tau,k}(\zeta_h^{n+1} - \zeta_S^{n+1})\|_{L^2(\Gamma)}^2 \\
& \leq C \sum_{j=n+1-k}^n \left(\epsilon_1^{-1} \|\bar{e}_{h,u}^j\|_{L^2(\Omega_S)}^2 + \epsilon_1 \|\nabla \bar{e}_{h,u}^j\|_{L^2(\Omega_S)}^2 \right) \\
(4.28) \quad & \quad + C \sum_{j=n+1-k}^n \left(\epsilon_1^{-1} \|\bar{e}_{h,\phi}^j\|_{L^2(\Omega_D)}^2 + \epsilon_1 \|\nabla \bar{e}_{h,\phi}^j\|_{L^2(\Omega_D)}^2 \right) + Ch^{2r+2}|\ln h|^2.
\end{aligned}$$

Summing up (4.26) for $n = k - 1, \dots, m$ and using the above two inequalities as well as the trace inequality, we obtain that (with $\epsilon_1 = \epsilon^2$)

$$\begin{aligned}
& \mathcal{E}_h^{m+1} + \sum_{n=k-1}^m \tau \left[\frac{1-\eta_k^2}{2} (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D} + (1-\eta_k^2) \nu \|\mathbb{D}(\bar{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}^2 \right] \\
& \leq \mathcal{E}_h^{k-1} + C\epsilon^{-1} (h^{2r+2} |\ln h|^2 + \tau^{2k}) \\
& \quad + C\epsilon^{-1} \sum_{n=0}^{k-1} \tau \left(\epsilon_1^{-1} \|\bar{e}_{h,u}^n\|_{L^2(\Omega_S)}^2 + \epsilon_1 \|\nabla \bar{e}_{h,u}^n\|_{L^2(\Omega_S)}^2 \right. \\
& \quad \quad \left. + \epsilon_1^{-1} \|\bar{e}_{h,\phi}^n\|_{L^2(\Omega_D)}^2 + \epsilon_1 \|\nabla \bar{e}_{h,\phi}^n\|_{L^2(\Omega_D)}^2 \right) \\
& \quad + C\epsilon^{-1} \sum_{n=k}^m \tau \left(\epsilon_1^{-1} \|\bar{e}_{h,u}^n\|_{L^2(\Omega_S)}^2 + \epsilon_1 \|\nabla \bar{e}_{h,u}^n\|_{L^2(\Omega_S)}^2 \right. \\
& \quad \quad \left. + \epsilon_1^{-1} \|\bar{e}_{h,\phi}^n\|_{L^2(\Omega_D)}^2 + \epsilon_1 \|\nabla \bar{e}_{h,\phi}^n\|_{L^2(\Omega_D)}^2 \right) \\
& \quad + C\epsilon \sum_{n=k-1}^m \tau \left(\|e_{h,\phi}^n\|_{L^2(\Omega_D)}^2 + \|\nabla e_{h,\phi}^n\|_{L^2(\Omega_D)}^2 \right) \\
& \quad + C\epsilon \sum_{n=k-1}^m \tau \left(\|\bar{e}_{h,u}^n\|_{L^2(\Omega_S)}^2 + \|\nabla \bar{e}_{h,u}^n\|_{L^2(\Omega_S)}^2 \right) \\
& \quad + C\epsilon \sum_{n=k-1}^m \tau \left(\|e_{h,\phi}^{n+1}\|_{L^2(\Omega_D)}^2 + \|\nabla e_{h,\phi}^{n+1}\|_{L^2(\Omega_D)}^2 \right) \\
& \quad + C\epsilon \sum_{n=k-1}^m \tau \left(\|\bar{e}_{h,u}^{n+1}\|_{L^2(\Omega_S)}^2 + \|\nabla \bar{e}_{h,u}^{n+1}\|_{L^2(\Omega_S)}^2 \right) \\
& \leq \left[C\epsilon^{-3} |\mathcal{I}_h^k|^2 + C\epsilon^{-1} (h^{2r+2} |\ln h|^2 + \tau^{2k}) \right] \\
& \quad + C\epsilon^{-3} \sum_{n=k}^m \tau \left(\|e_{h,\phi}^n\|_{L^2(\Omega_D)}^2 + \|\bar{e}_{h,u}^n\|_{L^2(\Omega_S)}^2 \right) \\
& \quad + C\epsilon \sum_{n=k-1}^m \tau \left(\|e_{h,\phi}^{n+1}\|_{L^2(\Omega_D)}^2 + \|\bar{e}_{h,u}^{n+1}\|_{L^2(\Omega_S)}^2 \right) \\
(4.29) \quad & + C\epsilon \sum_{n=k-1}^m \tau \left(\|\nabla e_{h,\phi}^{n+1}\|_{L^2(\Omega_D)}^2 + \|\nabla \bar{e}_{h,u}^{n+1}\|_{L^2(\Omega_S)}^2 \right).
\end{aligned}$$

By choosing a small ϵ , the last term on the right-hand side of (4.29) can be eliminated by the left-hand side, and (4.29) reduces to

$$\begin{aligned}
(4.30) \quad & \mathcal{E}_h^{m+1} + \sum_{n=k-1}^m \tau \left[\frac{1-\eta_k^2}{4} (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D} + \frac{(1-\eta_k^2)\nu}{2} \|\mathbb{D}(\bar{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}^2 \right] \\
& \leq C\epsilon^{-3} |\mathcal{I}_h^k|^2 + C\epsilon^{-1} (h^{2r+2} |\ln h|^2 + \tau^{2k}) + C\epsilon^{-3} \sum_{n=k}^m \tau \left(\|e_{h,\phi}^n\|_{L^2(\Omega_D)}^2 + \|\bar{e}_{h,u}^n\|_{L^2(\Omega_S)}^2 \right) \\
& \quad + C\epsilon \tau \left(\|e_{h,\phi}^{m+1}\|_{L^2(\Omega_D)}^2 + \|\bar{e}_{h,u}^{m+1}\|_{L^2(\Omega_S)}^2 \right) \\
& \leq C\epsilon^{-3} |\mathcal{I}_h^k|^2 + C\epsilon^{-1} (h^{2r+2} |\ln h|^2 + \tau^{2k}) + C\epsilon^{-3} \sum_{n=k}^m \tau \mathcal{E}_h^n + C\epsilon \tau \mathcal{E}_h^{m+1}.
\end{aligned}$$

By choosing ϵ small enough and applying the discrete Gronwall inequality, we obtain

$$(4.31) \quad \mathcal{E}_h^{m+1} + \sum_{n=k-1}^m \tau \left[\frac{1-\eta_k^2}{4} (\mathbb{K} \nabla e_{h,\phi}^{n+1}, \nabla e_{h,\phi}^{n+1})_{\Omega_D} + \frac{(1-\eta_k^2)\nu}{2} \|\mathbb{D}(\bar{e}_{h,u}^{n+1})\|_{L^2(\Omega_S)}^2 \right] \\ \leq C |\mathcal{I}_h^k|^2 + C(h^{2r+2} |\ln h|^2 + \tau^{2k}).$$

The proof of Theorem 2.1 is complete.

5. Conclusion. In this paper, the Stokes–Darcy system is decoupled by a noniterative, multiphysics domain decomposition method [15] with multistep BDFs for the time discretization, which allows us to introduce Ritz projections for the Stokes and Darcy equations, respectively, without involving interface terms. We have proved the corresponding L^∞ error estimates for the Stokes–Ritz projection under the stress boundary conditions in order to control the Ritz-projection error on the interface. By using these theoretical results and the multiplier technique of multistep BDFs, we have proved an almost optimal-order L^2 -norm convergence rate of the proposed method. As far as we know, both the L^∞ error estimate of the Stokes–Ritz projection (under the stress boundary condition) and the analysis of multistep BDFs with finite element spatial discretization for the Stokes–Darcy model are new contributions.

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